# An explicit theory for pulses in two component, singularly perturbed, reaction-diffusion equations

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**Abstract**

In recent years, methods have been developed to study the existence, stability and bifurcations of pulses in singularly perturbed reaction-diffusion equations in one space dimension, in the context of a number of explicit model problems, such as the Gray-Scott and the Gierer-Meinhardt equations. Although these methods are in principle of a general nature, their applicability a priori relies on the characteristics of these models. For instance, the slow reduced spatial problem is linear in the models considered in the literature. Moreover, the nonlinearities in the fast reduced spatial problem are of a very specific, polynomial, nature. These properties are crucially used, especially in the stability and bifurcation analysis. In this paper, we present an explicit theory for pulses in two-component singularly perturbed reaction-diffusion equations that significantly extends and generalizes existing methods.

*Dedicated to Klaus Kirchgässner, in gratitude for his inspiration and stimulation*

## 1 Introduction

The existing theory for the existence and stability of symmetric, stationary pulse solutions to two-component singularly perturbed reaction-diffusion equations has in essence been developed in the context of two explicit models, the Gray-Scott (GS) model for autocatalytic reactions [15, 7] and the Gierer-Meinhardt (GM) system modelling morphogenesis [14]. The (generalized) GM equation is directly included in the general class of two-component, singularly perturbed systems considered here,

$$\left\{ \begin{array}{rcrcccc} U_t &=& U_{xx} &-& [\mu U - \nu_1 F_1(U;\varepsilon)] &+& \frac{\nu_2}{\varepsilon} F_2(U,V;\varepsilon) \\ V_t &=& \varepsilon^2 V_{xx} &-& V &+& G(U,V;\varepsilon), \end{array} \right. \tag{1.1}$$

the particular structure of which emphasizes the new, generalized aspects of this system compared to the specific well-studied GS/GM-type models. More details on this specific form can be found in section 1.1.

In this paper, we consider equation (1.1) on the unbounded domain $\mathbb{R}$, so $U(x,t), V(x,t) : \mathbb{R}\times\mathbb{R}_{>0} \to \mathbb{R}_{>0}$; we restrict ourselves to positive solutions. Moreover, we assume that $\mu > 0$, $\nu_{1,2} \in \mathbb{R}$ and

$F_1 : \mathbb{R}_{>0} \to \mathbb{R}$, $F_2, G : \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ are nonlinear functions obeying mild regularity assumptions, see section 1.1. The parameter $0 < \varepsilon \ll 1$ is assumed to be asymptotically small, i.e. the results established in this paper will be valid for '$\varepsilon > 0$ small enough'. While strictly speaking not part of the domain of $F_2$ and $G$, the trivial background state $(U, V) \equiv (0, 0)$ is assumed to be asymptotically stable, see also Remark 1.6. The Gray-Scott equation can also be brought into the form (1.1) by a number of transformations that scale the magnitude of the patterns to $\mathcal{O}(1)$ with respect $\varepsilon$ and that shift the Gray-Scott background state $(1, 0)$ to the normalized state $(0, 0)$ in ( refe:PDEslow) [3, 9].

The model problem (1.1) can be considered as the most general (semilinear) two component, singularly perturbed reaction-diffusion system –see equations (1.3) and (1.4)– that may exhibit $\mathcal{O}(1)$ pulse patterns (Remark 1.2), *apart from an explicit co-dimension* 1 *condition on the structure of the linearized model near the trivial background state that determines the limiting behavior of the pulse.* This condition –that roughly states that near this background state the 'slow' $U$-component only couples into the 'fast' $V$-equation in a nonlinear way– has mainly been imposed for technical reasons; see however Remark 1.5. A derivation and more precise motivation of the model will be given in section 1.1, together with a list of specific assumptions on the parameters and nonlinearities in (1.1).

The class of equations covered by (1.1) significantly extends the GS and GM type models. In this paper, we will develop an explicit theory for the existence and the stability of symmetric, stationary pulse solutions to (1.1) that have positive $U$ and $V$-components and that have $\mathcal{O}(1)$ (sup-)norm with respect to $\varepsilon$ (Remark 1.2). We will especially highlight the effect of generalizing two –as it will turn out– quite restrictive properties shared by the GS and GM models. Firstly, these models do not allow for nonlinear behavior in $U$ in the slow $U$-equation outside the fast pulse region, i.e. the slow $U$-equations of the GS/GM models are linear in $U$ for $V = 0$. In other words, both the GS and the (generalized) GM equations correspond to system (1.1) with $\nu_1 = 0$ – the nonlinearity in the $U$-equation is decomposed in a $V$-independent term ($F_1$) and a term that vanishes at $V = 0$, hence $F_2(U, 0) = 0$ (see section 1.1 and especially assumption (A3)). In the literature, this linearity in the slow $U$-system is crucially exploited in the stability analysis of pulse solutions to both GS- as well as GM-type models: this analysis relies heavily on the fact that the stability problem can be solved explicitly in terms of exponential functions in the slow $U$-fields [3, 4, 18, 22, 23, 41]. Note that systems incorporating a slow nonlinearity ($\nu_1 \neq 0$) were already encountered in [26], although no pulse-type solutions were considered in this paper. Secondly, in almost all previous studies the nonlinear term $G(U, V)$ in (1.1) is a simple, explicit power of $V$ as function of $V$ (it is in fact quadratic in $V$ in the GS and the standard GM equation) – see [42, 27] for some exceptions involving a saturation term. This also forms an essential ingredient of the analysis, since it enables one to explicitly solve the fast reduced stability problem (see [3, 4] and section 3).

One can thus say that the existing methods for the explicit analysis of homoclinic pulses in two component, singularly perturbed reaction-diffusion equations are applicable to the subclass of (1.1) in which $\nu_1 = 0$ and $G(U, V) = g(U)V^d$ for $d > 1$ and some function $g(U)$ – see also Remark 1.2. The theory to be developed in this paper goes beyond these rather severe restrictions. Moreover, the richness of the novel phenomena introduced by the extended class (1.1) is shown by way of an explicit example in the companion paper [38] – see also Remark 1.1.

In section 2, the existence of stationary singular pulses for system (1.1) is established by the methods of geometric singular perturbation theory, under mild and natural assumptions; in particular, we assume that the fast $V$-system admits a homoclinic pulse solution. Similar to related results in [3, 7], pulses correspond to intersections of the slow unstable manifold $W_s^{\mathrm{u}}((0, 0))$ and a take off curve $T_o$ in the half-plane $\{(u,p) : u > 0\}$ associated to the reduced slow existence problem (i.e. $V \equiv 0$, $\varepsilon = 0$ and $U = u(x)$ in (1.1)). As a consequence, system (1.1) may in general exhibit various homoclinic pulse solutions – see Figure 2 in section 2. The precise existence result is summarized in Theorem 2.1. From section 3 onwards, the (linear) stability of a homoclinic pulse is analyzed using Evans function techniques. The slow reduced linear stability problem is no longer of constant coefficient type, as is the case in the GS/GM type models studied in the literature: both the slow and fast reduced linear problems have the structure of classical Sturm-Liouville problems. This fact is strongly used in the analysis. It is shown that the Evans function associated to the (spectral) stability of the pulse can be decomposed into a fast and a slow component. The main result of this analysis –which is obtained through a nonlocal eigenvalue problem (NLEP)– is Theorem 4.4, which provides an explicit expression for the slow component $t_{s,+}$ of the Evans function in terms of the nonlinearities $F_{1,2}$, $G$ of (1.1) and the leading order approximation of the pulse (as established by Theorem 2.1). Since it is established in Corollary 5.2 that all nontrivial eigenvalues correspond to zeroes of $t_{s,+}$, Theorem 4.4 thus provides an explicit analytical control over the stability of the pulses given by Theorem 2.1.

Even though the pulse is constructed in a most general setting under mild assumptions on the nonlinearities $F_{1,2}$ and $G$, a number of (relatively) simple instability results is obtained by detailed analysis of the function $t_{s,+}$ in the neighbourhood of known eigenvalues of the fast reduced problem, these results are presented in section 5. The instability of the homoclinic pulse can be established by determining the sign of certain explicit expressions (Corollary 5.7, Theorem 5.13). Some of these expressions can be interpreted and determined directly in terms of the existence problem, or more specifically, by considering the slow unstable manifold $W_s^{\mathrm{u}}((0,0))$ and the take off curve $T_o$ that establish the existence op the pulses (Theorem 2.1). In the linear $\nu_1 = 0$ case, $W_s^{\mathrm{u}}((0,0))$ always has positive $p(= u_x)$-coordinate so that $W_s^{\mathrm{u}}((0,0)) \cap T_o$ must lie in the positive quadrant of the slow reduced $\{(u,p) : u > 0\}$ half-plane. In general, $W_s^{\mathrm{u}}((0,0)) \cap T_o$ may have negative $p$-coordinates – in such cases the $U$-component of the pulse has a maximum on both sides the fast $V$-pulse, see Figure 4c. It is established in Corollary 5.7 that these pulses are unstable. Moreover, the sign of the relative slopes of the take-off curve $T_o$ with respect to the slow unstable manifold $W_s^{\mathrm{u}}((0,0))$ at their intersections also gives a direct instability criterion: this sign changes at successive intersections, but only those intersections with negative sign can be stable – see Lemma 5.12, Theorem 5.13 and Figure 8. Analysis of the slow component of the Evans function near the trivial eigenvalue $\lambda = 0$ reveals close relations between bifurcations in the existence problem and pulse instabilities, see Corollary 5.8 and Corollary 5.10.

Finally, in section 6, we discuss some implications of the general approach developed here .

**Remark 1.1** The present general results are both inspired by and reflected in the analysis in the companion paper [38], where the theory is developed in the explicit setting of a Gierer-Meinhardt problem with a 'slow nonlinearity':

$$\left\{ \begin{array}{rclcccc} U_t &=& U_{xx} &-& [\mu U - \nu_1 U^d] &+& \frac{\nu_2}{\varepsilon} V^2 \\ V_t &=& \varepsilon^2 V_{xx} &-& V &+& \frac{V^2}{U}. \end{array} \right. \tag{1.2}$$

For a specific system like this, it is possible to go beyond the previously mentioned instability results, especially since it is possible to get an even more explicit 'analytical control' over the reduced Sturm-Liouville problems associated to the stability of the pulses – in [38] a crucial role is played by associated Legendre functions. As a consequence, it is possible to obtain conditions in terms of the model parameters for which the homoclinic pulse is stable. Moreover, numerical analysis of the resulting Hopf bifurcations reveals rich nonlinear behaviour such as stable standing

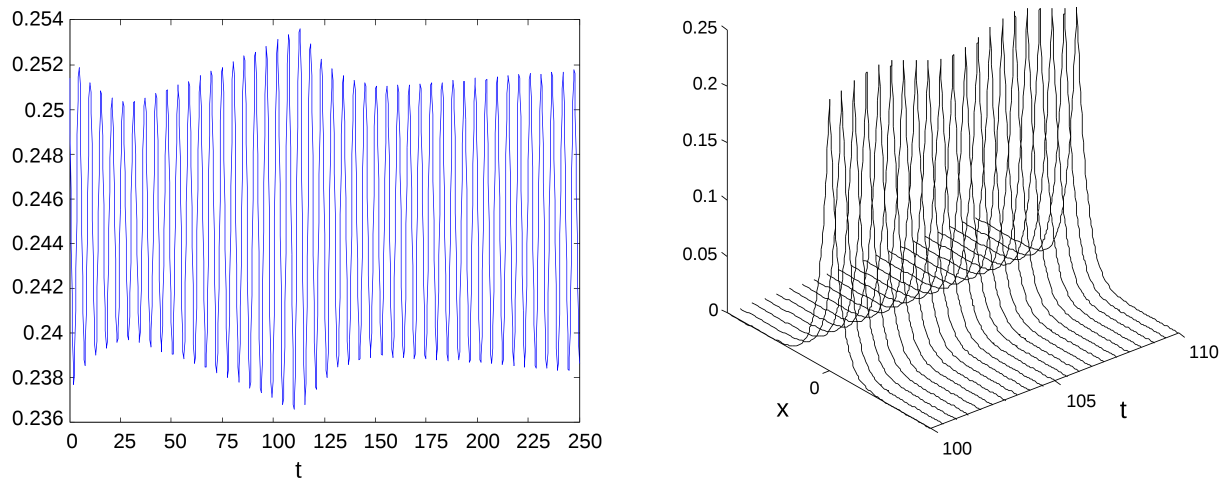



Figure 1: A stable oscillating pulse, as observed in the slowly nonlinear Gierer-Meinhardt system (1.2), studied in [38] (see also Remark 1.1). The left figure shows the position of the tip of the pulse as a function of time. In the right picture, the $u$-component of the pulse is shown in a space-time plot. These results were obtained by direct numerical simulation of the PDE system, for $(\mu, \nu_1, \nu_2, d) = (0.9, 1, 2, 2)$ and $\varepsilon = 0.02$.

localised pulses that bifurcate from the pulses considered here and of which the maximum oscillates up and down in a complex –periodic, quasi-periodic, chaotic– fashion, see Figure 1. This novel and intriguing behaviour has not been observed in the literature on GS/GM-type models. In a forthcoming paper [39], the nature of the Hopf bifurcation of pulses in system (1.2) is studied. It is established that this Hopf bifurcation can be both sub- and supercritical, as is expected in the general setting of system (1.1). The Hopf bifurcation in GM-type models is always subcritical, as is analytically confirmed in [39].

**Remark 1.2** In this paper we only consider pulse solutions for which the fast $V$-component makes one homoclinic excursion away from the stable rest state. Thus, we do not consider localised multi-pulse patterns that are also very common to GS/GM-type models [7, 3]. More importantly, we also do not consider pulse solutions of 'mesa' or FitzHugh-Nagumo type. Such pulses can be described as bi-heteroclinic (or multi-heteroclinic), since they consist of (at least) two heteroclinic jumps through the fast spatial field separated by a 'long' plateau in which the pattern evolves slowly (in space); see [16, 17, 20, 21, 25] and the references therein.

**Remark 1.3** The Schnakenberg model, the third standard model considered in the literature [34], is very similar to the GS and GM models, in the sense that the slow reduced system also does not contain nonlinearities and that the nonlinearity associated to $G(U, V)$ is again exactly quadratic as function of $V$. Although the Schnakenberg model does not have a trivial stable background state, it can be (and has been) studied by methods that are very similar to those developed for the GS and GM equation [19, 40].

### 1.1 The model

The most general two component reaction-diffusion system on the real line, i.e. for $\hat{x} \in \mathbb{R}$, reads

$$\left\{ \begin{array}{rcl} U_{\hat{t}} & = & d_U U_{\hat{x}\hat{x}} + a_{11}U + a_{12}V + H_1(U,V) \\ V_{\hat{t}} & = & d_V V_{\hat{x}\hat{x}} + a_{21}U + a_{22}V + H_2(U,V) \end{array} \right. \tag{1.3}$$

in which $H_{1,2}(U,V) : \mathbb{R}^2 \to \mathbb{R}$ are nonlinear terms that do not include linear components in $U$ or $V$. Stable pulse solutions must be bi-asymptotic to a spectrally stable 'trivial state' $(U,V) \equiv (\bar{U},\bar{V})$. It can be assumed, by a simple translation of $U$ and $V$, that $(\bar{U},\bar{V}) = (0,0)$ –which does not necessarily need to be a solution to (1.3), see Remark 1.6–. This trivial state is stable if $a_{11} + a_{22} < 0$ and $a_{11}a_{22} - a_{12}a_{21} > 0$. The system is assumed to be singularly perturbed, i.e. it is assumed that $U(\hat{x},\hat{t})$ is slowly varying as function of $\hat{x}$ compared to the (relatively fast) spatial variation of $V(\hat{x},\hat{t})$ – see also Remark 1.5. In other words, we assume that $0 < d_V \ll d_U$, or, without loss of generality, that $d_V = \varepsilon^2 \ll 1$, with $0 < \varepsilon \ll 1$ asymptotically small, and $d_U = 1$. This introduces the fast spatial variable $\hat{\xi} = \hat{x}/\varepsilon$, in which (1.3) has the form

$$\left\{ \begin{array}{rcl} \varepsilon^2 U_{\hat{t}} & = & U_{\hat{\xi}\hat{\xi}} + \varepsilon^2 \left[a_{11}U + a_{12}V + H_1(U,V)\right] \\ V_{\hat{t}} & = & V_{\hat{\xi}\hat{\xi}} + a_{21}U + a_{22}V + H_2(U,V). \end{array} \right. \tag{1.4}$$

Since $\hat{U}(x,t)$ is assumed to be bounded on $\mathbb{R}$, we formally conclude from the first equation in (1.4) that $U(\hat{\xi},\hat{t})$ must approach a constant value $\bar{U}$ in the limit $\varepsilon \to 0$; in other words, the singularly perturbed nature of (1.4) causes $U$ to be constant in leading order as function of the fast spatial variable $\hat{\xi}$. As a consequence, in the singular limit $\varepsilon \to 0$ the existence problem for stationary patterns reduces to a family of fast reduced existence problems for $V = v_f(\xi)$,

$$v_{f,\hat{\xi}\hat{\xi}} + a_{21}\bar{U} + a_{22}v_f + H_2(\bar{U},v_f) = 0, \tag{1.5}$$

parameterized by $\bar{U} \in \mathbb{R}$; note that this is an integrable planar system.

In this paper, we focus on the most simple pulse solutions: stationary solutions that are biasymptotic to the stable background state $(0,0)$ of (1.4), that are symmetric in $\hat{\xi}$ (or $\hat{x}$), and that only make one 'jump' through the fast field (which is to leading order described by (1.5)) – see also Remark 1.2. By the above asymptotic arguments, system (1.4) can only have such a pulse solution if there are values of $\bar{U}$ for which (1.5) has a 'fast' homoclinic orbit $v_{f,h}(\hat{\xi};\bar{U})$. The main co-dimension 1 assumption underlying the reduction of the most general system (1.3)/(1.4) to the model problem (1.1) is that this homoclinic solution is biasymptotic to the critical point $(v_f, v_{f,\hat{\xi}}) = (0,0)$ of (1.5), i.e. that $\lim_{\hat{\xi}\to\pm\infty} v_{f,h}(\hat{\xi};\bar{U}) = \lim_{\hat{\xi}\to\pm\infty} \frac{d}{d\hat{\xi}} v_{f,h}(\hat{\xi};\bar{U}) = 0$. In principle, this is quite a restrictive condition. Since $\bar{U} \neq 0$ in general, it directly implies that $a_{21}$ must be 0. Nevertheless, the methods developed in this paper can also be applied to systems for which $\lim_{\xi\to\pm\infty} v_{f,h}(\hat{\xi};\bar{U})$ depends on $\bar{U}$, and only approaches 0 on the slow spatial scale, as $\bar{U} \to 0$. However, the analysis does become more involved for those systems: outside the fast homoclinic jump region described by (1.5), the component $V$ will not be constant, but will evolve slowly (as function of $\hat{x}$), 'slaved' to the slow $U$-component – see Remark 1.5. To highlight the impact of allowing for fully general nonlinearities in (1.1) compared to the restricted cases of the GM and GS equations, we focus on a class of systems (1.3)/(1.4) with $a_{21} = 0$. In other words, we focus on the general class of two component, singularly perturbed, systems in which the slow component $U(\hat{x},\hat{t})$ only couples into the fast $V$-equation through the nonlinear term $H_2(U,V)$.

