An explicit example of a maximal 3-cyclically monotone operator with bizarre properties

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Abstract

Subdifferential operators of proper convex lower semicontinuous functions and, more generally, maximal monotone operators are ubiquitous in optimization and nonsmooth analysis. In between these two classes of operators are the maximal n-cyclically monotone operators. These operators were carefully studied by Asplund, who obtained a complete characterization within the class of positive semidefinite (not necessarily symmetric) matrices, and by Voisei, who presented extension theorems à la Minty.

All previous explicit examples of maximal n-cyclically monotone operators are maximal monotone; thus, they inherit the known good properties of maximal monotone operators. In this paper, we construct an explicit maximal 3-cyclically monotone operator with quite bizarre properties. This construction builds upon a recent, nonconstructive and Zorn's Lemma-based, example. Our operator possesses two striking properties that sets it far apart from both the maximal monotone operator and the subdifferential operator case: it is not maximal monotone and its domain, which is closed, fails to be convex. Indeed, the domain is the boundary of the unit diamond in the Euclidean plane. The path leading to this operator requires some new results that are interesting in their own right.

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1 Introduction

Throughout this paper, we assume that

$$X \text{ is a real Banach space with continuous dual } X^*\text{, dual pairing } \langle\cdot,\cdot\rangle\text{, and norm } \|\cdot\|. \tag{1}$$

Let A be a *set-valued operator* from X to X^*, i.e., $(\forall x \in X)\ Ax \subseteq X^*$ so that A is a mapping from X to the power set of X^*. We use the notation $A\colon X \rightrightarrows X^*$ and remark that A can be identified with its *graph*

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$\operatorname{gra} A := \big\{(x,x^*) \in X \times X^* \mid x^* \in Ax\big\}$. Now let $n \in \{2,3,\ldots\}$. Then A is *n-cyclically monotone* if

$$\left.\begin{array}{c}(a_1,a_1^*) \in \operatorname{gra} A,\\ \vdots\\ (a_n,a_n^*) \in \operatorname{gra} A\\ a_{n+1} := a_1\end{array}\right\} \quad \Rightarrow \quad \sum_{i=1}^{n} \langle a_{i+1} - a_i, a_i^* \rangle \leq 0. \tag{2}$$

We note that 2-monotonicity simplifies to

$$\big(\forall (x,x^*) \in \operatorname{gra} A\big)\big(\forall (y,y^*) \in \operatorname{gra} A\big) \quad \langle x-y, x^*-y^* \rangle \geq 0, \tag{3}$$

i.e., to ordinary *monotonicity*. *Cyclic monotonicity* describes the situation when A is m-cyclically monotone for every $m \in \{2,3,\ldots\}$. The operator A is *maximal n-cyclically monotone* if A is n-cyclically monotone and no proper extension (in the sense of inclusion of graphs) of A is n-cyclically monotone. Zorn's Lemma guarantees that every n-cyclically monotone operator admits a maximal n-cyclically monotone extension. At the one end of the spectrum of maximal n-cyclically monotone operators are the maximal 2-monotone, i.e., the *maximal monotone* operators. At the other end are the maximal cyclically monotone operators, which Rockafellar revealed to be the subdifferential operators of functions that are convex, lower semicontinuous, and proper [8]. Subdifferential and maximal monotone operators play a central role in optimization and nonsmooth analysis; see, e.g., [5, 9, 10, 11, 13].

Even though maximal monotone operators may fail to be subdifferential operators (consider, e.g., a nonzero skew linear operator), they often have properties similar to subdifferential operators. As an example, if X is a "nice" space (say reflexive) and A is maximal monotone, then the closure of the *domain* $\operatorname{dom} A := \big\{x \in X \mid Ax \neq \varnothing\big\}$ of A, similarly for *range* of A denoted by $\operatorname{ran} A := A(X) = \bigcup_{x\in X} Ax$, is convex; see, e.g., [6, 7, 11].

To test such properties for maximal n-cyclic monotonicity, concrete examples are needed. Although n-cyclic monotone operators were analyzed by Asplund [1] and by Voisei [12], the only concrete examples are matrices. (See also [3] for the special case of rotators in the Euclidean plane.) Since matrices are continuous operators with full domain, they are maximal monotone and thus have good properties. The existence of a maximal 3-cyclic monotone operator that is *not* maximal monotone was established recently in [2]. However, this operator was not explicitly known — the proof was based on Zorn's Lemma.

The goal of this note is to present an explicit *maximal 3-cyclically monotone operator that is* not *maximal monotone. This operator has another bizarre property that is in striking contrast to the maximal monotone operator and the subdifferential operator case: its domain is closed but* not *convex. In fact, convexity fails in a spectacular fashion — the operator's domain is the* boundary *of the unit diamond.*

The remainder of the paper is organized as follows. In Section 2, we present several new results on n-cyclically monotone operators that will aid us in the construction of the announced operator. Some of these results are quite interesting in their own right: Theorem 2.10 relates the recession cone of images to the normal cone of the closed convex hull of the domain, Theorem 2.12 is a characterization of maximal monotonicity within the class of maximal n-cyclically monotone operator that underlines the importance of convexity, and Theorem 2.14 gives precise information on the effect of adding a normal cone operator. The final Section 3 contains the actual construction of the operator. This is done in several steps, culminating in Theorem 3.10.

Our notation is standard; see, e.g., [13]. For a nonempty subset S of X, we use $\operatorname{conv} S$, $\overline{\operatorname{conv}}\, S$, $\operatorname{rec} S$, $\operatorname{int} S$, and $\operatorname{ri} S$ to denote its convex hull, its closed convex hull, its recession cone, its interior, and its relative interior, respectively. For a nonempty convex closed subset C of X and a point $x \in C$, the tangent and the normal cone of C at x are denoted by $T_C(x)$ and by $N_C(x)$, respectively. We write $\mathbb{R}_+ = \big\{\rho \in \mathbb{R} \mid \rho \geq 0\big\}$. Finally, the closed and relatively open line segments between two distinct vectors x and y in X are $[x, y] := \big\{(1-\lambda)x + \lambda y \mid 0 \leq \lambda \leq 1\big\}$ and $]x, y[:= \big\{(1-\lambda)x + \lambda y \mid 0 < \lambda < 1\big\}$, respectively.

2 Auxiliary results

Recognizing and constructing n-cyclically monotone operators

Let $A\colon X \rightrightarrows X^*$ be such that $\operatorname{gra} A$ contains only finitely many points. Given $n \in \{2, 3 \ldots\}$, it is conceptually simple to decide whether A is n-cyclically monotone: one has to check whether each of finitely many cyclic sums (as in (2)) is negative or zero. Our first result shows that if we are interested in all orders of cyclic monotonicity, then we only need to perform finitely many computations.

Theorem 2.1 *Let $A\colon X \rightrightarrows X^*$ and let $n \in \{2, 3, \ldots\}$. Suppose that A is n-cyclically monotone and that $\operatorname{gra} A$ contains exactly n points. Then A is cyclically monotone.*

Proof. Using strong induction on m, we show that A is m-cyclically monotone for every $m \geq n+1$. Suppose that A is $(m-1)$-cyclically monotone for some $m \geq n+1$ and take

$$\big\{(b_1, b_1^*), \ldots, (b_m, b_m^*)\big\} \subseteq \operatorname{gra} A. \tag{4}$$

We must show that

$$\sigma := \sum_{i=1}^{m} \langle b_{i+1} - b_i, b_i^* \rangle \leq 0, \quad \text{where } b_{m+1} := b_1. \tag{5}$$

Since $m \geq n+1$ but $\operatorname{gra} A$ contains only n points, there exist integers k and l such that

$$b_k = b_l \quad \text{and} \quad 1 \leq k < l \leq m. \tag{6}$$

We split σ into $\sigma_1 + \sigma_2$, where

$$\sigma_1 := \sum_{i=k}^{l-1} \langle b_{i+1} - b_i, b_i^* \rangle \quad \text{and} \quad \sigma_2 := \sum_{i=l}^{m} \langle b_{i+1} - b_i, b_i^* \rangle + \sum_{i=1}^{k-1} \langle b_{i+1} - b_i, b_i^* \rangle \tag{7}$$

are two cyclic sums, each of which contains at least one term and hence at most $(m-1)$ terms. Since A is $(m-1)$-cyclically monotone, we see that $\sigma_1 \leq 0$ and that $\sigma_2 \leq 0$. Therefore, $\sigma = \sigma_1 + \sigma_2 \leq 0$. ■

Remark 2.2 Rockafellar [8] (see also [13, Proposition 2.4.3]) showed that for every cyclically monotone operator A, there exists a convex lower semicontinuous and proper function $f\colon X \to \,]-\infty, +\infty]$ such that $\operatorname{gra} A \subseteq \operatorname{gra} \partial f$. In fact, if A is n-cyclically monotone and $\operatorname{gra} A$ contains exactly n points, then A is cyclically monotone (by Theorem 2.1). Rockafellar's proof, together with an argument similar to the proof of Theorem 2.1, shows that there exists a continuous polyhedral function f with $\operatorname{gra} A \subset \operatorname{gra} \partial f$.

