

# AN EXPLICIT SUB-WEYL BOUND FOR $\zeta(1/2 + it)$

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ABSTRACT. In this article we prove an explicit sub-Weyl bound for the Riemann zeta function  $\zeta(s)$  on the critical line  $s = 1/2 + it$ . In particular, we show that  $|\zeta(1/2 + it)| \leq 66.7t^{27/164}$  for  $t \geq 3$ . Combined, our results form the sharpest known bounds on  $\zeta(1/2 + it)$  for  $t \geq \exp(61)$ .

## 1. INTRODUCTION

An important open problem in analytic number theory is the growth rate of the Riemann zeta-function  $\zeta(s)$  on the critical line  $s = 1/2 + it$  as  $t \rightarrow \infty$ . The well-known Lindelöf Hypothesis asserts that  $\zeta(1/2 + it) \ll_{\epsilon} t^{\epsilon}$  for any  $\epsilon > 0$ . Among the consequences of the hypothesis are many profound results for prime number distributions. Although the Lindelöf Hypothesis is currently unproven, much effort have been expended to bound the zeta-function on the critical line, culminating in the current best-known bound of  $\zeta(1/2 + it) \ll_{\epsilon} t^{13/84+\epsilon}$  for any  $\epsilon > 0$ , due to Bourgain [Bou16].

In this article we are concerned with explicit bounds on  $\zeta(1/2 + it)$ . Such explicit bounds have been used to derive zero-free regions [For02; MTY22; Yan23], zero-density estimates [KLN18] and bounds on the argument of  $\zeta(s)$  on the critical line [Tru14; HSW21]. Recently, these results have in turn been used to obtain explicit theorems about prime distributions [KL14; CH21; Bro+21; CHL22; CHJ22; JY22; FKS22a; FKS22b], so there is substantial motivation to sharpen such bounds as much as possible. Nevertheless, known explicit bounds on  $\zeta(1/2 + it)$  currently lag far behind the asymptotically sharpest-known results. Only two types of explicit subconvexity results are known — the first being the classical van der Corput estimate of the form  $|\zeta(1/2 + it)| \leq At^{1/6} \log t$  for  $t \geq t_0$  for some absolute constants  $A$  and  $t_0$ . Such bounds are sometimes known as Weyl estimates because the exponent of  $1/6$  was first achieved via the Weyl–Littlewood–Hardy method. The sharpest estimate of this type is due to [HPY22], who built on the work of [CG04; Tru15; Hia16] to prove

$$|\zeta(1/2 + it)| \leq 0.618t^{1/6} \log t, \quad t \geq 3. \quad (1.1)$$

A second type of explicit bound, known sometimes as sub-Weyl estimates, was first made explicit by Patel [Pat21], who showed

$$|\zeta(1/2 + it)| \leq 307.098t^{27/164}, \quad t \geq 3. \quad (1.2)$$

Note that  $27/164 = 0.164\dots < 1/6$ . In particular, (1.2) is the best-known explicit bound for the zeta-function on the critical line for  $t \geq \exp(281)$ .

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*Date:* February 28, 2023.

*2010 Mathematics Subject Classification.* Primary 11L07, 11M06.

*Key words and phrases.* Van der Corput estimate, exponential sums, sub-Weyl bound, Riemann zeta function.

In this work we improve (1.2). Our main result is

**Theorem 1.1.** *For  $t \geq 3$ , we have*

$$|\zeta(1/2 + it)| \leq 66.7 t^{27/164}.$$

Theorem 1.1 represents the sharpest known bound on  $\zeta(1/2 + it)$  for  $t \geq \exp(105)$ . In §3.1 we show that still sharper bounds are possible for smaller  $t$ . Together, our results form the best known bound for  $t \geq \exp(61)$ . Therefore, (1.1) remains sharper at  $t \approx 3 \cdot 10^{12}$ , the verification height of the Riemann Hypothesis [PT21]. This is significant since bounds for  $\zeta(1/2 + it)$  near such values of  $t$  are used in multiple explicit results [HSW21; KLN18; For02].

On the other hand, sharp bounds on  $\zeta(1/2 + it)$  for larger values of  $t$  are useful for deriving explicit zero-free regions [For02; MTY22; Yan23], for improved bounds on  $S(T)$  [Tru14; HSW21], for refinements to Turing's method [Tru11; Tru16], and for asymptotically improved zero-density estimates [Tit86, Thm. 9.18]. For instance, following the method of [KLN18], we may use Theorem 1.1 to prove an explicit zero-density result of the form

$$N(\sigma, T) \ll T^{\frac{109}{41}(1-\sigma)} (\log T)^{4-3\sigma},$$

where  $N(\sigma, T)$  is the number of zeroes  $\rho = \beta + i\gamma$  of  $\zeta(s)$  with  $\sigma < \beta < 1$  and  $0 < \gamma < T$ .

**1.1. Approach and exponential sums.** As with all existing explicit bounds on  $\zeta(1/2 + it)$ , Theorem 1.1 relies on upper bounds on particular types of exponential sums, obtained via van der Corput's method of exponent pairs (for an exposition, see [GK91]). Roughly stated, let  $f(x)$  be a suitably well-defined and sufficiently smooth function satisfying  $f'(x) \approx yx^{-\sigma}$  for some  $y, \sigma > 0$ . If  $e(x) := \exp(2\pi ix)$ ,  $0 \leq k \leq 1/2 \leq l \leq 1$  and

$$S_f(a, N) := \sum_{a < n \leq a+N} e(f(n)) \ll \left(\frac{y}{N^\sigma}\right)^k N^l, \quad 0 < N \leq a,$$

then  $(k, l)$  is an exponent pair. For instance, from the trivial bound  $S_f(a, N) \ll N$  we see that  $(0, 1)$  is an exponent pair. The motivation for studying exponent pairs is highlighted by the result that if  $(k, l)$  is an exponent pair with  $k + 2l \geq 3/2$ , then

$$\zeta(1/2 + it) \ll t^{(2k+2l-1)/4} \log t,$$

see e.g. Phillips [Phi33].

The van der Corput method estimates  $S_f(a, N)$  by iteratively transforming it into simpler exponential sums, via two processes. The  $A$  process, also known as Weyl-differencing, expresses  $S_f(a, N)$  in terms of  $S_g(a, N)$ , where  $g(x)$  is a function of lower order than  $f(x)$  (and is hence easier to control). By applying the  $A$  process, we obtain that if  $(k, l)$  is an exponent pair, then so is

$$A(k, l) := \left(\frac{k}{2k+2}, \frac{k+l+1}{2k+2}\right).$$

The  $B$  process, also known as Poisson summation, expresses  $S_f(a, b)$  in terms of another exponential sum that is typically shorter. Using the  $B$  process, if  $(k, l)$  is an exponent pair, then so is

$$B(k, l) := \left(l - \frac{1}{2}, k + \frac{1}{2}\right).$$

Favourable exponential pairs and, by extension, good estimates of  $\zeta(1/2 + it)$ , can be obtained by beginning with the trivial  $(0, 1)$  exponent pair, then chaining together multiple applications of the  $A$  and  $B$  processes. The simplest van der Corput bound, such as (1.1), is obtained from the exponent pair  $AB(0, 1) = (1/6, 2/3)$ . On the other hand, bounds such as (1.2) and Theorem 1.1 can be obtained using  $ABA^3B(0, 1) = (11/82, 57/82)$ .

**1.2. Explicit exponent pairs.** Both the  $A$  and  $B$  processes have been made explicit. For the  $A$  process, we have the following lemma, due to [Yan23] which builds on the work of [CG04; PT15].

**Lemma 1.2** ([Yan23] Lem. 2.3). *Let  $f(x)$  be real-valued and defined on  $(a, a + N]$ , for some integers  $a, N$ . For all integers  $q > 0$ , we have*

$$(S_f(a, N))^2 \leq (N - 1 + q) \left( \frac{N}{q} + \frac{2}{q} \sum_{r=1}^{q-1} \left(1 - \frac{r}{q}\right) S_{g_r}(a, N - r) \right)$$

where  $g_r(x) := f(x + r) - f(x)$ .

A general explicit version of the  $B$  process was proved<sup>1</sup> in Karatsuba and Korolev [KK07], which relied on controlling the first four derivatives of the phase function  $f(x)$ . Patel [Pat21, Thm. 2.31] proved the following explicit Poisson summation formula, which only relied on the first three derivatives.

**Lemma 1.3** ([Pat21] Thm. 2.31). *Let  $f(x)$  be three times differentiable. Let  $f'(x)$  be decreasing in  $[a, b]$  and  $f'(b) = \alpha$ ,  $f'(a) = \beta$ . Further, let  $x_\nu$  be defined by*

$$f'(x_\nu) = \nu, \quad \alpha < \nu \leq \beta$$

Furthermore suppose that  $\lambda_2 \leq |f''(x)| \leq h_2\lambda_2$  and  $|f'''(x)| \leq h_3\lambda_3$ . Then

$$\left| \sum_{a < n \leq b} e(f(n)) - \sum_{\alpha < \nu \leq \beta} \frac{e(f(x_\nu) - \nu x_\nu - 1/8)}{|f''(x_\nu)|^{1/2}} \right| \leq \frac{40}{\sqrt{\pi}} \lambda_2^{-1/2} + A_1 \log(\beta - \alpha + 4) + A_2(b - a)\lambda_2^{1/5}\lambda_3^{1/5} + A_3.$$

where

$$A_1 = \frac{3 + 2h_2}{\pi}, \quad A_2 = \frac{8}{(6\pi^3)^{1/5}} h_2 h_3^{1/5}, \quad A_3 = \frac{1}{\pi} \left( 4\gamma + \log 2 + \pi + \frac{20}{7} \right),$$

and  $\gamma = 0.577\dots$  is the Euler-Mascheroni constant.

In practical application of van der Corput's method, we employ two tricks that frequently appear in the literature [CG04; PT15; Hia16; Pat21; HPY22; Yan23]. First, in the  $B(0, 1)$  process it is often helpful to replace the Poisson summation step with a second-derivative test that uses the Kuzmin–Landau lemma. This substitution preserves the original goal of shortening the exponential sum under consideration, without generating problematic secondary error terms that typically arise when applying Poisson summation. Second, to minimise tedium we typically apply an  $A^n B$  block as a single operation instead of  $n + 1$  separate operations. The following lemma, due to [Yan23], incorporates both of these modifications, which we will make extensive use of in this work.

<sup>1</sup>We note here that in this general explicit version of  $B$  process derived by Karatsuba and Korolev, one of the “lower” order term,  $K_2$ , in their assertion could grow larger than the main-term given by the sum,  $\sum c(n)Z(n)$  if  $f''$  is small.

**Lemma 1.4** (Explicit  $k$ th derivative test). *Let  $a, N$  be integers with  $N > 0$ . Let  $f(x)$  be equipped with  $k \geq 3$  continuous derivatives, with  $f^{(k)}$  monotonic, and suppose that  $\lambda_k \leq |f^{(k)}(x)| \leq h\lambda_k$  for all  $x \in (a, a + N]$  and some  $\lambda_k > 0$  and  $h > 1$ . Then, for all  $\eta > 0$ , we have*

$$S_f(a, N) \leq A_k h^{2/K} N \lambda_k^{1/(2K-2)} + B_k N^{1-2/K} \lambda_k^{-1/(2K-2)}$$

where  $K = 2^{k-1}$ , and

$$A_3 := \sqrt{\frac{1}{\eta h} + \frac{32}{15\sqrt{\pi}} \sqrt{\eta + \lambda_0^{1/3}} + \frac{1}{3} (\eta + \lambda_0^{1/3}) \lambda_0^{1/3} \delta_3}, \quad B_3 := \frac{\sqrt{32}}{\sqrt{3}\pi^{1/4}\eta^{1/4}} \delta_3,$$

$$\lambda_0 := \left( \frac{1}{\eta} + \frac{32\eta^{1/2}h}{15\sqrt{\pi}} \right)^{-3} \quad \delta_3 := \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{3}{8} \pi^{1/2} \eta^{3/2}}}.$$

and  $A_k, B_k$  for  $k \geq 4$  are defined recursively via

$$A_{k+1}(\eta, h) := \delta_k \left( h^{-1/K} + \frac{2^{19/12}(K-1)}{\sqrt{(2K-1)(4K-3)}} A_k(\eta, h)^{1/2} \right), \quad (1.3)$$

$$B_{k+1}(\eta) := \delta_k \frac{2^{3/2}(K-1)}{\sqrt{(2K-3)(4K-5)}} B_k(\eta)^{1/2}, \quad (1.4)$$

$$\delta_k := \sqrt{1 + \frac{2}{2337^{1-2/K}} \left( \frac{9\pi}{1024} \eta \right)^{1/K}}. \quad (1.5)$$

*Proof.* Follows by combining [Yan23, Lem. 2.4] and [Yan23, Lem. 2.5]  $\square$

**1.3. Sources of improvement.** We briefly review the main sources of improvement of Theorem 1.1 over (1.2), in case similar methods may be applied in other settings. Our first source of improvement originates from an interval-based argument as follows. An intermediary result in the argument of Patel [Pat21] produces a bound of the form

$$|\zeta(1/2 + it)| \leq A(t_0, t) t^{27/164}, \quad t \geq t_0$$

where  $A(t_0, t)$  is a bounded function that is decreasing in  $t_0$  and increasing in  $t$ . This immediately implies the bound  $|\zeta(1/2 + it)| \leq A_0 t^{27/164}$  for  $t \geq t_0$ , where

$$A_0 := \lim_{t \rightarrow \infty} A(t_0, t).$$

However, in our application  $A_0$  is typically large unless we take  $t_0$  to be very large, which defeats the purpose of obtaining an explicit bound holding for all  $t \geq 3$ . Instead, we may proceed as follows. For  $t_0 < t_1 < \dots < t_n = \infty$ , we apply

$$A(t_j, t) \leq A(t_j, t_{j+1}), \quad t_j \leq t \leq t_{j+1}$$

and so, for all  $t \geq t_0$ ,

$$|\zeta(1/2 + it)| \leq A_1 t^{27/164}, \quad A_1 := \max_{0 \leq j \leq n-1} A(t_j, t_{j+1}).$$

The central idea is to choose the  $t_j$ 's so that  $A(t_j, t_{j+1})$  is never too large. For instance, we take  $t_{n-1}$  to be sufficiently large so that

$$\lim_{t \rightarrow \infty} A(t_{n-1}, t)$$

is of an acceptable size. For more details and computation, see §3.1.