Since $a_{21} = 0$, the assumption that the trivial state $(U(\hat{\xi},\hat{t}), V(\hat{\xi},\hat{t}) \equiv (0,0)$ is spectrally stable reduces to $a_{11} < 0$ and $a_{22} < 0$. By introducing $t = -a_{22}\hat{t}$ and $\xi = \sqrt{-a_{22}}\hat{\xi}$, equation (1.4) can now be written as

$$\left\{ \begin{array}{rcccccc} \varepsilon^2 U_t & = & U_{\xi\xi} & + & \varepsilon^2 [-\mu U & + & F(U,V;\varepsilon)] \\ V_t & = & V_{\xi\xi} & - & V & + & G(U,V;\varepsilon) \end{array} \right.$$

with

$$\mu = \frac{a_{11}}{a_{22}} > 0,\ F(U,V;\varepsilon) = -\frac{1}{a_{22}}\left[a_{12}V + H_1(U,V)\right],\ G(U,V;\varepsilon) = -\frac{1}{a_{22}}H_1(U,V).$$

Next, we decompose $F(U,V;\varepsilon)$ into a part that depends only on $U$ and a part that is 0 if $V = 0$,

$$F(U,V;\varepsilon) = F(U,0;\varepsilon) + [F(U,V;\varepsilon) - F(U,0;\varepsilon)] \stackrel{\text{def}}{=} \nu_1 F_1(U;\varepsilon) + \frac{\nu_2}{\varepsilon}F_2(U,V;\varepsilon), \tag{1.6}$$

where $\nu_{1,2} \in \mathbb{R}$ (not necessarily $\mathcal{O}(1)$ in $\varepsilon$) have been introduced to control the relative impact of the nonlinear, non-GS/GM term $F_1(U)$ and the nonlinear coupling term $F_2(U,V)$. Hence, we arrive at (1.1) written in the fast spatial variable $\xi$,

$$\left\{ \begin{array}{rcccccc} \varepsilon^2 U_t & = & U_{\xi\xi} & - & \varepsilon^2[\mu U - \nu_1 F_1(U;\varepsilon)] & + & \varepsilon\nu_2 F_2(U,V;\varepsilon) \\ V_t & = & V_{\xi\xi} & - & V & + & G(U,V;\varepsilon). \end{array} \right. \tag{1.7}$$

Apart from the condition on the (non)appearance of terms that are linear in $U$ in the $V$-equation, (Remarks 1.5 and 3.1), the model problem (1.1) can thus be seen as a general two component, singularly perturbed model, in which $\mathcal{O}(1)$ pulses can exist. A priori, one could argue that the term $\nu_2/\varepsilon$ in (1.6) also introduces a further restriction, but this is not the case since $\nu_2$ will be allowed to be $\mathcal{O}(\varepsilon)$ in the analysis. The $F_2$-term in (1.1)/(1.7) has been artificially 'blown up' by a factor of $1/\varepsilon$ for clarity of presentation – which can be explained most clearly by looking at (1.1). The fast $V$ component enters into the slow $U$-equation of (1.1) through an asymptotically large term of $\mathcal{O}(1/\varepsilon)$ – as is also the case in the GS, generalized GM and Schnakenberg models. Since $V(x,t)$ is strongly localized to a domain of size $\mathcal{O}(\varepsilon)$ in the $x$-scaling, this is quite natural: if the interaction term in the $U$-equation would be smaller, then the direct impact of $V$ on the evolution of $U$ would be asymptotically small. As was already remarked, this situation can, and will, be studied by considering $|\nu_2| \ll 1$ in (1.7), see Corollary 5.15. It will be found that (1.7) may have pulse solutions in this case, but that these pulse must be unstable: (1.7) in essence decouples into two scalar equations, the coupling is not strong enough to counteract the unstable eigenvalues of the scalar $U,V$-subsystems. In other words, by artificially 'blowing up' the $F_2$-term in (1.1), we automatically focus on the most relevant region in the parameter space associated to (1.1).

Since we have introduced ambiguities by the introduction of $\nu_{1,2}$ in (1.6), and since we so far not discussed the precise nature of the nonlinear terms, we now list the basic assumptions we impose on the parameters $\mu, \nu_1, \nu_2$ and the nonlinearities $F_1, F_2, G$ in (1.1)/(1.7) in the subsequent analysis:

**Definition 1.4** *A statement of the form '$f(x) \rightsquigarrow c \cdot g(x)$ as $x \to x_0$' is true whenever the limit $\lim_{x \to x_0} \frac{1}{g(x)} f(x) = c$ exists and is well-defined.*

**(A1)** $\mu$, $\nu_{1,2}$ are real and nonsingular in $\varepsilon$; furthermore, $\mu > 0$.

**(A2)** • $F_1(U;\varepsilon) \rightsquigarrow U^{f_1}$ as $U \downarrow 0$ for some $f_1 > 1$;
and $F_1$ is smooth both on its domain and as a function of $\varepsilon$.

**(A3)** Writing $F_2(U,V;\varepsilon) = F_{2,1}(U;\varepsilon)\,V + F_{2,2}(U,V;\varepsilon)$,
• $F_{2,1}(U;\varepsilon) \rightsquigarrow \tilde{F}_{2,1}(\varepsilon)\,U^{\gamma_1}$ as $U \downarrow 0$ for some $\gamma_1 \geq 0$ and $\tilde{F}_{2,1}(\varepsilon) \in \mathbb{R}$;
• $F_{2,2}(U,V;\varepsilon) \rightsquigarrow \tilde{F}_{2,2,u}(V;\varepsilon)\,U^{\alpha_1}$ as $U \downarrow 0$ for some $\alpha_1 \in \mathbb{R}$;
• $F_{2,2}(U,V;\varepsilon) \rightsquigarrow \tilde{F}_{2,2,v}(U;\varepsilon)\,V^{\beta_1}$ as $V \to 0$ for some $\beta_1 > 1$;
and $F_2$ is smooth both on its domain and as a function of $\varepsilon$.

**(A4)** • $G(U,V;\varepsilon) \rightsquigarrow \tilde{G}_u(V;\varepsilon)\,U^{\alpha_2}$ as $U \downarrow 0$ for some $\alpha_2 \in \mathbb{R}$;
• $G(U,V;\varepsilon) \rightsquigarrow \tilde{G}_v(U;\varepsilon)\,V^{\beta_2}$ as $V \to 0$ for some $\beta_2 > 1$;
and $G$ is smooth both on its domain and as a function of $\varepsilon$.

Assumption (A2) defines $F_1(U;\varepsilon)$ and $\nu_1$ uniquely, while $F_2(U,V;\varepsilon)$ and $\nu_2$ are not (yet), but will be uniquely defined with assumption (B3). The possibly singular behavior of the functions $F_{2,2}(U,V;\varepsilon)$ and $G(U,V;\varepsilon)$ for $U$ and/or $V$ small (assumptions (A3,4)) is in accordance with the behavior of the nonlinearities in the generalized GM model, see also Remark 3.1. In fact, the generalized GM model corresponds to (1.1)/(1.7) with

$$\nu_1 = 0, \nu_2 = 1, F_{2,1}(U;\varepsilon) \equiv 0, F_{2,2}(U,V;\varepsilon) = U^{\alpha_1}V^{\beta_1}, G(U,V;\varepsilon) = U^{\alpha_2}V^{\beta_2}, \beta_1, \beta_2 > 1. \qquad (1.8)$$

**Remark 1.5** As was already noted, if $a_{21} \neq 0$, it follows from (1.5) that the fast $V$-component of the homoclinic pattern $(U_h(\xi), V_h(\xi))$ does not go to 0 as $\xi$ leaves the fast field, but instead will be 'slaved' to the slowly evolving $U$-component and thus only approaches 0 on the slow spatial scale. It has been shown for a model problem [6] that such a situation can be studied along the lines of the present approach. Thus, letting go of the condition $a_{21} = 0$ a priori mostly introduces additional technicalities (see also Remark 3.1). However, allowing $a_{21}$ to be $\neq 0$ may possibly generate more than just 'additional technicalities'. A linear $U$-term in the $V$-equation may introduce the possibility of having homoclinic pulse patterns with spatially oscillating, i.e. non-monotonously, decaying 'tails'. We are not aware of any analytical, or even numerical, study of this type of localized patterns in singularly perturbed reaction diffusion equations. At the introduction of the asymptotically large $\nu_2/\varepsilon$ pre-factor in (1.6), we argued that the fast $V$-component must couple in an asymptotically strong fashion into the slow $U$-equation. If $V$ is slowly varying, and thus no longer at leading order constant (i.e. 0) outside the fast field, one has to think carefully about the magnitude and/or impact of the nonlinear coupling term $F_2(U,V)$ in the $U$-equation. Thus, our choice to impose the co-dimension 1 condition $a_{21} = 0$ is motivated by our preference to avoid 'additional technicalities', however, this more technical case may exhibit novel phenomena and/or may eventually ask for the development of a novel theoretical approach.

**Remark 1.6** We explicitly allow the nonlinearities $F_2$ and $G$ to be singular in $U$ as $U \downarrow 0$, see assumptions (A3) and (A4) on the exponents $\alpha_{1,2}$. This implies that $(U,V) = (0,0)$ is not necessarily a solution to (1.1). However, the specific form of (1.1) was derived from (1.3) based on considerations on the stability of the trivial state. While strictly speaking this line of reasoning loses validity for singular $F_2$ and $G$, the specific context of the pulse construction (see section 2) allows for a more 'loose' notion of stability of the trivial state. Since it will turn out that that $V = 0$ to exponential order long before $U \downarrow 0$ in $\hat{x}$, it is only necessary that $\lim_{U\to 0}\lim_{V\to 0} H_{1,2}(U,V) = 0$, which follows from assumptions (A3) and (A4).
As to questions concerning nonlinear stability, the presence of a singularity as $U \to 0$ does have an important influence on the treatment of the subject; however, these questions fall outside the scope of this paper. A more elaborate discussion on the influence of singular terms on well-posedness and nonlinear stability can be found in [3], Remark 1.3.

## 2 The existence of pulses

In this section, we study the existence of positive, symmetric, stationary pulse solutions $(U(\xi,t),V(\xi,t) = (U_h(\xi),V_h(\xi))$ to (1.7) (or equivalently (1.1)). In the fast spatial coordinate $\xi$, the associated ODE takes the form

$$\left\{\begin{array}{lcl} u_\xi & = & \sqrt{\varepsilon}p \\ p_\xi & = & \sqrt{\varepsilon}\,[-\nu_2 F_2(u,v;\varepsilon) + \varepsilon[\mu u - \nu_1 F_1(u;\varepsilon)]] \\ v_\xi & = & q \\ q_\xi & = & v - G(u,v;\varepsilon) \end{array}\right. \tag{2.1}$$

This equation inherits the reversibility symmetry of (1.7) in the form of

$$\xi \to -\xi,\ \ p \to -p,\ \ q \to -q. \tag{2.2}$$

Especially since we focus on symmetric pulses, this symmetry will play a crucial role in the forthcoming analysis. The singularly perturbed system (2.1) has a family of integrable planar ODEs as fast reduced limit,

$$v_{f,\xi\xi} = v_f - G(u_0,v_f;0) \ \text{ or } \left\{\begin{array}{lcl} v_{f,\xi} & = & q_f \\ q_{f,\xi} & = & v_f - G(u_0,v_f;0) \end{array}\right. ,\ u_0 > 0 \tag{2.3}$$

with integrals

$$\mathcal{H}_v(u_0) = \frac{1}{2}q_f^2 - \frac{1}{2}v_f^2 + \int_0^{v_f} G(u_0,\tilde{v};0)\,d\tilde{v}, \tag{2.4}$$

parameterized by $u_0$. Note that by the assumption (A4) on $G(u,v;\varepsilon)$ (section 1.1), $(v_f,q_f) = (0,0)$ is a critical point of saddle type for all $u_0$. The following additional assumption on $G(u,v;0)$ will be used throughout the paper.

**(A5)** *For all $u_0 > 0$ there exists a positive solution $v_{f,h}(\xi;u_0)$ to* (2.3) *which is homoclinic to* $(v_f,q_f) = (0,0)$.

Assumption (A5) implies that, for all for all $u_0 > 0$, the level set $\{\mathcal{H}_v(u_0) = 0\}$ (2.4) through the saddle point $(0,0)$ must intersect the $v$-axis at $v_M > 0$. If there are multiple intersections, $v_M$ is defined uniquely as the smallest (positive) solution. Due to the translation invariance, $v_{f,h}(\xi;u_0)$ is not yet determined uniquely as function of $\xi$. Since we consider symmetric pulses in the paper, we fix $v_{f,h}(\xi;u_0)$ by assuming that

$$v_{f,h}(0;u_0) = v_M,\ \ \frac{d}{d\xi}v_{f,h}(0;u_0) = 0. \tag{2.5}$$

It is essential for the existence of (positive, symmetric, stationary) pulse solutions $(U_h(\xi)),V_h(\xi))$ to (1.7) that there are open regions in $u_0$ for which (2.3) has homoclinic solutions to $(0,0)$: the fast component $V_h(\xi)$ is to leading order determined by an orbit $v_{f,h}(\xi;u_0)$ for a certain $u_0 = u_*$ (see [3, 7] and the subsequent analysis). It is in principle not necessary that such a $u_0$-region includes the full positive half line. Therefore (A5) is not a crucial assumption to the fullest extent, in the sense that the theory developed here can be straightforwardly extended to equations of the type (1.7) that do not satisfy this condition for all $u_0 > 0$. However, if (A5) is not satisfied, then especially the bifurcation analysis would become much more involved, since homoclinic orbits will appear and disappear as $u_*$ approaches a boundary of one of these regions. These additional bifurcations are not relevant for the method, but do severely diminish the transparency of presentation.

The structure of this section is as follows: we first present an intuitive sketch of the geometrical procedure by which the existence of pulse can be established (that is strongly based on [3]). Based on this, we then formulate our main existence result (Theorem 2.1).

By assumption (A4) on $G(U,V;\varepsilon)$, system (2.1) has a two-dimensional invariant, normally hyperbolic (slow) manifold $\mathcal{M}$, given by

$$\mathcal{M} = \{(u,p,v,q) : v = q = 0, u > 0\}, \tag{2.6}$$

where we restrict ourselves to the positive $u$-half space since we have allowed $G(U,V;\varepsilon)$ to be singular at $U = 0$ ((A4), [3]). By Fenichel theory [11, 12], $\mathcal{M}$ must have (three-dimensional) stable and unstable manifolds, $\mathcal{W}^{\rm s}(\mathcal{M})$ and $\mathcal{W}^{\rm u}(\mathcal{M})$, that are $\mathcal{O}(\sqrt{\varepsilon})$ close to the (three-dimensional) stack of level sets $\{\mathcal{H}_v(u_0) = 0\}$ (2.4) associated to the fast reduced limit (2.3). Note that this also implies that both $\mathcal{W}^{\rm s}(\mathcal{M})$ and $\mathcal{W}^{\rm u}(\mathcal{M})$ must intersect the hyperplane $\{q = 0\}$ transversally.

The pulse patterns $(U_h(\xi), V_h(\xi))$ considered here (Remark 1.2) correspond to homoclinic orbits $\Gamma_h(\xi) = (u_h(\xi), p_h(\xi), v_h(\xi), q_h(\xi))$ to the critical point $(0,0,0,0)$ of (2.1). These orbits must be contained in the intersection $\mathcal{W}^{\rm s}(\mathcal{M}) \cap \mathcal{W}^{\rm u}(\mathcal{M})$ of the stable and unstable manifolds $\mathcal{W}^{\rm s}(\mathcal{M})$ and $\mathcal{W}^{\rm u}(\mathcal{M})$ of $\mathcal{M}$. These manifolds may (and most often will) have countably many (two-dimensional) intersections [3]. Here, we restrict ourselves to the first intersections of $\mathcal{W}^{\rm s}(\mathcal{M})$ and $\mathcal{W}^{\rm u}(\mathcal{M})$ on which the most simple, one-circuit, homoclinic orbits lie (Remark 1.2). It can be shown by a straightforward Melnikov calculation that the two-dimensional first intersections $\mathcal{I}^{+1} = \mathcal{W}^{\rm u}(\mathcal{M}) \cap \{q = 0\}$ and $\mathcal{I}^{-1} = \mathcal{W}^{\rm s}(\mathcal{M}) \cap \{q = 0\}$ must intersect in a one-dimensional manifold

$$\mathcal{I}^{+1} \cap \mathcal{I}^{-1} = \{(u_0, 0, v_{f,h}(0;u_0) + \mathcal{O}(\sqrt{\varepsilon}), 0); u_0 > 0\} \subset \{p = q = 0\},$$

parameterized by $u_0$ (see [3]). To each $u_0 > 0$ corresponds a solution $\Gamma(\xi;u_0) = (u(\xi;u_0), p(\xi;u_0), v(\xi;u_0), q(\xi;u_0))$ to (2.1) that is biasymptotic to $\mathcal{M}$ (with $\Gamma(0;u_0) \in \mathcal{I}^{+1} \cap \mathcal{I}^{-1}$). Note that this is a natural result: the intersection corresponds to symmetric solutions $\Gamma(\xi;u_0)$ to (2.2); their components $u(\xi;u_0)$ and $v(\xi;u_0)$ are even as function of $\xi$ and have a local extremum at $\xi = 0$. Moreover, if it exists, the homoclinic orbit $\Gamma_h(\xi)$ must correspond to one of the orbits $\Gamma(\xi;u_0)$, i.e. $\Gamma_h(\xi) = \Gamma(\xi;u_*)$ for a certain $u_* > 0$.

Since the $u$- and $p$-coordinates only vary slowly in (2.1), the $u$- and $p$-components of each orbit $\Gamma(\xi;u_0) \in \mathcal{W}^{\rm s}(\mathcal{M}) \cap \mathcal{W}^{\rm u}(\mathcal{M})$ remain to leading order constant during the passage of $\Gamma(\xi;u_0)$ through the fast field. To determine $u_*$, it is necessary to compute the accumulated change $\Delta u(u_0)$ in $u(\xi;u_0)$ and $\Delta p(u_0)$ in $p(\xi;u_0)$ during a 'jump' of $\Gamma(\xi;u_0)$ through the fast field. To do so, we first give a more precise definition of the fast field,

$$I_f \stackrel{\rm def}{=} \left[-\frac{1}{\varepsilon^{\frac{1}{4}}}, \frac{1}{\varepsilon^{\frac{1}{4}}}\right]. \tag{2.7}$$

The boundary of $I_f$ has been placed at the transition zone in which $|\xi| = \varepsilon^{-\frac{1}{4}} \gg 1$ and $|x| = \varepsilon^{\frac{3}{4}} \ll 1$, the precise location of $\partial I_f$ is not essential [3, 5]. In particular the quantity $\Delta p(u_0)$ plays an important role in the analysis, and can be determined as

$$
\begin{array}{rcl}
\Delta p(u_0) & = & \int_{-\varepsilon^{-\frac{1}{4}}}^{\varepsilon^{-\frac{1}{4}}} p_\xi \, \mathrm{d}\xi \\
& = & \sqrt{\varepsilon} \int_{-\varepsilon^{-\frac{1}{4}}}^{\varepsilon^{-\frac{1}{4}}} [-\nu_2 F_2(u,v;\varepsilon) + \varepsilon[\mu u - \nu_1 F_1(u;\varepsilon)]] \, \mathrm{d}\xi \\
& = & -\nu_2 \sqrt{\varepsilon} \int_{-\varepsilon^{-\frac{1}{4}}}^{\varepsilon^{-\frac{1}{4}}} F_2(u(\xi;u_0), v(\xi;u_0);\varepsilon) \, \mathrm{d}\xi + \mathcal{O}(\varepsilon^{\frac{5}{4}}) \\
& = & -\nu_2 \sqrt{\varepsilon} \int_{-\varepsilon^{-\frac{1}{4}}}^{\varepsilon^{-\frac{1}{4}}} F_2(u_0, v_{f,h}(\xi;u_0);0) \, \mathrm{d}\xi + \mathcal{O}(\varepsilon^{\frac{3}{4}}) \\
& = & -\nu_2 \sqrt{\varepsilon} \int_{-\infty}^{\infty} F_2(u_0, v_{f,h}(\xi;u_0);0) \, \mathrm{d}\xi + \mathcal{O}(\varepsilon^{\frac{3}{4}}),
\end{array}
\tag{2.8}
$$

where we have used the regular perturbation result that both $|u(\xi;u_0) - u_0|$ and $|v(\xi;u_0) - v_{f,h}(\xi;u_0)|$ are $\mathcal{O}(\sqrt{\varepsilon})$ for $\xi \in I_f$ with $v_{f,h}(\xi;u_0)$ the homoclinic solution of the fast reduced limit system (2.3); note also that $F_2(u_0, v_{f,h}(\xi;u_0);0)$ decays exponentially in $\xi$ as $|\xi| \to \infty$ (since $F_2(u,0;0) = 0$ (A3) and $v_{f,h}(\xi;u_0)$ decays exponentially). We define

$$
D_p(u_0) = \int_{-\infty}^{\infty} F_2(u_0, v_{f,h}(\xi;u_0);0) \, \mathrm{d}\xi, \tag{2.9}
$$

so that $\Delta p(u_0) = -\nu_2 \sqrt{\varepsilon} D_p(u_0) + o(\varepsilon^{\frac{3}{4}})$. Hence, $p(\xi;u_0) = \mathcal{O}(\sqrt{\varepsilon})$ in $I_f$, which implies that

$$
\Delta u(u_0) = \int_{-\varepsilon^{-\frac{1}{4}}}^{\varepsilon^{-\frac{1}{4}}} u_\xi \, \mathrm{d}\xi = \sqrt{\varepsilon} \int_{-\varepsilon^{-\frac{1}{4}}}^{\varepsilon^{-\frac{1}{4}}} p(\xi;u_0) \, \mathrm{d}\xi = \mathcal{O}(\varepsilon^{\frac{3}{4}}), \tag{2.10}
$$

i.e. that $u(\xi;u_0)$ does not vary at $\mathcal{O}(\sqrt{\varepsilon})$. We can now remove the remaining ambiguities involving the sign of the product of $\nu_2$ and $F_2$ by determining the leading order behaviour of $F_2$ by gauging