Remark 2.3 To construct a maximal n-cyclically monotone operator that is not maximal monotone, it is natural to start with an operator A such that A is n-cyclically monotone, but A is not $(n+1)$-cyclically monotone, and $\operatorname{gra} A$ contains $n+1$ points. (Starting with n or fewer points is not advisable because one might end up with a cyclically monotone extension; see Theorem 2.1.) Indeed, this clue helped us construct the bizarre monotone operator in Section 3; see Remark 3.2.

We now provide some basic results concerning the construction and characterization of n-cyclically monotone operators. The proofs of the first two results are omitted since they are straightforward.

Proposition 2.4 *Let $A\colon X \rightrightarrows X^*$ and let $B\colon X \rightrightarrows X^*$ be n-cyclically monotone operators, for some $n \in \{2,3,\ldots\}$. Then $A+B$ is n-cyclically monotone.*

Proposition 2.5 *Let $A\colon X \rightrightarrows X^*$, let $n \in \{2,3,\ldots\}$, and let $B\colon X \rightrightarrows X^*$ be such that its graph is the norm×weak* closure of $\operatorname{gra} A$. Then A is n-cyclically monotone $\Leftrightarrow$ B is n-cyclically monotone.*

The next result is part of the folklore.

Proposition 2.6 *Let $n \in \{2,3,\ldots\}$ and let $S_1 \times \cdots \times S_n \subseteq Y^n$, where Y is a real vector space. Then $\operatorname{conv}(S_1 \times \cdots \times S_n) = (\operatorname{conv} S_1) \times \cdots \times (\operatorname{conv} S_n)$.*

Corollary 2.7 *Let $n \in \{2,3,\ldots\}$, let $S_1,\ldots,S_n$ be subsets of X^*, and let, for every $i \in \{1,\ldots,n\}$, $y_i \in \operatorname{conv}(S_i)$. Then there exist finitely many reals $\lambda_1,\ldots,\lambda_m$ in $[0,1]$ such that $\lambda_1+\cdots+\lambda_m = 1$ and there exists, for every $i \in \{1,\ldots,n\}$, subsets $\{s_{i,k}\}_{1\le k\le m}$ of S_i such that $y_i = \lambda_1 s_{i,1} + \cdots + \lambda_m s_{i,m}$.*

Corollary 2.7 has the following useful consequence.

Proposition 2.8 *Let $A\colon X \rightrightarrows X^*$, let $n \in \{2,3,\ldots\}$, and set $B\colon X \rightrightarrows X^*\colon x \mapsto \operatorname{conv}(Ax)$. Then A is n-cyclically monotone $\Leftrightarrow$ B is n-cyclically monotone.*

Proof. "$\Leftarrow$": Clear. "$\Rightarrow$": Assume that A is n-cyclically monotone, and take $(x_1,y_1^*),\ldots,(x_n,y_n^*)$ in $\operatorname{gra} B$. By Corollary 2.7, there exist convex coefficients $\lambda_1,\ldots,\lambda_m$ and sets $\{x_{i,k}^*\}_{1\le k\le m}$ in Ax_i such that

$$\big(\forall i \in \{1,\ldots,n\}\big) \quad y_i^* = \sum_{k=1}^{m} \lambda_k x_{i,k}^*. \tag{8}$$

Set $x_{n+1} := x_1$. Because A is n-cyclically monotone, we have $(\forall k \in \{1,\ldots,m\})$ $0 \ge \sum_{i=1}^n \langle x_{i+1}-x_i, x_{i,k}^*\rangle$. Multiply the k^{th} inequality by λ_k, and add all resulting inequalities to deduce that $0 \ge \sum_k \lambda_k \sum_i \langle x_{i+1}-x_i, x_{i,k}^*\rangle = \sum_i \langle x_{i+1}-x_i, \sum_k \lambda_k x_{i,k}^*\rangle = \sum_i \langle x_{i+1}-x_i, y_i^*\rangle$. The other direction is obvious. ■

Properties of maximal n-cyclically monotone operators

We first recall the following result which deals with the extensibility of n-cyclic monotone operators.

Fact 2.9 [2, Proposition 2.7] *Let $A\colon X \rightrightarrows X^*$ be n-cyclically monotone for some $n \in \{2,3,\ldots\}$, let $(x,x^*) \in X \times X^*$, and define $B\colon X \rightrightarrows X^*$ via $\operatorname{gra} B := \operatorname{gra} A \cup \{(x,x^*)\}$. Then (x,x^*) is n-cyclically monotonically related to $\operatorname{gra} A$, i.e., $\operatorname{gra} B$ is n-cyclically monotone $\Leftrightarrow$*

$$\sup_{\substack{(a_1,a_1^*)\in \operatorname{gra} A \\ \vdots \\ (a_{n-1},a_{n-1}^*)\in \operatorname{gra} A}} \left(\sum_{i=1}^{n-2} \langle a_{i+1}-a_i, a_i^*\rangle\right) + \langle x-a_{n-1}, a_{n-1}^*\rangle + \langle a_1 - x, x^*\rangle \le 0. \tag{9}$$

Theorem 2.10 *Let $A\colon X \rightrightarrows X^*$ be maximal n-cyclically monotone for some $n \in \{2, 3, \ldots\}$, let $x \in \operatorname{dom} A$, and set $C := \overline{\operatorname{conv}} \operatorname{dom} A$. Then $\operatorname{rec}(Ax) = N_C(x)$.*

Proof. Take $x^* \in Ax$, and $(a_1, a_1^*), \ldots, (a_{n-1}, a_{n-1}^*)$ in $\operatorname{gra} A$.

"$N_C(x) \subseteq \operatorname{rec}(Ax)$": Take $y^* \in N_C(x)$. Since A is n-cyclically monotone, we have

$$\sum_{i=1}^{n-2} \langle a_{i+1} - a_i, a_i^* \rangle + \langle x - a_{n-1}, a_{n-1}^* \rangle + \langle a_1 - x, x^* \rangle \le 0. \tag{10}$$

Now $a_1 \in \operatorname{dom} A \subseteq C$, thus

$$\langle a_1 - x, y^* \rangle \le 0. \tag{11}$$

Adding (10) and (11), we obtain

$$\sum_{i=1}^{n-2} \langle a_{i+1} - a_i, a_i^* \rangle + \langle x - a_{n-1}, a_{n-1}^* \rangle + \langle a_1 - x, x^* + y^* \rangle \le 0. \tag{12}$$

By Fact 2.9, $(x, x^*{+}y^*)$ is n-cyclically monotonically related to $\operatorname{gra} A$. As A is maximal n-cyclically monotone, we deduce that $x^* + y^* \in Ax$. Therefore, $Ax + y^* \subseteq Ax$ and hence $y^* \in \operatorname{rec}(Ax)$.