A second source of improvement comes from using the improved  $k$ th derivative test (Lemma 1.4) which makes use of the trivial bound to increase its sharpness. For details, see [Yan23, §2].

A third source of improvement arises from applying a sharpened version of the Poisson summation formula (i.e. using Lemma 2.3 in place of Lemma 1.3). In our application, the error terms introduced in estimating the stationary phase approximation to an exponential sum can be significant.

Lastly, in bounding long exponential sums, we make use of geometrically-sized intervals so that there are  $O(\log t)$  subintervals instead of  $O(t^A)$  subintervals, for some fixed  $A > 0$ . Since the method of proof is unable to detect cancellation between terms of two different subintervals, having less divisions is beneficial.

## 2. IMPROVED POISSON SUMMATION FORMULAE

In this section we prove a sharpened version of Lemma 1.3, which is useful since error terms arising from Poisson summation formulae are significant in our application. The main result of this section (Lemma 2.3) is an explicit van der Corput  $B$  process.

We begin by recalling some useful results, starting with bounds on exponential integrals. If  $f'$  is continuous and  $|f'(x)| \geq \lambda_1 > 0$  for  $\alpha \leq x \leq \beta$ , then by Rogers [Rog05, Lem. 3]

$$\left| \int_{\alpha}^{\beta} e(f(x)) dx \right| \leq \frac{1}{\pi \lambda_1}. \quad (2.1)$$

In addition, a corollary of a result due to Kershner [Ker35; Ker38] is that if  $f''$  is continuous and  $|f''(x)| \geq \lambda_2 > 0$  for  $\alpha \leq x \leq \beta$ , then

$$\left| \int_{\alpha}^{\beta} e(f(x)) dx \right| \leq \frac{1.343}{\sqrt{\lambda_2}}. \quad (2.2)$$

The constant 1.343 is sharp (up to rounding) and has an exact representation in terms of Fresnel integrals, however for our purposes the decimal approximation is sufficient.<sup>2</sup> The same result was also proved with a constant of  $4\sqrt{2/\pi} = 3.191\dots$  in Titchmarsh [Tit86, Lem. 4.4] and  $4/\sqrt{\pi} = 2.256\dots$  in [Rog05, Eqn. (3)].

Throughout, we let  $\psi(x)$  denote the digamma function, defined as the logarithmic derivative of the gamma function, i.e.  $\psi(x) = \Gamma'(x)/\Gamma(x)$ . We briefly recall that for  $x > 0$ , we have

$$\psi(x) < \log x - \frac{1}{2x}. \quad (2.3)$$

The digamma function has the following series representation, valid for all  $x > -1$  (see e.g. [AS13, §6.3.16])

$$\psi(1+x) = -\gamma + \sum_{n=1}^{\infty} \frac{x}{n(n+x)}. \quad (2.4)$$

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<sup>2</sup>Using an arbitrary-precision numerical integration package, we find that the variables  $\mu_0$  and  $\gamma_0$  appearing in the main result of [Ker35] appear to be  $\mu_0 = -0.7266\dots$  and  $\gamma_0 = 3.3643\dots$  instead of the stated values  $\mu_0 = -0.725\dots$  and  $\gamma_0 = 3.327\dots$  respectively.

Finally, we recall the following upper bound on harmonic numbers. For  $x \geq 2$ , we have

$$\sum_{n \leq x} \frac{1}{n} = \log x + \gamma - \frac{\{x\} - 1/2}{x} + O^*\left(\frac{1}{8x^2}\right) \leq \log x + \gamma + \frac{9}{32}, \quad (2.5)$$

where, here and throughout,  $A = O^*(B)$  means  $|A| \leq B$ . The equality is due to [MV73] and the inequality follows from  $x \geq 2$ .

We begin by approximating an exponential sum with a sum of exponential integrals in Lemma 2.1 below, which makes explicit a result of van der Corput [Cor21]. As a small technical detail, the range of the second sum in the below lemma is typically taken to be  $(\alpha - \eta, \beta + \eta)$  for arbitrary  $\eta \in (0, 1)$  — see e.g. [Tit86, Lem. 4.7]. In our presentation, we fix  $\eta = 1/2$ , which greatly simplifies the arguments that follow. This result may be compared to [Pat21, Lem. 2.26].

**Lemma 2.1.** *Let  $f(x)$  be real valued, with a continuous and steadily decreasing derivative  $f'(x)$  in  $(a, b)$ , and let  $f'(b) = \alpha, f'(a) = \beta$ . Then*

$$\left| \sum_{a < n \leq b} e(f(n)) - \sum_{\alpha - \frac{1}{2} < m < \beta + \frac{1}{2}} \int_a^b e(f(x) - mx) dx \right| \leq \frac{3}{\pi} \log(\beta - \alpha + 2) + 4.$$

*Proof.* Assume without loss of generality that  $-\frac{1}{2} < \alpha \leq \frac{1}{2}$ , for otherwise we may replace  $f(x)$  with  $f(x) - kx$  for a suitable integer  $k$ . Using Euler–Maclaurin summation (see [Tit86, Eqn. (2.12)]), we have

$$\sum_{a < n \leq b} e(f(n)) = \int_a^b e(f(x)) dx + 2\pi i \int_a^b \left(x - [x] - \frac{1}{2}\right) f'(x) e(f(x)) dx + R(a, b) \quad (2.6)$$

where

$$R(a, b) = \left(a - [a] - \frac{1}{2}\right) e(f(a)) - \left(b - [b] - \frac{1}{2}\right) e(f(b))$$

so that  $|R(a, b)| \leq 1$ . Meanwhile, for all non-integer  $x$ , we have

$$2\pi i \left(x - [x] - \frac{1}{2}\right) = -2i \sum_{m=1}^{\infty} \frac{\sin 2\pi mx}{m} = \sum_{m=1}^{\infty} \frac{e(-mx) - e(mx)}{m}$$

so that

$$\begin{aligned} & 2\pi i \int_a^b \left(x - [x] - \frac{1}{2}\right) f'(x) e(f(x)) dx \\ &= \sum_{m=1}^{\infty} \frac{1}{m} \int_a^b f'(x) e(f(x) - mx) dx - \sum_{m=1}^{\infty} \frac{1}{m} \int_a^b f'(x) e(f(x) + mx) dx = S_1 - S_2, \end{aligned}$$

say. We have

$$S_1 = \sum_{m=1}^{\infty} \frac{1}{m} \int_a^b \frac{f'(x)}{f'(x) - m} (f'(x) - m) e(f(x) - mx) dx = \frac{1}{2\pi i} \sum_{m=1}^{\infty} \frac{1}{m} \int_a^b \frac{f'(x)}{f'(x) - m} d(e(f(x) - mx))$$

and similarly

$$S_2 = \frac{1}{2\pi i} \sum_{m=1}^{\infty} \frac{1}{m} \int_a^b \frac{f'(x)}{f'(x) + m} d(e(f(x) + mx)).$$

By the second mean-value theorem, there exists  $c \in (a, b)$  such that

$$\int_a^b \frac{f'(x)}{f'(x) + m} d(e(f(x) + mx)) = \frac{f'(a)}{f'(a) + m} \int_a^c d(e(f(x) + mx)) + \frac{f'(b)}{f'(b) + m} \int_c^b d(e(f(x) + mx)).$$

However

$$\left| \int_a^b d(e(f(x) + mx)) \right| = |e(f(b) + mb) - e(f(a) + ma)| \leq 2$$

so

$$\left| \int_a^b \frac{f'(x)}{f'(x) + m} d[e(f(x) + mx)] \right| \leq 2 \left| \frac{f'(a)}{f'(a) + m} \right| + 2 \left| \frac{f'(b)}{f'(b) + m} \right| = \frac{2|\alpha|}{|m + \alpha|} + \frac{2|\beta|}{|m + \beta|}.$$

Therefore,

$$\pi |S_2| \leq \sum_{m=1}^{\infty} \left( \frac{|\alpha|}{m|\alpha + m|} + \frac{|\beta|}{m|\beta + m|} \right)$$

First, since  $\beta \geq \alpha > -1/2$ , we have  $\beta + m > 0$  and

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{|\beta|}{m|\beta + m|} &= \left| \sum_{m=1}^{\infty} \frac{\beta}{m(\beta + m)} \right| \leq \max\{-(\psi(1/2) + \gamma), \psi(\beta + 1) + \gamma\} \\ &< \log(\beta + 1) + 3 \log 2, \end{aligned} \quad (2.7)$$

where the first inequality follows from (2.4) for  $\beta \geq 0$  and via a separate evaluation for  $-1/2 < \beta < 0$ . Similarly, since  $|\alpha| \leq 1/2$ ,

$$\sum_{m=1}^{\infty} \frac{|\alpha|}{m|\alpha + m|} \leq \max\{-(\psi(1/2) + \gamma), \psi(\alpha + 1) + \gamma\} \leq 2 \log 2$$

hence

$$|S_2| \leq \frac{\log(\beta + 1) + 5 \log 2}{\pi}. \quad (2.8)$$

Now consider  $S_1$ . Let  $M := \max\{1, \beta + 1/2\}$  and

$$S_1 = \sum_{1 \leq m < M} + \sum_{m \geq M} = S_{11} + S_{12}. \quad (2.9)$$

We have

$$\pi |S_{12}| \leq \sum_{m \geq M} \left( \frac{|\alpha|}{m(m - \alpha)} + \frac{|\beta|}{m(m - \beta)} \right).$$

If  $n = \lfloor m - \beta \rfloor$ , then  $m(m - \beta) \geq n(n + \beta)$ . Furthermore note that  $n \geq 1$  for all  $m \geq M + 1$ , and that there is one integer in  $[M, M + 1)$ , say  $m_0$ . Therefore, since  $\beta > -1/2$  and by (2.7),

$$\sum_{m \geq M} \frac{|\beta|}{m(m - \beta)} \leq \frac{|\beta|}{m_0(m_0 - \beta)} + \sum_{n=1}^{\infty} \frac{|\beta|}{n(n + \beta)} \quad (2.10)$$

If  $\beta < 0$ , then  $m_0(m_0 - \beta) \geq M(M - \beta) > 1$ , and hence the first term on the RHS is at most  $1/2$ . Meanwhile using the same argument as (2.7), the sum on the RHS is bounded by  $-\psi(1/2) - \gamma = 2 \log 2$ . On the other hand if  $\beta \geq 0$ , then by the same argument used in (2.7)

$$\sum_{m \geq M} \frac{|\beta|}{m(m - \beta)} \leq \frac{\beta}{\frac{1}{2}(\beta + \frac{1}{2})} + \psi(\beta + 1) + \gamma \leq \log(\beta + 1) + \gamma + 2.$$

In either case the RHS of (2.10) is at most  $\log(\beta + 1) + 1/2 + 3 \log 2$ . Similarly, writing  $n' = \lfloor m - \alpha \rfloor$ , so that  $n' \geq 1$  for all  $m \geq M + 1$  (since  $\beta \geq \alpha$ ), and using  $|\alpha| \leq 1/2$ ,

$$\begin{aligned} \sum_{m \geq M} \frac{|\alpha|}{m(m - \alpha)} &\leq \frac{|\alpha|}{m_0(m_0 - \alpha)} + \sum_{n'=1}^{\infty} \frac{|\alpha|}{n'(n' + \alpha)} \\ &\leq \max\{1/2 + 2 \log 2, 1 + \psi(\alpha + 1) + \gamma\} \leq 1/2 + 2 \log 2. \end{aligned}$$

Thus

$$|S_{12}| \leq \frac{\log(\beta + 1) + 1 + 5 \log 2}{\pi}. \quad (2.11)$$

We now divide our argument into the following two cases.