**(B3)** $D_p(u) \rightsquigarrow 1 \cdot u^{d_p}$ as $u \downarrow 0$ for some $d_p \in \mathbb{R}$;

see the discussion immediately below Theorem 2.1 for a motivation of this definition. From the above, it follows that the orbits $\Gamma(\xi;u_0)$ 'take off' from $\mathcal{M}$ $\mathcal{O}(\varepsilon^{\frac{3}{4}})$ close to the curve

$$
T_o = \left\{ p = \tfrac{1}{2}\nu_2 \sqrt{\varepsilon} D_p(u), u > 0 \right\} \subset \mathcal{M} \tag{2.11}
$$

and 'touch down' again on its symmetrical image

$$
T_d = \left\{ p = -\tfrac{1}{2}\nu_2 \sqrt{\varepsilon} D_p(u), u > 0 \right\} \subset \mathcal{M}. \tag{2.12}
$$

The curve $T_o$, respectively $T_d$, represents the leading order approximation of the collection of base points of the Fenichel fibers in $\mathcal{W}^{\mathrm{u}}(\mathcal{M})$, resp. $\mathcal{W}^{\mathrm{s}}(\mathcal{M})$, that are elements of $\mathcal{W}^{\mathrm{u}}(\mathcal{M}) \cap \mathcal{W}^{\mathrm{s}}(\mathcal{M})$ – see [3] for more details. Hence, the slow evolution of $\Gamma(\xi;u_0) \subset \mathcal{W}^{\mathrm{u}}(\mathcal{M}) \cap \mathcal{W}^{\mathrm{s}}(\mathcal{M})$ after, respectively before its jump through the fast field, i.e. for $\xi > \varepsilon^{-\frac{1}{4}}$ resp. $\xi < -\varepsilon^{-\frac{1}{4}}$, is to leading order governed by a solution of the flow on (the invariant manifold) $\mathcal{M}$ that has $(u_0, p_0) \in T_d$, resp. $\in T_o$, as boundary (initial, resp. end) conditions. Since $F_2(u,0;\varepsilon) \equiv 0$ (assumption (A3)), the flow on $\mathcal{M}$ is governed by

$$
u_{s,xx} = \mu u_s - \nu_1 F_1(u_s;\varepsilon), \quad \text{or} \quad \left\{ \begin{array}{rcl} u_{s,x} & = & p_s \\ p_{s,x} & = & \mu u_s - \nu_1 F_1(u_s;\varepsilon) \end{array} \right., \; u > 0, \tag{2.13}
$$

where $x$ is the original slow spatial coordinate of (1.1) (i.e. $x = \varepsilon\xi$). Equation (2.13) is integrable with integral

$$
\mathcal{H}_u(\varepsilon) = \frac{1}{2}p_s^2 - \frac{1}{2}\mu u_s^2 + \nu_1 \int_0^{u_s} F_1(\tilde{u};\varepsilon) \, d\tilde{u}. \tag{2.14}
$$

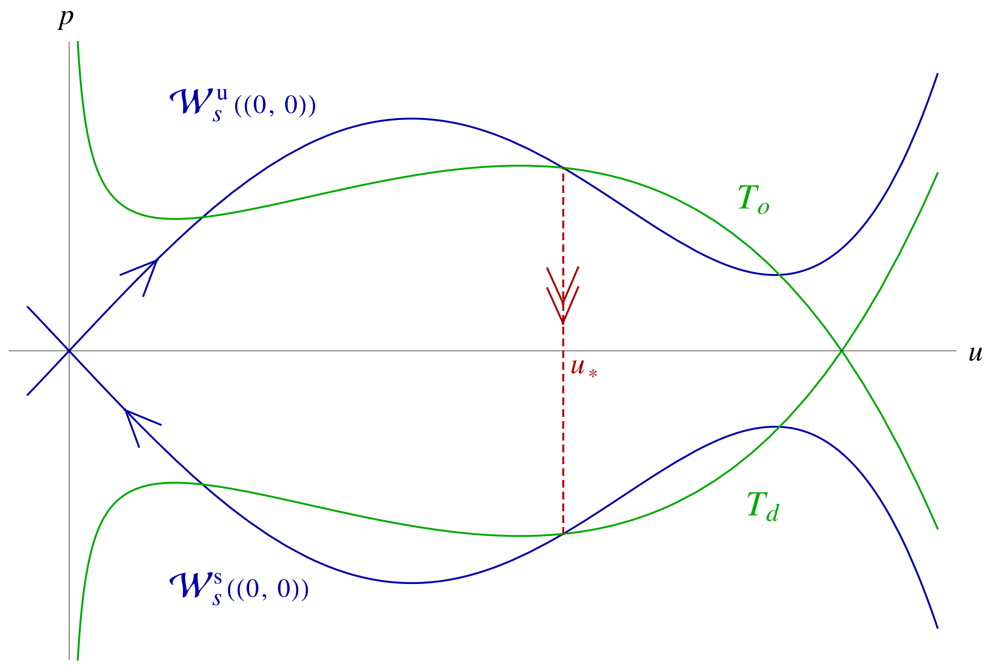



Figure 2: The dynamics on the slow manifold $\mathcal{M}$, governed by (2.13). The jump through the fast field is indicated by a red dashed line.

Since $|x| = \varepsilon^{\frac{3}{4}}$ on $\partial I_f$, the above boundary conditions can (to leading order) be considered as conditions on $(u_s(x), p_s(x))$ at $x = 0$. Note that (2.13) does still depend on $\varepsilon$, i.e. it is not the slow reduced limit associated (2.1): in the form $(u_s(x), p_s(x), 0, 0)$, the solution of (2.13) also is an exact solution of (2.1), in the slow variable $x$ since $\mathcal{M}$ is invariant for the full system (2.1). In fact, since $u_x = \frac{1}{\varepsilon} u_\xi = \frac{1}{\sqrt{\varepsilon}} p$ in (2.1), $p_s$ in (2.13) corresponds to $\frac{1}{\sqrt{\varepsilon}} p$ in (2.1), so that the boundary conditions on $(u_s(x), p_s(x))$ correspond in leading order to

$$(u_s(0), p_s(0)) = \left(u_0, \pm\tfrac{1}{2}\nu_2 D_p(u_0)\right). \tag{2.15}$$

By the stability of the background state $(U, V) \equiv (0, 0)$ and assumption (A2), the critical point $(0, 0)$ of (2.13) is a saddle point with (one dimensional) stable, and unstable, manifolds $\mathcal{W}_s^{\mathrm{s}}((0, 0); \varepsilon)$ and $\mathcal{W}_s^{\mathrm{u}}((0, 0); \varepsilon)$. Near $\mathcal{M}$, the orbits $\Gamma(\xi; u_0) \subset \mathcal{W}^{\mathrm{u}}(\mathcal{M}) \cap \mathcal{W}^{\mathrm{s}}(\mathcal{M})$ are with exponential accuracy governed by solutions $u_s(x)$ of (2.13) that satisfy the boundary conditions $(u_s(0), p_s(0)) \in T_{o,d}$, hence it follows that $\Gamma(\xi; u_0)$ is homoclinic to $(0, 0, 0, 0)$ if $u_0 = u_* > 0$ is such that $\Gamma(\xi; u_*)$ takes off from $\mathcal{W}_s^{\mathrm{u}}((0, 0))$ and touches down at $\mathcal{W}_s^{\mathrm{s}}((0, 0))$. In other words, the homoclinic orbit $\Gamma_h(\xi)$ corresponds to a $\Gamma(\xi; u_*)$ with $u_*$ determined as the $u$-coordinate of an intersection of $T_o$ and $\mathcal{W}_s^{\mathrm{u}}((0, 0))$; note that $T_o \cap\ mathcal W_s^{\mathrm{u}}((0, 0))$ and $T_d \cap \mathcal{W}_s^{\mathrm{s}}((0, 0))$ have the same $u$-coordinates by the symmetry (2.2) – see Figure 2.

The manifolds $\mathcal{W}_s^{\mathrm{u}}((0, 0))$ resp. $\mathcal{W}_s^{\mathrm{s}}((0, 0))$ are by definition spanned by the solutions $(u_s^{\mathrm{u}}(x; \varepsilon), p_s^{\mathrm{u}}(x; \varepsilon))$ resp. $(u_s^{\mathrm{s}}(x; \varepsilon), p_s^{\mathrm{s}}(x; \varepsilon))$ of (2.13). Note that $u_s^{\mathrm{s}}(x) = u_s^{\mathrm{u}}(-x)$ and $p_s^{\mathrm{s}}(x) = -p_s^{\mathrm{u}}(-x)$ by the reversibility symmetry. As with the definition of the fast reduced homoclinic orbit $v_{f,h}(\xi; u_0)$ – see (2.5) – we need to be more precise here and eliminate the translational invariance from the orbit $(u_s^{\mathrm{u}}(x; \varepsilon), p_s^{\mathrm{u}}(x; \varepsilon))$. This can be done by fixing the location of the point $x = 0$ as $(u_s^{\mathrm{u}}(0; \varepsilon), p_s^{\mathrm{u}}(0; \varepsilon)) = (u_0^{\mathrm{u}}, p_0^{\mathrm{u}}) \in \mathcal{W}_s^{\mathrm{u}}((0, 0))$; now $(u_s^{\mathrm{u}}(x; \varepsilon), p_s^{\mathrm{u}}(x; \varepsilon))$ and therefore $(u_s^{\mathrm{s}}(x; \varepsilon), p_s^{\mathrm{s}}(x; \varepsilon))$ are uniquely determined as solutions of (2.13). Note that the precise position of the point $(u_0^{\mathrm{u}}, p_0^{\mathrm{u}}) \in \mathcal{W}_s^{\mathrm{u}}((0, 0))$ is in general not relevant. However, in an explicit setting, a natural choice for $(u_0^{\mathrm{u}}, p_0^{\mathrm{u}})$ often presents itself; see the discussion on the relative configurations of $\mathcal{W}_s^{\rm u}((0,0))$ and $\mathcal{W}_s^{\rm s}((0,0))$ following the statement of Theorem 2.1.

As manifolds $\mathcal{W}_s^{\rm u}((0,0))$ and $W_s^{\rm s}((0,0))$ cannot cross to the negative half-plane (in $u$), both $\mathcal{W}_s^{\rm u}((0,0))$ and $\mathcal{W}_s^{\rm s}((0,0))$ are subsets of $\{\mathcal{H}_u = 0\} \cap \{u \geq 0\}$. A necessary, leading order condition on the critical value(s) $u_*$ for which $\Gamma(\xi; u_*)$ is homoclinic to $(0,0,0,0)$ can be obtained by combining (2.15) with (2.14) and setting $\varepsilon$ to 0, i.e. by imposing that $(u_s(0;0), p_s(0;0)) \in \{\mathcal{H}_u(0) = 0\}$, yielding

$$\mu u^2 - 2\nu_1 \int_0^u F_1(\tilde{u}; 0)\, d\tilde{u} = \tfrac{1}{4}\nu_2^2\, D_p^2(u) = \tfrac{1}{4}\nu_2^2 \left[\int_{-\infty}^{\infty} F_2(u, v_{f,h}(\xi; u); 0)\, \mathrm{d}\xi\right]^2. \tag{2.16}$$

Since this relation does neither distinguish between $T_o$ and $T_d$ nor between $W_s^{\rm u}((0,0))$ and $W_s^{\rm s}((0,0))$, there may be solutions $u_{*,j}$ of this equation that do not correspond to homoclinic orbits $\Gamma_h(\xi)$. Using (2.15) we define $p_* = +\frac{1}{2}\nu_2 D_p(u_*)$.

As a final prerequisite for the upcoming theorem, we combine the above defined solutions $(u_s^{\rm u}(x;\varepsilon), p_s^{\rm u}(x;\varepsilon))$ which span the slow unstable manifold $\mathcal{W}_s^{\rm u}((0,0))$ with the (possibly multiple) solution(s) $u_*$ of (2.16) by introducing translational shift(s) $x_*$, for which the following leading order expression holds:

$$(u_s^{\rm u}(-x_*;0), p_s^{\rm u}(-x_*;0)) = (u_*, p_*) = (u_s^{\rm s}(x_*;0), -p_s^{\rm s}(x_*;0)). \tag{2.17}$$

Note that the value of the shift(s) $x_*$ is directly related to the choice of $(u_0^{\rm u}, p_0^{\rm u})$.

**Theorem 2.1** *Assume that conditions (A1-5) and (B3) hold and let $\varepsilon > 0$ be small enough. Let $K$ be the number of non-degenerate solutions $u = u_{*,k} > 0$ of (2.16) such that $(u_{*,k}, p_{*,k}) = (u_{*,k}, \frac{1}{2}\nu_2 D_p(u_{*,k})) \in \mathcal{W}_s^{\rm u}((0,0);0)$. If $K = 0$ then there are no symmetric, positive, one-circuit homoclinic solutions to $(0,0,0,0)$ in (2.1). If $K \neq 0$, there are $K$ distinct positive, symmetric, one-circuit homoclinic orbits $\Gamma_{h,k}(\xi) = (u_{h,k}(\xi), p_{h,k}(\xi), v_{h,k}(\xi), q_{h,k}(\xi)) \subset \mathcal{W}^{\rm u}(\mathcal{M}) \cap \mathcal{W}^{\rm s}(\mathcal{M})$, $k = 1, 2, ..., K$, in (2.1) with internal reflection point $\xi = 0$, so that $\Gamma_{h,k}(0) = (u_{h,k}(0), 0, v_{h,k}(0), 0)$. In the fast field, $\Gamma_{h,k}(\xi)$ is to leading order determined by the homoclinic solution $v_{f,h}(\xi; u_{*,k})$ of (2.3): there is an $\mathcal{O}(1)$ constant $C_1 > 0$ such that*

$$\left|u_{h,k}(\xi) - u_{*,k}\right|, \left|p_{h,k}(\xi)\right|, \left|v_{h,k}(\xi) - v_{f,h}(\xi; u_{*,k})\right|, \left|q_{h,k}(\xi) - \tfrac{d}{d\xi} v_{f,h}(\xi; u_{*,k})\right| < C_1\sqrt{\varepsilon} \text{ for } \xi \in I_f \tag{2.18}$$

*cf. (2.7). In the slow field, $\Gamma_{h,k}(\xi)$ approaches $\mathcal{W}_s^{\rm u}((0,0);\varepsilon) \subset \mathcal{M}$, respectively $\mathcal{W}_s^{\rm s}((0,0);\varepsilon) \subset \mathcal{M}$ exponentially fast for $\xi \to -\infty$, resp. $\xi \to \infty$: there exist $\mathcal{O}(1)$ constants $C_{2,3} > 0$ such that*

- *$\left|v_{h,k}(\xi)\right|, \left|q_{h,k}(\xi)\right| < C_2 e^{-C_3|\xi|}$ for $\xi \in \mathbb{R}\backslash I_f$;*
- *there are shifts $x_{*,k} \in \mathbb{R}$ and solutions $\left(u_{*,k}^{\rm u}(x), p_{*,k}^{\rm u}(x)\right) = \left(u_s^{\rm u}(x - x_{*,k}), p_s^{\rm u}(x - x_{*,k})\right)$ of (2.13), such that $\left(u_{*,k}^{\rm u}(-\varepsilon^{\frac{3}{4}}), p_{*,k}^{\rm u}(-\varepsilon^{\frac{3}{4}})\right) = \left(u_{h,k}(-\varepsilon^{-\frac{1}{4}}), \frac{1}{\sqrt{\varepsilon}} p_{h,k}(-\varepsilon^{-\frac{1}{4}})\right) = \left(u_{*,k} + \mathcal{O}(\sqrt{\varepsilon}), p_{*,k} + \mathcal{O}(\sqrt{\varepsilon})\right)$ and*

$$\left|u_{h,k}(\xi) - u_{*,k}^{\rm u}(\varepsilon\xi)\right|, \left|\tfrac{1}{\sqrt{\varepsilon}} p_{h,k}(\xi) - p_{*,k}^{\rm u}(\varepsilon\xi)\right| < C_2 e^{C_3\xi} \text{ for } \xi < -\varepsilon^{-\frac{1}{4}}; \tag{2.19}$$

- *$\left(u_{*,k}^{\rm s}(\varepsilon^{\frac{3}{4}}), p_{*,k}^{\rm s}(\varepsilon^{\frac{3}{4}})\right) = \left(u_{h,k}(\varepsilon^{-\frac{1}{4}}), \frac{1}{\sqrt{\varepsilon}} p_{h,k}(\varepsilon^{-\frac{1}{4}})\right) = \left(u_{*,k} + \mathcal{O}(\sqrt{\varepsilon}), -p_{*,k} + \mathcal{O}(\sqrt{\varepsilon})\right)$ with $\left(u_{*,k}^{\rm s}(x), p_{*,k}^{\rm s}(x)\right) = \left(u_{*,k}^{\rm u}(-x), -p_{*,k}^{\rm u}(-x)\right)$ and*

$$\left|u_{h,k}(\xi) - u_{*,k}^{\rm s}(\varepsilon\xi)\right|, \left|\tfrac{1}{\sqrt{\varepsilon}} p_{h,k}(\xi) - p_{*,k}^{\rm s}(\varepsilon\xi)\right| < C_2 e^{-C_3\xi} \text{ for } \xi > \varepsilon^{-\frac{1}{4}}. \tag{2.20}$$

*The orbits $\Gamma_{h,k}(\xi)$ correspond to the homoclinic pulse patterns $\left(U_{h,k}(\xi), V_{h,k}(\xi)\right)$ in (1.7) that are symmetric with respect to $\xi = 0$ through $U_{h,k}(\xi) = u_{h,k}(\xi)$, $V_{h,k}(\xi) = v_{h,k}(\xi)$, $k = 1, ..., K$.*

See also Figure 2.

**Proof:** The essential ingredients of the proof have already been sketched above. The fact that (2.1) concerns a more general class of systems than the generalized GM model does not influence the geometric approach, therefore, we refer to [3] for the full details. □

The (implicit) definition of the signs of $F_2(U,V)$ and $\nu_2$ in assumption (B3) implies that $T_o \subset \{u_s \geq 0, p_s \leq 0\}$ for $u_s$ small enough and $\nu_2 < 0$. In the case that $\mathcal{W}_s^{\mathrm{u}}((0,0)) \subset \{u_s \geq 0, p_s \geq 0\}$ this for instance immediately implies that $T_o$ and $\mathcal{W}_s^{\mathrm{u}}((0,0))$ cannot have intersections near the saddle $(0,0)$ if $\nu_2 < 0$. In fact, it follows that there cannot be homoclinic pulse patterns in this case, i.e. $\mathcal{W}_s^{\mathrm{u}}((0,0)) \subset \{u_s \geq 0, p_s \geq 0\}$ and $\nu_2 < 0$, if it is known that expression $D_p(u)$ cannot change sign – which is the case for both the GS and the (generalized) GM models, see [7, 3] and section 5.3. Thus, the definition of the signs of $F_2(U,V)$ and $\nu_2$ through (B3a) provides a more direct insight in the relevance of a solution $u_*$ of (2.16), since it gauges the relative positions of $T_o$ and $\mathcal{W}_s^{\mathrm{u}}((0,0))$ as function of $\nu_2$.

Clearly, the condition that $(u_{*,k}, p_{*,k}) \in \mathcal{W}_s^{\mathrm{u}}((0,0);0)$ is central to the construction of the pulse pattern $\big(U_{h,k}(\xi), V_{h,k}(\xi)\big)$. Therefore, it is relevant to note that there are two distinct configurations. If $\mathcal{W}_s^{\mathrm{u}}((0,0)) \cap \{p_s = 0\} = \emptyset$, then clearly $\mathcal{W}_s^{\mathrm{u}}((0,0)) \cap \mathcal{W}_s^{\mathrm{s}}((0,0)) = \emptyset$ and $\mathcal{W}_s^{\mathrm{u}}((0,0)) \subset \{u_s \geq 0, p_s \geq 0\}$ (and $\mathcal{W}_s^{\mathrm{s}}((0,0)) \subset \{u_s \geq 0, p_s \leq 0\}$). On the other hand, if $\mathcal{W}_s^{\mathrm{u}}((0,0)) \cap \{p_s = 0\} \neq \emptyset$, then (by the symmetry) $\mathcal{W}_s^{\mathrm{u}}((0,0))$ and $\mathcal{W}_s^{\mathrm{s}}((0,0))$ must have the same, unique, intersection $u_M > 0$ with the $u_s$-axis and thus merge in a homoclinic orbit to $(0,0)$ – note that this can only happen in the non-GS/GM case with $\nu_1 \neq 0$. In this case it is natural to determine $\big(u_s^{\mathrm{u}}(x;\varepsilon), p_s^{\mathrm{u}}(x;\varepsilon)\big)$ uniquely by choosing $x = 0$ as the location of the internal reflection point of the homoclinic orbit, i.e. to set $\big(u_s^{\mathrm{u}}(0;\varepsilon), p_s^{\mathrm{u}}(0;\varepsilon)\big) = (u_0^{\mathrm{u}}, p_0^{\mathrm{u}}) = (u_M, 0)$. Once this gauge choice is made, the sign of $x_*$ determines whether the jump through the fast field occurs before or after the slow component of the pulse passes through the maximum of the slow homoclinic orbit, i.e. whether the jump is downwards ($x_* > 0$) or upwards ($x_* < 0$); see Figure 4a resp. 4c for an illustration of these two configurations in the context of the model (1.2) studied in the companion paper [38]. It will be shown in section 5 (Corollary 5.7) that the second configuration is always unstable.