"$\operatorname{rec}(Ax) \subset N_C(x)$": Let $z^* \in \operatorname{rec}(Ax)$. Then $(\forall \rho > 0)\ x^* + \rho z^* \in Ax$ and hence

$$\sum_{i=1}^{n-2} \langle a_{i+1} - a_i, a_i^* \rangle + \langle x - a_{n-1}, a_{n-1}^* \rangle + \langle a_1 - x, x^* + \rho z^* \rangle \le 0, \tag{13}$$

which, after dividing by ρ and letting $\rho \to +\infty$, yields $\langle a_1 - x, z^* \rangle \le 0$. Since a_1 is an arbitrary point in $\operatorname{dom} A$, it follows that $z^* \in N_C(x)$. ■

Corollary 2.11 *Suppose that X is finite-dimensional and let $A\colon X \rightrightarrows X^*$ be maximal n-monotone for some $n \in \{2, 3, \ldots\}$. Then either $\operatorname{dom} A$ is unbounded or $\operatorname{ran} A$ is unbounded. Consequently, $\operatorname{gra} A$ contains infinitely many points.*

Proof. Assume to the contrary that both $\operatorname{dom} A$ and $\operatorname{ran} A$ are bounded. We claim that

$$\operatorname{dom} A \text{ is closed.} \tag{14}$$

Indeed, take a sequence $(a_n)_{n\in\mathbb{N}}$ in $\operatorname{dom} A$ such that $a_n \to a \in X$. Suppose $(\forall n \in \mathbb{N})\ a_n^* \in Aa_n$. Since $(a_n^*)_{n\in\mathbb{N}}$ is bounded and X^* is finite-dimensional, we assume that $(a_n^*)_{n\in\mathbb{N}}$ converges to $a^* \in X^*$. The n-cyclic maximality of A and Proposition 2.5 imply that $\operatorname{gra} A$ is closed. Thus $(a, a^*) \in \operatorname{gra} A$ and hence $a \in \operatorname{dom} A$, which verifies (14). Since $\operatorname{dom} A$ is bounded and closed, [9, Theorem 17.2] implies that $C := \overline{\operatorname{conv}} \operatorname{dom} A = \operatorname{conv} \operatorname{dom} A$. Next, [9, Corollary 18.3.1 and Corollary 18.5.3] yield an extreme point x of C such that $x \in \operatorname{dom} A$. Clearly, $x \in \operatorname{bdry} C$ and thus, by [9, Theorem 23.4], $N_C(x)$ is unbounded. In view of Theorem 2.10, $\operatorname{rec}(Ax)$ is unbounded and so is Ax. This is absurd since we assumed that $\operatorname{dom} A$ and $\operatorname{ran} A$ are bounded. ■

The following result, which is of use in Section 3, underlines the importance of convexity.

Theorem 2.12 *Suppose that X is finite-dimensional and let $A\colon X \rightrightarrows X^*$ be maximal n-cyclically monotone for some $n \in \{2, 3, \ldots\}$. Then A is maximal monotone $\Leftrightarrow \operatorname{ri} \operatorname{conv} \operatorname{dom} A \subseteq \operatorname{dom} A$.*

Proof. "$\Rightarrow$": This implication is due to Minty [6]. "$\Leftarrow$": Combine [4, Proposition 2.2(iii) and Corollary 2.8] with [2, Corollary 2.15]. ■

Extension of n-cyclically monotone operators

Proposition 2.13 *Let $A\colon X \rightrightarrows X^*$, let $n \in \{2,3,\ldots\}$, let $(x,x^*) \in X \times X^*$, and set $B\colon X \rightrightarrows X^*\colon x \mapsto \operatorname{conv}(Ax)$. Then (x,x^*) is n-cyclically monotonically related to $\operatorname{gra} A \Leftrightarrow (x,x^*)$ is n-cyclically monotonically related to $\operatorname{gra} B$.*

Proof. "$\Leftarrow$": is clear. "$\Rightarrow$": Use (9) and argue analogously to the proof of Proposition 2.8. ■

We now demonstrate that adding a normal cone operator may be useful in order to restrict the domain of n-cyclically monotone extensions.

Theorem 2.14 *Let $A\colon X \rightrightarrows X^*$, let $n \in \{2,3,\ldots\}$, let $(x,x^*) \in X \times X^*$, and set $C := \overline{\operatorname{conv}} \operatorname{dom} A$. Then (x,x^*) is n-cyclically monotonically related to $\operatorname{gra}(A+N_C)$ if and only if*

$$(x,x^*) \text{ is } n\text{-cyclically monotonically related to } \operatorname{gra} A \text{ and} \tag{15}$$

$$x \in \bigcap_{a \in \operatorname{dom} A} \big(a + T_C(a)\big). \tag{16}$$

Proof. Take $(a_1,a_1^*),\ldots,(a_{n-1},a_{n-1}^*)$ in $\operatorname{gra} A$.

"$\Rightarrow$": By Fact 2.9, we have $\forall (a_i,a_i^*) \in \operatorname{gra} A$ with $i = 1,\ldots,n-1$,

$$\sup\left(\sum_{i=1}^{n-2}\langle a_{i+1}-a_i, a_i^* + N_C(a_i)\rangle + \langle x - a_{n-1}, a_{n-1}^* + N_C(a_{n-1})\rangle + \langle a_1 - x, x^*\rangle\right) \leq 0. \tag{17}$$

Since 0 belongs to each normal cone $N_C(a_i)$, we obtain

$$\sum_{i=1}^{n-2}\langle a_{i+1}-a_i, a_i^*\rangle + \langle x - a_{n-1}, a_{n-1}^*\rangle + \langle a_1 - x, x^*\rangle \leq 0. \tag{18}$$

Using again Fact 2.9, we see that this implies (15). Now assume that $a_1 = \cdots = a_{n-1}$. Then (17) becomes $\sup\langle x - a_1, a_1^* + N_C(a_1) - x^*\rangle \leq 0$; equivalently,

$$\sup\langle x - a_1, N_C(a_1)\rangle \leq \langle x - a_1, x^* - a_1^*\rangle. \tag{19}$$

Since $N_C(a_1)$ is a cone, we must have $\sup\langle x - a_1, N_C(a_1)\rangle \leq 0$. Therefore, $x - a_1 \in T_C(a_1)$ and hence $x \in a_1 + T_C(a_1)$, which verifies (16).

"$\Leftarrow$": By (15) and Fact 2.9, $\forall (a_i,a_i^*) \in \operatorname{gra} A$ with $i = 1,\cdots,n-1$ we have

$$\sum_{i=1}^{n-2}\langle a_{i+1}-a_i, a_i^*\rangle + \langle x - a_{n-1}, a_{n-1}^*\rangle + \langle a_1 - x, x^*\rangle \leq 0. \tag{20}$$

Take, for every $i \in \{1,\ldots,n-1\}$, $y_i^* \in N_C(a_i)$. Then

$$\sum_{i=1}^{n-2}\langle a_{i+1}-a_i, y_i^*\rangle \leq 0. \tag{21}$$

By (16), $x - a_{n-1} \in T_C(a_{n-1})$ and thus

$$\langle x - a_{n-1}, y^*_{n-1}\rangle \leq 0. \tag{22}$$

Adding (20)–(22), we obtain

$$\sum_{i=1}^{n-2}\langle a_{i+1} - a_i, a^*_i + y^*_i\rangle + \langle x - a_{n-1}, a^*_{n-1} + y^*_{n-1}\rangle + \langle a_1 - x, x^*\rangle \leq 0. \tag{23}$$

Then (x, x^*) is n-cyclically monotonically related to $\operatorname{gra}(A + N_C)$ by Fact 2.9. ■

We now turn to the condition (16). The following general result will be used in Section 3.

Theorem 2.15 *Suppose that X is a Hilbert space and let S be a nonempty subset of X such that $C := \operatorname{conv} S$ is closed. Then*

$$\bigcap_{s\in S}\big(s + T_C(s)\big) = C. \tag{24}$$

Proof. For every $s \in S$, we have $C - s \subseteq \overline{\operatorname{cone}}(C - s) = T_C(s)$ and hence $C \subseteq s + T_C(s)$. Thus

$$C \subseteq \bigcap_{s\in S}\big(s + T_C(s)\big). \tag{25}$$

Now take $x \in X \smallsetminus C$. To conclude the proof, it suffice to find a point $s \in S$ such that

$$x \notin s + T_C(s). \tag{26}$$

Denote by $P_C x$ the projection of x onto the nonempty closed convex set C. Then $x \neq P_C x$ and the halfspace

$$H := \big\{y \in X \mid \langle y - P_C x, x - P_C x\rangle \leq 0\big\} \tag{27}$$

contains C. We claim that

$$S \cap \operatorname{bdry} H \neq \varnothing. \tag{28}$$

Otherwise, $S \subseteq \operatorname{int} H \Rightarrow C = \operatorname{conv} S \subseteq \operatorname{int} H \Rightarrow P_C x \in \operatorname{int} H$, which is absurd since $P_C x \in \operatorname{bdry} H$. Thus (28) is true, and we take $s \in S \cap \operatorname{bdry} H$. Then

$$s \in S \quad \text{and} \quad \langle s - P_C x, x - P_C x\rangle = 0. \tag{29}$$

and hence $(\forall c \in C)\ \langle c - s, x - P_C x\rangle = \langle c - P_C x, x - P_C x\rangle + \langle P_C x - s, x - P_C x\rangle \leq 0$. Thus

$$C - s \subseteq K := \big\{y \in X \mid \langle y, x - P_C x\rangle \leq 0\big\}. \tag{30}$$