*Case 1:*  $\beta \leq 1/2$ . Then,  $M = 1$  and  $S_{11}$  is an empty sum. Then, we have (vacuously)

$$\left| S_1 - \sum_{1 \leq m < \beta + 1/2} \int_a^b e(f(x) - mx) dx \right| = |S_{12}|, \quad (2.12)$$

since the sum on the LHS is empty.

*Case 2:*  $\beta > 1/2$ . Then,  $M = \beta + 1/2$  and we let

$$S_{11} = S_3 + \sum_{1 \leq m < \beta + 1/2} \int_a^b e(f(x) - mx) dx, \quad (2.13)$$

where

$$\begin{aligned} S_3 &:= \sum_{1 \leq m < \beta + 1/2} \frac{1}{m} \int_a^b (f'(x) - m) e(f(x) - mx) dx \\ &= \left| \sum_{1 \leq m < \beta + 1/2} \frac{1}{2\pi m i} e(f(x) - mx) \right|_a^b. \end{aligned}$$

Therefore,

$$|S_3| \leq \frac{1}{\pi} \sum_{1 \leq m < \beta + 1/2} \frac{1}{m} < \frac{1}{\pi} \left( \log(\beta + 1) + \gamma + \frac{9}{32} \right),$$

where in the last inequality we have used (2.5) if  $\beta + 1/2 \geq 2$ , and a direct evaluation if  $\beta + 1/2 < 2$ . It follows that in this case, from combining (2.9) and (2.13), that

$$\left| S_1 - \sum_{1 \leq m < \beta + 1/2} \int_a^b e(f(x) - mx) dx \right| \leq |S_3| + |S_{12}| \quad (2.14)$$

Therefore, in either case, by collecting (2.6), (2.8), (2.11), (2.12) and (2.14) we have

$$\sum_{a < n \leq b} e(f(n)) = \sum_{0 \leq m < \beta + 1/2} \int_a^b e(f(x) - mx) dx + R_1$$



where

$$\begin{aligned} |R_1| &\leq |S_2| + |S_{12}| + |S_3| + 1 \\ &\leq \frac{\log(\beta + 1) + 5 \log 2}{\pi} + \frac{\log(\beta + 1) + 1 + 5 \log 2}{\pi} \\ &\quad + \frac{1}{\pi} \left( \log(\beta + 1) + \gamma + \frac{9}{32} \right) + 1 \\ &< \frac{3}{\pi} \log(\beta + 1) + 4. \end{aligned}$$

To complete the argument we note that the assumption that  $-1/2 < \alpha \leq 1/2$  implies that  $\beta + 1 < \beta - \alpha + 2$ ,<sup>3</sup> and that sums over  $[0, \beta + 1/2)$  are equivalent to sums over  $(\alpha - 1/2, \beta + 1/2)$ .  $\square$

Next, we require a lemma related to the principle of stationary phase, which approximates an exponential integral. The traditional presentation of this result (see e.g. [Tit86, Lem. 4.6]) has a main error term of size  $O(\lambda_2^{-4/5} \lambda_3^{1/5})$ , where  $\lambda_2, \lambda_3$  are respectively the orders of the second and third derivative of the phase function. In the following lemma, we make explicit an argument of Phillips [Phi33] to bound this error term to  $O(\lambda_2^{-1} \lambda_3^{1/3})$ , which is smaller in our application. We also record that Heath-Brown [HB83] has shown that under suitable conditions, the main error term may be removed completely. However, since the error term is already of an acceptable size, we do not pursue such an optimisation here.

**Lemma 2.2.** *Let  $f(x)$  be real and three times differentiable, satisfying  $f'' < 0$ ,*

$$0 < \lambda_2 \leq |f''(x)| \leq h_2 \lambda_2, \quad |f'''(x)| \leq h_3 \lambda_3, \quad x \in (a, b).$$

*Furthermore, suppose  $f'(c) = 0$  for some  $c \in [a, b]$ . Then,*

$$\left| \int_a^b e(f(x)) dx - \frac{e(f(c) - 1/8)}{|f''(c)|^{1/2}} \right| \leq \frac{2 \cdot 3^{2/3}}{\pi^{2/3}} h_3^{1/3} \lambda_2^{-1} \lambda_3^{1/3} + \frac{1}{\pi} \left( \frac{1}{|f'(a)|} + \frac{1}{|f'(b)|} \right).$$

*Proof.* Suppose first that  $c \in [a + \delta, b - \delta]$ , for some fixed  $\delta > 0$  to be chosen later. Let

$$\int_a^b e(f(x)) dx = \int_a^{c-\delta} + \int_{c-\delta}^{c+\delta} + \int_{c+\delta}^b = I_1 + I_2 + I_3, \quad (2.15)$$

say. Since  $f'' < 0$ , for all  $x \in [a, c - \delta]$  we have by the mean-value theorem

$$f'(x) \geq f'(c - \delta) = f'(c) - \delta f''(\xi) \geq \delta \lambda_2$$

for some  $\xi \in [c - \delta, c]$ . Similarly,  $|f'(x)| \geq \delta \lambda_2$  for all  $x \in [c + \delta, b]$ . Via (2.1), we have

$$|I_1|, |I_3| \leq \frac{1}{\pi \delta \lambda_2}. \quad (2.16)$$

Let

$$g(x) := f(x + c) - f(c) - \frac{1}{2} x^2 f''(c)$$

so that, for all  $x$  there exists some  $\xi \in (c, x + c)$  such that

$$g(x) = f(c) + x f'(c) + \frac{x^2}{2} f''(c) + \frac{x^3}{6} f'''(\xi) - f(c) - \frac{x^2}{2} f''(c) \leq \frac{x^3}{6} h_3 \lambda_3. \quad (2.17)$$

<sup>3</sup>This inequality can be readily sharpened (the constant of 2 is chosen for cosmetic purposes). In any case, for our application the constant term makes no difference to the final result.

Hence

$$\begin{aligned} \int_{c-\delta}^{c+\delta} e(f(x)) dx &= e(f(c)) \int_{c-\delta}^{c+\delta} e\left(\frac{(x-c)^2}{2} f''(c)\right) e\left(f(x) - f(c) - \frac{(x-c)^2}{2} f''(c)\right) dx \\ &= e(f(c)) \int_{-\delta}^{\delta} e\left(\frac{x^2}{2} f''(c)\right) dx + e(f(c)) \int_{-\delta}^{\delta} e\left(\frac{x^2}{2} f''(c)\right) (e(g(x)) - 1) dx. \end{aligned} \quad (2.18)$$

However,

$$\begin{aligned} \int_{-\delta}^{\delta} e\left(\frac{x^2}{2} f''(c)\right) dx &= \frac{1}{|\pi f''(c)|^{1/2}} \int_0^{\pi \delta^2 |f''(c)|} \frac{e^{-iu}}{\sqrt{u}} du \\ &= \frac{1}{|\pi f''(c)|^{1/2}} \left( \int_0^{\infty} \frac{e^{-iu}}{\sqrt{u}} du - \int_{\pi \delta^2 |f''(c)|}^{\infty} \frac{e^{-iu}}{\sqrt{u}} du \right) \\ &= \frac{e(-1/8)}{|\pi f''(c)|^{1/2}} + O^*\left(\frac{1}{\pi \delta |f''(c)|}\right) \end{aligned}$$

so that

$$\left| e(f(c)) \int_{-\delta}^{\delta} e\left(\frac{x^2}{2} f''(c)\right) dx - \frac{e(f(c) - 1/8)}{|f''(c)|^{1/2}} \right| \leq \frac{1}{\pi \delta \lambda_2}. \quad (2.19)$$

We will now bound the modulus of

$$I := \int_{-\delta}^{\delta} e\left(\frac{x^2}{2} f''(c)\right) (e(g(x)) - 1) dx.$$

First, suppose that  $\delta \leq h/(\lambda_2 \delta)$  for some arbitrary constant  $h > 0$  to be chosen later. Then via the trivial bound, we have

$$|I| \leq \int_{-\delta}^{\delta} |e(g(x)) - 1| dx \leq 4\delta \leq \frac{4h}{\delta \lambda_2}. \quad (2.20)$$

Assume now that  $\delta > h/(\lambda_2 \delta)$ . Then

$$I = \int_{-\delta}^{-h/(\lambda_2 \delta)} + \int_{-h/(\lambda_2 \delta)}^{h/(\lambda_2 \delta)} + \int_{h/(\lambda_2 \delta)}^{\delta} = I_4 + I_5 + I_6,$$

say. First, consider  $I_5$ . Using the trivial bound, we have

$$|I_5| \leq \int_{-h/(\lambda_2 \delta)}^{h/(\lambda_2 \delta)} |e(g(x)) - 1| dx \leq \frac{4h}{\delta \lambda_2}. \quad (2.21)$$

Next, consider  $I_6$ . Letting  $\lambda = f''(c)/2$ , and integrating by parts, we have

$$\begin{aligned} I_6 &= \int_{h/(\lambda_2 \delta)}^{\delta} e(\lambda x^2) (e(g(x)) - 1) dx = \int_{h/(\lambda_2 \delta)}^{\delta} 4\pi i \lambda x e^{2\pi i \lambda x^2} \frac{e(g(x)) - 1}{4\pi i \lambda x} dx \\ &= \left[ e(\lambda x^2) \frac{e(g(x)) - 1}{4\pi i \lambda x} \right]_{h/(\lambda_2 \delta)}^{\delta} - \int_{h/(\lambda_2 \delta)}^{\delta} e(\lambda x^2) \frac{d}{dx} \left( \frac{e(g(x)) - 1}{4\pi i \lambda x} \right) dx \end{aligned} \quad (2.22)$$

However,

$$\frac{d}{dx} \left( \frac{e^{2\pi i g(x)} - 1}{x} \right) = \frac{2\pi i x g'(x) e(g(x)) - (e(g(x)) - 1)}{x^2}$$

and for some  $\xi \in (c, x + c)$ , we have

$$g'(x) = f'(x + c) - xf''(c) = \left( f'(c) + xf''(c) + \frac{x^2}{2} f'''(\xi) \right) - xf''(c) \leq \frac{x^2}{2} h_3 \lambda_3. \quad (2.23)$$

Furthermore, we use the identity

$$e(x) = \int_0^{2\pi x} i e^{it} dt + 1$$

to obtain using (2.17) that

$$|e(g(x)) - 1| = \left| \int_0^{2\pi g(x)} i e^{it} dt \right| \leq 2\pi g(x) \leq \frac{\pi}{3} x^3 h_3 \lambda_3. \quad (2.24)$$

This implies that, by combining (2.23) and (2.24),

$$|2\pi i x g'(x) e(g(x)) - (e(g(x)) - 1)| \leq 2\pi \frac{x^3}{2} h_3 \lambda_3 + \frac{\pi}{3} x^3 h_3 \lambda_3 = \frac{4\pi}{3} x^3 h_3 \lambda_3,$$

and thus

$$\left| \int_{h/(\lambda_2 \delta)}^{\delta} e(\lambda x^2) \frac{d}{dx} \left( \frac{e(g(x)) - 1}{4\pi i \lambda x} \right) dx \right| \leq \frac{h_3 \lambda_3}{3|\lambda|} \int_{h/(\lambda_2 \delta)}^{\delta} x dx = \frac{h_3 \lambda_3}{6|\lambda|} \left( \delta^2 - \frac{h^2}{(\delta \lambda_2)^2} \right). \quad (2.25)$$

Meanwhile, once again using (2.24), and the triangle inequality,

$$\left| \left[ e(\lambda x^2) \frac{e(g(x)) - 1}{4\pi i \lambda x} \right]_{h/(\lambda_2 \delta)}^{\delta} \right| \leq \frac{h_3 \lambda_3}{12|\lambda|} \left( \delta^2 + \frac{h^2}{(\delta \lambda_2)^2} \right), \quad (2.26)$$

and so, collecting (2.22), (2.25) and (2.26), and using  $|\lambda| = |f''(c)|/2 \geq \lambda_2/2$ ,

$$|I_6| \leq \frac{h_3 \lambda_3}{4|\lambda|} \delta^2 - \frac{h_3 \lambda_3 h^2}{12(\delta \lambda_2)^2} < \frac{h_3}{2} \frac{\lambda_3}{\lambda_2} \delta^2. \quad (2.27)$$

We bound  $I_4$  in the same way. Therefore, collecting (2.21) and (2.27) we find

$$|I| \leq \frac{h_3 \lambda_3 \delta^2}{\lambda_2} + \frac{4h}{\delta \lambda_2}, \quad (2.28)$$

in the case where  $\delta > h/(\lambda_2 \delta)$ . However, since the bound (2.28) is strictly greater than (2.20), we conclude (2.28) in fact holds for all  $\delta > 0$ . Combining this with (2.16), (2.18), (2.19), we find that

$$\left| \int_a^b e(f(x)) dx - \frac{e(f(c) - 1/8)}{|f''(c)|^{1/2}} \right| \leq \frac{h_3 \lambda_3 \delta^2}{\lambda_2} + \left( 4h + \frac{3}{\pi} \right) \frac{1}{\lambda_2 \delta}.$$