If $\mathcal{W}_s^{\mathrm{u}}((0,0)) \cap \{p_s = 0\} = \emptyset$, there is no natural unique way to gauge the choice of $\big(u_s^{\mathrm{u}}(0;\varepsilon), p_s^{\mathrm{u}}(0;\varepsilon)\big)$. This is undesirable since in extension the value (and sign) of $x_*$ (2.17) is not fixed. This will turn out to be the cause of technical complications in some parts of the stability analysis, see section 5.1. However, the following Lemma allows us to make an unambiguous gauge choice for $\big(u_s^{\mathrm{u}}(0;\varepsilon), p_s^{\mathrm{u}}(0;\varepsilon)\big) = (u_0^{\mathrm{u}}, p_0^{\mathrm{u}})$ in either case. The idea is to alter the vector field defined by $F_{1,2}$ and $G$ beyond a certain $u$-value, 'bending' the unstable and stable slow manifolds towards each other such that they do intersect.

**Lemma 2.2** *Without loss of generality, we may assume that* $\mathcal{W}_s^{\mathrm{u}}((0,0)) \cap \{p_s = 0\} \neq \emptyset$ *and therefore choose* $\big(u_s^{\mathrm{u}}(0;\varepsilon), p_s^{\mathrm{u}}(0;\varepsilon)\big) = (u_0^{\mathrm{u}}, p_0^{\mathrm{u}}) = (u_M, 0)$*. This fixes the sign of* $x_*$ *as* $sgn(x_*) = sgn(p_*)$*.*

**Proof.** Given the functions $F_{1,2}$ and $G$ for which assumptions A(1-5) and B3 hold, consider an open neighbourhood $\mathcal{U} \in \mathbb{R}^4$ of the set $\big\{(u,p,v,q) \in \mathbb{R}^4 \,|\, u \leq u_{*,K}\big\}$ where $u_{*,K}$ is largest solution to (2.16), see Theorem 2.1. For each function trio $(\tilde{F}_{1,2}, \tilde{G}) \in \mathcal{O}mega$ with

$$\mathcal{O}mega = \Big\{\hat{F}_{1,2} \text{ and } \hat{G} \text{ are smooth and } (\tilde{F}_{1,2}(\mathcal{U}), \tilde{G}(\mathcal{U})) = (F_{1,2}(\mathcal{U}), G(\mathcal{U}))\Big\},$$

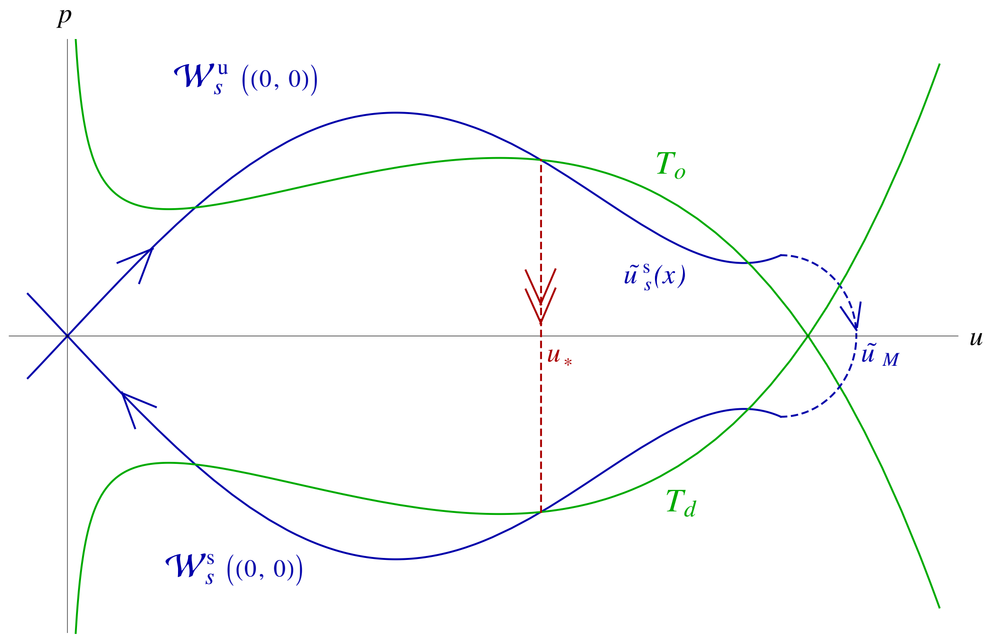



Figure 3: The slow manifold with its dynamics as in Figure 2, but altered for $u > u_*$ in such a way that the (new) slow stable and unstable manifolds $\tilde{\mathcal{W}}_s^{\mathrm{s}}((0,0))$ and $\tilde{\mathcal{W}}_u^{\mathrm{s}}((0,0))$ intersect at $(\tilde{u}_M, 0)$.

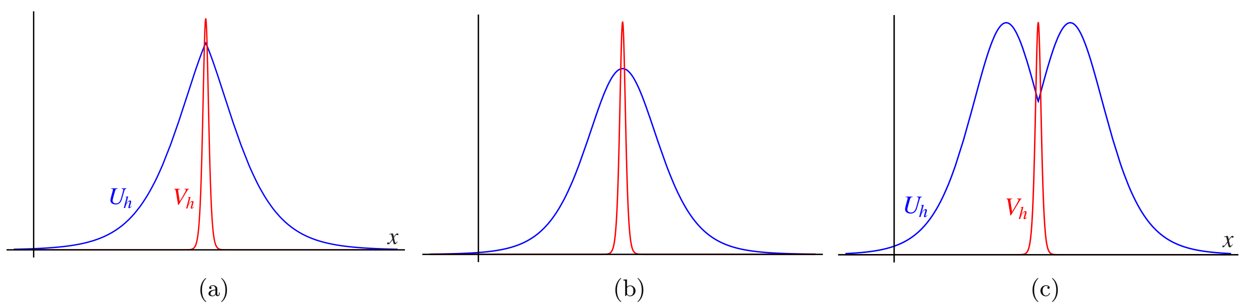



Figure 4: The stationary homoclinic pulse $\Gamma_h(x) = (U_h(x), V_h(x))$ as a solution to (1.2) for $x_* > 0$ (4a), $x_* = 0$ (4b) and $x_* < 0$ (4c), as studied in the companion paper [38].

Theorem 2.1 can be applied in the same way and will yield the same results. Now, we alter the original vector field defined by $F_{1,2}$ and $G$, i.e. pick a suitable function trio $(\tilde{F}_{1,2}, \tilde{G}) \in \mathcal{O}mega$ such that the associated slow stable manifold $\tilde{\mathcal{W}}_s^{\mathrm{s}}((0,0))$ actually does intersect the $u$-axis beyond $u_{*,K}$ and therefore coincides with $\tilde{\mathcal{W}}_u^{\mathrm{s}}((0,0))$ by symmetry. In the new, altered vector field, the function $\tilde{u}_s^{\mathrm{s}}(x)$ defines a homoclinic orbit, see Figure 3. We make the natural choice $(\tilde{u}_0^{\mathrm{u}}, \tilde{p}_0^{\mathrm{u}}) = (\tilde{u}_M, 0)$, see the discussion after Theorem 2.1; this also redefines $x_*$ accordingly as $\tilde{x}_*$. From the choice $\tilde{p}_0^{\mathrm{u}} = \tilde{p}_0^{\mathrm{s}} = 0$ it follows that $\tilde{x}_* > 0$ when $\tilde{p}_* > 0$ and vice versa, see (2.17). □

Note that the modification of $F_{1,2}$ and $G$ may induce new intersections of $\tilde{W}_s^{\mathrm{u}}((0,0))$ and $T_o$, see Figure 3. These intersections are artificial and –of course– do not correspond to homoclinic orbits in the original system.

Finally, we formulate a result on the occurrence of homoclinic saddle node bifurcations, which will especially be relevant in the upcoming stability analysis. We again refer to [3] for (details on the geometry behind) its proof.

**Corollary 2.3** *Assume that conditions (A1-5) and (B3) hold and let $\varepsilon > 0$ be small enough. Assume that $u = u_{*,sn} > 0$ is a degenerate solution of (2.16), i.e. that both (2.16) and its u-derivative*

$$2\mu u - 2\nu_1 F_1(u;0) = \tfrac{1}{2}\nu_2^2\, D_p(u)\, \frac{d}{du} D_p(u) \tag{2.21}$$

*hold for a certain parameter combination $(\mu_{sn}, \nu_{1,sn}, \nu_{2,sn})$ to leading order in $\varepsilon$. Assume furthermore that $(u_{*,sn}, p_{*,sn}) = (u_{*,sn}, \frac{1}{2}\nu_2 D_p(u_{*,sn})) \in \mathcal{W}^{\mathrm{u}}_s((0,0);0)$ and that $u_{*,sn}$ is a quadratic zero of (2.16). Then the parameter combination $(\mu_{sn}, \nu_{1,sn}, \nu_{2,sn})$ determines a saddle node bifurcation of homoclinic orbits: by changing one of the parameters $\mu$, $\nu_1$, or $\nu_2$ (and keeping the other two fixed), two distinct homoclinic orbits $\Gamma_{h,l}(\xi)$ and $\Gamma_{h,l+1}(\xi)$ of (2.1) merge and annihilate each other.*

**Remark 2.4** It the forthcoming stability analysis – see especially sections 3.2 and 5 – it will be necessary to have a measure for the decay rate of $v_{f,h}(\xi;u_*)$ and $v_{f,h}(\xi;u_{*,k})$ as $\xi \to \pm\infty$. It follows from (2.3) in combination with assumption (A4) that $v_{f,h}(\xi;u_*)$ decays like $e^{\mp\xi}$ for $\xi \to \pm\infty$. Therefore, we define $v_{f,\infty}$ by

$$v_{f,h}(\xi;u_*) \rightsquigarrow v_{f,\infty}(u_*)\, e^{\mp\xi} \quad \text{as} \quad \xi \to \pm\infty. \tag{2.22}$$

Note that $v_{f,\infty} \neq 0$ is determined uniquely, since $v_{f,h}(\xi;u_*)$ has been determined uniquely (2.5); it has the same value for $\xi \to \pm\infty$ since $v_{f,h}(\xi;u_*)$ is even as function of $\xi$ (2.2), (2.5). Likewise, we define $u_{s,\infty}$ by

$$u^{\mathrm{s}}_s(x;\varepsilon) \rightsquigarrow u_{s,\infty}\, e^{-\sqrt{\mu}x} \quad \text{as} \quad x \to \infty, \tag{2.23}$$

where $u^{\mathrm{s}}_s(x;\varepsilon)$ is the nonzero solution to (2.13) that spans the stable manifold $\mathcal{W}^{\mathrm{s}}_s((0,0);\varepsilon)$ – note that the limit exists by assumption (A2).

# 3 Linearization and the reduced problems

In the forthcoming sections we consider the stability of one of the $K$ homoclinic pulse patterns in (1.1) or (1.7) – Theorem 2.1 –, denoted by either $(U_h(x), V_h(x))$ or $(U_h(\xi), V_h(\xi))$.

## 3.1 The linear stability problem

With a small abuse of notation, re-introduce $u(\xi)$ and $v(\xi)$ by

$$U(\xi,t) = U_h(\xi) + u(\xi)e^{\lambda t}, \;\; V(\xi,t) = V_h(x) + v(\xi)e^{\lambda t}, \tag{3.1}$$

with $\lambda \in \mathbb{C}$. The linearized stability of $(U_h(\xi), V_h(\xi))$ is thus determined by

$$\left\{ \begin{array}{rcllllll} \varepsilon^2\lambda u &=& u_{\xi\xi} &-& \varepsilon^2[\mu u - \nu_1 \frac{dF_1}{dU}(U_h)u] &+& \varepsilon\nu_2\frac{\partial F_2}{\partial U}(U_h,V_h)u &+& \varepsilon\nu_2\frac{\partial F_2}{\partial V}(U_h,V_h)v \\ \lambda v &=& v_{\xi\xi} &-& v &+& \frac{\partial G}{\partial U}(U_h,V_h)u &+& \frac{\partial G}{\partial V}(U_h,V_h)v \end{array} \right. \tag{3.2}$$

which can also be written as a system by introducing the vector $\phi(\xi) = (u(\xi), p(\xi), v(\xi), q(\xi))^T$, with $p = \frac{1}{\sqrt{\varepsilon}}\, u_\xi$ and $q = v_\xi$ as

$$\dot{\phi} = \mathcal{A}(\xi;\lambda,\varepsilon)\phi, \tag{3.3}$$

where the dot represents $\frac{d}{d\xi}$. Here,

$$\mathcal{A}(\xi;\lambda,\varepsilon)=\begin{pmatrix} 0 & \sqrt{\varepsilon} & 0 & 0 \\ \sqrt{\varepsilon}\left(-\nu_2\frac{\partial F_2}{\partial U}(U_h,V_h)+\varepsilon\left[\mu+\lambda-\nu_1\frac{dF_1}{dU}(U_h)\right]\right) & 0 & \sqrt{\varepsilon}\left(-\nu_2\frac{\partial F_2}{\partial V}(U_h,V_h)\right) & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\partial G}{\partial U}(U_h,V_h) & 0 & \lambda+1-\frac{\partial G}{\partial V}(U_h,V_h) & 0 \end{pmatrix}. \tag{3.4}$$

It follows from the smoothness and decay rates in $V$ as $V\to 0$ assumed in (A1,3) that $\frac{\partial F_2}{\partial V}(U,0)=F_{2,1}(U)+\frac{\partial F_{2,2}}{\partial V}(U,0)=F_{2,1}(U)$ and that $\frac{\partial F_2}{\partial U}(U,0)=0$. Likewise, by (A1,4), $\frac{\partial G}{\partial U}(U,0)=\frac{\partial G}{\partial V}(U,0)=0$. Since $V_h(\xi)$ becomes exponentially small as $\xi$ approaches the boundaries $\xi=\pm\varepsilon^{-\frac{1}{4}}$ of the fast field $I_f$, by the approximation results (2.19), (2.20) on $U_h(\xi)$ for $|\xi|>\varepsilon^{-\frac{1}{4}}$ (Theorem 2.1) and by the reversibility symmetry (2.2), it follows that $\mathcal{A}(\xi;\lambda,\varepsilon)$ approaches the intermediate, slowly varying matrix

$$\mathcal{A}_s(\varepsilon\xi;\lambda,\varepsilon)=\begin{pmatrix} 0 & \sqrt{\varepsilon} & 0 & 0 \\ \varepsilon\sqrt{\varepsilon}\left[(\mu+\lambda)-\nu_1\frac{dF_1}{dU}(u_*^{\mathrm{s}}(|\varepsilon\xi|))\right] & 0 & -\nu_2\sqrt{\varepsilon}F_{2,1}(u_*^{\mathrm{s}}(|\varepsilon\xi|)) & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \lambda+1 & 0 \end{pmatrix} \tag{3.5}$$

in the slow field $|\xi|>\varepsilon^{-\frac{1}{4}}$ – see section 3.4. It clearly also follows from (2.19), (2.20) (and assumptions (A1-4)) that there are positive $\mathcal{O}(1)$ constants $\tilde{C}_2$ and $\tilde{C}_3$ such that

$$\|\mathcal{A}(\xi;\lambda,\varepsilon)-\mathcal{A}_s(\varepsilon\xi;\lambda,\varepsilon)\|\le\tilde{C}_2e^{-\tilde{C}_3|\xi|}\ \text{ for }\ |\xi|>\varepsilon^{-\frac{1}{4}}. \tag{3.6}$$

Both matrices $\mathcal{A}(\xi;\lambda,\varepsilon),\mathcal{A}_s(x;\lambda,\varepsilon)$ approach the constant coefficient matrix

$$\mathcal{A}_\infty(\lambda,\varepsilon)=\begin{pmatrix} 0 & \sqrt{\varepsilon} & 0 & 0 \\ \varepsilon\sqrt{\varepsilon}(\lambda+\mu) & 0 & -\nu_2\sqrt{\varepsilon}F_{2,1}(0) & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \lambda+1 & 0 \end{pmatrix} \tag{3.7}$$

as $\xi,x\to\pm\infty$ (A3). Due to the block diagonal, upper triangular structure of $\mathcal{A}_\infty(\lambda)$, its eigenvalues $\{\pm\Lambda_f,\pm\varepsilon\Lambda_s\}$ with

$$\Lambda_f(\lambda)=\sqrt{1+\lambda},\ \ \Lambda_s(\lambda)=\sqrt{\mu+\lambda}. \tag{3.8}$$

are not influenced by the coupling term $-\nu_2\sqrt{\varepsilon}F_{2,1}(0)$. Hence, it follows that $\mathrm{Re}\,\Lambda_f(\lambda)>\mathrm{Re}\,\varepsilon\Lambda_s(\lambda)$ outside an $\mathcal{O}(\varepsilon)$ neighborhood of the essential spectrum

$$\sigma_e=\{\lambda\in\mathbb{R}:\lambda\le\ \max\,(-\mu,-1)\}\subset\mathbb{C} \tag{3.9}$$

associated to the linear stability problem (3.2)/(3.3) – recall that $\sigma_e$ corresponds to those values of $\lambda$ for which one of the $\Lambda_{f,s}(\lambda)$'s is purely imaginary [32]. The impact of the coupling term $-\nu_2\sqrt{\varepsilon}F_{2,1}(0)$ on the eigenvectors of $\mathcal{A}_\infty(\lambda)$ is at most of $\mathcal{O}(\sqrt{\varepsilon})$ as long as $\lambda$ is not $\mathcal{O}(\sqrt{\varepsilon})$ close to $\sigma_e$:

$$\begin{aligned} E_{f,\pm}(\lambda,\varepsilon) &= \left(-\tfrac{\nu_2\varepsilon F_{2,1}(0;\varepsilon)}{1+\lambda-\varepsilon^2(\lambda+\mu)}\,,\,\mp\tfrac{\nu_2\sqrt{\varepsilon}F_{2,1}(0;\varepsilon)}{1+\lambda-\varepsilon^2(\lambda+\mu)}\sqrt{1+\lambda}\,,\,1\,,\,\pm\sqrt{1+\lambda}\right)^T, \\ E_{s,\pm}(\lambda,\varepsilon) &= \left(1\,,\,\pm\sqrt{\varepsilon}\sqrt{\mu+\lambda}\,,\,0\,,\,0\right)^T. \end{aligned} \tag{3.10}$$

The essential difference between the present stability analysis and the existing literature on the stability of pulses in GS/GM-type models [3, 4, 18, 22, 23] is made explicit by the terms $\nu_1\frac{dF_1}{dU}(u_*^{\mathrm{s}}(|\varepsilon\xi|))$

and $-\nu_2\sqrt{\varepsilon}F_{2,1}(u_*^{\rm s}(|\varepsilon\xi|))$ of $\mathcal{A}_{\rm s}(\varepsilon\xi;\lambda)$, i.e. by the fact that there is an intermediate slowly varying matrix between $\mathcal{A}(\xi;\lambda,\varepsilon)$ and $\mathcal{A}_\infty(\lambda\varepsilon)$: in general the matrix $\mathcal{A}(\xi;\lambda,\varepsilon)$ thus does not approach its constant coefficient limit state $\mathcal{A}_\infty(\lambda,\varepsilon)$ exponentially fast on the fast spatial scale. In other words, the GS/GM-type models are (very!) special in the sense that $\nu_1 = 0$ and $F_{2,1}(0) = 0$, so that there is an exponentially accurate estimate like (3.6) on $\|\mathcal{A}(\xi;\lambda,\varepsilon) - \mathcal{A}_\infty(\lambda,\varepsilon)\|$ for $\xi \in \mathbb{R}\backslash I_f$. This fact is crucially used in the stability analysis: it allows one to solve (3.3) outside $I_f$ with an exponential accuracy in terms of simple exponentials (based on (3.8), (3.10)). Moreover, this behavior is also central to the construction of an Evans function $\mathcal{D}(\lambda;\varepsilon)$ associated to (3.3) and it subsequent decomposition into a slow and a fast Evans function [1, 3, 4].