Since K is a closed convex cone, we deduce that $T_C(s) = \overline{\operatorname{cone}}(C - s) \subseteq K$. Thus

$$s + T_C(s) \subseteq s + K. \tag{31}$$

If $x \in s + K$, then $x - s \in K$ and hence (using once more (29)) we would obtain the absurdity

$$0 < \|x - P_C x\|^2 = \langle x - P_C x, x - P_C x\rangle + \langle P_C x - s, x - P_C x\rangle = \langle x - s, x - P_C x\rangle \leq 0. \tag{32}$$

Thus $x \notin s + K$. By (31), $x \notin s + T_C(s)$ and therefore (26) holds, which completes the proof. ■

Corollary 2.16 *Suppose that X is finite-dimensional, let S be a nonempty bounded closed subset of X, and set $C := \overline{\operatorname{conv}}\, S$. Then*

$$\bigcap_{s\in S} \big(s + T_C(s)\big) = C. \tag{33}$$

Proof. By [9, Theorem 17.2], $\operatorname{conv} S$ is closed. The result thus follows from Theorem 2.15. ■

Remark 2.17 Corollary 2.16 fails if the assumption that S be bounded or the assumption that S be closed is omitted: (i) If $S := \big\{(x,y) \in \mathbb{R}^2 \mid |x|y = 1\big\}$, then S is nonempty, unbounded, and closed, and the set $C := \overline{\operatorname{conv}}\, S$ is equal to the closed upper halfplane; therefore $(\forall s \in S)\ T_C(s) = \mathbb{R}^2$ and (33) fails. (ii) If $S := \big\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\big\}$, then S is nonempty, bounded, but not closed, and (33) fails again.

3 The bizarre example

Introducing A and its graph closure B

From now on, $X = \mathbb{R}^2$. We shall construct a maximal 3-cyclically monotone operator $\mathbf{M}$ with remarkable properties. This will require several steps. Let us denote the *full unit diamond* by D, i.e.,

$$D := \big\{(x,y) \in \mathbb{R}^2 \mid |x| + |y| \le 1\big\}. \tag{34}$$

The extreme points of D are

$$\mathbf{b}_1 := (1,0),\ \ \mathbf{b}_2 := (0,1),\ \ \mathbf{b}_3 := (-1,0),\ \text{ and }\ \mathbf{b}_4 := (0,-1). \tag{35}$$

Now define

$$\begin{aligned}
&\mathbf{a}_1 \colon [0,1] \to [\mathbf{b}_1,\mathbf{b}_2] \colon t \mapsto (1-t,t), && \mathbf{a}_1^* \colon [0,1] \to X \colon t \mapsto (-t,1-t),\\
&\mathbf{a}_2 \colon [0,1] \to [\mathbf{b}_2,\mathbf{b}_3] \colon t \mapsto (-t,1-t), && \mathbf{a}_2^* \colon [0,1] \to X \colon t \mapsto (-1,0),\\
&\mathbf{a}_3 \colon [0,1] \to [\mathbf{b}_3,\mathbf{b}_4] \colon t \mapsto (t-1,-t), && \mathbf{a}_3^* \colon [0,1] \to X \colon t \mapsto (t-1,t-2),\\
&\mathbf{a}_4 \colon [0,1] \to [\mathbf{b}_4,\mathbf{b}_1] \colon t \mapsto (t,t-1), && \mathbf{a}_4^* \colon [0,1] \to X \colon t \mapsto (0,-1).
\end{aligned}$$

Since $(\forall t \in [0,1])\ \langle \mathbf{a}_1(t), \mathbf{a}_1^*(t)\rangle = 0$, $\langle \mathbf{a}_2(t), \mathbf{a}_2^*(t)\rangle = t$, $\langle \mathbf{a}_3(t), \mathbf{a}_3^*(t)\rangle = 1$, and $\langle \mathbf{a}_4(t), \mathbf{a}_4^*(t)\rangle = 1-t$, we obtain

$$(\forall i \in \{1,2,3,4\})(\forall j \in \{1,2,3,4\})(\forall s \in [0,1]) \quad [0,1] \to \mathbb{R}^3 \colon t \mapsto \begin{pmatrix} \langle \mathbf{a}_i(t), \mathbf{a}_i^*(t)\rangle \\ \langle \mathbf{a}_i(s), \mathbf{a}_j^*(t)\rangle \\ \langle \mathbf{a}_i(t), \mathbf{a}_j^*(s)\rangle \end{pmatrix} \text{ is affine.} \tag{36}$$

Define $\mathbf{A} \colon \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ via

$$\operatorname{gra} \mathbf{A} := \bigcup_{i\in\{1,2,3,4\},\, t\in]0,1[} \big\{\big(\mathbf{a}_i(t),\ \mathbf{a}_i^*(t)\big)\big\}. \tag{37}$$

Note that the domain of $\mathbf{A}$ consists of the four relatively open line segments in the boundary of the unit diamond D, i.e.,

$$\operatorname{dom} \mathbf{A} =]\mathbf{b}_1,\mathbf{b}_2[\,\cup\,]\mathbf{b}_2,\mathbf{b}_3[\,\cup\,]\mathbf{b}_3,\mathbf{b}_4[\,\cup\,]\mathbf{b}_4,\mathbf{b}_1[\,. \tag{38}$$

Let $\mathbf{B}$ be the graph closure of $\mathbf{A}$, i.e., $\mathbf{B}$ is defined by

$$\operatorname{gra} \mathbf{B} := \overline{\operatorname{gra} \mathbf{A}}. \tag{39}$$

Since

$$(\forall i \in \{1,2,3,4\}) \quad [0,1] \to \mathbb{R}^4\colon t \mapsto (\mathbf{a}_i(t), \mathbf{a}_i^*(t)) \text{ is continuous,} \tag{40}$$

it is clear that

$$\operatorname{gra} \mathbf{B} = \bigcup_{i \in \{1,2,3,4\}, t \in [0,1]} \big\{\big(\mathbf{a}_i(t),\, \mathbf{a}_i^*(t)\big)\big\}. \tag{41}$$

Hence

$$\operatorname{dom} \mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2] \cup [\mathbf{b}_2, \mathbf{b}_3] \cup [\mathbf{b}_3, \mathbf{b}_4] \cup [\mathbf{b}_4, \mathbf{b}_1], \tag{42}$$

i.e., the domain of $\mathbf{B}$ is the boundary of the unit diamond D. Define $\mathbf{B} \smallsetminus \mathbf{A}\colon \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ via

$$\operatorname{gra}(\mathbf{B} \smallsetminus \mathbf{A}) := (\operatorname{gra} \mathbf{B}) \smallsetminus (\operatorname{gra} \mathbf{A}) \tag{43}$$

and observe that

$$\operatorname{gra}(\mathbf{B} \smallsetminus \mathbf{A}) = \big\{\big(\mathbf{a}_i(t), \mathbf{a}_i^*(t)\big) \mid i \in \{1,2,3,4\},\ t \in \{0,1\}\big\} \tag{44}$$

contains only *finitely many* — in fact, at most eight — points. Proposition 2.5 states that

$$\mathbf{A} \text{ is 3-cyclically monotone if and only if } \mathbf{B} \text{ is.} \tag{45}$$

Set

$$\mu := \sup_{\text{each } (\mathbf{x}_i, \mathbf{x}_i^*) \in \operatorname{gra} \mathbf{B}} \langle \mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_1^* \rangle + \langle \mathbf{x}_3 - \mathbf{x}_2, \mathbf{x}_2^* \rangle + \langle \mathbf{x}_1 - \mathbf{x}_3, \mathbf{x}_3^* \rangle. \tag{46}$$