However, since  $h > 0$  is arbitrary, we take the limit as  $h \rightarrow 0^+$  and choose

$$\delta = \left( \frac{3}{\pi h_3} \right)^{1/3} \lambda_3^{-1/3}$$

to balance the two terms on the RHS. This choice gives

$$\left| \int_a^b e(f(x)) dx - \frac{e(f(c) - 1/8)}{|f''(c)|^{1/2}} \right| \leq \frac{2 \cdot 3^{2/3}}{\pi^{2/3}} h_3^{1/3} \lambda_2^{-1} \lambda_3^{-1/3}. \quad (2.29)$$

If  $c + \delta > b$ , then we instead bound  $I_3$  using

$$|I_3| = \left| \int_b^{c+\delta} e(f(x)) dx \right| \leq \frac{1}{\pi |f'(b)|}, \quad (2.30)$$

which follows from (2.1) since for all  $x \in [b, c+\delta]$ , we have  $0 = f'(c) > f'(b) \geq f'(x)$ , as  $b > c$  and  $f'' < 0$ . Similarly, if  $c - \delta < a$ , we instead bound  $I_1$  using

$$|I_1| = \left| \int_{c-\delta}^a e(f(x)) dx \right| \leq \frac{1}{\pi |f'(a)|}. \quad (2.31)$$

The desired result follows from combining (2.29), (2.30) and (2.31).  $\square$

**Lemma 2.3** (Improved Poisson summation formula). *Let  $f(x)$  be three times differentiable. Let  $f'(x)$  be decreasing in  $[a, b]$  and  $f'(b) = \alpha$ ,  $f'(a) = \beta$ . For all integer  $\nu \in (\alpha, \beta]$ , let  $x_\nu$  be defined by  $f'(x_\nu) = \nu$ . Furthermore suppose that  $\lambda_2 \leq |f''(x)| \leq h_2 \lambda_2$  and  $|f'''(x)| \leq h_3 \lambda_3$ . Then*

$$\begin{aligned} & \left| \sum_{\alpha < n \leq b} e(f(n)) - \sum_{\alpha < \nu \leq \beta} \frac{e(f(x_\nu) - \nu x_\nu - 1/8)}{|f''(x_\nu)|^{1/2}} \right| \\ & \leq \frac{4.686}{\sqrt{\lambda_2}} + \frac{2 \cdot 3^{2/3}}{\pi^{2/3}} h_2 h_3^{1/3} (b-a) \lambda_3^{1/3} + \frac{5}{\pi} \log(\beta - \alpha + 2) + 6. \end{aligned}$$

*Proof.* We use Lemma 2.2 to obtain

$$\begin{aligned} & \sum_{\alpha + \frac{1}{2} < \nu < \beta - \frac{1}{2}} \int_a^b e(f(x) - \nu x) dx = \sum_{\alpha + \frac{1}{2} < \nu < \beta - \frac{1}{2}} \frac{e(f(x_\nu) - \nu x_\nu - 1/8)}{|f''(x_\nu)|^{1/2}} \\ & + \sum_{\alpha + \frac{1}{2} < \nu < \beta - \frac{1}{2}} \frac{2 \cdot 3^{2/3}}{\pi^{2/3}} h_3^{1/3} \lambda_2^{-1} \lambda_3^{1/3} + \sum_{\alpha + \frac{1}{2} < \nu < \beta - \frac{1}{2}} \frac{1}{\pi} \left( \frac{1}{|f'(a) - \nu|} + \frac{1}{|f'(b) - \nu|} \right) \\ & = S_1 + S_2 + S_3. \end{aligned} \quad (2.32)$$

Since there is at most one integer each in the intervals  $(\alpha, \alpha + \frac{1}{2}]$  and  $[\beta - \frac{1}{2}, \beta]$ , and  $|f''(x_\nu)|^{1/2} \geq \lambda_2^{1/2}$ , we have

$$S_1 = \sum_{\alpha < \nu \leq \beta} \frac{e(f(x_\nu) - \nu x_\nu - 1/8)}{|f''(x_\nu)|^{1/2}} + O^* \left( \frac{2}{\sqrt{\lambda_2}} \right). \quad (2.33)$$

Next, since  $|f''(x)| \leq h_2 \lambda_2$ , we have  $\beta - \alpha = f'(a) - f'(b) \leq (b-a) h_2 \lambda_2$ ,

$$|S_2| < \frac{2 \cdot 3^{2/3}}{\pi^{2/3}} h_3^{1/3} (\beta - \alpha) \lambda_2^{-1} \lambda_3^{1/3} \leq \frac{2 \cdot 3^{2/3}}{\pi^{2/3}} h_2 h_3^{1/3} (b-a) \lambda_3^{1/3}. \quad (2.34)$$

Finally, consider  $S_3$ . For all  $a \leq \nu \leq b$ , we have  $|f'(a) - \nu| = \beta - \nu$  and  $|f'(b) - \nu| = \nu - \alpha$ . Furthermore, the  $k$ th smallest integer in the interval  $(\alpha + 1/2, \beta + 1/2)$  is bounded below by  $\alpha + k - 1/2$ . Therefore,

$$\sum_{\alpha + \frac{1}{2} < \nu < \beta - \frac{1}{2}} \frac{1}{\nu - \alpha} < \sum_{1 \leq n < \beta - \alpha - \frac{1}{2}} \frac{1}{n - \frac{1}{2}} < \psi(\beta - \alpha) - \psi(1/2)$$

where  $\psi(x)$  is the digamma function. Similarly

$$\sum_{\alpha + \frac{1}{2} < \nu < \beta - \frac{1}{2}} \frac{1}{\beta - \nu} < \psi(\beta - \alpha) - \psi(1/2).$$

Therefore, using  $\psi(x) \leq \log x$  for  $x > 0$ ,

$$|S_3| \leq \frac{2}{\pi}\psi(\beta - \alpha) - \frac{2}{\pi}\psi(1/2) < \frac{2}{\pi}\log(\beta - \alpha) + 2. \quad (2.35)$$

Finally, since the intervals  $(\alpha - 1/2, \alpha + 1/2]$  and  $[\beta - 1/2, \beta + 1/2)$  contain at most one integer each, and using (2.2),

$$\sum_{\alpha + \frac{1}{2} < \nu < \beta - \frac{1}{2}} \int_a^b e(f(x) - \nu x) dx = \sum_{\alpha - \frac{1}{2} < \nu < \beta + \frac{1}{2}} \int_a^b e(f(x) - \nu x) dx + O^*\left(\frac{2.686}{\sqrt{\lambda_2}}\right). \quad (2.36)$$

The desired result follows upon applying Lemma 2.1 and collecting (2.32), (2.33), (2.34), (2.35) and (2.36).  $\square$

### 3. PROOF OF THEOREM 1.1

This section contains the proof of our main result. We derive an upper bound on  $\zeta(1/2 + it)$  using the Riemann–Siegel formula, which allows us to express  $\zeta(1/2 + it)$  in terms of an exponential sum of length  $O(t^{1/2})$ . This enables us to readily apply the techniques of the previous sections to produce a non-trivial estimate of  $\zeta(1/2 + it)$ . The main ingredients this step are the explicit  $A$ ,  $B$  and  $A^k B(0, 1)$  processes, given by Lemma 1.2, 2.3 and 1.4 respectively.

To begin, we recall the following result, due to Hiary [Hia16], which is a consequence of the Riemann–Siegel formula.

**Lemma 3.1.** *For all  $t \geq 200$ ,*

$$|\zeta(1/2 + it)| \leq 2 \left| \sum_{1 \leq n \leq \lfloor \sqrt{t/(2\pi)} \rfloor} n^{-1/2 - it} \right| + 1.48t^{-1/4} + 0.127t^{-3/4} \quad (3.1)$$

*Proof.* See Hiary [Hia16, Lem. 2.1] and also Gabcke [Gab79].  $\square$

We use a similar approach as [Tit86, Thm. 5.18] to evaluate the main sum on the RHS of (3.1). We divide the sum into three subsums and bound each individually. Firstly, for  $n \ll t^{27/82}$ , we use the triangle inequality and the trivial bound. Secondly, for  $t^{27/82} \ll n \ll t^{7/17}$  we use Lemma 1.4 with  $k = 4$ . Lastly, for  $t^{7/17} \ll n \ll t^{1/2}$ , we use the  $ABA^3B(0, 1)$  process (Lemma 3.4 below).

We remark that the last subsum, taken over  $t^{7/17} \ll n \ll t^{1/2}$ , is by far the most significant. In fact, the second sum can be bounded to be  $\ll t^{19/119}$  which is  $o(t^{27/164})$ . Additionally, we have the freedom make the first subsum as small as we please, by appropriately choosing the boundary between the first and second subsums. Therefore, in what follows we will expend the most effort in bounding the third subsum.

To begin, let  $h_1, h_2 > 1$ , and  $\theta_1, \theta_2, \theta_3 > 0$  be scaling parameters to be chosen later. Define

$$a_k := \left\lfloor h_1^{-k} \sqrt{\frac{t}{2\pi}} \right\rfloor, \quad k = 0, 1, \dots, K \quad (3.2)$$

where

$$K := K(t) = \left\lceil \frac{\frac{3}{34} \log t - \log(\theta_2 \sqrt{2\pi})}{\log h_1} \right\rceil. \quad (3.3)$$

Note that this choice of  $K$  guarantees that

$$a_K \leq h_1^{-K} \sqrt{\frac{t}{2\pi}} \leq \theta_2 t^{7/17}. \quad (3.4)$$

Similarly, let

$$a'_r = \left\lfloor h_2^{-r} \theta_2 t^{7/17} \right\rfloor, \quad r = 0, 1, \dots, R, \quad (3.5)$$

$$R := R(t) = \left\lceil \frac{\frac{115}{1394} \log t - \log(\theta_3/\theta_2)}{\log h_2} \right\rceil \quad (3.6)$$

so that  $a'_0 \leq a_K$ . These parameters are chosen so that  $a'_R \leq \theta_3 t^{27/82}$ .

We thus divide

$$\begin{aligned} \sum_{1 \leq n \leq \lfloor \sqrt{t/(2\pi)} \rfloor} n^{-1/2-it} &= \sum_{1 \leq n \leq a'_R} + \sum_{a'_R < n \leq a_K} + \sum_{a_K < n \leq \lfloor \sqrt{t/(2\pi)} \rfloor} \\ &= S_1 + S_2 + S_3, \end{aligned} \quad (3.7)$$

say. The next few lemmas are used to bound each of the three subsums.

**Lemma 3.2.** *For  $t \geq t_0$  and  $\theta_3 > 0$ , we have*

$$|S_1| \leq C_0 t^{27/164} - \sqrt{2}, \quad C_0 := 2 \sqrt{\theta_3 \left( 1 + \frac{1}{2t_0^{27/82}} \right)}.$$

*Proof.* Recall that  $a'_R \leq \theta_3 t^{27/82}$ , so that, by the triangle inequality and the trivial bound,

$$\begin{aligned} \left| \sum_{1 \leq n \leq a'_R} n^{-1/2-it} \right| &\leq \sum_{n=1}^{a'_R} \frac{1}{\sqrt{n}} \leq \int_{1/2}^{a'_R+1/2} \frac{dx}{x^{1/2}} \leq \int_{1/2}^{\theta_3 t^{27/82}+1/2} \frac{dx}{x^{1/2}} \\ &\leq 2 \sqrt{\theta_3 t^{27/82} + \frac{1}{2}} - \sqrt{2} \leq C_0 t^{27/164} - \sqrt{2}, \end{aligned}$$

for all  $t \geq t_0$ . Here, the second inequality follows from the convexity of  $x^{-1/2}$  and Jensen's inequality, since

$$n^{-1/2} = \left( \int_{n-1/2}^{n+1/2} x dx \right)^{-1/2} \leq \int_{n-1/2}^{n+1/2} \frac{dx}{x^{1/2}}.$$

□

**Lemma 3.3.** *Suppose  $0 < t_0 \leq t \leq t_1$ ,  $h_2 > 1$  and  $\eta_2, \theta_2, \theta_3 > 0$ . Then*

$$|S_2| \leq D_3 t^{19/119} + D_4 t^{71/476}.$$

where

$$\begin{aligned} h_3 &:= \frac{h_2}{1 - \frac{h_2}{\theta_2} t^{-27/82}}, \\ D_3 &:= \frac{3^{1/14}}{\pi^{1/14}} A_4(\eta_2, h_3) h_3^{5/7} (h_3 - 1) \theta_2^{3/14} \frac{1 - h_2^{-3R(t_1)/14}}{h_2^{3/14} - 1}, \\ D_4 &:= \frac{\pi^{1/14}}{3^{1/14}} B_4(\eta_2) h_3^{2/7} (h_3 - 1)^{3/4} \theta_2^{15/28} \frac{1 - h_2^{-15R(t_1)/28}}{h_2^{15/28} - 1}. \end{aligned}$$