In this paper, the role of $\mathcal{A}_\infty(\lambda,\varepsilon)$ will be taken over by the slow intermediate matrix $\mathcal{A}_s(\varepsilon\xi;\lambda,\varepsilon)$ for $\xi \notin I_f$. The construction of the Evans function $\mathcal{D}(\lambda)$ associated to (3.3) will also be based on the matrix $\mathcal{A}_s(\varepsilon\xi;\lambda,\varepsilon)$. This Evans function will be decomposed in a slow and a fast component using the fast exponential estimate (3.6) – see section 4. In the construction of $\mathcal{D}(\lambda)$, the role of the simple exponentials associated to $\mathcal{A}_\infty(\lambda,\varepsilon)$ will be taken over by the fundamental intermediate solutions of the linear system associated to $\mathcal{A}_s(\varepsilon\xi;\lambda,\varepsilon)$. This system will be studied in section 3.4. However, we will first study the fast reduced limit systems associated to (3.2)/(3.3).

**Remark 3.1** The assumption that $\beta_2 > 1$ in (A4) excludes the possibility of having terms like $UV$ in the $V$ equation, i.e. the nonlinear term $G(U,V)$ of the $V$-equation is not allowed to be like its counterpart $F_2(U,V)$ in the $U$-equation (A3). Similar to the effect of a linear term in $U$ in the $V$-equation for the existence problem – see section 1.1 and Remark 1.5 – terms like $UV$ in the $V$-equation will lead to slowly varying terms in the fast stability equation. Once again (Remark 1.5), this can in principle be handled, see for instance [5] in which an explicit (Ginzburg-Landau type) system with a coupling term the type $UV$ has been analyzed along the lines of the present approach. However, since it introduces additional technicalities (and thus obscures the presentation), we refrain from going into the details.

### 3.2 The homogeneous fast reduced Sturm-Liouville problem

It follows from (2.18) in Theorem 2.1 that the linear stability problem (3.2) reduces in the region $I_f$ and in the limit $\varepsilon \to 0$ to the fast reduced limit problem

$$\lambda v = v_{\xi\xi} - v + \frac{\partial G}{\partial U}(u_*, v_{f,h}(\xi;u_*))\,u(0) + \frac{\partial G}{\partial V}(u_*, v_{f,h}(\xi;u_*))\,v, \quad \xi \in \mathbb{R}, \tag{3.11}$$

where we have used that $u(\xi)$ only varies slowly and thus approaches a constant value $u(0)$ in $I_f$ in this limit. Equation (3.11) is an inhomogeneous Sturm-Liouville problem. In this section, we study the associated homogeneous problem

$$(\mathcal{L}_f(\xi) - \lambda)\,w \stackrel{\rm def}{=} w_{\xi\xi} + \left[\frac{\partial G}{\partial V}(u_*, v_{f,h}(\xi;u_*)) - (1+\lambda)\right] w = 0, \;\text{ with } \lim_{\xi\to\pm\infty} w(\xi) = 0. \tag{3.12}$$

In the NLEP analysis of the Evans function associated to the stability of a pulse in GS/GM-type models [3, 4], the homogeneous fast reduced linearized stability problem (3.12) has a very special form: as function of $V$, $G(U,V)$ simply behaves as $V^{\beta_1}$ (with $\beta_2 > 1$ in the (generalized) GM setting (1.8) and $\beta_2 = 2$ for the GS and the standard GM model). As a consequence, (3.12) can be solved exactly (in terms of hypergeometric functions [3, 4] or associated Legendre functions, see the companion paper [38]). This fact is an essential ingredient of the NLEP analysis in this type of models. Of course this is quite a special, and thus a priori restrictive feature of the GS/GM-type models. As was already remarked in the introduction, this restriction forms the second main ingredient of our motivation to develop the present more general (stability) theory.

To do so, we first note that for functions $G(U,V)$ as described by assumption (A4) and $v_{f,h}(\xi;u_*)$ as homoclinic solution to (2.3), equation (3.12) has the form of a classical (singular) Sturm-Liouville eigenvalue problem. The following lemma summarizes results on this type of problems in the literature (see for instance [36]).

**Lemma 3.2** *Let $H:\mathbb{R}_{\geq 0}\to\mathbb{R}$ be such that the differential equation $w_{xx}=\rho\, w-H(w)$, $\rho>0$ has a solution $w_h$ which is homoclinic to $(w,w_x)=(0,0)$, and write $h(x)=H'(w_h(x))$. For a differential operator of the form $\mathcal{L}(x)=\frac{d^2}{dx^2}+h(x)-\rho$, consider the eigenvalue problem $[\mathcal{L}(x)-\lambda]\,w=0$ with boundary conditions $\lim_{x\to\pm\infty}w(x)=0$. Moreover, define $\Lambda=\sqrt{\rho+\lambda}$; $\arg(\Lambda)\in(-\frac{\pi}{2},\frac{\pi}{2}]$. Then the following holds:*

*(i) There is a finite number of real eigenvalues $\lambda_j$, $j=0,1,\cdots,J$ for which $\lambda_0>0$, $\lambda_1=0$ and $0>\lambda_2>\cdots>\lambda_J>-\rho$. Equivalently, there is a finite number of real eigenvalues $\Lambda_j$ for which $\Lambda_0>\sqrt{\rho}$, $\Lambda_1=\sqrt{\rho}$ and $\sqrt{\rho}>\Lambda_2>\cdots>\Lambda_J>0$.*

*(ii) The associated eigenfunctions $w_j(x)$ have $j$ distinct zeroes and are even resp. odd as a function of $x$ if $j$ is even resp. odd. Moreover, $\frac{d}{dx}w_h(x)$ is an eigenfunction for $\lambda_1=0$ (or $\Lambda_1=1$); in other words, $w_1(x)\in span\left\{\frac{d}{dx}w_h(x)\right\}$.*

*(iii) The eigenfunctions $w_j(x)$, $j=0,\cdots,J$ form an orthogonal set:*

$$\langle w_j,w_k\rangle=\int_{-\infty}^{\infty}w_j(x)w_k(x)\,dx=0\ \ \textit{for}\ \ j\neq k,\quad \textit{and}\ \ \|w_j\|_2=\sqrt{\langle w_j,w_j\rangle}\neq 0;$$

*these eigenfunctions can be determined uniquely by the condition*

$$w_j(x)\rightsquigarrow 1\cdot e^{-\Lambda_j x}\quad \textit{as}\quad x\to\infty \tag{3.13}$$

*(iv) The spectrum associated to the eigenvalue problem $[\mathcal{L}(x)-\lambda]\,w=0$ is given by $\sigma_\lambda=(-\infty,-\rho)\cup\{\lambda_0,\cdots,\lambda_J\}$ or equivalently $\sigma_\Lambda=i\mathbb{R}_{>0}\cup\{\Lambda_0,\cdots,\Lambda_J\}$.*

*(v) For every $\lambda\notin\sigma_\lambda$, there is a unique solution $w_\lambda^{\rm R}(x)$ (which depends smoothly on $\lambda$) such that*

$$w_\lambda^{\rm R}(x)\rightsquigarrow 1\cdot e^{-\Lambda x}\quad \textit{as}\quad x\to\infty. \tag{3.14}$$

*Moreover, the pair $\{w_\lambda^{\rm R},w_\lambda^{\rm L}\}$ with $w_\lambda^{\rm L}(x)=w_\lambda^{\rm R}(-x)$ spans the solution space of the eigenvalue problem $[\mathcal{L}(x)-\lambda]\,w=0$.*

For (3.12) we can apply the above Lemma with $\rho=1$, obtaining a set of fast eigenvalues $\lambda_{f,j}$ and their associated eigenfunctions $w_{f,j}(\xi)$. Moreover, we observe that for $\rho=1$, $\Lambda=\Lambda_f$ (3.8).

Next, we consider the Wronskian

$$\mathcal{W}(\lambda)\stackrel{\rm def}{=}\det\begin{pmatrix} w_\lambda^{\rm L}(\xi) & w_\lambda^{\rm R}(\xi)\\ \frac{d}{d\xi}w_\lambda^{\rm L}(\xi) & \frac{d}{d\xi}w_\lambda^{\rm R}(\xi)\end{pmatrix} \tag{3.15}$$

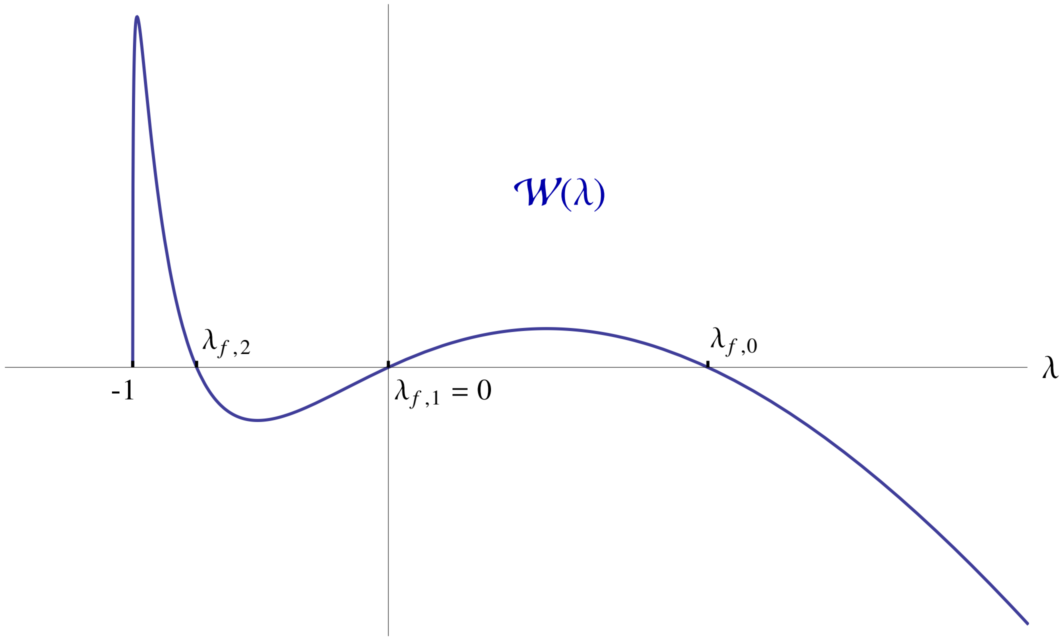



Figure 5: A sketch of the Wronskian $\mathcal{W}(\lambda)$ associated to (3.12) in the case of the model problem (1.2), for $\lambda \in \mathbb{R}$.

associated to (3.12). For notational convenience we only consider $\mathcal{W}$ as function of $\lambda$ here and in the upcoming lemma. In the forthcoming analysis we will however often switch between the equivalent expressions $\mathcal{W}(\lambda)$ and $\mathcal{W}(\Lambda_f)$. This Wronskian can be defined as a smooth, in fact analytic, function of $\lambda$ for all $\lambda \in \mathbb{C}$ outside the (closure of the) essential spectrum associated to (3.12), i.e. for $\lambda \notin (-\infty, -1]$, but including the (eigen)values $\lambda = \lambda_{f,j}$ (Lemma 3.2), by setting $\mathcal{W}(\lambda_{f,j}) = 0$, $j = 0, ..., J$ [36]. Note that $\mathcal{W}(\lambda)$ is in fact an Evans function [1]. In combination with Lemma 3.2, the following result on $\mathcal{W}(\lambda)$ enables us to generalize the GS/GM-type hypergeometric functions approach to the present setting.

**Lemma 3.3** *Let $\mathcal{W}(\lambda)$ be the Wronskian associated to (3.12) and let $\lambda \notin (-\infty, -1]$, then*

$$\mathcal{W}(\lambda) \rightsquigarrow (-1)^{j+1} \|w_{f,j}\|_2^2 (\lambda - \lambda_{f,j}) \ \text{ as } \ \lambda \to \lambda_{f,j}, \ j = 0, ..., J.$$

See Figure 5 for a sketch of a $\mathcal{W}(\lambda)$ for real $\lambda > -1$.

**Proof.** Since we know that $\mathcal{W}(\lambda)$ is a smooth function of $\lambda$ near its zeroes $\lambda_{f,j}$, the proof can be based on a (finite) Taylor expansion of $\mathcal{W}(\lambda_{f,j} + \delta)$ for $\delta = \lambda - \lambda_{f,j} \in \mathbb{C}$ small. To do so, we first need to approximate $w_\lambda^{\rm R}(\xi)$ for $\lambda = \lambda_{f,j} + \delta$. Therefore, we introduce the (regular) approximation

$$w^{\rm R}_{\lambda_{f,j}+\delta}(\xi) = w_{f,j}(\xi) + \delta w_{1,j}(\xi) + \mathcal{R}(\xi; \delta), \tag{3.16}$$

in which $\mathcal{R}(\xi; \delta)$ represents the error terms. This expansion can in general not give a valid approximation of $w^{\rm R}_{\lambda_{f,j}+\delta}(\xi)$ for $\xi \to \infty$. However, it follows directly from Poincaré's expansion theorem (see for instance [37]) that for every $\rho \in [0, 1)$ there is a positive $\mathcal{O}(1)$ constant $C_\rho$ such that

$$|w^{\rm R}_{\lambda_{f,j}+\delta}(\xi) - (w_{f,j}(\xi) + \delta w_{1,j}(\xi))| = |\mathcal{R}(\xi; \delta)| < C_\rho \delta^{2(1-\rho)}, \tag{3.17}$$

for $|\xi| < \mathcal{O}(\delta^{-\rho})$. Note that the standard (and natural) result that $\mathcal{R}(\xi;\delta)| = \mathcal{O}(\delta^2)$ on $\mathcal{O}(1)$ $\xi$-intervals corresponds to the case $\rho = 0$ in (3.17). To determine the leading order correction $w_{1,j}(\xi)$, we substitute (3.16) into (3.12) and obtain the inhomogeneous problem

$$(\mathcal{L}_f(\xi) - \lambda_{f,j})\, w_{1,j} = w_{f,j}(\xi) + \mathcal{O}\left(\delta^{1-2\rho}\right)$$

(3.12) on the domain $|\xi| < \mathcal{O}(\delta^{-\rho})$. It is clear that for the above to be a leading order expression, $\rho < \frac{1}{2}$ must hold. This equation cannot have a solution that is bounded on $\mathbb{R}$, since the operator $\mathcal{L}_f(\xi) - \lambda$ is not invertible at $\lambda = \lambda_{f,j}$ and the inhomogeneous term $b(\xi) = w_{f,j}(\xi)$ clearly does not satisfy the solvability condition $\langle b, w_{f,j}\rangle = \langle w_{f,j}, w_{f,j}\rangle = 0$, see also section 3.3. However, this is not a problem: we are constructing an approximation of a solution $w^{\mathrm{R}}_\lambda(\xi)$ and this solution need not be bounded on $\mathbb{R}$ for $\lambda \neq \lambda_{f,j}$ (Lemma 3.2). Since $w_{f,j}(\xi)$ is a solution of the homogeneous problem, we apply the variation of constants method, i.e. we introduce the unknown function $c_j(\xi)$ by $w_{1,j}(\xi) = c_j(\xi) w_{f,j}(\xi)$ and obtain an equation for $c_j$:

$$\ddot{c}_j w_{f,j} + 2\dot{c}_j \dot{w}_{f,j} = w_{f,j}.$$

This implies that

$$\dot{c}_j(\xi) = \frac{1}{w^2_{f,j}(\xi)} \left[\int_0^\xi w^2_{f,j}(\eta)\, d\eta + c_{1,j}\right],$$

where $c_{1,j}$ is a constant of integration. Writing $c_{1,j} = \hat{c}_{1,j} - \int_0^\infty w^2_{f,j}(\eta)\, d\eta$, we investigate the behaviour of $\dot{c}_j(\xi)$ as $\xi \to \delta^{-\rho}$. From Lemma 3.2, we know that $w_{f,j}(\xi) \rightsquigarrow e^{-\Lambda_{f,j}\delta^{-\rho}}$ as $\xi \to \delta^{-\rho}$. Therefore,

$$\begin{aligned} \dot{c}_j(\xi) \rightsquigarrow\ & e^{2\Lambda_{f,j}\delta^{-\rho}} \left[\int_0^{\delta^{-\rho}} w^2_{f,j}(\eta)\, d\eta - \int_0^\infty w^2_{f,j}(\eta)\, d\eta + \hat{c}_{1,j}\right] = \left[-\int_{\delta^{-\rho}}^\infty w^2_{f,j}(\eta)\, d\eta + \hat{c}_{1,j}\right] e^{2\Lambda_{f,j}\delta^{-\rho}} \\ & = -\frac{1}{2\Lambda_{f,j}} + \hat{c}_{1,j} e^{2\Lambda_{f,j}\delta^{-\rho}} \end{aligned} \tag{3.18}$$

as $\xi \to \delta^{-\rho}$. Since the solution $w^{\mathrm{R}}_{\lambda_{f,j}+\delta}(\xi)$ (3.16) does not grow exponentially as $\xi \to \delta^{-\rho}$ (3.14), it necessarily follows that $w_{1,j}(\xi)$ does neither. Therefore, $c_j(\xi)$ can at most grow as $\frac{1}{w_{f,j}}$, which is as $e^{\Lambda_{f,j}\xi}$. From this, it follows that $\hat{c}_{1,j} = 0$ and therefore

$$c_{1,j} = -\int_0^\infty w^2_{f,j}(\eta)\, d\eta \ \text{ so that } \ \dot{c}_j(\xi) = -\frac{1}{w^2_{f,j}(\xi)} \int_\xi^\infty w^2_{f,j}(\eta)\, d\eta. \tag{3.19}$$

We now return to the Wronskian (3.15). Since $w^{\mathrm{R}}_{\lambda_{f,j}+\delta}(\xi) = w_{f,j}(\xi)\,(1 + \delta c_j(\xi)) + \mathcal{R}(\xi;\delta)$, we can use Lemma 3.2 (ii),(v) to obtain

$$\begin{aligned} w^{\mathrm{R}}_{\lambda_{f,j}+\delta}(\xi) &= w_{f,j}(\xi)\,(1 + \delta c_j(\xi)) + \mathcal{R}(\xi;\delta), & (3.20)\\ w^{\mathrm{L}}_{\lambda_{f,j}+\delta}(\xi) &= (-1)^j w_{f,j}(\xi)\,(1 + \delta c_j(-\xi)) + \mathcal{R}(-\xi;\delta), & (3.21)\\ \tfrac{\mathrm{d}}{\mathrm{d}\xi} w^{\mathrm{R}}_{\lambda_{f,j}+\delta}(\xi) &= \tfrac{\mathrm{d}w_{f,j}}{\mathrm{d}\xi}(\xi)\,(1 + \delta c_j(\xi)) + \delta w_{f,j}(\xi) \tfrac{\mathrm{d}c_j}{\mathrm{d}\xi}(\xi) + \tfrac{\mathrm{d}\mathcal{R}}{\mathrm{d}\xi}(\xi;\delta), & (3.22)\\ \tfrac{\mathrm{d}}{\mathrm{d}\xi} w^{\mathrm{L}}_{\lambda_{f,j}+\delta}(\xi) &= (-1)^j \left[\tfrac{\mathrm{d}w_{f,j}}{\mathrm{d}\xi}(\xi)\,(1 + \delta c_j(-\xi)) - \delta w_{f,j}(\xi) \tfrac{\mathrm{d}c_j}{\mathrm{d}\xi}(-\xi)\right] - \tfrac{\mathrm{d}\mathcal{R}}{\mathrm{d}\xi}(-\xi;\delta), & (3.23) \end{aligned}$$

Since $w^{\mathrm{L/R}}_\lambda(\xi)$ depends smoothly on $\lambda$ (cf. Lemma 3.2), the Poincaré expansion theorem can be applied to $\frac{\mathrm{d}}{\mathrm{d}\xi} w^{\mathrm{R}}_{\lambda_{f,j}+\delta}$ to obtain the result that for every $\hat{\rho} \in [0,1)$ there is a $C_{\hat{\rho}}$ such that $|\frac{\mathrm{d}\mathcal{R}}{\mathrm{d}\xi}(\xi;\delta)| < C_{\hat{\rho}} \delta^{2(1-\hat{\rho})}$. Choosing $\hat{\rho} = \rho < \frac{1}{2}$ enables us to treat $\frac{\mathrm{d}\mathcal{R}}{\mathrm{d}\xi}$ as a higher order term. Using the above expansions for the Wronskian, we obtain

$$\begin{aligned}\mathcal{W}(\lambda_{f,j}+\delta) &= (-1)^j \left(w_{f,j}\tfrac{\mathrm{d}w_{f,j}}{\mathrm{d}\xi} - \tfrac{\mathrm{d}w_{f,j}}{\mathrm{d}\xi} w_{f,j}\right)(1+\delta c_j(\xi)+\delta c_j(-\xi)) \\ &\quad + \delta\,(-1)^j w_{f,j}^2(\xi)\left[\tfrac{\mathrm{d}c_j}{\mathrm{d}\xi}(\xi) + \tfrac{\mathrm{d}c_j}{\mathrm{d}\xi}(-\xi)\right] + \mathcal{O}(\delta^2) \\ &= \delta\,(-1)^j w_{f,j}^2(\xi)\left[\tfrac{\mathrm{d}c_j}{\mathrm{d}\xi}(\xi) + \tfrac{\mathrm{d}c_j}{\mathrm{d}\xi}(-\xi)\right] + \mathcal{O}(\delta^2), \end{aligned} \tag{3.24}$$

in which we refrained from explicitly writing down all $\mathcal{O}(\delta^2) = \mathcal{O}(|\lambda - \lambda_{f,j}|^2)$ correction terms. Using (3.19), we see that

$$w_{f,j}^2(\xi)\left[\tfrac{\mathrm{d}c_j}{\mathrm{d}\xi}(\xi) + \tfrac{\mathrm{d}c_j}{\mathrm{d}\xi}(-\xi)\right] = -\int_{\xi}^{\infty} w_{f,j}^2(\eta)\,\mathrm{d}\eta - \int_{-\xi}^{\infty} w_{f,j}^2(\eta)\,\mathrm{d}\eta = -\int_{-\infty}^{\infty} w_{f,j}^2(\eta)\,\mathrm{d}\eta = -\|w_{f,j}\|_2^2$$

using again Lemma 3.2 (ii). □

Clearly, the Wronskian $\mathcal{W}(\lambda)$ has an extremum for $\lambda \in \mathbb{R}$ between two successive eigenvalues. Based on the previous Lemma it can easily be established that this extremum is a maximum between $\lambda_{2j+1} < \lambda_{2j}$ and a minimum between $\lambda_{2j} < \lambda_{2j-1}$. The following Lemma determines the limit behavior of $\mathcal{W}(\lambda)$ for $\lambda \in \mathbb{R}$ large, see also Figure 5.