Then $\mathbf{B}$ is 3-cyclically monotone if and only if $\mu \leq 0$. Because of (40) and (41), the supremum in (46) is *attained*, say

$$\mu = \langle \mathbf{a}_j(\bar{s}) - \mathbf{a}_i(\bar{r}), \mathbf{a}_i^*(\bar{r}) \rangle + \langle \mathbf{a}_k(\bar{t}) - \mathbf{a}_j(\bar{s}), \mathbf{a}_j^*(\bar{s}) \rangle + \langle \mathbf{a}_i(\bar{r}) - \mathbf{a}_k(\bar{t}), \mathbf{a}_k^*(\bar{t}) \rangle, \tag{47}$$

where $\{i,j,k\} \subset \{1,2,3,4\}$ and $\{\bar{r}, \bar{s}, \bar{t}\} \subset [0,1]$. Then

$$\mu = \max_{r \in [0,1]} \langle \mathbf{a}_j(\bar{s}) - \mathbf{a}_i(r), \mathbf{a}_i^*(r) \rangle + \langle \mathbf{a}_k(\bar{t}) - \mathbf{a}_j(\bar{s}), \mathbf{a}_j^*(\bar{s}) \rangle + \langle \mathbf{a}_i(r) - \mathbf{a}_k(\bar{t}), \mathbf{a}_k^*(\bar{t}) \rangle, \tag{48}$$

and the function we maximize over $r \in [0,1]$ in (48) is *affine* (by (36)); thus, this function has a maximizer in $\{0,1\}$. Therefore, without loss of generality, we assume that $\bar{r} \in \{0,1\}$ and analogously that $\bar{s} \in \{0,1\}$ and $\bar{t} \in \{0,1\}$. Hence

$$\mu = \max_{\text{each } (\mathbf{x}_i, \mathbf{x}_i^*) \in (\operatorname{gra} \mathbf{B}) \smallsetminus (\operatorname{gra} \mathbf{A})} \langle \mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_1^* \rangle + \langle \mathbf{x}_3 - \mathbf{x}_2, \mathbf{x}_2^* \rangle + \langle \mathbf{x}_1 - \mathbf{x}_3, \mathbf{x}_3^* \rangle, \tag{49}$$

which implies that

$$\mathbf{B} \text{ is 3-cyclically monotone if and only if } \mathbf{B} \smallsetminus \mathbf{A} \text{ is.} \tag{50}$$

Combining (45) with (50), we have thus established the following result.

Proposition 3.1 *The following are equivalent.*

(i) $\mathbf{A}$ *is 3-cyclically monotone.*

(ii) $\mathbf{B}$ *is 3-cyclically monotone.*

(iii) $\mathbf{B} \smallsetminus \mathbf{A}$ *is 3-cyclically monotone.*

Proposition 3.1 shows that the verification of 3-cyclic monotonicity of $\mathbf{B}$, the graph of which has infinitely many points, can be reduced to the verification of 3-cyclic monotonicity of $\mathbf{B} \smallsetminus \mathbf{A}$, the graph of which has only finitely many points. We will tackle this verification in the next subsection.

Verifying that B is 3-cyclically monotone

Recall the expression for the graph of $\mathbf{B} \smallsetminus \mathbf{A}$ given in (44). It turns out that this graph contains only six different points:

$$\operatorname{gra}(\mathbf{B} \smallsetminus \mathbf{A}) = \bigcup_{i\in\{1,2,\ldots,6\}} \big\{(\mathbf{b}_i, \mathbf{b}_i^*)\big\}, \tag{51}$$

where

$$\begin{aligned}
\mathbf{b}_1 &= (1,0) = \mathbf{a}_1(0), & \mathbf{b}_1^* &:= (0,1) = \mathbf{a}_1^*(0), && (52)\\
\mathbf{b}_2 &= (0,1) = \mathbf{a}_2(0) = \mathbf{a}_1(1), & \mathbf{b}_2^* &:= (-1,0) = \mathbf{a}_2^*(0) = \mathbf{a}_1^*(1), && (53)\\
\mathbf{b}_3 &= (-1,0) = \mathbf{a}_3(0) & \mathbf{b}_3^* &:= (-1,-2) = \mathbf{a}_3^*(0), && (54)\\
\mathbf{b}_4 &= (0,-1) = \mathbf{a}_4(0) = \mathbf{a}_3(1), & \mathbf{b}_4^* &:= (0,-1) = \mathbf{a}_4^*(0) = \mathbf{a}_3^*(1), && (55)\\
\mathbf{b}_5 &:= (-1,0) = \mathbf{a}_2(1), & \mathbf{b}_5^* &:= (-1,0) = \mathbf{a}_2^*(1), && (56)\\
\mathbf{b}_6 &:= (1,0) = \mathbf{a}_4(1), & \mathbf{b}_6^* &:= (0,-1) = \mathbf{a}_4^*(1). && (57)
\end{aligned}$$

Remark 3.2 We wish to point out that the operator with graph $\bigcup_{i\in\{1,2,3,4\}}\{(\mathbf{b}_i, \mathbf{b}_i^*)\}$ was utilized in [2] and [4]. (This operator, and hence $\mathbf{B}$, is *not* 4-cyclically monotone.) The construction presented in this section is the result of our effort to find a concrete maximal 3-cyclically monotone extension of this operator.

To tackle 3-cyclic monotonicity, we set — analogously to (46) —

$$\sigma := \sup_{\text{each } (\mathbf{x}_i,\mathbf{x}_i^*)\in\operatorname{gra}(\mathbf{B}\smallsetminus\mathbf{A})} \langle \mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_1^*\rangle + \langle \mathbf{x}_3 - \mathbf{x}_2, \mathbf{x}_2^*\rangle + \langle \mathbf{x}_1 - \mathbf{x}_3, \mathbf{x}_3^*\rangle. \tag{58}$$

Observe that $\mathbf{B} \smallsetminus \mathbf{A}$ is 3-cyclically monotone if and only if $\sigma \leq 0$. Since $\operatorname{gra}(\mathbf{B} \smallsetminus \mathbf{A})$ contains only finitely many points (see (51)), the supremum in (58) is attained. In fact, we write equivalently

$$\sigma = \max_{\{i,j,k\}\subset\{1,2,\ldots,6\}} \langle \mathbf{b}_j - \mathbf{b}_i, \mathbf{b}_i^*\rangle + \langle \mathbf{b}_k - \mathbf{b}_j, \mathbf{b}_j^*\rangle + \langle \mathbf{b}_i - \mathbf{b}_k, \mathbf{b}_k^*\rangle. \tag{59}$$

This formulation requires us to compute $6^3 = 216$ sums. We computed σ using the following (nonoptimized) code written in `GNU Octave`. (See `http://www.octave.org` for further information on this freely available software. The adaption of this code to other programming languages is straightforward.)

```
b=[1 0;0 1;-1 0;0 -1;-1 0; 1 0];
bstar = [0 1;-1 0;-1 -2;0 -1;-1 0;0 -1];
sigma = -Inf;
for i=1:6  for j=1:6  for k=1:6
 t1 = (b(j,:)-b(i,:))*bstar(i,:)';
 t2 = (b(k,:)-b(j,:))*bstar(j,:)';
 t3 = (b(i,:)-b(k,:))*bstar(k,:)';
 sigma=max(sigma,t1+t2+t3);
end end end
disp(sigma);
```

The code yields $\sigma = 0$. (Note that we are not in any danger of having to deal with round-off errors: the vectors in $\operatorname{gra}(\mathbf{B} \smallsetminus \mathbf{A})$ all have small *integer entries*, which implies that the sums in (59) are all integers as well). Therefore, $\mathbf{B} \smallsetminus \mathbf{A}$ is 3-cyclically monotone. In view of Proposition 3.1, we have verified the following result.

Proposition 3.3 *The operator* $\mathbf{B}$ *is 3-cyclically monotone.*

Introducing C and M

We now set

$$\mathbf{C}\colon \mathbb{R}^2 \rightrightarrows \mathbb{R}^2 \colon x \mapsto \operatorname{conv}(\mathbf{B}x). \tag{60}$$

It is clear that (recall (42))

$$\operatorname{dom}\mathbf{C} = \operatorname{dom}\mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2] \cup [\mathbf{b}_2, \mathbf{b}_3] \cup [\mathbf{b}_3, \mathbf{b}_4] \cup [\mathbf{b}_4, \mathbf{b}_1], \tag{61}$$

and that (combine Proposition 3.3 and Proposition 2.8)

$$\mathbf{C} \text{ is 3-cyclically monotone.} \tag{62}$$

The operator $\mathbf{C}$ is identical to the operator $\mathbf{B}$, except (see (52)–(57)) for $\mathbf{B}(\mathbf{b}_1) = \{\mathbf{b}_1^*, \mathbf{b}_6^*\} \subset \mathbf{C}(\mathbf{b}_1) = [\mathbf{b}_1^*, \mathbf{b}_6^*]$ and for $\mathbf{B}(\mathbf{b}_3) = \{\mathbf{b}_3^*, \mathbf{b}_5^*\} \subset \mathbf{C}(\mathbf{b}_3) = [\mathbf{b}_3^*, \mathbf{b}_5^*]$. Finally, we set

$$\mathbf{M} := \mathbf{C} + N_D, \tag{63}$$

where N_D is the normal cone operator of the full unit diamond D. Since $N_D = \partial\iota_D$ is a subdifferential operator, it is cyclically monotone and, in particular, 3-cyclically monotone. In view of Proposition 2.4 and (61), we obtain the following result.