*Proof.* With  $a'_r$  as defined in (3.5), we have

$$\frac{a'_r}{a'_{r+1}} \leq \frac{h_2^{-r} \theta_2 t^{7/17}}{h_2^{-(r+1)} \theta_2 t^{7/17} - 1} \leq h_3.$$

since

$$\begin{aligned} h_2^{-(r+1)} \theta_2 t^{7/17} &\geq h_2^{-\left(\frac{115}{1394} \log t - \log(\theta_3/\theta_2)\right)/\log h_2 - 1} \theta_2 t^{7/17} \\ &= \frac{\theta_2}{h_2} t^{-115/1394} \frac{\theta_3}{\theta_2} \theta_2 t^{7/17} \geq \frac{\theta_3}{h_2} t_0^{27/82} \end{aligned}$$

Applying Lemma 1.4 with  $k = 4$ ,  $h = h_3^4$ ,  $a = a'_r$ ,  $b = a'_{r-1} \leq h_3 a$  and

$$f(x) = -\frac{t}{2\pi} \log x, \quad \lambda_4 = \frac{3t}{\pi(h_3 a'_r)^4},$$

we obtain, for any  $\eta_2 > 0$ ,

$$\begin{aligned} S_f(a'_r, a'_{r-1} - a'_r) &\leq A_4(\eta_2, h_3^4) h_3 (h_3 - 1) a'_r \left( \frac{3t}{\pi(h_3 a'_r)^4} \right)^{1/14} \\ &\quad + B_4(\eta_2) (h_3 - 1)^{3/4} a_r'^{3/4} \left( \frac{3t}{\pi(h_3 a'_r)^4} \right)^{-1/14} \\ &= D_1 a_r'^{5/7} t^{1/14} + D_2 a_r'^{29/28} t^{-1/14} \end{aligned}$$

where

$$D_1(h_3) := \frac{3^{1/14}}{\pi^{1/14}} A_4 h_3^{5/7} (h_3 - 1), \quad D_2(h_3) := \frac{\pi^{1/14}}{3^{1/14}} B_4 h_3^{2/7} (h_3 - 1)^{3/4}.$$

Next, by partial summation

$$\begin{aligned} \left| \sum_{a'_r < n \leq a'_{r-1}} n^{-1/2-it} \right| &\leq a_r'^{-1/2} \max_{L \in (a'_r, a'_{r-1}]} S_f(a'_r, L - a'_r) \\ &\leq D_1(h_3) a_r'^{3/14} t^{1/14} + D_2(h_3) a_r'^{15/28} t^{-1/14} \end{aligned}$$

so that, combined with

$$a'_r \leq h_2^{-r} \theta_2 t^{7/17},$$

we obtain

$$\begin{aligned} \left| \sum_{a'_R < n \leq a'_0} n^{-1/2-it} \right| &\leq \sum_{r=1}^R \left| \sum_{a'_r < n \leq a'_{r-1}} n^{-1/2-it} \right| \\ &\leq D_1 t^{1/14} \sum_{r=1}^R \left( h_2^{-r} \theta_2 t^{7/17} \right)^{3/14} + D_2 t^{-1/14} \sum_{r=1}^R \left( h_2^{-r} \theta_2 t^{7/17} \right)^{15/28} \\ &= D_1 \theta_2^{3/14} t^{19/119} \sum_{r=1}^R h_2^{-3r/14} + D_2 \theta_2^{15/28} t^{71/476} \sum_{r=1}^R h_2^{-15r/28} \\ &= D_1 \theta_2^{3/14} \frac{1 - h_2^{-3R/14}}{h_2^{3/14} - 1} t^{19/119} + D_2 \theta_2^{15/28} \frac{1 - h_2^{-15R/28}}{h_2^{15/28} - 1} t^{71/476}. \end{aligned}$$

The result follows from  $R(t) \leq R(t_1)$ .  $\square$

**Lemma 3.4.** *Let  $t \geq t_0 > 0$  and  $\theta_1, \theta_2, \eta_1 > 0$ ,  $1 < h \leq 2$  be arbitrary constants. Suppose  $a, b$  satisfy  $\theta_2 t^{7/17} < a < b \leq ha$  and  $q_0 := \theta_1 \theta_2^{7/17} t_0^{65/697} \geq 2$ . Then*

$$\left| \sum_{a < n \leq b} n^{-it} \right| \leq C_1(h) a^{23/41} t^{11/82} + C_2(h) a^{147/328} t^{61/328} \\ + E_1(h) a^{169/164} t^{-15/82} + E_2(h) a^{59/123} t^{5/41} + E_3(h) a^{1/2}.$$

where

$$C_1(h) := \alpha \left( \frac{\theta_1^{-1}(h-1)}{1-q_0^{-1}} + 0.4750 \theta_1^{11/30} A_5 \left( \eta_1, \frac{76545\sqrt{2}}{107264} h^9 \right) h^{21/8} (h-1) \right)^{1/2},$$

$$C_2(h) := \alpha \left( 0.2531 \theta_1^{61/120} B_5(\eta_1) h^{3/2} (h-1)^{7/8} \right)^{1/2},$$

$$E_1(h) := \alpha (12.496\sqrt{\pi})^{1/2} h^{3/4} \left( \theta_1 - \frac{1}{\theta_2 t_0^{100/697}} \right)^{-1/4},$$

$$E_2(h) := \alpha \left( \frac{9}{14} \theta_1^{1/3} \left( \frac{4.465(h-1)^{1/3}}{\pi^{4/3} \theta_2^{1/3} t_0^{7/51}} + \frac{6}{\pi} h^3 (h-1) \right) \right)^{1/2},$$

$$E_3(h) := \alpha \left( 6 + \frac{5}{\pi} \log 2 \right)^{1/2}, \quad \alpha = \sqrt{h-1 + \frac{\theta_1}{\theta_2^{5/41} t_0^{222/697}}},$$

and  $A_5, B_5$  are functions defined in Lemma 1.4.

*Proof.* Let

$$f(x) := -\frac{t}{2\pi} \log x, \quad a \leq x \leq b,$$

so that

$$\sum_{a < n \leq b} n^{-it} = S_f(a, b-a).$$

Also, let

$$g_r(x) := f(x+r) - f(x) = -\frac{t}{2\pi} \log \left( 1 + \frac{r}{x} \right), \quad a \leq x \leq b-r,$$

$$\beta := g'_r(a), \quad \alpha := g'_r(b-r).$$

Note that since  $b \leq ha$ , we have  $\beta \leq h\alpha$ . Furthermore, let  $x_\nu$  be such that  $g'_r(x_\nu) = \nu$ , i.e.

$$x_\nu := \frac{1}{2} \sqrt{r^2 + \frac{2tr}{\pi\nu}} - \frac{r}{2}.$$

Finally, define

$$\phi_r(\nu) := g_r(x_\nu) - \nu x_\nu.$$



3.0.1. *Applying the  $A^3B(0, 1)$  process.* We begin by considering the exponential sum

$$S_{\phi_r}(\alpha, \beta - \alpha) = \sum_{\alpha < n \leq \beta} e(\phi_r(n)),$$

which is an intermediary sum encountered prior to applying the final  $AB$  process. We bound this sum using the 5th derivative test, which corresponds to the  $A^3B(0, 1)$  process. Via a direct computation, we have

$$\begin{aligned} |\phi_r^{(5)}(\nu)| &= \frac{3t\sqrt{r}}{2\sqrt{\pi}\nu^{9/2}(\pi r\nu + 2t)^{7/2}} (8\pi^3 r^3 \nu^3 + 36\pi^2 r^2 t \nu^2 + 60\pi r \nu t^2 + 35t^3) \\ &= \frac{3}{2\sqrt{\pi}(x+2)^{7/2}} (8x^3 + 36x^2 + 60x + 35) \frac{(tr)^{1/2}}{\nu^{9/2}} \end{aligned} \quad (3.8)$$

where  $x := \pi r \nu / t$ . For all  $\alpha \leq \nu \leq \beta$ , we have  $x \in (0, 1/4]$ , since

$$\frac{\pi r \beta}{t} = \frac{\pi r g'_r(a)}{t} = \frac{1}{2} \frac{r^2}{a(a+r)} \leq \frac{1}{4}$$

as  $h \leq 2$  implies  $r \leq a$ , and  $x^2/(a(a+x))$  is increasing for  $x > 0$ .<sup>4</sup> Therefore,

$$\frac{6704}{2187} \leq \frac{8x^3 + 36x^2 + 60x + 35}{(x+2)^{7/2}} \leq \frac{35}{8\sqrt{2}}. \quad (3.9)$$

Furthermore,

$$\frac{tr}{2\pi a^2} \geq \frac{tr}{2\pi a(a+r)} = g'_r(a) = \beta \geq \nu \geq \alpha = g'_r(b-r) = \frac{tr}{2\pi b(b-r)} \geq \frac{tr}{2\pi h^2 a^2}$$

so that

$$(2\pi)^{9/2} \frac{a^9}{t^4 r^4} \leq \frac{(tr)^{1/2}}{\nu^{9/2}} \leq (2\pi)^{9/2} h^9 \frac{a^9}{t^4 r^4}. \quad (3.10)$$

Combining (3.8), (3.9) and (3.10), for  $\nu \in [\alpha, \beta]$ , we have

$$\lambda_5 \leq |\phi_r^{(5)}(\nu)| \leq h_5 \lambda_5 \quad (3.11)$$

where

$$\lambda_5 = \frac{53632\sqrt{2}\pi^4}{729} \frac{a^9}{t^4 r^4}, \quad h_5 := \frac{76545\sqrt{2}}{107264} h^9.$$

Meanwhile, by the mean-value theorem we have  $\beta - \alpha \leq (b - a - r)g''(\xi)$  for some  $\xi \in [a, b - r]$ , and so by directly computing  $g''(x)$  we have

$$\beta - \alpha < a(h-1) \frac{tr}{2\pi} \frac{2a+r}{a^2(a+r)^2} \leq \frac{h-1}{\pi} \frac{tr}{a^2} \quad (3.12)$$

We apply Lemma 1.4 with  $k = 5$ ,  $f = \phi_r$ ,  $N = b - a$  and  $h = h_5$  to obtain, using (3.11) and (3.12),

$$\begin{aligned} |S_{\phi_r}(\alpha, \beta - \alpha)| &\leq A_5(\eta_1, h_5) h_5^{1/8} (\beta - \alpha) \lambda_5^{1/30} + B_5(\eta_1) (\beta - \alpha)^{7/8} \lambda_5^{-1/30} \\ &< c_1 a^{-17/10} (tr)^{13/15} + c_2 a^{-41/20} (tr)^{121/120} \end{aligned} \quad (3.13)$$

for any  $\eta_1 > 0$ , where

$$c_1 = A_5(\eta_1, h_5) b_1 h^{9/8} (h-1), \quad c_2 = B_5(\eta_1) b_2 (h-1)^{7/8}$$

---

<sup>4</sup>In fact a much sharper inequality can be applied here, since we ultimately take  $r = o(a)$ . However, such optimisations do not appear to affect the final result.

and

$$b_1 := \frac{1}{\pi} \left( \frac{53632\sqrt{2}\pi^4}{729} \right)^{1/30} \left( \frac{76545\sqrt{2}}{107264} \right)^{1/8},$$

$$b_2 := \frac{1}{\pi^{7/8}} \left( \frac{729}{53632\sqrt{2}\pi^4} \right)^{1/30}.$$

Note that the leading term of (3.13) corresponds to the  $A^3B(0, 1) = (1/30, 13/15)$  exponent pair.