**Lemma 3.4** *Let $\mathcal{W}(\lambda)$ be the Wronskian associated to (3.12) and let $\lambda \in \mathbb{R}\backslash(-\infty,-1]$, then*

$$\mathcal{W}(\lambda) \rightsquigarrow -2\sqrt{\lambda} \;\; \textit{as} \;\; \lambda \to +\infty.$$

**Proof.** Define $\delta = 1/\Lambda_f > 0$ ($\Lambda_f \in \mathbb{R}$). It can be shown by the methods of the above proof that for $\delta$ small enough, i.e. $\Lambda_f > 0$ large enough,

$$w_\lambda^{\mathrm{R}}(\xi) = e^{-\Lambda_f \xi}(1+\mathcal{O}(\delta)), \;\; \text{and} \;\; w_\lambda^{\mathrm{L}}(\xi) = e^{\Lambda_f \xi}(1+\mathcal{O}(\delta)).$$

on an $\mathcal{O}(1)$ $\xi$-domain $\supset \{\xi = 0\}$. Hence, for $\Lambda_f$ large enough,

$$\mathcal{W}(\Lambda_f) = \det\begin{pmatrix} e^{\Lambda_f\xi}(1+\mathcal{O}(\delta)) & e^{-\Lambda_f\xi}(1+\mathcal{O}(\delta)) \\ \Lambda_f e^{\Lambda_f\xi}(1+\mathcal{O}(\delta)) & -\Lambda_f e^{-\Lambda_f\xi}(1+\mathcal{O}(\delta))\end{pmatrix} = -2\Lambda_f(1+\mathcal{O}(\delta)),$$

which is equivalent to the statement of the lemma by the definition of $\Lambda_f$ (3.8). □

### 3.3 The inhomogeneous fast reduced Sturm-Liouville problem

Since the inhomogeneous problem (3.11) is linear (and can thus be rescaled), we define $v_{\mathrm{in}}(\xi;\lambda)$ as the bounded solution of

$$(\mathcal{L}_f(\xi) - \lambda)\,v = -\frac{\partial G}{\partial U}(u_*, v_{f,h}(\xi;u_*)). \tag{3.25}$$

Note that this is only possible if $u(0) \neq 0$; the situation where $u(0) = 0$ will be treated in section 5 (which is related to the case $B_-(\lambda) = 0$ there). It follows from the general theory on Sturm-Liouville problems that $v_{\mathrm{in}}(\xi;\lambda)$ is uniquely determined for $\lambda \notin \sigma_f$ ([36]). Since $\{w_\lambda^{\mathrm{L}}(\xi), w_\lambda^{\mathrm{R}}(\xi)\} = \{w_\lambda^{\mathrm{R}}(-\xi), w_\lambda^{\mathrm{R}}(\xi)\}$ span the solution space associated to the homogeneous problem (Lemma 3.2), $v_{\mathrm{in}}(\xi;\lambda)$ can be determined explicitly (in terms of $w_\lambda^{\mathrm{R}}(\pm\xi)$).

**Lemma 3.5** *The bounded solution of (3.25) is given by* $v_{in}(\xi;\lambda) = A(\xi)w_\lambda^{\rm R}(\xi) + A(-\xi)w_\lambda^{\rm R}(-\xi)$, *with*

$$A(\xi) = A(\xi;\lambda) = -\frac{1}{\mathcal{W}(\lambda)}\int_{-\infty}^{\xi} \frac{\partial G}{\partial U}(u_*, v_{f,h}(\tilde{\xi};u_*))\, w_\lambda^{\rm R}(-\tilde{\xi})\, d\tilde{\xi}. \tag{3.26}$$

Note that it immediately follows from this expression and assumption (A4) in combination with the properties of $v_{f,h}(\xi;u_*)$ that $v_{\rm in}(\xi;\lambda)$ decays exponentially fast to 0 as $\xi \to \pm\infty$ (and as $\xi$ approaches the boundaries of $I_f$ (2.7)).

**Proof.** By the variation of constants approach, we introduce the unknown functions $A^{\rm L/R}(\xi)$ by $v_{\rm in}(\xi) = A^{\rm L}(\xi)w_\lambda^{\rm L}(\xi) + A^{\rm R}(\xi)w_\lambda^{\rm R}(\xi)$. Substitution in (3.25) yields

$$\dot{A}^{\rm L/R} = \frac{\mp 1}{\mathcal{W}(\lambda)}\frac{\partial G}{\partial U}(u_*, v_{f,h}(\xi;u_*))w_\lambda^{\mp}(\xi),$$

so that

$$A^{\rm L/R}(\xi) = A^{\rm L/R}(0) \mp \frac{1}{\mathcal{W}(\lambda)}\int_0^{\xi} \frac{\partial G}{\partial U}(u_*, v_{f,h}(\tilde{\xi};u_*))w_\lambda^{\mp}(\tilde{\xi})\, d\tilde{\xi}.$$

Both the operator $\mathcal{L}_f(\xi)$ and the inhomogeneous term in (3.25) are even as function of $\xi$. This implies that also $v_{\rm in}(\xi;\lambda)$ must be even, so that $A^{\rm R}(\xi) = A^{\rm L}(-\xi) \stackrel{\rm def}{=} A(\xi)$ and $A^{\rm R}(0) = A^{\rm L}(0)$. A straightforward analysis yields that $v_{\rm in}(\xi)$ can only be bounded if

$$A(0) = -\frac{1}{\mathcal{W}(\lambda)}\int_{-\infty}^{0} \frac{\partial G}{\partial U}(u_*, v_{f,h}(\tilde{\xi};u_*))w_\lambda^{\rm L}(\tilde{\xi})\, d\tilde{\xi},$$

which is a converging integral by assumption (A4). □

A priori, there is a singularity in the solutions $v_{\rm in}(\xi;\lambda)$ as $\lambda \to \lambda_{f,j}$, due to the fact that $(\mathcal{L}_f(\xi) - \lambda)$ is not invertible at $\lambda_{f,j}$ (and that thus $\mathcal{W}(\lambda_{f,j}) = 0$, Lemma 3.3). However, by the Fredholm alternative, (3.25) will have solutions for $\lambda = \lambda_{f,j}$ with $j$ odd, since $w_{f,j}(\xi)$ is odd as function of $\xi$ (Lemma 3.2) and the (even) inhomogeneity of (3.25) thus satisfies the solvability condition.

**Corollary 3.6** *For $j$ even,*

$$v_{in}(\xi;\lambda) \rightsquigarrow \left(\frac{w_{f,j}(\xi)}{\|w_{f,j}\|_2^2}\int_{-\infty}^{\infty} \frac{\partial G}{\partial U}(u_*, v_{f,h}(\tilde{\xi};u_*))\, w_{f,j}(\tilde{\xi})\, d\tilde{\xi}\right) \cdot \frac{1}{\lambda - \lambda_{f,j}} \quad as \quad \lambda \to \lambda_{f,j}, \tag{3.27}$$

*while* $\lim_{\lambda \to \lambda_{f,j}} v_{in}(\xi;\lambda)$ *exists for $j$ odd.*

**Proof.** Using the fact that $w_{f,j}(\xi)$ is even/odd as function of $\xi$ for $j$ even/odd, identity (3.27) can be obtained directly by combining Lemma's 3.3 and 3.5, both for $j$ even and for $j$ odd – in the latter case, the integral in (3.27) vanishes. □

It will be necessary to also have an explicit characterization of $v_{\rm in}(\xi;\lambda)$ for $\lambda$ near $\lambda_{f,1}$, the crucial (odd) case $j = 1$ for which $\lambda_{f,1} = 0$.

**Lemma 3.7** *For $\lambda = \lambda_{f,1} = 0$, $v_{in}(\xi;\lambda)$ is not uniquely determined: here,*

$$v_{in}(\xi;0) = \frac{\partial}{\partial u} v_{f,h}(\xi;u)|_{u=u_*} + C\dot{v}_{f,h}(\xi;u_*), \tag{3.28}$$

*in which $C \in \mathbb{R}$ is a free parameter.*

It is also possible to obtain leading order approximations of $v_{\rm in}(\xi;\lambda)$ for $\lambda$ near $\lambda_{f,j}$ with $j \geq 3$ odd. However, we refrain from going into these details.

**Proof.** The fact that $\frac{\partial}{\partial u} v_{f,h}(\xi;u)|_{u=u_*}$ is a solution of (3.25) follows immediately from taking the derivative with respect to the parameter $u$ (or $u_0$) in (2.3). Uniqueness is lost by adding the kernel $\dot{v}_{f,h}(\xi;u_*)$ associated to the operator $\mathcal{L}_f(\xi)$. □

### 3.4 The intermediate, slowly varying problem

Consider the intermediate (linear) problem

$$\dot{\psi} = A_s(\varepsilon\xi;\lambda,\varepsilon)\psi \tag{3.29}$$

to the right of $I_f$, that is for $\xi > \varepsilon^{-\frac{1}{4}}$. Its solution space is four-dimensional. For $\lambda \notin \sigma_e$ (3.9), one can decompose the basis of this space into two fast solutions $\psi_{f,\pm}(\xi;\lambda,\varepsilon)$ that vary with $\xi$ and two slowly varying solutions $\psi_{s,\pm}(\varepsilon\xi;\lambda,\varepsilon)$. The fast solutions $\psi_{f,\pm}(\xi;\lambda,\varepsilon)$ are at leading order determined by the lower diagonal $2\times 2$ block of $A_s(\varepsilon\xi;\lambda,\varepsilon)$ (3.5) and thus at leading order determined by their $(v,q)$-components: the $(u,p)$-components are only weakly driven by the asymptotically small coupling term $-\nu_2\sqrt{\varepsilon}F_{2,1}(u_{s,*}(\varepsilon\xi))$ in (3.5). The existence of a fast converging solution $\psi_{f,-}$ with an exponential decay that is governed by the most negative eigenvalue $-\Lambda_f$ of the limiting matrix $\mathcal{A}_\infty(\lambda,\varepsilon)$ (3.8) follows directly by classical methods (see also [1, 13]). Note that $\psi_{f,-}$ is uniquely determined up to a normalization constant (see below). This is not the case for its fast diverging counterpart $\psi_{f,+}$, of which the growth is determined by $+\Lambda_f$ for $\xi$ large enough (one can for instance add a multiple of $\psi_{f,-}$ to $\psi_{f,+}$). However, its existence can be settled by the same methods as for $\psi_{f,-}$. In fact, $\psi_{f,+}$ can be chosen such that for $x$ large, i.e. for $\xi \gg \frac{1}{\varepsilon}$, its decomposition with respect to the four basis solutions of the limiting constant coefficient problem associated to $\mathcal{A}_\infty(\lambda,\varepsilon)$ (section 3.1) does not contain 'slow' behavior (governed by the eigenvalues $\pm\varepsilon\Lambda_s$). Nevertheless $\psi_{f,+}$ needs to be chosen from a family of options, also after normalization. Note that these assertions are all standard within the framework of the Evans function approach – see [1, 13, 3, 4]. Note also that in general $\psi_{f,+}(\xi) \neq \psi_{f,-}(-\xi)$.

In the upcoming analysis , it will be convenient to normalize the solutions $\psi_{f,\pm}(\xi;\lambda,\varepsilon)$ as

$$\psi_{f,\pm}(\xi;\lambda,\varepsilon) \rightsquigarrow E_{f,\pm}(\lambda;\varepsilon)\, e^{\pm\Lambda_f\xi} \quad \text{as} \quad \xi \to \infty \tag{3.30}$$

with $\Lambda_f$ and $E_{f,\pm}(\lambda;\varepsilon)$ as defined in (3.8), (3.10).

It follows from the structure of $A_s(\varepsilon\xi;\lambda,\varepsilon)$ (3.5) that the slow solutions $\psi_{s,\pm}$ have trivial $v$ and $q$ components, so that $\psi_{s,\pm}(\varepsilon\xi;\lambda,\varepsilon) = (u_{s,\pm}(\varepsilon\xi;\lambda,\varepsilon), p_{s,\pm}(\varepsilon\xi;\lambda,\varepsilon), 0, 0)^T$. Note that $\psi_{s,\pm}$ is considered here as function of the slowly varying spatial variable $x = \varepsilon\xi$; more specifically, $p_{s,\pm}$ is defined as $\frac{d}{dx}u_{s,\pm}$. By construction, and by the approximations of Theorem 2.1, $u_{s,\pm}(x;\lambda,\varepsilon)$ is a solution of

$$u_{xx} - \left[(\mu+\lambda) - \nu_1 \frac{dF_1}{dU}\left(u_*^{\rm s}(x)\right)\right] u = 0 \;\; \text{for} \;\; x > \varepsilon^{\frac{3}{4}}. \tag{3.31}$$

Since

$$u_*^{\rm s}(x) = u_*^{\rm u}(-x) = u_s^{\rm u}(-x-x_*) = u_s^{\rm s}(x+x_*) = u_s^{\rm s}(y),$$

with $y = x + x_*$ (section 2 and Theorem 2.1), (3.31) can be rewritten as

$$(\mathcal{L}_s(y) - \lambda)\,\hat{u} \stackrel{\rm def}{=} \hat{u}_{yy} + \left[\nu_1 \frac{dF_1}{dU}(u_s^{\rm s}(y)) - (\mu+\lambda)\right]\hat{u} = 0 \;\; \text{for} \;\; y > y_*, \tag{3.32}$$

where $y_* = x_* + \varepsilon^{\frac{3}{4}}$. Except for the condition on $y$, this is a Sturm-Liouville problem of the type (3.12). As a consequence, Lemma 3.2 can be applied to (3.32) with $\rho$ replaced by $\mu$ so that $\Lambda = \Lambda_s$. Hence, for $\lambda \notin \sigma_e$ (3.9) we can define the converging function $\hat{u}_{s,-}(y;\lambda,\varepsilon)$ as the solution of (3.32) that satisfies

$$\hat{u}_{s,-}(y;\lambda,\varepsilon) \rightsquigarrow 1 \cdot e^{-\Lambda_s x} = e^{\Lambda_s x_*} \cdot e^{-\Lambda_s y} \quad \text{as} \quad y \to \infty. \tag{3.33}$$

Note that it is not necessary to exclude the values of $\lambda$ that are eigenvalues for the full problem, i.e. (3.32) with $y \in \mathbb{R}$: in that case $\hat{u}_{s,-}$ can be defined as the (normalized) eigenfunction. Of course, $u_{s,-}$ is related to $\hat{u}_{s,-}$ through $u_{s,-}(x;\lambda,\varepsilon) = \hat{u}_{s,-}(x + x_*;\lambda,\varepsilon)$. Its diverging counterpart $\hat{u}_{s,+}(y;\lambda,\varepsilon)$ can be obtained by the same methods as above, or as in the proof of Lemma 3.2 (note that the existence of these diverging solutions is not a part of this Lemma). As above, the diverging solution is again not uniquely determined and in general not equal to $\hat{u}_{s,-}(-y;\lambda,\varepsilon)$. In fact, this is impossible at an eigenvalue of the full problem, since in that case $\hat{u}_{s,-}(-y;\lambda,\varepsilon)$ does not grow exponentially as $y \to \infty$. For future reference, we gauge the diverging solution $\hat{u}_{s,+}$ such that

$$\hat{u}_{s,+}(y;\lambda,\varepsilon) \rightsquigarrow 1 \cdot e^{+\Lambda_s x} \quad \text{as} \quad y \to \infty. \tag{3.34}$$

Both basis functions $\psi_{s,\pm}(x;\lambda,\varepsilon)$ can now be defined for $\lambda \notin \sigma_e$; recall that $\psi_{s,\pm}(x;\lambda,\varepsilon)$ are only defined for $x > \varepsilon^{\frac{3}{4}}$. As above, we assume that $\psi_{s,\pm}(x;\lambda,\varepsilon)$ are normalized such that

$$\psi_{s,\pm}(x;\lambda,\varepsilon) \rightsquigarrow E_{s,\pm}(\lambda;\varepsilon)\, e^{\pm\Lambda_s x} \quad \text{as} \quad x \to \infty \tag{3.35}$$

(3.8), (3.10); note that this is equivalent to (3.33) for $\psi_{-,s}(x;\lambda)$.

Since the matrix $A_s$ is symmetric in $\varepsilon\xi$, the above solutions $\psi_{f,\pm}(\xi;\lambda,\varepsilon), \psi_{s,\pm}(\varepsilon\xi;\lambda,\varepsilon)$ can be used to define their equivalent counterparts to the left of $I_f$, i.e. for $\xi < -\varepsilon^{-\frac{1}{4}}$, by using the reflection $\xi \to -\xi$. This fact will be exploited in the next section where the Evans function will be constructed.

# 4 The Evans function and the NLEP procedure

## 4.1 The construction of the Evans function

The Evans function, which is complex analytic outside the essential spectrum – see [32], [1] and the references therein – associated to system (3.3) can be defined by

$$\mathcal{D}(\lambda,\varepsilon) = \det\left[\phi_i(\xi;\lambda,\varepsilon)\right] \tag{4.1}$$

where the functions $\phi_i$, $i = 1,2,3,4$ satisfy boundary conditions at $\pm\infty$ (see below) and span the solution space of (3.3). The eigenvalues of (3.4) outside $\sigma_e$ coincide with the roots of $\mathcal{D}(\lambda,\varepsilon)$, including multiplicities.

**Lemma 4.1** *For all $\lambda \in \mathbb{C} \setminus \sigma_e$, there are solutions $\phi_f^{\mathrm{L/R}}(\xi;\lambda,\varepsilon)$ and $\phi_s^{\mathrm{L/R}}(\xi;\lambda,\varepsilon)$ to* (3.3) *such that* *the set* $\left\{\phi_f^{\mathrm{L/R}}(\xi;\lambda,\varepsilon), \phi_s^{\mathrm{L/R}}(\xi;\lambda,\varepsilon)\right\}$ *spans the solution space of* (3.3) *and*

$$\phi_f^{\mathrm{L}}(\xi;\lambda,\varepsilon) \rightsquigarrow E_{f,+}\, e^{\Lambda_f \xi} \quad as \quad \xi \to -\infty \tag{4.2a}$$
$$\phi_f^{\mathrm{R}}(\xi;\lambda,\varepsilon) \rightsquigarrow E_{f,-}\, e^{-\Lambda_f \xi} \quad as \quad \xi \to \infty \tag{4.2b}$$
$$\phi_s^{\mathrm{L}}(\xi;\lambda,\varepsilon) \rightsquigarrow E_{s,+}\, e^{\varepsilon\Lambda_s \xi} \quad as \quad \xi \to -\infty \tag{4.2c}$$
$$\phi_s^{\mathrm{R}}(\xi;\lambda,\varepsilon) \rightsquigarrow E_{s,-}\, e^{-\varepsilon\Lambda_s \xi} \quad as \quad \xi \to \infty \tag{4.2d}$$

*Moreover, there exist analytic transmission functions* $t_{f,+}(\lambda,\varepsilon)$ *and* $t_{s,+}(\lambda,\varepsilon)$ *such that*

$$\phi_f^{\mathrm{L}}(\xi;\lambda,\varepsilon) \rightsquigarrow t_{f,+}(\lambda,\varepsilon)\, E_{f,+}\, e^{\Lambda_f \xi} \quad as \quad \xi \to \infty \tag{4.3a}$$
$$\phi_s^{\mathrm{L}}(\xi;\lambda,\varepsilon) \rightsquigarrow t_{s,+}(\lambda,\varepsilon)\, E_{s,+}\, e^{\varepsilon\Lambda_s \xi} \quad as \quad \xi \to \infty \tag{4.3b}$$

*where* $t_{s,+}(\lambda,\varepsilon)$ *is only defined if* $t_{f,+}(\lambda,\varepsilon) \neq 0$. *These choices, when possible, determine* $\phi_f^{\mathrm{L/R}}$ *and* $\phi_s^{\mathrm{L}}$ *uniquely.*

**Proof.** Although the linearized system (3.3) is not identical to its counterpart in [3], exactly the same arguments as in [3] can be applied here. Therefore, we refer to [3] for the details of the proof.□

The relation between the functions $\phi_{f/s}^{\mathrm{L/R}}$ defined in the above Lemma and the functions $\psi_{f/s,\pm}$ defined in section 3.4 will be specified in the next section. Using this relation, an explicit leading order expression for the slow transmission function $t_{s,+}(\lambda)$ will be derived.