Proposition 3.4 *The operator* $\mathbf{M}$ *is 3-cyclically monotone with*

$$\operatorname{dom}\mathbf{M} = (\operatorname{dom}\mathbf{C}) \cap (\operatorname{dom} N_D) = \operatorname{dom}\mathbf{C} = [\mathbf{b}_1, \mathbf{b}_2] \cup [\mathbf{b}_2, \mathbf{b}_3] \cup [\mathbf{b}_3, \mathbf{b}_4] \cup [\mathbf{b}_4, \mathbf{b}_1]. \tag{64}$$

Deriving the extension inequalities for M

We aim to show that $\mathbf{M}$ is *maximal* 3-cyclically monotone. To this end, let

$$(\mathbf{x}, \mathbf{x}^*) = \big((x,y),(u,v)\big) \in \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4 \tag{65}$$

be 3-cyclically monotonically related to $\mathbf{M}$, i.e.,

$$\big\{\big((x,y),(u,v)\big)\big\} \cup (\operatorname{gra}\mathbf{M}) \;\text{ is 3-cyclically monotone.} \tag{66}$$

We must show that

$$\big((x,y),(u,v)\big) \in \operatorname{gra}\mathbf{M}. \tag{67}$$

Since $\operatorname{dom}\mathbf{C}$ is nonempty, bounded and closed, with $D = \overline{\operatorname{conv}}\operatorname{dom}\mathbf{C}$, Corollary 2.16 yields

$$\bigcap_{\mathbf{z}\in\operatorname{dom}\mathbf{C}} \big(\mathbf{z} + T_D(\mathbf{z})\big) = D. \tag{68}$$

Now (68) and Theorem 2.14 show that (66) is equivalent to

$$\big\{\big((x,y),(u,v)\big)\big\} \cup (\operatorname{gra}\mathbf{C}) \quad \text{is 3-cyclically monotone} \quad \text{and} \quad (x,y) \in D. \tag{69}$$

In view of (34) and Proposition 2.13, we write (69) equivalently as

$$|x| + |y| \leq 1 \;\text{ and }\; \big\{\big((x,y),(u,v)\big)\big\} \cup (\operatorname{gra}\mathbf{B}) \text{ is 3-cyclically monotone.} \tag{70}$$

Using Fact 2.9, we note that the second condition in (70) is equivalent to

$$\sup_{\text{each } (\mathbf{x}_i,\mathbf{x}_i^*)\in \operatorname{gra}\mathbf{B}} \langle \mathbf{x}_2-\mathbf{x}_1,\mathbf{x}_1^*\rangle + \langle (x,y)-\mathbf{x}_2,\mathbf{x}_2^*\rangle + \langle \mathbf{x}_1-(x,y),(u,v)\rangle \le 0. \tag{71}$$

Arguing as in the proof that led to the equivalence (50), we obtain that (71) is equivalent to

$$\max_{\text{each } (\mathbf{x}_i,\mathbf{x}_i^*)\in \operatorname{gra}(\mathbf{B}\smallsetminus\mathbf{A})} \langle \mathbf{x}_2-\mathbf{x}_1,\mathbf{x}_1^*\rangle + \langle (x,y)-\mathbf{x}_2,\mathbf{x}_2^*\rangle + \langle \mathbf{x}_1-(x,y),(u,v)\rangle \le 0. \tag{72}$$

In view of (51) and (52)–(57), we obtain the following result.

Proposition 3.5 *The point* $\big((x,y),(u,v)\big)\in\mathbb{R}^4$ *is 3-cyclically monotonically related to* $\operatorname{gra}\mathbf{M}$ *if and only if* $(x,y)\in D$, *i.e.,*

$$|x|+|y|\le 1, \tag{73}$$

and the extension inequalities

$$\big(\forall \{i,j\}\subset\{1,2,\dots,6\}\big)\quad \langle \mathbf{b}_j-\mathbf{b}_i,\mathbf{b}_i^*\rangle + \langle (x,y)-\mathbf{b}_j,\mathbf{b}_j^*\rangle \le \langle (x,y)-\mathbf{b}_i,(u,v)\rangle \tag{74}$$

hold.

Listing the extension inequalities for M

We sort the inequalities (74) parameterized by $\{i,j\}\subset\{1,2,\dots,6\}$ into two groups: $i=j$ and $i\neq j$. The case when $i=j$ in (74) results in the six inequalities

$$(\forall i\in\{1,2,\dots,6\})\quad \langle (x,y)-\mathbf{b}_i,\mathbf{b}_i^*\rangle \le \langle (x,y)-\mathbf{b}_i,(u,v)\rangle, \tag{75}$$

i.e.,

$$y \le (x-1)u+yv, \tag{76}$$
$$-x \le xu+(y-1)v, \tag{77}$$
$$-x-1-2y \le (x+1)u+yv, \tag{78}$$
$$-y-1 \le xu+(y+1)v, \tag{79}$$
$$-x-1 \le (x+1)u+yv, \tag{80}$$
$$-y \le (x-1)u+yv. \tag{81}$$

We now turn to the case when $i\neq j$ in (74). Since $\mathbf{b}_1=\mathbf{b}_6$, $\mathbf{b}_3=\mathbf{b}_5$, $\mathbf{b}_2^*=\mathbf{b}_5^*$, and $\mathbf{b}_4^*=\mathbf{b}_6^*$, each of the sets $\{\mathbf{b}_1,\dots,\mathbf{b}_6\}$ and $\{\mathbf{b}_1^*,\dots,\mathbf{b}_6^*\}$ contains actually only four elements. If $i\neq j$, but $\mathbf{b}_i=\mathbf{b}_j$ or $\mathbf{b}_i^*=\mathbf{b}_j^*$, then we obtain one of the inequalities previously considered in (76)–(81). Thus, the remaining inequalities have the form

$$i\neq j,\quad \mathbf{b}_i\neq\mathbf{b}_j \ \text{ and } \ \mathbf{b}_i^*\neq\mathbf{b}_j^*. \tag{82}$$

Therefore, we do not have to consider the extension inequalities corresponding to

$$\{i,j\}\in\big\{\{1,6\},\{3,5\},\{2,5\},\{4,6\}\big\}. \tag{83}$$

The remaining extension inequalities (74) are

$$
\begin{array}{lrr}
i=1,\ j=2: & 1-x \le (x-1)u+yv, & (84)\\
i=2,\ j=1: & -1+y \le xu+(y-1)v, & (85)\\
i=1,\ j=3: & -1-x-2y \le (x-1)u+yv, & (86)\\
i=3,\ j=1: & -2+y \le (x+1)u+yv, & (87)\\
i=1,\ j=4: & -2-y \le (x-1)u+yv, & (88)\\
i=4,\ j=1: & -1+y \le xu+(y+1)v, & (89)\\
i=1,\ j=5: & -1-x \le (x-1)u+yv, & (90)\\
i=5,\ j=1: & -2+y \le (x+1)u+yv, & (91)\\
i=2,\ j=3: & -x-2y \le xu+(y-1)v, & (92)\\
i=3,\ j=2: & -3-x \le (x+1)u+yv, & (93)\\
i=2,\ j=4: & -1-y \le xu+(y-1)v, & (94)\\
i=4,\ j=2: & -2-x \le xu+(y+1)v, & (95)\\
i=2,\ j=6: & -1-y \le xu+(y-1)v, & (96)\\
i=6,\ j=2: & -1-x \le (x-1)u+yv, & (97)\\
i=3,\ j=4: & -y \le (x+1)u+yv, & (98)\\
i=4,\ j=3: & -2-x-2y \le xu+(y+1)v, & (99)\\
i=3,\ j=6: & -2-y \le (x+1)u+yv, & (100)\\
i=6,\ j=3: & -1-x-2y \le (x-1)u+yv, & (101)\\
i=4,\ j=5: & -2-x \le xu+(y+1)v, & (102)\\
i=5,\ j=4: & -2-y \le (x+1)u+yv, & (103)\\
i=5,\ j=6: & -2-y \le (x+1)u+yv, & (104)\\
i=6,\ j=5: & -1-x \le (x-1)u+yv. & (105)
\end{array}
$$

We have verified the following result.