3.0.2. *Applying the B process.* Equipped with a bound for  $S_{\phi_r}$ , we apply the  $B$  process (Lemma 2.2) with  $f = g$ . The end result of this subsection is an explicit  $BA^3B(0, 1) = (11/30, 8/15)$  exponent pair. To do this we first require a few intermediary results. To begin, note that

$$g_r''(x_\nu) \geq g_r''(b-r) = \frac{tr}{2\pi} \frac{2b-r}{b^2(b-r)^2} \geq \frac{tr}{\pi} \frac{1}{(b-r/2)^3} > \frac{tr}{\pi b^3} \geq \frac{tr}{\pi h^3 a^3}.$$

Here, the second inequality follows from the arithmetic-geometric means inequality. Therefore, by partial summation and using (3.13),

$$\left| \sum_{\alpha < \nu \leq \beta} \frac{e(\phi_r(\nu))}{|g_r''(x_\nu)|^{1/2}} \right| \leq \pi^{1/2} h^{3/2} \frac{a^{3/2}}{(tr)^{1/2}} \max_{\alpha < L \leq \beta} S_{\phi_r}(\lfloor \alpha \rfloor, L) \quad (3.14)$$

$$\leq c_3 a^{-1/5} (tr)^{11/30} + c_4 a^{-11/20} (tr)^{61/120}$$

where

$$c_3 = b_1 \pi^{1/2} A_5 h^{21/8} (h-1), \quad c_4 = b_2 \pi^{1/2} B_5 h^{3/2} (h-1)^{7/8}.$$

Additionally, note that

$$\frac{tr}{\pi h^3 a^3} \leq |g_r''(x)| = \frac{tr(2x+r)}{2\pi x^2(x+r)^2} \leq \frac{tr}{\pi a^3},$$

and thus

$$\lambda_2 \leq |g_r''(x)| < h_2 \lambda_2, \quad \lambda_2 := \frac{tr}{\pi h^3 a^3}, \quad h_2 := h^3. \quad (3.15)$$

Similarly,

$$|g_r'''(x)| = \frac{tr}{\pi} \frac{3x^2 + 3xr + r^2}{x^3(x+r)^3} \in \left[ \frac{3tr}{\pi h^4 a^4}, \frac{3tr}{\pi a^4} \right],$$

$$\lambda_3 \leq |g_r'''(x)| \leq h_3 \lambda_3, \quad \lambda_3 := \frac{3tr}{\pi h^4 a^4}, \quad h_3 := h^4. \quad (3.16)$$

Applying Lemma 2.2 and using (3.14), (3.15) and (3.16), we finally obtain

$$|S_{g_r}(a, b-r-a)| \leq c_3 a^{-1/5} (tr)^{11/30} + c_4 a^{-11/20} (tr)^{61/120} + E \quad (3.17)$$

where, from Lemma 2.3, the error term  $E$  satisfies

$$|E| \leq 4.686 \lambda_2^{-1/2} + \frac{5}{\pi} \log(\beta - \alpha + 2) + \frac{2 \cdot 3^{2/3}}{\pi^{2/3}} h_2 h_3^{1/3} (b-a) \lambda_3^{1/3} + 6.$$

Setting

$$T_1 = 4.686 \lambda_2^{-1/2}, \quad T_2 = \frac{5}{\pi} \log(\beta - \alpha + 2),$$

$$T_3 = \frac{2 \cdot 3^{2/3}}{\pi^{2/3}} h_2 h_3^{1/3} (b-a) \lambda_3^{1/3}, \quad T_4 = 6,$$

and substituting (3.15) and (3.16), we have

$$T_1 = 4.686\sqrt{\pi}h^{3/2}\frac{a^{3/2}}{(tr)^{1/2}}, \quad T_3 \leq \frac{6}{\pi}h^3(h-1)\frac{(tr)^{1/3}}{a^{1/3}}. \quad (3.18)$$

Furthermore, since  $\log(2+x) \leq \log 2 + 0.893x^{1/3}$  for all  $x > 0$ , we have, using (3.12),

$$\log(2+\beta-\alpha) \leq \log 2 + 0.893(\beta-\alpha)^{1/3} \leq \log 2 + \frac{0.893}{a^{1/3}}\left(\frac{h-1}{\pi}\frac{tr}{a}\right)^{1/3}.$$

This implies, from  $a \geq \theta_2 t^{7/17} \geq \theta_2 t_0^{7/17}$ , that

$$T_2 \leq \frac{5}{\pi}\left(\log 2 + \frac{0.893}{\theta_2^{1/3}t_0^{7/51}}\left(\frac{h-1}{\pi}\right)^{1/3}\frac{(tr)^{1/3}}{a^{1/3}}\right). \quad (3.19)$$

Combining (3.17), (3.18) and (3.19), we have

$$\begin{aligned} |S_{g_r}(a, b-r-a)| &\leq c_3 a^{-1/5}(tr)^{11/30} + c_4 a^{-11/20}(tr)^{61/120} \\ &\quad + E_4 \frac{a^{3/2}}{(tr)^{1/2}} + E_5 \frac{(tr)^{1/3}}{a^{1/3}} + E_6, \end{aligned} \quad (3.20)$$

where

$$\begin{aligned} E_4 &:= 4.686\sqrt{\pi}h^{3/2}, \quad E_5 := \frac{0.893}{\theta_2^{1/3}t_0^{7/51}}\frac{5}{\pi}\left(\frac{h-1}{\pi}\right)^{1/3} + \frac{6}{\pi}h^3(h-1), \\ E_6 &:= 6 + \frac{5}{\pi}\log 2. \end{aligned}$$

**3.0.3. Applying the  $A$  process.** To complete the proof we apply the  $A$  process once more to obtain the exponent pair  $ABA^3B(0, 1) = (11/82, 57/82)$ . To do so we rely on the following inequality, which can be found in e.g. [Pat21]:

$$\sum_{r=1}^q \left(1 - \frac{r}{q}\right) r^s \leq \frac{q^{1+s}}{(1+s)(2+s)}, \quad -1 < s \leq 1, \quad (3.21)$$

for all integers  $q \geq 1$ . Applying this formula, and using (3.20), we obtain

$$\begin{aligned} \frac{2}{q} \sum_{r=1}^{q-1} \left(1 - \frac{r}{q}\right) |S_{g_r}(a, b-r-a)| &\leq \frac{1800}{2911} c_3 a^{-1/5}(qt)^{11/30} + \frac{28800}{54481} c_4 a^{-11/20}(qt)^{61/120} \\ &\quad + \frac{8}{3} E_4 \frac{a^{3/2}}{(tq)^{1/2}} + \frac{9}{14} E_5 \frac{(tq)^{1/3}}{a^{1/3}} + E_6. \end{aligned}$$

We use this in Lemma 1.2, together with  $b-a \leq (h-1)a$ , to obtain

$$\begin{aligned} |S_f(a, b-a)|^2 &\leq \left(h-1 + \frac{q}{a}\right) \left(\frac{(h-1)a^2}{q} + \frac{1800}{2911} c_3 a^{4/5}(qt)^{11/30}\right. \\ &\quad \left. + \frac{28800}{54481} c_4 a^{9/20}(qt)^{61/120} + \frac{8}{3} E_4 \frac{a^{5/2}}{(tq)^{1/2}} + \frac{9}{14} E_5 a^{2/3}(tq)^{1/3} + E_6 a\right). \end{aligned} \quad (3.22)$$

We choose  $q = \lfloor \theta_1 a^{36/41} t^{-11/41} \rfloor$  for some  $\theta_1 > 0$  to be chosen later, so that

$$\theta_1 a^{36/41} t^{-11/41} - 1 \leq q \leq \theta_1 a^{36/41} t^{-11/41}.$$

Now, with  $q_0$  defined in the lemma statement, we have

$$q_0 = \theta_1 \theta_2^{7/17} t_0^{65/697} \leq \theta_1 a^{36/41} t^{-11/41} \leq q + 1$$

and hence by assumption,  $q \geq 1$ . Observe that the following inequalities hold

$$\begin{aligned} \frac{a^2}{q} &\leq \frac{a^2}{\theta_1 a^{36/41} t^{-11/41} - 1} \leq \frac{\theta_1^{-1}}{1 - q_0^{-1}} \cdot a^{46/41} t^{11/41}, \\ \frac{q}{a} &\leq \frac{\theta_1 a^{36/41} t^{-11/41}}{a} \leq \frac{\theta_1}{(\theta_2 t^{7/17})^{5/41} t^{11/41}} \leq \frac{\theta_1}{\theta_2^{5/41} t_0^{222/697}}, \\ \frac{a^{5/2}}{(qt)^{1/2}} &\leq \frac{1}{(\theta_1 a^{36/41} t^{-11/41} - 1)^{1/2}} a^{5/2} t^{-1/2} \leq \left( \theta_1 - a^{-36/41} t^{11/41} \right)^{-1/2} a^{169/82} t^{-15/41} \\ &\leq \left( \theta_1 - \frac{1}{\theta_2 t_0^{100/697}} \right)^{-1/2} a^{169/82} t^{-15/41}, \\ a^{2/3} (tq)^{1/3} &\leq \theta_1^{1/3} a^{118/123} t^{10/41}. \end{aligned}$$

Using the above inequalities, we obtain

$$\begin{aligned} |S_f(a, b-a)|^2 &\leq \left( h - 1 + \frac{\theta_1}{\theta_2^{5/41} t_0^{222/697}} \right) \left( \left( \frac{\theta_1^{-1}(h-1)}{1 - q_0^{-1}} + \frac{1800}{2911} \theta_1^{11/30} c_3 \right) a^{46/41} t^{11/41} \right. \\ &+ \left( \frac{28800}{54481} \theta_1^{61/120} c_4 \right) a^{147/164} t^{61/164} + \frac{8}{3} E_4 \left( \theta_1 - \frac{1}{\theta_2 t_0^{100/697}} \right)^{-1/2} a^{169/82} t^{-15/41} \\ &\left. + \frac{9}{14} \theta_1^{1/3} a^{118/123} t^{10/41} + E_6 a \right) \end{aligned}$$

Taking square roots of both sides, applying  $\sqrt{x_1 + \dots + x_n} \leq \sqrt{x_1} + \dots + \sqrt{x_n}$ , and substituting the values of  $c_3$ ,  $c_4$ ,  $E_4$ ,  $E_5$  and  $E_6$ , the desired result follows.  $\square$

**Lemma 3.5.** *Let  $100 \leq t_0 \leq t \leq t_1$ ,  $h > 1$  and  $\theta_1, \theta_2, \theta_3, \eta_1 > 0$  be constants jointly satisfying the conditions of Lemma 3.4. Furthermore assume that  $h_0 := h/(1 - \theta_1 t_0^{-7/17}) \in (1, 2]$ . Then*

$$|S_3| \leq C_4(t_0, t_1, h, \eta_1, \theta_1, \theta_2, \theta_3) t^{27/164}$$

where

$$\begin{aligned} C_4 &:= C_1(h_0) \mu_1 \left( \frac{5}{82} \right) + C_2(h_0) \mu_2 \left( \frac{17}{328} \right) h^{17K(t_1)/328} t_0^{-3/656} \\ &+ E_1(h_0) \mu_1 \left( \frac{87}{164} \right) t_0^{-27/328} + E_2(h_0) \mu_2 \left( \frac{5}{246} \right) h^{5K(t_1)/246} t_0^{-13/246} + E_3(h_0) K(t_1) t_0^{-27/164}, \end{aligned} \quad (3.23)$$

$$\mu_1(\alpha) := \frac{1}{(2\pi)^{\alpha/2}} \frac{1 - h^{-\alpha K(t_1)}}{h^\alpha - 1}, \quad \mu_2(\alpha) := \mu_1(\alpha) \left( 1 - \frac{h}{\theta_2 t_0^{7/17}} \right)^{-\alpha} \quad (3.24)$$

and  $C_1$ ,  $C_2$ ,  $E_1$ ,  $E_2$  and  $E_3$  are as defined in Lemma 3.4. Furthermore, for all  $t \geq t_0$ ,

$$|S_3| \leq C_5(t_0, h, \eta_1, \theta_1, \theta_2, \theta_3) t^{27/164}$$

where

$$\begin{aligned} C_5 := & C_1(h_0)\mu_3\left(\frac{5}{82}\right) + C_2(h_0)\mu_4\left(\frac{17}{328}\right) + E_1(h_0)\mu_3\left(\frac{87}{164}\right)t_0^{-27/328} \\ & + E_2(h_0)\mu_4\left(\frac{5}{246}\right)t_0^{-427/8364} + E_3(h_0)\left(\frac{\frac{3}{34}\log t_0 - \log(\theta_2\sqrt{2\pi})}{\log h} + 1\right)t_0^{-27/164}, \end{aligned} \quad (3.25)$$

$$\mu_3(\alpha) := \frac{1}{(2\pi)^{\alpha/2}(h^\alpha - 1)}, \quad \mu_4(\alpha) := \left(1 - \frac{h}{\theta_2 t_0^{7/17}}\right)^{-\alpha} \frac{(h/\theta_2)^\alpha - \sqrt{2\pi}t_0^{-3/34}}{1 - h^{-\alpha}}. \quad (3.26)$$

*Proof.* With  $a_k$  as defined in (3.2), we have

$$\frac{h^{-(k-1)}\sqrt{\frac{t}{2\pi}}}{h^{-k}\sqrt{\frac{t}{2\pi}} - 1} \leq h \left(\frac{1}{1 - a_k^{-1}}\right) < h_0,$$

say. Therefore, we may apply Lemma 3.4, with  $h = h_0$ ,  $a = a_k$ ,  $b = a_{k-1}$  and  $t_0 \leq t$  to obtain, via partial summation,

$$\begin{aligned} \left| \sum_{a_k < n \leq a_{k-1}} n^{-1/2-it} \right| & \leq a_k^{-1/2} \max_{a_k < L \leq a_{k-1}} S_f(a_k, L - a_k) \\ & \leq C_1 a^{5/82} t^{11/82} + C_2 a^{-17/328} t^{61/328} + E_1 a^{87/164} t^{-15/82} \\ & \quad + E_2 a^{-5/246} t^{5/41} + E_3, \end{aligned}$$

and thus

$$\begin{aligned} |S_3| & \leq \sum_{k=1}^K \left| \sum_{a_k < n \leq a_{k-1}} n^{-1/2-it} \right| \leq C_1 t^{11/82} \sum_{k=1}^K a_k^{5/82} + C_2 t^{61/328} \sum_{k=1}^K a_k^{-17/328} \\ & \quad + E_1 t^{-15/82} \sum_{k=1}^K a_k^{87/164} + E_2 t^{5/41} \sum_{k=1}^K a_k^{-5/246} + E_3 K. \end{aligned} \quad (3.27)$$