The Evans function can be determined by taking the limit $\xi \to \infty$ of the determinant of the functions defined in Lemma 4.1, since the Evans function itself does not depend on $\xi$; the latter can be established by combining Abel's theorem with the fact that the trace of $\mathcal{A}(\xi;\lambda,\varepsilon)$ vanishes. This yields using (3.8) and (3.10)

$$\begin{aligned}
\mathcal{D}(\lambda,\varepsilon) &= \det\left[\left\{\phi_f^{\mathrm{L}}, \phi_f^{\mathrm{R}}, \phi_s^{\mathrm{L}}, \phi_s^{\mathrm{R}}\right\}\right] = \lim_{\xi\to\infty} \det\left[\left\{\phi_f^{\mathrm{L}}, \phi_f^{\mathrm{R}}, \phi_s^{\mathrm{L}}, \phi_s^{\mathrm{R}}\right\}\right] \\
&= \lim_{\xi\to\infty} \det\left[\left\{t_{f,+}(\lambda,\varepsilon)\, E_{f,+}\, e^{\Lambda_f\xi},\, E_{f,-}\, e^{-\Lambda_f\xi},\, t_{s,+}(\lambda,\varepsilon)\, E_{s,+}\, e^{\varepsilon\Lambda_s\xi},\, E_{s,-}\, e^{-\varepsilon\Lambda_s\xi}\right\}\right] \\
&= \lim_{\xi\to\infty} t_{f,+}(\lambda,\varepsilon) t_{s,+}(\lambda,\varepsilon) \det\left[\{ E_{f,+},\, E_{f,-},\, E_{s,+},\, E_{s,-}\}\right] \\
&= 4\varepsilon\, t_{f,+}(\lambda,\varepsilon)\, t_{s,+}(\lambda,\varepsilon) \sqrt{1+\lambda}\sqrt{\mu+\lambda}.
\end{aligned} \tag{4.4}$$

**Corollary 4.2** *The set of eigenvalues of* (3.4) *is contained in the union of the sets of roots of* $t_{f,+}(\lambda,\varepsilon)$ *and* $t_{s,+}(\lambda,\varepsilon)$.

Note that, due to the fact that $t_{s,+}(\lambda,\varepsilon)$ only defined when $t_{f,+}(\lambda,\varepsilon) \neq 0$, the Evans function $\mathcal{D}(\lambda,\varepsilon)$ doesn't necessarily vanish when $t_{f,+}(\lambda,\varepsilon) = 0$. This is called the 'resolution to the NLEP paradox' in [3] and [4]. Referring to [3], we recall that the roots of $t_{f,+}(\lambda,\varepsilon)$ are to leading order given by the eigenvalues of the homogeneous fast eigenvalue problem (3.12), so

**Lemma 4.3** *There are unique* $\lambda_j(\varepsilon) \in \mathbb{R}$ *such that* $\lim_{\varepsilon\to 0}\lambda_j(\varepsilon) = \lambda_{f,j}$ *and* $t_{f,+}(\lambda_j(\varepsilon),\varepsilon) = 0$ *with multiplicity 1 for* $j = 0, ..., J$.

**Proof.** This statement (and, as a consequence, its proof) is analogous to that of [3], Lemma 4.1. From Lemma 3.3 we know that the Wronksian $\mathcal{W}(\lambda)$ defined there –which is an Evans function itself for the fast problem (3.12)– vanishes at $\lambda = \lambda_{f,j}$ in a nondegenerate way. By construction, $t_{f,+}(\lambda_j(\varepsilon), \varepsilon)$ approaches $\mathcal{W}(\lambda)$ as $\varepsilon \to 0$; see [1, 13, 4] for the technical details. □

Hence, the eigenvalues of (3.12) are to leading order zeroes of the fast component of the Evans function $\mathcal{D}(\lambda, \varepsilon)$ given in (4.4) and thus in principle candidates for being a zero of the full Evans function.

## 4.2 The NLEP procedure

Since in the slow field $\xi > |\varepsilon^{-\frac{1}{4}}|$ the matrix $\mathcal{A}(\xi; \lambda, \varepsilon)$ (3.4) is with exponential accuracy (3.6) given by $\mathcal{A}_s(\varepsilon\xi; \lambda, \varepsilon)$ (3.5), we can conclude that

$$\phi_s^{\mathrm{R}}(\xi; \lambda, \varepsilon) = \psi_{s,-}(\varepsilon\xi; \lambda, \varepsilon) \quad \text{for} \quad \xi > \varepsilon^{-\frac{1}{4}} \tag{4.5}$$

by combining (3.35) with Lemma 4.1. By the reversibility symmetry,

$$\phi_s^{\mathrm{L}}(\xi; \lambda, \varepsilon) = \psi_{s,-}(-\varepsilon\xi; \lambda, \varepsilon) \quad \text{for} \quad \xi < -\varepsilon^{-\frac{1}{4}}. \tag{4.6}$$

Both approximations are valid with exponential accuracy. Moreover, from the second part of Lemma 4.1, we can infer that in the right slow field we can approximate $\phi_s^{\mathrm{L}}$ to exponential accuracy as

$$\phi_s^{\mathrm{L}}(\xi; \lambda, \varepsilon) = t_{s,+}(\lambda, \varepsilon)\, \psi_{s,+}(\varepsilon\xi; \lambda, \varepsilon) + t_{s,-}(\lambda, \varepsilon)\, \psi_{s,-}(\varepsilon\xi; \lambda, \varepsilon) \quad \text{for} \quad \xi > \varepsilon^{-\frac{1}{4}}. \tag{4.7}$$

The additional transmission function $t_{s,-}$ needs to be introduced since the asymptotic behaviour of $\phi_s^{\mathrm{L}}$ in the right slow field is only determined by its slow growth, see Lemma 4.1. This normalization choice does not exclude the possibility that $\phi_{s,\mathrm{L}}$ has a slowly decaying component in the right slow field. Since $\{\psi_{f,\pm}, \psi_{s,\pm}\}$ form a basis of the solution space of the right slow field to exponential accuracy, the slowly decaying component can be represented by $\psi_{s,-}$. Note that since the solution is approximated to exponential accuracy, the possible presence of a fast decaying component is incorporated in this exponential error estimate.

Using the above approximations, an explicit leading order expression for the transmission function $t_{s,+}$ can be determined. Recall that from section 4.1 it is known that $\lambda$ is a zero of the Evans function, and thus an eigenvalue of (3.3), if $t_{s,+}(\lambda, \varepsilon) = 0$. The Theorem below can therefore be considered as the main result of section 4 and therefore as one of the main results of this paper.

**Theorem 4.4** *Let $\varepsilon > 0$ be small enough. Define $B_\pm$ and $B'_\pm$ by*

$$B_\pm(\lambda) = \lim_{\varepsilon \to 0} \hat{u}_{s,\pm}(y_*; \lambda, \varepsilon), \; B'_\pm(\lambda) = \lim_{\varepsilon \to 0} \frac{d}{dy} \hat{u}_{s,\pm}(y; \lambda, \varepsilon)\Big|_{y=y_*}, \tag{4.8}$$

*then, up to corrections of $\mathcal{O}(\varepsilon^{\frac{3}{4}})$,*

$$t_{s,+}(\lambda) = -\frac{B_-^2}{\Lambda_s} \left\{ \frac{B'_-}{B_-} + \frac{1}{2}\nu_2 \int_{-\infty}^{\infty} \left[ \frac{\partial F_2}{\partial U}(u_*, v_{f,h}(\xi; u_*)) + \frac{\partial F_2}{\partial V}(u_*, v_{f,h}(\xi; u_*))\, v_{in}(\xi) \right] d\xi \right\}, \tag{4.9}$$

*with $v_{in}(\xi; \lambda)$ as given in Lemma 3.5.*

**Proof.** Let $u_s^{\rm L}$ be the $u$-component of $\phi_s^{\rm L}$. The approximations (4.6) and (4.7) of $u_s^{\rm L}(\xi;\lambda,\varepsilon)$ are valid outside $I_f$ (2.7). Since $I_f$ has a $\mathcal{O}(\varepsilon^{-\frac{1}{4}})$ width in $\xi$ and $u_\xi = \varepsilon p = \mathcal{O}(\varepsilon)$, it follows that $u_s^{\rm L}(\xi;\lambda,\varepsilon)$ can at most change an amount of $\mathcal{O}(\varepsilon^{\frac{3}{4}})$ over $I_f$. Hence, taking the limits $\xi \uparrow -\varepsilon^{-\frac{1}{4}}$ in (4.6) and $\xi \downarrow \varepsilon^{-\frac{1}{4}}$ in (4.7) yields, in combination with de definition of $y$ (section 3.4),

$$\hat{u}_{s,-}(y_*;\lambda) = t_{s,+}(\lambda)\hat{u}_{s,+}(y_*;\lambda) + t_{s,-}(\lambda)\hat{u}_{s,-}(y_*;\lambda) + \mathcal{O}(\varepsilon^{\frac{3}{4}}).$$

From this, we obtain a first relation between $t_+(\lambda)$ and $t_-(\lambda)$:

$$B_-(\lambda) = t_+(\lambda)B_+(\lambda) + t_-(\lambda)B_-(\lambda) + \mathcal{O}(\varepsilon^{\frac{3}{4}}). \tag{4.10}$$

A second leading order relation between $t_+(\lambda)$ and $t_-(\lambda)$ can be obtained by studying the accumulated change in $\frac{\rm d}{{\rm d}\xi}u_s^{\rm L}(\xi;\lambda)$ over $I_f$. According to (3.2) and by Theorem 2.1,

$$\begin{array}{rcl}\Delta_f\left(\frac{\rm d}{{\rm d}\xi}u_s^{\rm L}\right) & = & \int_{I_f}\frac{{\rm d}^2}{{\rm d}\xi^2}u_s^{\rm L}\,{\rm d}\xi \\ & = & -\varepsilon\nu_2\int_{I_f}\left[\frac{\partial F_2}{\partial U}(u_h(\xi),v_h(\xi))u_s^{\rm L}(\xi) + \frac{\partial F_2}{\partial V}(u_h(\xi),v_h(\xi))v_s^{\rm L}(\xi) + \mathcal{O}(\varepsilon)\right]{\rm d}\xi \\ & = & -\varepsilon\nu_2\int_{I_f}\left[\frac{\partial F_2}{\partial U}(u_*,v_{f,h}(\xi;u_*))u_s^{\rm L}(\xi) + \frac{\partial F_2}{\partial V}(u_*,v_{f,h}(\xi;u_*))v_s^{\rm L}(\xi)\right]{\rm d}\xi + \mathcal{O}(\varepsilon^{7/4}),\end{array}$$

where $v_s^{\rm L}$ is the $v$-component of $\phi_s^{\rm L}$. For $\xi \in I_f$, we know that $u_s^{\rm L}$ is constant to leading order. Using (4.10), we see that $u_s^{\rm L}(\xi) = B_- + \mathcal{O}(\varepsilon^{\frac{3}{4}})$ for $\xi \in I_f$. The second equation of (3.2) – which describes the evolution of $v_s^{\rm L}$ – can therefore be written as

$$\left(\mathcal{L}_f(\xi) - \lambda\right)v_s^{\rm L} = -B_-\frac{\partial G}{\partial U}(u_*,v_{f,h}(\xi;u_*)) + \mathcal{O}(\varepsilon^{\frac{3}{4}}),$$

which implies that (see section 3.3)

$$v_s^{\rm L}(\xi;\lambda,\varepsilon) = B_-(\lambda)v_{\rm in}(\xi;\lambda) + \mathcal{O}(\varepsilon^{\frac{3}{4}}),$$

so that $v_{+,s}^{\rm L}(\xi;\lambda)$ is explicitly known (Lemma 3.5) to leading order. As a consequence,

$$\begin{array}{rcl}\Delta_f\left(\frac{\rm d}{{\rm d}\xi}u_s^{\rm L}\right) & = & -\varepsilon\nu_2 B_-\int_{I_f}\left[\frac{\partial F_2}{\partial U}(u_*,v_{f,h}(\xi)) + \frac{\partial F_2}{\partial V}(u_*,v_{f,h}(\xi))v_{\rm in}(\xi)\right]{\rm d}\xi + \mathcal{O}(\varepsilon^{7/4}) \\ & = & -\varepsilon\nu_2 B_-\int_{-\infty}^{\infty}\left[\frac{\partial F_2}{\partial U}(u_*,v_{f,h}(\xi)) + \frac{\partial F_2}{\partial V}(u_*,v_{f,h}(\xi))v_{\rm in}(\xi)\right]{\rm d}\xi + \mathcal{O}(\varepsilon^{7/4})\end{array} \tag{4.11}$$

by the convergence properties of $v_{f,h}(\xi;u_*)$ and $v_{\rm in}(\xi;\lambda)$ in combination with assumption (A3) – note that this same combination also implies that the integral converges. Of course, this accumulated change in $\frac{\rm d}{{\rm d}\xi}u_{+,s}^{\rm L}(\xi;\lambda)$ must also be reflected by the leading order approximations (4.6) and (4.7) as $\xi \uparrow -\varepsilon^{-\frac{1}{4}}$ respectively $\xi \downarrow \varepsilon^{-\frac{1}{4}}$. Combining (4.11) with (4.6) and (4.7) yields

$$\begin{array}{rcl}\Delta_s\left(\frac{\rm d}{{\rm d}\xi}u_{+,s}^{\rm L}\right) & = & \lim_{\xi\downarrow\varepsilon^{-\frac{1}{4}}}\frac{\rm d}{{\rm d}\xi}\left[t_{s,+}\hat{u}_{s,+}(\varepsilon\xi + x_*) + t_{s,-}\hat{u}_{s,-}(\varepsilon\xi + x_*)\right] - \lim_{\xi\uparrow-\varepsilon^{\frac{1}{4}}}\frac{\rm d}{{\rm d}\xi}\hat{u}_{s,-}(-\varepsilon\xi + x_*) \\ & = & \varepsilon\left[t_{s,+}(\lambda)B_+'(\lambda) + t_{s,-}(\lambda)B_-'(\lambda) + B_-'(\lambda)\right] + O(\varepsilon^{7/4}).\end{array} \tag{4.12}$$

The second relation between $t_-(\lambda)$ and $t_+(\lambda)$ follows by identifying (4.11) and (4.12).
Finally, the term $B_+B_-' - B_-B_+'$ obtained by combining (4.10) with (4.12) and solving for $t_{s,+}$ can be simplified by recognizing it as the Wronksian associated to (3.32) for the solutions $\hat{u}_{s,\pm}$, evaluated at $y = y_*$. Using Abel's theorem, we see that the Wronskian associated (3.32) is constant

in $y$. Its value can therefore be determined by taking the limit $y \to \infty$, using (3.33) and (3.34). Thus,

$$B_+ B'_- - B_- B'_+ = \lim_{y\to\infty} B_+ B'_- - B_- B'_+ = -2\Lambda_s.$$

Identity (4.9) can now be obtained by combining relation (4.12) with (4.10), using the above simplification for $B_+ B'_- - B_- B'_+$. □

The expression for $t_{s,+}$ (4.9) can be studied in the 'linear' limit, yielding the following result:

**Corollary 4.5** *Both as $\nu_1 \to 0$ and in the limit of large positive $y_*$, the roots of $t_{s,+}(\lambda)$ are to leading order given by the solutions to*

$$\frac{\nu_2}{2}\int_{-\infty}^{\infty}\left[\frac{\partial F_2}{\partial U}(u_*, v_{f,h}(\xi; u_*)) + \frac{\partial F_2}{\partial V}(u_*, v_{f,h}(\xi; u_*))\, v_{in}(\xi)\right] d\xi = \sqrt{\mu+\lambda}. \tag{4.13}$$

Note that the 'linear limit' (4.13) indeed coincides with [3], expression (4.11), which determines the (nontrivial) zeroes of an Evans function associated to the stability of pulses in a 'linear' generalized GM-type system (i.e. $F_1(U;\varepsilon) \equiv 0$, $F_2(U,V;\varepsilon) = U^{\alpha_1}V^{\beta_1}$, $G(U,V;\varepsilon) = U^{\alpha_2}V^{\beta_2}$).

**Proof.** We approximate $B_\pm$, $B'_\pm$ for large $y_* > 0$ using (3.33) and (3.34), yielding

$$B_-(\lambda) \rightsquigarrow e^{-\Lambda_s y_*} \quad \text{and} \quad B'_-(\lambda) \rightsquigarrow -\Lambda_s\, e^{-\Lambda_s y_*} \quad \text{as} \quad y_* \to \infty. \tag{4.14}$$

From (3.32), it follows that the limit $\nu_1 \to 0$ also yields the 'linear limit', i.e. the solutions $\hat{u}_{s,\pm}$ become pure exponentials. Therefore, any zero of $t_{s,+}$ in either of these limits comes from a solution of equation (4.13). □

# 5 Implications of Theorem 4.4: (in)stability results

The explicit leading order expression for $t_{s,+}(\lambda)$ established in the previous section and stated in Theorem 4.4 can be interpreted in certain limiting situations, such as near the known fast eigenvalues $\lambda_{f,j}$ of the homogeneous problem (3.12) or for certain parameter limits. In this section, a number of results of this type will be stated, leading to a number of explicit (in)stability results for the full problem (3.3).

**Lemma 5.1** *For $\lambda$ close to $\lambda_{f,0}$, we can describe the leading order behaviour of $t_{s,+}$ as*

$$t_{s,+}(\lambda) \rightsquigarrow -\frac{\nu_2 B_-(\lambda_{f,0})^2}{2\Lambda_s(\lambda_{f,0})} \cdot \frac{T}{\lambda - \lambda_{f,0}} \quad \text{as} \quad \lambda \to \lambda_{f,0}, \tag{5.1}$$

*where*

$$T = \left(\int_{-\infty}^{\infty}\frac{\partial F_2}{\partial V}(u_*, v_{f,h}(\xi;u_*))\,\frac{w_{f,0}(\xi)}{\|w_{f,0}\|_2}\,d\xi\right)\left(\int_{-\infty}^{\infty}\frac{\partial G}{\partial U}(u_*, v_{f,h}(\xi;u_*))\,\frac{w_{f,0}(\xi)}{\|w_{f,0}\|_2}\,d\xi\right). \tag{5.2}$$

**Proof.** By Corollary 3.6 we see that $v_{\text{in}}(\xi;\lambda)$ is singular in $\lambda$ as $\lambda \to \lambda_{f,0}$. Combining (3.27) with (4.9), we can describe the leading order behaviour of the slow transmission function as (5.1) with the constant $T$ is given by (5.2). □

**Corollary 5.2** *Let $\varepsilon > 0$ be sufficiently small. The nontrivial roots of the Evans function $\mathcal{D}(\lambda,\varepsilon)$ are determined by $t_{s,+}(\lambda,\varepsilon)$ (4.9). In other words, $t_{s,+}$ determines the stability of the pulse $\Gamma_h$ as defined in Theorem 2.1.*

**Proof.** By Lemma 4.3, the fast transmission function $t_{f,+}(\lambda,\varepsilon)$ has a single zero at $\lambda = \lambda_{f,0}$ to leading order in $\varepsilon$. Since $t_{f,+}(\lambda)$ is smooth and it approximates the Wronskian $\mathcal{W}(\lambda)$ of Lemma 3.3 (see Lemma 4.3), we can approximate it linearly as $t_{f,+}(\lambda,\varepsilon) \rightsquigarrow \hat{t}_{f,+}\cdot(\lambda-\lambda_{f,0})+\mathcal{O}(\varepsilon)$ as $\lambda \to \lambda_{f,0}$, with $\hat{t}_{f,+} = -\|w_{f,0}\|_2^2 \neq 0$. This would suggest that $\lambda_{f,0}$ is a zero of the full Evans function (4.4). However, combining the results of Lemma 5.1 with the fact that $\Lambda_s(\lambda_{f,0}) = \sqrt{\mu+\lambda_{f,0}}$, we see that the Evans function (4.4) behaves to leading order in $\varepsilon$ as

$$\mathcal{D}(\lambda,\varepsilon) \rightsquigarrow 2\,\varepsilon\,\nu_2\,\|w_{f,0}\|_2^2 T\sqrt{1+\lambda_{f,0}}\,B_-(\lambda_{f,0})^2 \quad \text{as} \quad \lambda \to \lambda_{f,0}. \tag{5.3}$$

We see that $\mathcal{D}(\lambda_{f,0},\varepsilon) = 0$ if and only if $\nu_2 T B_-(\lambda_{f,0})^2 = 0$. Thus, the possibility of an eigenvalue at $\lambda = \lambda_{f,0}$ is determined by $t_{s,+}(\lambda)$, not by $t_{f,+}(\lambda)$. □

Note that in general $\lambda_{f,0}$ is thus not (close to) an eigenvalue of the full problem. This –again– relates directly to the resolution of the NLEP paradox [3, 4]. The first, positive eigenvalue $\lambda_{f,0}$ of the fast homogeneous problem (3.12) is a zero of the Evans function (4.4) and therefore an eigenvalue of the full problem (3.3) if and only if

$$\nu_2\,T\,B_-(\lambda_{f,0}) = 0, \tag{5.4}$$

where $T$ as defined in (5.2); therefore, (5.4) determines a condition on the parameters of (1.1). Moreover, the relevance of more detailed insight in the behaviour in general and the roots in particular of $B_-(\lambda)$ is apparent.