Proposition 3.6 *The point* $\big((x,y),(u,v)\big) \in \mathbb{R}^4$ *is 3-cyclically monotonically related to* $\operatorname{gra}\mathbf{M}$ *if and only if* $|x|+|y| \le 1$ *and the 28 inequalities* (76)–(81) *and* (84)–(105) *hold.*

Simplifying the extension inequalities

The 28 inequalities referred to in Proposition 3.6 are clearly not easy to handle. Fortunately, we are able to reduce this system of inequalities significantly. The key is to observe that the 28 inequalities have only four different right-hand sides, namely

$$(x-1)u+yv,\quad xu+(y-1)v,\quad (x+1)u+yv,\quad \text{and}\quad xu+(y+1)v, \tag{106}$$

and that

$$|x|+|y| \le 1. \tag{107}$$

The right-hand side $(x-1)u+yv$

The corresponding inequalities are (76), (81), (84), (86)=(101), (88), and (90)=(97)=(105), leaving us with the following set of left-hand sides:

$$\{y, -y, 1-x, -1-x-2y, -2-y, -1-x\}. \tag{108}$$

Clearly, $-1-x \leq 1-x$ and also $-2-y \leq y$ (recall (107)). Deleting two redundant left-hand sides, we have reduced the set to $\{y, -y, 1-x, -1-x-2y\}$. Moreover, by (107), we also have $y \leq 1-x$ and $-1-x-2y \leq -y \leq 1-x$. Therefore, all inequalities with right-hand side $(x-1)u+yv$ reduce to the *single* inequality

$$1-x \leq (x-1)u+yv. \tag{109}$$

The right-hand side $xu+(y-1)v$

The corresponding inequalities are (77), (85), (92), and (94)=(96), with the set of left-hand sides

$$\{-x, -1+y, -x-2y, -1-y\}. \tag{110}$$

Now $x \pm y \leq x+|y| \leq 1 \Rightarrow -1 \pm y \leq -x$; thus, we are left with $\max\{-x, -x-2y\} \leq xu+(y-1)v$, which we rewrite as

$$|y|-y-x \leq xu+(y-1)v. \tag{111}$$

The right-hand side $(x+1)u+yv$

The corresponding inequalities are (78), (80), (87)=(91), (93), (98), and (100)=(103)=(104). The set of left-hand sides is

$$\{-x-1-2y, -x-1, -2+y, -3-x, -y, -2-y\}. \tag{112}$$

Now (107) reveals that $-y$ is the largest element in this set. Therefore, we are left with the inequality

$$-y \leq (x+1)u+yv. \tag{113}$$

The right-hand side $xu+(y+1)v$

The corresponding inequalities are (79), (89), (95)=(102), and (99). The set of left-hand sides is

$$\{-y-1, -1+y, -2-x, -2-x-2y\}. \tag{114}$$

Again (107) is very useful; it allows to decide that $-2-x \leq -1+y$ and that $-2-x-2y \leq -1-y$. Thus we are left with $\max\{-1+y, -1-y\} \leq xu+(y+1)v$, i.e., with

$$-1+|y| \leq xu+(y+1)v. \tag{115}$$

Altogether, we have achieved a reduction from 29 inequalities to 5.

Proposition 3.7 *The point* $\big((x, y), (u, v)\big) \in \mathbb{R}^4$ *is 3-cyclically monotonically related to* $\operatorname{gra} \mathbf{M}$ *if and only if the following 5 inequalities hold.*

$$|x| + |y| \le 1, \tag{116}$$
$$1 - x \le (x - 1)u + yv, \tag{117}$$
$$|y| - y - x \le xu + (y - 1)v, \tag{118}$$
$$-y \le (x + 1)u + yv, \tag{119}$$
$$-1 + |y| \le xu + (y + 1)v. \tag{120}$$

Excluding the interior of the unit diamond

In this section, we assume that (x, y) lies in the interior of the unit diamond, i.e.,

$$|x| + |y| < 1, \tag{121}$$

and that (117)–(120) hold. We will consider cases and each time arrive at a contradiction.

Case $y = 0$**.**

On the one hand, (117) becomes $1 - x \le (x - 1)u \Leftrightarrow u \le -1$. On the other hand, (119) yields $0 \le (x + 1)u \Leftrightarrow 0 \le u$. Altogether, we have obtained the absurdity $u \le -1 < 0 \le u$.

Case $y > 0$**.**

In this case, (117)–(119) become

$$1 - x \le (x - 1)u + yv, \tag{122}$$
$$-x \le xu + (y - 1)v, \tag{123}$$
$$-y \le (x + 1)u + yv. \tag{124}$$

Since $0 < y < 1$, we solve (122)–(124) for v:

$$\frac{1 - x + u - xu}{y} \le v, \tag{125}$$
$$\frac{x + xu}{1 - y} \ge v, \tag{126}$$
$$\frac{-y - u - xu}{y} \le v. \tag{127}$$

Combining yields

$$\frac{1 - x + u - xu}{y} \le \frac{x + xu}{1 - y}, \tag{128}$$
$$\frac{-y - u - xu}{y} \le \frac{x + xu}{1 - y}. \tag{129}$$

Now (128)–(129) are equivalent to the two inequalities

$$
\begin{aligned}
(1-y)(1-x+u-xu) &\le y(x+xu), && (130)\\
(1-y)(-y-u-xu) &\le y(x+xu), && (131)
\end{aligned}
$$

which we rewrite as

$$
\begin{aligned}
1-x-y &\le -u(1-x-y), && (132)\\
-y(1+x-y) &\le u(1+x-y). && (133)
\end{aligned}
$$

By (121), we have $1-y\pm x>0$; hence, (132)–(133) yield $1\le -u$ and $-y\le u$. Therefore, $1\le -u\le y$, which contradicts (121).

Case $y<0$.

In this case, (117), (119), and (120) become

$$
\begin{aligned}
1-x &\le (x-1)u+yv, && (134)\\
-y &\le (x+1)u+yv, && (135)\\
-1-y &\le xu+(1+y)v. && (136)
\end{aligned}
$$

Since $-1<y<0$, we solve (134)–(136) for v:

$$
\begin{aligned}
\frac{1-x+u-xu}{y} &\ge v, && (137)\\
\frac{-y-xu-u}{y} &\ge v, && (138)\\
\frac{-1-y-xu}{1+y} &\le v. && (139)
\end{aligned}
$$

Combining yields

$$
\begin{aligned}
\frac{-1-y-xu}{1+y} &\le \frac{1-x+u-xu}{y}, && (140)\\
\frac{-1-y-xu}{1+y} &\le \frac{-y-xu-u}{y}. && (141)
\end{aligned}
$$

Next, (140)–(141) are equivalent to

$$
\begin{aligned}
-(1+y)(1+y-x) &\ge u(1+y-x), && (142)\\
u(1+y+x) &\ge 0. && (143)
\end{aligned}
$$

By (121), we have $1+y\pm x>0$; thus, (142)–(143) results in $-(1+y)\ge u$ and $u\ge 0$. Altogether, $0\ge -u\ge 1+y$, which contradicts (121).

Therefore, we have obtained the following refinement of Proposition 3.7.

Proposition 3.8 *The point* $\big((x,y),(u,v)\big) \in \mathbb{R}^4$ *is 3-cyclically monotonically related to* $\operatorname{gra}\mathbf{M}$ *if and only if the following hold.*

$$\begin{aligned} |x|+|y| &= 1, && (144)\\ 1-x &\le (x-1)u+yv, && (145)\\ |y|-y-x &\le xu+(y-1)v, && (146)\\ -y &\le (x+1)u+yv, && (147)\\ -1+|y| &\le xu+(y+1)v. && (148) \end{aligned}$$

Revisiting the boundary of the unit diamond

We now assume that $\big((x,y),(u,v)\big)$ is 3-cyclically monotonically related to $\operatorname{gra}\mathbf{M}$, i.e., (144)–(148) hold (by Proposition 3.8). We shall show that $(u,v) \in \mathbf{M}(x,y)$, and we do this by considering 8 cases (the 4 extreme points of D, and the 4 relatively open line segments in the boundary of D).

Case $(x,y) = \mathbf{b}_1 = (1,0)$**.**

Then (145)–(148) simplify to

$$\begin{aligned} 0 &\le u, && (149)\\ -u-1 \le v &\le u+1; && (150) \end{aligned}$$

equivalently, to $(u,v) \in [(0,1),(0,-1)] + \mathbb{R}_+(1,1) + \mathbb{R}_+(1,-1) = \operatorname{conv}(\mathbf{B}(1,0)) + N_D(1,0) = \mathbf{M}(1,0)$.

Case $(x,y) = \mathbf{b}_2 = (0,1)$**.**

Then (145)–(148) are equivalent to

$$\max\{u+1,-u-1\} \le v, \qquad (151)$$

i.e., $(u,v) \in (-1,0) + \mathbb{R}_+(-1,1) + \mathbb{R}_+(1,1) = \mathbf{B}(0,1) + N_D(0,1) = \mathbf{M}(0,1)$.