Since

$$h^{-k}\sqrt{\frac{t}{2\pi}} - 1 \leq a_k \leq h^{-k}\sqrt{\frac{t}{2\pi}}$$

we have, for any  $\alpha > 0$  and  $t_0 \leq t \leq t_1$ ,

$$\sum_{k=1}^K a_k^\alpha \leq \sum_{k=1}^K \left(h^{-k}\sqrt{\frac{t}{2\pi}}\right)^\alpha = \left(\frac{t}{2\pi}\right)^{\alpha/2} \sum_{k=1}^K (h^{-\alpha})^k = \mu_1(\alpha)t^{\alpha/2} \quad (3.28)$$

and, again for  $\alpha > 0$  and  $t_0 \leq t \leq t_1$ ,

$$\begin{aligned} \sum_{k=1}^K a_k^{-\alpha} & \leq \sum_{k=1}^K \left(h^{-k}\sqrt{\frac{t}{2\pi}} - 1\right)^{-\alpha} = \left(1 - h^k\sqrt{\frac{2\pi}{t}}\right)^{-\alpha} \sum_{k=1}^K \left(h^{-k}\sqrt{\frac{t}{2\pi}}\right)^{-\alpha} \\ & < \left(1 - \frac{h}{\theta_2 t_0^{7/17}}\right)^{-\alpha} \left(\frac{t}{2\pi}\right)^{-\alpha/2} \frac{h^{\alpha K} - 1}{1 - h^{-\alpha}} = \mu_2(\alpha)h^{\alpha K(t_1)}t^{-\alpha/2}. \end{aligned} \quad (3.29)$$

Substituting these into (3.27), we obtain the estimate

$$|S_3| \leq C_1 \mu_1 \left( \frac{5}{82} \right) t^{27/164} + C_2 \mu_2 \left( \frac{17}{328} \right) h^{17K(t_1)/328} t^{105/656} \\ + E_1 \mu_1 \left( \frac{87}{164} \right) t^{27/328} + E_2 \mu_2 \left( \frac{5}{246} \right) h^{5K(t_1)/246} t^{55/492} + E_3 K(t_1),$$

which gives (3.23) from  $t \geq t_0$ , and forms the main bound for  $S_3$  for  $t$  in finite intervals  $[t_0, t_1]$ . To obtain a bound holding for all  $[t_0, \infty)$ , we use

$$h^K < h^{(\frac{3}{34} \log t - \log(\theta_2 \sqrt{2\pi})) / \log h + 1} = \frac{h}{\theta_2 \sqrt{2\pi}} t^{3/34}$$

to continue the argument from (3.28) and (3.29) to obtain, for  $t \geq t_0$ ,

$$\sum_{k=1}^K a_k^\alpha < \mu_3(\alpha),$$

$$\sum_{k=1}^K a_k^{-\alpha} < \left( 1 - \frac{h}{\theta_2 t_0^{7/17}} \right)^{-\alpha} \frac{(h/\theta_2)^\alpha - \sqrt{2\pi} t_0^{-3/34}}{1 - h^{-\alpha}} t^{-7\alpha/17} = \mu_4(\alpha) t^{-7\alpha/17}.$$

Equation (3.25) then follows from substituting these estimates into (3.27) and using  $t \geq t_0$ .  $\square$

**3.1. Computations.** For each row  $(\log t_0, \log t_1, h_1, h_2, \eta_1, \eta_2, \theta_1, \theta_2, \theta_3, A)$  of Table 1, we substitute the relevant parameter values into Lemma 3.2, 3.3 and 3.5 to verify, in each case, that

$$|\zeta(1/2 + it)| \leq At^{27/164}, \quad t_0 \leq t \leq t_1.$$

Upon inspection, we have  $A \leq 66.7$  in each case, which proves Theorem 1.1 for  $\exp(60) \leq t \leq \exp(875)$ . These parameters are found via a stochastic optimisation routine so are not necessarily globally optimal, however they suffice for justifying an upper bound on the constant factor in Theorem 1.1.

In addition, by taking  $t_0 = \exp(875)$ ,  $\theta_1 = 1.14283$ ,  $\theta_2 = 261658$ ,  $\theta_3 = 2.53087 \cdot 10^{-11}$ ,  $h_1 = 1.01563$ ,  $h_2 = 1.00270$ ,  $\eta_1 = 1.59875$  and  $\eta_2 = 0.828895$  in Lemma 3.2, 3.3 and (3.25), we obtain

$$|\zeta(1/2 + it)| \leq 66.7 t^{27/164}, \quad t \geq \exp(875).$$

Note that in the application of Lemma 3.3, we take the limit as  $t_1 \rightarrow \infty$ . This implies Theorem 1.1 for  $t \geq \exp(875)$ .

For small values of  $t$ , we use the following bound

$$|\zeta(1/2 + it)| \leq 0.478013 t^{1/6} \log t + 3.853165 t^{1/6} - 2.914229, \quad t \geq 10^{12}, \quad (3.30)$$

proved in §3.4 of [HPY22]. This estimate covers the range  $10^{12} \leq t < \exp(60)$ . Finally, for  $3 \leq t < 10^{12}$ , we use the classical van der Corput estimate (1.1). This completes the proof of Theorem 1.1. Lastly, we note that the bounds in Table 1 improve on both (3.30) and (1.2) for all  $t \geq \exp(60.6)$ , and is thus the sharpest known bound on  $\zeta(1/2 + it)$  in this range.

## 4. CONCLUSION AND FUTURE WORK

Theorem 1.1 represents the first of many successively sharper sub-Weyl bounds of the form  $\zeta(1/2 + it) \ll_{\epsilon} t^{\theta + \epsilon}$  obtainable from van der Corput's method. The next few values of  $\theta$ , due to [Phi33], [Tit42], [Min49], [Han63] and [Kol82] respectively, are

$$\frac{229}{1392}, \quad \frac{19}{116}, \quad \frac{15}{92}, \quad \frac{6}{37}, \quad \frac{35}{216}.$$

The first result,  $\theta = 229/1392$ , can be obtained via the exponent pair

$$ABA^3BA^2BA^2B(0, 1) = \left( \frac{97}{696}, \frac{20}{29} \right)$$

and can thus be made explicit using a similar but longer version of the arguments presented in this paper. Exponents starting from  $\theta = 19/116$ , however, rely on estimates of higher-dimensional exponential sums. For example, in the two-dimensional case, the function  $g_r(x) = f(x + r) - f(x)$  encountered in the  $A$  process (Lemma 1.2) can be treated as a function of two variables,  $r$  and  $x$ . Such a sum can be estimated using two-dimensional analogs of the  $A$  and  $B$  processes.

The main obstacles to computing an explicit version of such results are difficulties with the two-dimensional Poisson summation formula. In the two-dimensional analog of the  $B$  process, the factor  $|f''(x_{\nu})|^{-1/2}$  appearing in Lemma 2.3 is replaced by the Hessian of  $f$ , defined by

$$Hf(x, y) := \det \begin{bmatrix} \partial_{xx}f & \partial_{xy}f \\ \partial_{yx}f & \partial_{yy}f \end{bmatrix}.$$

However, if  $Hf$  vanishes within the rectangle of summation, as is the case when bounding  $\zeta(1/2 + it)$ , it can be difficult to control the transformed sum. Successful implementations [Tit35; Min49] of two-dimensional exponent pairs rely on elaborate arguments to isolate problematic regions within the summation rectangle, and applying the trivial bound in those regions instead. Explicit versions of higher-dimensional Poisson summation formulae will be investigated in a future article.

## ACKNOWLEDGEMENTS

We would like to thank Timothy S. Trudgian and Ghaith A. Hiary for their continuous support and helpful suggestions throughout the writing of this paper.

TABLE 1. Parameter values used in the proof of Theorem 1.1.

$\log t_0$	$\log t_1$	$h_1$	$h_2$	$\eta_1$	$\eta_2$	$\theta_1$	$\theta_2$	$\theta_3$	$A$
60	65	1.02932	1.06726	1.72183	1.02275	0.957426	0.180062	0.172999	37.10
65	70	1.02739	1.06227	1.65356	1.08576	0.938332	0.189995	0.121681	38.09
70	75	1.02561	1.05961	1.76719	1.03304	0.926944	0.194964	0.0811119	39.05
75	80	1.0231	1.05617	1.94747	1.04071	0.927416	0.20589	0.0549285	39.98
80	85	1.02302	1.05429	2.20976	1.03294	0.915828	0.223422	0.0383106	40.88
85	90	1.02248	1.05174	1.98074	1.00763	0.917169	0.230051	0.0265495	41.75
90	95	1.02221	1.04803	1.94585	1.02266	0.908887	0.254651	0.0203407	42.60
95	100	1.02211	1.04783	2.12321	1.04079	0.93665	0.290334	0.0137463	43.41
100	105	1.02165	1.04598	1.93655	1.06102	0.906136	0.305624	0.00865144	44.19
105	110	1.02191	1.0438	2.03193	0.948977	0.920571	0.320444	0.0076111	44.96
110	115	1.02142	1.04532	1.98343	1.01929	0.931888	0.364937	0.00567012	45.70
115	120	1.02014	1.04372	2.08479	0.988021	0.931926	0.391814	0.00426777	46.41
120	125	1.02155	1.03987	2.02632	0.972639	0.932056	0.427062	0.00308127	47.10
125	130	1.02124	1.0422	1.90876	1.02384	0.919159	0.44983	0.00146827	47.76
130	135	1.02096	1.03965	1.93958	1.0108	0.926399	0.502711	0.00196936	48.42
135	140	1.02133	1.03736	2.0425	0.9705	0.935329	0.549766	0.00129695	49.05
140	145	1.02187	1.03816	1.90804	1.00432	0.931849	0.592399	0.000924586	49.66
145	150	1.02183	1.03717	1.99931	0.968481	0.934159	0.652989	0.00104845	50.26
150	155	1.02183	1.03826	1.88929	0.950307	0.943886	0.718618	0.000853561	50.84
155	160	1.02048	1.03617	1.93121	0.98237	0.938295	0.738158	0.000619765	51.39
160	165	1.02138	1.03453	1.96189	0.977865	0.935631	0.857581	0.00064869	51.94
165	170	1.02094	1.03251	1.89843	0.988454	0.940228	0.909567	0.000521171	52.46
170	175	1.02158	1.03544	1.89217	0.98871	0.950508	1.00477	0.000414171	52.97
175	180	1.0214	1.03158	1.93452	1.04548	0.939419	1.03725	0.000314426	53.47
180	185	1.02039	1.03179	2.03717	0.972755	0.95516	1.18322	0.000517973	53.97
185	190	1.02096	1.03431	1.91394	0.982296	0.942759	1.28455	0.000322386	54.43
190	195	1.02108	1.02999	1.92547	0.97085	0.954305	1.36094	0.000475754	54.90
195	200	1.02121	1.03205	1.9824	0.949768	0.944675	1.51919	0.000385305	55.33
200	205	1.02125	1.02682	1.95701	0.982448	0.947591	1.63942	0.000229165	55.75
205	210	1.0212	1.02869	1.89377	0.953281	0.952798	1.70281	0.000313579	56.18
210	215	1.0209	1.03072	1.99679	0.979303	0.953546	1.92024	0.000228668	56.58
215	220	1.02096	1.02656	1.95651	0.928927	0.967107	2.09168	0.000336118	56.98
220	225	1.02094	1.02878	1.89319	0.939551	0.954351	2.33168	0.000284262	57.36
225	230	1.02105	1.02806	2.06086	0.965398	0.954194	2.5612	0.000398031	57.74
230	235	1.01954	1.02547	2.04008	0.929068	0.954843	2.67558	0.000256528	58.09
235	240	1.02038	1.02633	1.8313	0.941395	0.963688	3.04059	0.000209508	58.43
240	245	1.02026	1.02293	1.92062	0.939602	0.967721	3.18847	0.000202481	58.77
245	250	1.02015	1.02567	1.91183	0.908205	0.973884	3.49943	0.000335679	59.11
250	255	1.02167	1.02194	1.82234	0.926453	0.961316	3.67918	0.000399662	59.44
255	260	1.0207	1.02546	1.85731	0.981673	0.972605	4.17292	0.000240777	59.73
260	265	1.02011	1.02388	2.09485	0.969026	0.965687	4.65096	0.000192086	60.02
265	270	1.02066	1.02359	1.96056	0.893437	0.969203	4.83374	0.000251274	60.32
270	275	1.0201	1.02217	1.83113	0.987098	0.964569	5.25223	0.000238956	60.60
275	280	1.02004	1.02147	1.8859	0.958011	0.982108	5.73906	0.000261819	60.87