Case $(x,y) = \mathbf{b}_3 = (-1,0)$**.**

Then (145)–(148) simplify to

$$\begin{aligned} u &\le -1, && (152)\\ u-1 \le v &\le -u-1; && (153) \end{aligned}$$

equivalently, to $(u,v) \in [(-1,0),(-1,-2)] + \mathbb{R}_+(-1,1) + \mathbb{R}_+(-1,-1) = \operatorname{conv}(\mathbf{B}(-1,0)) + N_D(-1,0) = \mathbf{M}(-1,0)$.

Case $(x,y) = \mathbf{b}_4 = (0,-1)$**.**

Then (145)–(148) are equivalent to

$$v \le -|u| - 1, \tag{154}$$

i.e., $(u,v) \in (0,-1) + \mathbb{R}_+(-1,-1) + \mathbb{R}_+(1,-1) = \mathbf{B}(0,-1) + N_D(0,-1) = \mathbf{M}(0,-1)$.

Case $(x,y) = (1-t,t) = \mathbf{a}_1(t) \in \,]\mathbf{b}_1, \mathbf{b}_2[$**, where** $0 < t < 1$**.**

Then (145) and (146) yield $v \ge u+1$ and $v \le u+1$, respectively. Hence $v = u+1$. Then (147) becomes $u \ge -t$, which is stronger than $u \ge -1$, i.e., (148). Thus write $u = -t + \rho$, where $\rho \in \mathbb{R}_+$, and therefore $(u,v) = (-t+\rho, -t+\rho+1) = (-t,-t+1) + \rho(1,1) \in \mathbf{a}_1^*(t) + N_D(x,y) = \mathbf{A}(x,y) + N_D(x,y) = \mathbf{M}(x,y)$.

Case $(x,y) = (-t,1-t) = \mathbf{a}_2(t) \in \,]\mathbf{b}_2, \mathbf{b}_3[$**, where** $0 < t < 1$**.**

Here (146) and (147) result in $v \le -u-1$ and $v \ge -u-1$, respectively. Thus $v = -u-1$. Then (148) becomes $u \le t-1$, which is weaker than (145), i.e., $u \le -1$. Hence write $u = -1-\rho$, where $\rho \in \mathbb{R}_+$, and therefore $(u,v) = (-1-\rho, 1+\rho-1) = (-1,0) + \rho(-1,1) \in \mathbf{a}_2^*(t) + N_D(x,y) = \mathbf{A}(x,y) + N_D(x,y) = \mathbf{M}(x,y)$.

Case $(x,y) = (t-1,-t) = \mathbf{a}_3(t) \in \,]\mathbf{b}_3, \mathbf{b}_4[$**, where** $0 < t < 1$**.**

This time, (147) and (148) give $v \le u-1$ and $v \ge u-1$, respectively. Hence $v = u-1$. Now (145) corresponds to $u \le -1+t$, which is stronger than $u \le 0$, which is the counterpart of (146). We write $u = -1+t-\rho$, where $\rho \in \mathbb{R}_+$, and observe that $(u,v) = (t-1-\rho, t-2-\rho) = (t-1,t-2) + \rho(-1,-1) \in \mathbf{a}_3^*(t) + N_D(x,y) = \mathbf{A}(x,y) + N_D(x,y) = \mathbf{M}(x,y)$.

Case $(x,y) = (t,t-1) = \mathbf{a}_4(t) \in \,]\mathbf{b}_4, \mathbf{b}_1[$**, where** $0 < t < 1$**.**

Here (145) and (148) yield $v \le -u-1$ and $v \ge -u-1$, respectively. Hence $v = -u-1$. Now (146) becomes $u \ge -t$, which is weaker than $u \ge 0$, which is the outcome of (147). Hence $(u,v) = (u,-u-1) \in (0,-1) + \mathbb{R}_+(1,-1) = \mathbf{a}_4^*(t) + N_D(x,y) = \mathbf{A}(x,y) + N_D(x,y) = \mathbf{M}(x,y)$.

Let us record the outcome of this subsection in the following result.

Proposition 3.9 *The point* $\big((x,y),(u,v)\big) \in \mathbb{R}^4$ *is 3-cyclically monotonically related to* $\operatorname{gra}\mathbf{M}$ *if and only if it belongs to* $\operatorname{gra}\mathbf{M}$ *already.*

Summary

Theorem 3.10 *The operator*

$$\mathbf{M}\colon \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$$
$$\mathbf{x} \mapsto \begin{cases} [(0,1),(0,-1)] + \mathbb{R}_+(1,1) + \mathbb{R}_+(1,-1), & \text{if } \mathbf{x} = (1,0); \\ (-t, 1-t) + \mathbb{R}_+(1,1), & \text{if } \mathbf{x} = (1-t, t) \text{ and } 0 < t < 1; \\ (-1,0) + \mathbb{R}_+(-1,1) + \mathbb{R}_+(1,1), & \text{if } \mathbf{x} = (0,1); \\ (-1,0) + \mathbb{R}_+(-1,1), & \text{if } \mathbf{x} = (-t, 1-t) \text{ and } 0 < t < 1; \\ [(-1,0),(-1,-2)] + \mathbb{R}_+(-1,1) + \mathbb{R}_+(-1,-1), & \text{if } \mathbf{x} = (-1,0); \\ (t-1, t-2) + \mathbb{R}_+(-1,-1), & \text{if } \mathbf{x} = (t-1, -t) \text{ and } 0 < t < 1; \\ (0,-1) + \mathbb{R}_+(-1,-1) + \mathbb{R}_+(1,-1), & \text{if } \mathbf{x} = (0,-1); \\ (0,-1) + \mathbb{R}_+(1,-1), & \text{if } \mathbf{x} = (t, t-1) \text{ and } 0 < t < 1; \\ \varnothing, & \text{otherwise} \end{cases} \tag{155}$$

is maximal 3-cyclically monotone and its domain

$$\operatorname{dom} \mathbf{M} = \big\{(x,y) \in \mathbb{R}^2 \mid |x| + |y| = 1\big\} \tag{156}$$

is equal to the boundary *of the unit diamond. In stark contrast,* $\mathbf{M}$ *is* not *maximal monotone; in fact, every maximal monotone extension of* $\mathbf{M}$ *must have its domain equal to the full unit diamond* $\big\{(x,y) \in \mathbb{R}^2 \mid |x| + |y| \leq 1\big\}$.

Proof. Proposition 3.9 implies that $\mathbf{M}$ is maximal 3-cyclically monotone. Since $(\operatorname{ri} D) \cap (\operatorname{dom} \mathbf{M}) = (\operatorname{int} D) \cap (\operatorname{bdry} D) = \varnothing$, it is clear from Theorem 2.12 that $\mathbf{M}$ is not maximal monotone. Now let $\mathbf{N}$ be a maximal monotone extension of $\mathbf{M}$, and take $(\mathbf{x}, \mathbf{x}^*) \in \operatorname{gra} \mathbf{N}$. Then $(\mathbf{x}, \mathbf{x}^*)$ is monotonically related to $\mathbf{C} + N_D$. By Theorem 2.14 and Corollary 2.16, $\mathbf{x} \in D$. Thus

$$\operatorname{dom} \mathbf{N} \subseteq D. \tag{157}$$

Since $\mathbf{N}$ is an extension of $\mathbf{M}$, we clearly have

$$\operatorname{bdry} D = \operatorname{dom} \mathbf{M} \subseteq \operatorname{dom} \mathbf{N}, \tag{158}$$

which implies $D = \operatorname{conv} \operatorname{bdry} D \subseteq \operatorname{conv} \operatorname{dom} \mathbf{N}$ and thus

$$\operatorname{int} D \subseteq \operatorname{int} \operatorname{conv} \operatorname{dom} \mathbf{N}. \tag{159}$$

Now [11, Theorem 18.3] states $\operatorname{int} \operatorname{dom} \mathbf{N} = \operatorname{int} \operatorname{conv} \operatorname{dom} \mathbf{N}$. Hence (159) implies that

$$\operatorname{int} D \subseteq \operatorname{int} \operatorname{dom} \mathbf{N}. \tag{160}$$

Combining (158) and (160) yields

$$D \subseteq \operatorname{dom} \mathbf{N}. \tag{161}$$

In view of (157) and (161), we conclude that $\operatorname{dom} \mathbf{N} = D$. ■

Remark 3.11 It would be interesting to find a more systematic way of constructing a maximal n-cyclically monotone operator that is not maximal monotone.

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