280	285	1.02017	1.02028	1.82159	0.976256	0.96597	5.91678	0.000223948	61.13
285	290	1.02029	1.02019	1.86161	0.930287	0.977082	6.75422	0.000270409	61.39
290	295	1.02025	1.0215	1.8654	0.93816	0.976725	7.47305	0.000230413	61.63
295	300	1.02004	1.02012	1.80343	0.90056	0.981492	7.99716	0.000137621	61.86
300	305	1.01958	1.02158	1.7938	0.917443	0.980441	9.15625	0.000194688	62.10
305	310	1.02009	1.01906	1.82009	0.962434	0.981321	9.66325	0.000221732	62.32
310	315	1.02048	1.02047	1.80586	0.925325	0.984564	10.3824	0.000170812	62.53
315	320	1.0198	1.02051	2.01424	0.912211	0.987193	11.9473	0.000322156	62.76
320	325	1.02049	1.01903	1.89547	0.935573	0.983211	12.4283	0.000243574	62.95
325	330	1.02015	1.01867	1.96936	0.920717	0.985354	13.1599	0.000243177	63.14
330	335	1.01998	1.01839	1.75777	0.923855	0.989922	14.3557	0.000337639	63.34
335	340	1.0198	1.01668	1.95299	0.922568	0.997515	16.5761	0.000132965	63.49
340	345	1.0202	1.01716	1.81734	0.94091	0.998342	17.9692	0.000255942	63.68
345	350	1.01961	1.01542	1.79264	0.964049	0.992996	19.85	0.00022936	63.84
350	355	1.01963	1.0165	1.86902	0.928483	0.996867	20.2785	0.000225968	64.00
355	360	1.02034	1.01579	1.91329	0.935197	0.999189	23.4266	0.000160669	64.15
360	365	1.01909	1.01585	1.82083	0.890385	1.00133	24.8655	0.00020917	64.30
365	370	1.01971	1.01676	1.77505	0.936947	1.00886	27.9853	0.000273506	64.45
370	375	1.02008	1.01541	1.80961	0.898	1.01433	30.7508	0.000247706	64.59
375	380	1.0194	1.01554	1.96358	0.895426	1.0133	31.265	0.000248272	64.72
380	385	1.01983	1.01418	1.79329	0.932697	1.00831	34.8777	0.000273313	64.85
385	390	1.01977	1.01589	1.84736	0.91237	1.01851	38.5851	0.000242406	64.96
390	395	1.02013	1.01406	1.77664	0.934046	1.00421	42.763	0.000194332	65.07
395	400	1.01939	1.01407	1.82445	0.903297	1.01028	46.6084	0.000318785	65.20
400	405	1.01904	1.01305	1.825	0.899607	1.01356	52.1417	0.000220817	65.29
405	410	1.0189	1.01201	1.88238	0.912005	1.01941	53.1663	0.000211265	65.39
410	415	1.01878	1.01242	1.84523	0.934599	1.01315	61.3463	0.000231803	65.49
415	420	1.01899	1.01342	1.79699	0.898858	1.02272	62.7726	0.000243697	65.59
420	425	1.01885	1.01242	1.75512	0.900257	1.02809	70.4483	0.000182461	65.66
425	430	1.01924	1.01234	1.7811	0.940522	1.01604	73.6136	0.000204745	65.75
430	435	1.01942	1.01163	1.75987	0.887557	1.01677	85.4154	0.000200687	65.83
435	440	1.01918	1.01177	1.77652	0.85801	1.03222	94.0339	0.000137152	65.89
440	445	1.01931	1.01146	1.80958	0.877476	1.0233	99.6689	0.000145403	65.97
445	450	1.0189	1.01039	1.82757	0.84405	1.02366	111.324	0.000187234	66.04
450	455	1.01814	1.01226	1.6686	0.939848	1.03289	116.834	0.0001905	66.10
455	460	1.01958	1.01329	1.7525	0.850712	1.03058	130.957	0.000269602	66.17
460	465	1.01852	1.0109	1.796	0.903805	1.02878	139.055	0.000208527	66.22
465	470	1.01893	1.0108	1.75859	0.894058	1.03424	151.974	0.000475389	66.30
470	475	1.01912	1.01125	1.71125	0.853318	1.04161	164.259	0.000282763	66.33
475	480	1.01916	1.00968	1.77909	0.89025	1.04003	187.133	0.00029613	66.38
480	485	1.0178	1.01075	1.80647	0.898548	1.04088	210.376	0.00016592	66.40
485	490	1.01826	1.00939	1.74844	0.900994	1.04706	217.358	0.000288872	66.46
490	495	1.01802	1.0088	1.7787	0.87362	1.05188	243.335	0.000207585	66.48
495	500	1.01834	1.00975	1.79853	0.930467	1.0486	277.44	0.000151062	66.51
500	505	1.01867	1.01099	1.82223	0.892651	1.05582	277.559	0.000276115	66.55
505	510	1.0187	1.00994	1.80285	0.899901	1.04805	309.723	0.000206119	66.57

510	515	1.01851	1.00857	1.76958	0.897381	1.05071	336.08	0.000216335	66.59
515	520	1.01841	1.00972	1.74506	0.867681	1.04449	358.983	0.000129439	66.60
520	525	1.01808	1.00909	1.69862	0.91292	1.04574	395.643	0.00014666	66.62
525	530	1.01775	1.00811	1.83159	0.881166	1.05596	472.358	0.000228591	66.65
530	535	1.01775	1.00761	1.78934	0.899516	1.05953	462.286	0.000296956	66.67
535	540	1.01804	1.00799	1.75446	0.864478	1.05182	531.88	0.000272301	66.68
540	545	1.01797	1.00778	1.71041	0.969014	1.05986	533.438	0.000323617	66.70
545	550	1.01814	1.00869	1.71209	0.867872	1.05977	612.157	0.000172334	66.68
550	555	1.01798	1.0077	1.79251	0.931142	1.05695	700.06	0.000188567	66.69
555	560	1.01809	1.00941	1.69138	0.919723	1.06404	731.948	0.000180617	66.69
560	565	1.01797	1.00911	1.68662	0.864885	1.06303	884.433	0.000261075	66.70
565	570	1.01805	1.00675	1.69631	0.874895	1.07243	977.767	0.000189268	66.68
570	575	1.01791	1.00703	1.61088	0.903648	1.06953	1008.6	0.000260523	66.69
575	580	1.01717	1.00681	1.71789	0.84638	1.07915	1090.48	0.000214881	66.68
580	585	1.01789	1.00627	1.75529	0.862506	1.07199	1161.28	0.000137629	66.65
585	590	1.01769	1.00851	1.72491	0.81516	1.07599	1243.56	0.000237294	66.66
590	595	1.01737	1.00637	1.67399	0.87291	1.07789	1496.72	0.000244104	66.64
595	600	1.01778	1.00634	1.6905	0.892796	1.07244	1480.77	0.000267985	66.63
600	605	1.01737	1.00657	1.67695	0.909403	1.08182	1757.68	0.000242804	66.61
605	610	1.01719	1.00573	1.74453	0.884353	1.08245	1852.2	0.000313867	66.60
610	615	1.01787	1.00628	1.67817	0.924263	1.09403	2201.2	0.000225138	66.57
615	620	1.01779	1.00529	1.71989	0.89541	1.10734	2358.51	0.000269884	66.56
620	625	1.01766	1.00591	1.68201	0.898451	1.09211	2557.47	0.00023344	66.53
625	630	1.01739	1.00679	1.63793	0.953277	1.08654	2762.17	0.000242996	66.51
630	635	1.01721	1.00683	1.65337	0.870998	1.0922	2875.57	0.000338419	66.49
635	640	1.01726	1.00489	1.69949	0.844695	1.09723	3334.1	0.000222066	66.45
640	645	1.0169	1.0052	1.72567	0.843783	1.08135	3258.7	0.000165915	66.41
645	650	1.01764	1.00539	1.70071	0.916906	1.09723	3603.31	0.000177251	66.38
650	655	1.01739	1.0043	1.56953	0.951441	1.09117	4037.06	0.000149961	66.35
655	660	1.01748	1.00401	1.63269	0.886415	1.09295	4648.06	0.000186047	66.32
660	665	1.01649	1.0049	1.72418	0.886985	1.09507	4673.86	0.00017414	66.29
665	670	1.01674	1.00599	1.67976	0.869346	1.10091	5572.25	0.000177106	66.25
670	675	1.01663	1.0046	1.61937	0.83382	1.1006	5675.14	0.000245665	66.22
675	680	1.01694	1.00428	1.65215	0.872436	1.11171	6423.04	0.00011767	66.16
680	685	1.01709	1.00442	1.66715	0.866925	1.11875	7423.57	0.000142691	66.13
685	690	1.01638	1.00529	1.58343	0.863125	1.11827	8229.43	0.000242883	66.11
690	695	1.01616	1.0052	1.63037	0.85535	1.1079	8876.13	0.00018326	66.06
695	700	1.01678	1.00377	1.60685	0.847676	1.1222	9005.87	0.000360569	66.04
700	705	1.01663	1.00434	1.6478	0.832542	1.10678	11011.9	0.000259432	65.98
705	710	1.01626	1.00427	1.6748	0.858014	1.13075	11374.6	0.000161921	65.92
710	715	1.01702	1.00388	1.66823	0.904501	1.12282	12579.1	0.000253355	65.89
715	720	1.01654	1.0052	1.57025	0.91045	1.12799	13030.7	0.000164892	65.83
720	725	1.01554	1.00395	1.61751	0.875735	1.12081	14543.9	0.000163032	65.79
725	730	1.01616	1.0048	1.61618	0.894992	1.13861	16398.8	0.000164328	65.74
730	735	1.01616	1.0052	1.58148	0.881893	1.13236	18388.7	0.000250531	65.70
735	740	1.01633	1.00404	1.58224	0.884335	1.12985	21073.1	0.000240423	65.65

740	745	1.0158	1.00347	1.62662	0.823366	1.12091	22637.4	0.00019436	65.59
745	750	1.01726	1.00319	1.55616	0.837796	1.13674	22591	0.000274638	65.56
750	755	1.01604	1.00424	1.66054	0.886899	1.13739	26085.9	0.000176809	65.49
755	760	1.01606	1.00397	1.63481	0.866679	1.12632	29306.6	0.000171531	65.44
760	765	1.0165	1.00346	1.67097	0.847691	1.13233	29419	0.00019725	65.39
765	770	1.01591	1.00381	1.47213	0.900812	1.14082	34118.8	0.000185818	65.33
770	775	1.01603	1.00414	1.59268	0.928957	1.15879	38353.4	0.000156928	65.27
775	780	1.01618	1.00278	1.49228	0.8967	1.14947	38574.2	0.000226631	65.23
780	785	1.01611	1.00268	1.56406	0.877608	1.14836	47867.9	0.000207654	65.17
785	790	1.0164	1.00312	1.51167	0.858272	1.16038	50635.3	0.00028385	65.12
790	795	1.01537	1.00393	1.64687	0.861995	1.15151	55750.3	0.000200208	65.06
795	800	1.0153	1.0029	1.47289	0.887874	1.16342	60283.2	0.000154351	64.99
800	805	1.01527	1.00401	1.55174	0.865362	1.1537	66111.4	0.000195508	64.94
805	810	1.01574	1.0029	1.5166	0.902223	1.17073	74262.8	0.000225813	64.89
810	815	1.01522	1.00339	1.60073	0.876369	1.15792	77118.3	0.000266209	64.83
815	820	1.01601	1.00238	1.66365	0.906645	1.15121	84767.1	0.000161823	64.76
820	825	1.01583	1.00271	1.6235	0.928793	1.16244	107264	0.000234246	64.71
825	830	1.01563	1.00315	1.5685	0.859089	1.15991	106225	0.000187707	64.64
830	835	1.01536	1.00383	1.53532	0.868098	1.16856	112590	0.000132111	64.57
835	840	1.01556	1.0038	1.56455	0.926623	1.17023	134713	0.000235556	64.53
840	845	1.01502	1.00246	1.51063	0.874939	1.1639	132092	0.000207851	64.47
845	850	1.01563	1.00206	1.53822	0.86811	1.17549	152832	0.000178814	64.40
850	855	1.01498	1.00283	1.63544	0.845347	1.16767	148866	0.000158165	64.34
855	860	1.0154	1.00294	1.52587	0.895686	1.16311	186060	0.000141116	64.27
860	865	1.01506	1.00221	1.59709	0.850172	1.1766	192984	0.000238358	64.22
865	870	1.01519	1.00193	1.54244	0.88004	1.17385	222438	0.000186894	64.15
870	875	1.01542	1.00225	1.52497	0.851648	1.18563	242664	0.00028923	64.10

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