## 5.1 The structure of $B_-(\lambda)$

Recalling the definition of $B_-(\lambda)$ (4.8), we see that the roots of $B_-(\lambda)$ are directly related to the structure of $\hat{u}_{s,-}$ as a function of $\lambda$; also recall that $\hat{u}_{s,-}$ is the solution of (3.32) that decreases exponentially as $y \to \infty$, see (3.33).

Consider the slow eigenvalue problem (3.32). Following the classical approach of [36], we introduce the polar coordinate transformation

$$\hat{u}(y) = r(y)\,\cos\theta(y), \quad \hat{u}_y(y) = r(y)\,\sin\theta(y), \tag{5.5}$$

where $r(y) > 0$. Using the consistency condition

$$r(y)\,\sin\theta(y) = \hat{u}_y = \frac{d}{dy}\hat{u} = r'(y)\,\cos\theta(y) - r(y)\theta'(y)\,\sin\theta(y), \tag{5.6}$$

the second order equation (3.32) can be transformed into the system

$$r' = \left[1 + \mu + \lambda - \nu_1 \frac{dF_1}{dU}(u_s^{\mathrm{s}}(y))\right] r \cos\theta \sin\theta \tag{5.7}$$

$$\theta' = \left[1 + \mu + \lambda - \nu_1 \frac{dF_1}{dU}(u_s^{\mathrm{s}}(y))\right] \cos^2\theta - 1. \tag{5.8}$$

Since $r$ is a strictly positive function, we can identify the zeroes of $\hat{u}(y)$ by studying $\theta(y)$:

$$\hat{u}(y) = 0 \quad \mathcal{L}ongleftrightarrow \quad \theta(y) = \frac{1}{2}\pi + k\pi, \quad k \in \mathbb{Z}. \tag{5.9}$$

Moreover, we can establish an ordering principle for $\theta$, as stated in the following Lemma:

**Lemma 5.3** *Let $\hat{u}_1$, $\hat{u}_2$ be solutions to* (3.32) *for real $\lambda_1$ resp. $\lambda_2$ outside the essential spectrum* (3.9). *Assume $\theta_2(y_0) < \theta_1(y_0)$ for some $y_0 \in (y_*, \infty)$, where $\theta_{1,2}$ are related to $\hat{u}_{1,2}$ by* (5.5). *If $\lambda_2 > \lambda_1$, then $\theta_2(y) < \theta_1(y)$ for all $y_* \leq y \leq y_0$.*

**Proof.** Introduce $\Delta\lambda = \lambda_2 - \lambda_1 > 0$. Using (5.8), we can deduce a differential equation for the difference $\theta_1 - \theta_2$:

$$(\theta_1 - \theta_2)' = \left[1 + \mu + \lambda_1 - \nu_1 \frac{dF_1}{dU}(u_s^{\mathrm{s}}(y))\right] \frac{\left(\cos^2\theta_1 - \cos^2\theta_2\right)}{\theta_1 - \theta_2} (\theta_1 - \theta_2) - \Delta\lambda \cos^2\theta_2 \tag{5.10}$$

This equation has the form $u' = f\,u - h$, where $u = \theta_1 - \theta_2$ and $h = \Delta\lambda \cos^2\theta_2 \geq 0$. By introducing $F(y) = \int_y^{y_0} f(\eta) d\eta$, we see that $e^F\,(u' - f\,u) = \frac{d}{dy}\left(e^F u\right) = -h\,e^F \leq 0$. Since $e^F u$ is decreasing and $e^{F(y_0)} u(y_0) = u(y_0) > 0$ since $\theta_1(y_0) - \theta_2(y_0) > 0$, we can conclude that $e^{F(y)} u(y) > 0$ and hence $u(y) > 0$ for all $y_* \leq y \leq y_0$. □

We use Lemma 5.3 to establish a similar ordering result for $\hat{\theta}_{s,-}(y; \lambda, \varepsilon)$, which is the 'angular function' associated to $\hat{u}_{s,-}(y; \lambda, \varepsilon)$ through the polar transformation (5.5). Once again we use the fact that we can approximate $\hat{\theta}_{s,-}$ by an exponential for large values of $y$ (see (3.31)), by taking $y_0$ arbitrarily large.

**Lemma 5.4** *Consider real $\lambda_1, \lambda_2 \notin \sigma_e$* (3.9). *If $\lambda_2 > \lambda_1$, then $\theta_{s,-}(y; \lambda_2, \varepsilon) < \theta_{s,-}(y; \lambda_1, \varepsilon)$ for all $y \in (y_*, \infty)$.*

**Proof.** Since $\hat{u}_{s,-}(y; \lambda, \varepsilon) \rightsquigarrow e^{\Lambda_s x_*} e^{-\Lambda_s y}$ as $y \to \infty$ (3.33) and therefore $\frac{d}{dy}\hat{u}_{s,-} \rightsquigarrow -\Lambda_s\, e^{\Lambda_s x_*} e^{-\Lambda_s y}$ as $y \to \infty$, it follows that

$$r_{s,-} \rightsquigarrow \sqrt{1 + \lambda + \mu}\, e^{\sqrt{\lambda+\mu} x_*} e^{-\sqrt{\lambda+\mu}\, y} \quad \text{as} \quad y \to \infty,$$

$$\cos\theta_{s,-} \rightsquigarrow \frac{1}{\sqrt{1 + \lambda + \mu}} \quad \text{as} \quad y \to \infty,$$

$$\sin\theta_{s,-} \rightsquigarrow -\frac{\sqrt{\lambda + \mu}}{\sqrt{1 + \lambda + \mu}} \quad \text{as} \quad y \to \infty$$

so $\tan\theta_{s,-} \rightsquigarrow -\sqrt{\lambda + \mu}$ as $y \to \infty$. Since we consider $\lambda \in \mathbb{R}$ outside the essential spectrum (3.9), we know that $0 < \frac{1}{\sqrt{1+\lambda+\mu}} < 1$ so $\theta_{s,-}(\mathrm{mod}\, 2\pi) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ as $y \to \infty$. The angle variable $\theta$ is still defined up to a multiple of $2\pi$: we gauge $\theta_{s,-}$ such that $\theta_{s,-} \in (-\frac{\pi}{2}, \frac{\pi}{2})$ as $y \to \infty$. Since the tangent is strictly increasing on $(-\frac{\pi}{2}, \frac{\pi}{2})$, it follows that if $\lambda_2 > \lambda_1$ then $-\sqrt{\lambda_2 + \mu} < -\sqrt{\lambda_1 + \mu}$ and therefore $\tan\theta_{s,-}(y;\lambda_1;\varepsilon) - \tan\theta_{s,-}(y;\lambda_2;\varepsilon) \rightsquigarrow K_1 > 0$ as $y \to \infty$; we can conclude that $\theta_{s,-}(y;\lambda_1,\varepsilon) - \theta_{s,-}(y;\lambda_2,\varepsilon) \rightsquigarrow K_2 > 0$ as $y \to \infty$. Lemma 5.3 can now be used to establish the statement of the Lemma. □

The ordering principle from Lemma 5.3 can be combined with the eigenfunction hierarchy from Lemma 3.2. This is only possible if the slow eigenvalue problem (3.31) can be extended to the entire real line, i.e. when $u_s^{\mathrm{s}}(x)$ is bounded for $x \to \pm\infty$. Invoking the result of Lemma 2.2, we see that we may assume $\mathcal{W}_s^{\mathrm{s}}((0,0)) \cap \{p_s = 0\} \neq \emptyset$ without loss of generality, and that as a consequence $u_s^{\mathrm{s}}(x)$ may be assumed to be bounded, since the function $u_s^{\mathrm{s}}(x)$ describes a homoclinic orbit on the slow manifold $\mathcal{M}$. This allows us to use Lemma 3.2 on (the extended) eigenvalue problem (3.31), introducing the slow eigenvalues $\lambda_{s,j}$ with their associated eigenfunctions $w_{s,j}$ from Lemma 3.2.

**Lemma 5.5** *Assume without loss of generality that $\mathcal{W}_s^{\mathrm{s}}((0,0)) \cap \{p_s = 0\} \neq \emptyset$. If $\lambda_{s,j+1} < \lambda < \lambda_{s,j}$ then the associated function $\hat{u}_{s,-}(y;\lambda;\varepsilon)$ has at least $j$ and at most $j+1$ zeroes as a function of $y$. Furthermore, if $0 < \lambda < \lambda_{s,0}$, then $\hat{u}_{s,-}(y;\lambda;\varepsilon) > 0$ if $y > 0$. Secondly, if $\lambda_{s,0} \leq \lambda$, then $\hat{u}_{s,-}(y;\lambda;\varepsilon) > 0$ for all $y \in \mathbb{R}$.*

**Proof.** Since $\mathcal{W}_s^{\mathrm{s}}((0,0)) \cap \{p_s = 0\} \neq \emptyset$, $u_s^{\mathrm{s}}(x)$ is bounded. This allows us to apply Lemma 3.2 in full, introducing the slow eigenvalues $\lambda_{s,j}$ with associated eigenfunctions $w_{s,j}$. From Lemma 3.2 (ii) and (iii), it follows that $w_{s,1}(y) = -\frac{d}{dy}u_s^{\mathrm{s}}(y)$ for $\lambda_{s,1} = 0$. Moreover, $w_{s,j}(y)$ has $j$ distinct zeroes – in particular, $w_{s,0}(y)$ is positive (Lemma 3.2 (iii)) and never zero. Using the fact that $w_{s,0}(y)$ is even, we can reason analogously to the proof of Lemma 5.4 and conclude that $\theta_{s,0}(y) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ for all $y$ – we use the same gauge for $\theta_{s,j}$ as that for $\theta_{s,-}$ in the proof of Lemma 5.4.

Furthermore, evaluating (5.8) at the 'critical' $\theta$-values from (5.9), we see that

$$\theta(y) = \frac{1}{2}\pi + k\pi, \quad k \in \mathbb{Z} \qquad \mathcal{L}ongleftrightarrow \qquad \theta'(y) = -1. \tag{5.11}$$

The function $\theta(y)$ thus crosses each 'critical' value $\theta(y) = \frac{1}{2}\pi + k\pi$, $k \in \mathbb{Z}$ only once, and in a transversal way. Since $w_{s,1}(y)$ is odd, we can infer analogously to the proof of Lemma 5.4 that $\cos\theta_{s,1} \rightsquigarrow -\frac{1}{\sqrt{1+\mu}}$ and $\sin\theta_{s,1} \rightsquigarrow -\frac{\sqrt{\mu}}{\sqrt{1+\mu}}$ as $y \to -\infty$. This means that $\theta_{s,1}\,(\mathrm{mod} 2\pi) \in (\pi, \frac{3}{2}\pi)$ as $y \to -\infty$. Using the fact that $w_{s,1}(y)$ is has only one zero and therefore $\theta_{s,1}$ crosses the line $\theta = \frac{1}{2}\pi$ only once, we see that the gauge choice allows us to omit the "mod$2\pi$", yielding $\theta_{s,1} \in (\pi, \frac{3}{2}\pi)$ as $y \to -\infty$. Using Lemmas 5.3 and 5.4 (extended to the entire real line), we conclude that for all $\lambda_{s,1} = 0 < \lambda < \lambda_{s,0}$, the function $w_{s,\lambda}^{\mathrm{R}}(y)$ has at most one zero, see Figure 6. Furthermore, since we know that $\theta_{s,1}$ crosses the line $\theta = \frac{1}{2}\pi$ exactly at $y = 0$ (with slope $-1$), the aforementioned zero of $w_{s,\lambda}^{\mathrm{R}}(y)$ can only occur for negative values of $y$. Moreover, analogous reasoning can be applied to every pair $(\lambda_{s,j}, \lambda_{s,j+1})$: if $\lambda_{s,j+1} < \lambda < \lambda_{s,j}$, then $w_{s,\lambda}^{\mathrm{R}}(y)$ has at least $j$ and at most $j+1$ zeroes. Note that the above also implies that for $\lambda_{s,0} < \lambda$, the function $w_{s,\lambda}^{\mathrm{R}}(y)$ is never zero. Identification of $w_{s,\lambda}^{\mathrm{R}}(y)$ with $\hat{u}_{s,-}(y;\lambda,\varepsilon)$ yields the Lemma. □

The result of Lemma 5.5 can used to make a statement about $B_-(\lambda)$:

**Lemma 5.6** *If $y_* > 0$, then $B_-(\lambda) \neq 0$ for all $\lambda \geq 0$. If $y_* \leq 0$, then there is a $\lambda \geq 0$ for which $B_-(\lambda) = 0$.*

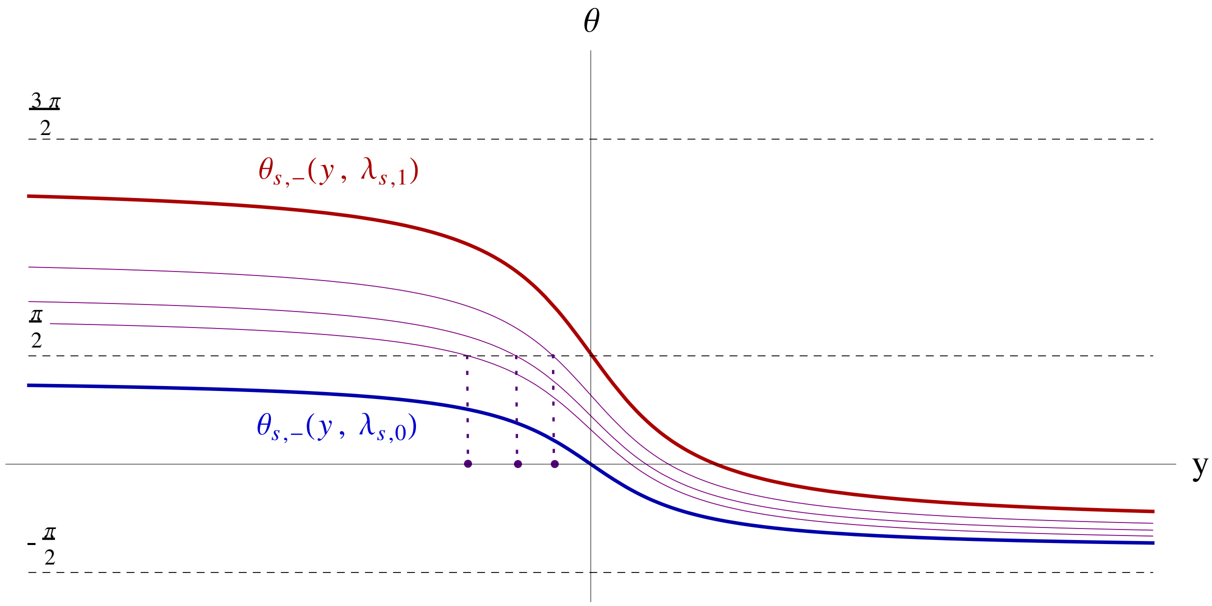



Figure 6: The ordering of $\theta_{s,-}(y;\lambda)$, depicted for $\lambda_1 \leq \lambda \leq \lambda_0$. The intersections with the line $\theta = \frac{\pi}{2}$ are also indicated; for these values of $\lambda$, each of these intersections is to the left of $y = 0$.

Of course, this has an immediate consequence for pulses for which $x_* < 0$ (see Figure 4c).

**Corollary 5.7** *Let $F_{1,2}$ and $G$ be such that $\mathcal{W}^{\mathrm{s}}_{s}((0,0)) \cap \{p_s = 0\} \neq \emptyset$, and assume that $x_* < 0$. Let $\Gamma_h$ be a pulse solution of* (1.7) *with $x_* < 0$ (Theorem 2.1). Then $\Gamma_h$ is unstable.*

**Proof of Lemma 5.6.**
$\mathbf{y_* > 0}$ : Since $B_-(\lambda) = \lim_{\varepsilon\to 0} \hat{u}_{s,-}(y_*;\lambda,\varepsilon)$, the last statement of Lemma 5.5 applies for all values of $y_*$ if $\lambda \geq \lambda_{s,0}$. For $0 \leq \lambda < \lambda_{s,0}$, the second statement of Lemma 5.5 makes sure that whenever $y_* > 0$, $\hat{u}_{s,-}(y_*;\lambda,\varepsilon) \neq 0$ and therefore $B_-(\lambda) \neq 0$.

$\mathbf{y_* \leq 0}$ : Consider $\lambda \geq 0$; we set out to prove that for every $y_* \leq 0$ there is a $\lambda \geq 0$ such that $\lim_{\varepsilon\to 0} \hat{u}_{s,-}(y_*;\lambda,\varepsilon) = 0$. Define the zero-set $\mathcal{U}_0 = \{(\lambda, y_0) \,|\, \lim_{\varepsilon\to 0} \hat{u}_{s,-}(y_0;\lambda,\varepsilon) = 0\}$. Using Lemma 5.5 and the previously proven results for $y_* > 0$, we know that $(\mathcal{U}_0 \cap \{(\lambda, y_0) \,|\, \lambda \geq 0\}) \subset [0, \lambda_{s,0}) \times (-\infty, 0]$. By the polar coordinate transformation (5.5), we see that $\mathcal{U}_0 = \Theta_0$, where $\Theta_0 = \left\{(\lambda, y_0) \,|\, \lim_{\varepsilon\to 0} \theta_{s,-}(y_0,\lambda,\varepsilon) = \frac{\pi}{2}\right\}$. Taking the derivative with respect to $\lambda$ of the defining equation $\lim_{\varepsilon\to 0} \theta_{s,-}(y_0;\lambda,\varepsilon) = \frac{\pi}{2}$ yields $\frac{\partial\theta_{s,-}}{\partial y}\frac{dy_0}{d\lambda} + \frac{\partial\theta_{s,-}}{\partial\lambda} = 0$. For $(\lambda, y_0) \in \Theta_0$ we have $\frac{\partial\theta_{s,-}}{\partial y} = -1$ (5.11), so $\frac{dy_0}{d\lambda} = \frac{\partial\theta_{s,-}}{\partial\lambda}$ for $(\ lambda, y_0) \in \Theta_0$. Now consider $(\hat{\lambda}, \hat{y}_0) \in \Theta_0$ and take $0 < \delta \ll 1$ small enough. Using the smoothness of $\theta_{s,-}$ as a function of $\lambda$, we can write $\theta_{s,-}(\hat{y}_0;\hat{\lambda}+\delta,\varepsilon) = \theta_{s,-}(\hat{y}_0;\hat{\lambda},\varepsilon) + \delta\,\frac{\partial\theta_{s,-}}{\partial\lambda}(\hat{y}_0;\hat{\lambda},\varepsilon) + \mathcal{O}(\delta^2)$. Using the extension of Lemma 5.4 to the entire real line, we know that $\theta_{s,-}(\hat{y}_0;\hat{\lambda}+\delta,\varepsilon) < \theta_{s,-}(\hat{y}_0;\hat{\lambda},\varepsilon)$. Taking the limit $\varepsilon \to 0$ yields $\frac{\pi}{2} + \delta \lim_{\varepsilon\to 0} \frac{\partial\theta_{s,-}}{\partial\lambda}(\hat{y}_0;\hat{\lambda},\varepsilon) + \mathcal{O}(\delta^2) < \frac{\pi}{2}$ so $\lim_{\varepsilon\to 0} \frac{\partial\theta_{s,-}}{\partial\lambda}(\hat{y}_0;\hat{\lambda},\varepsilon) < 0$ for all $(\hat{\lambda}, \hat{y}_0) \in \Theta_0$. This implies that $\frac{dy_0}{d\lambda} < 0$ on $\Theta_0$. Therefore, $\Theta_0$ is a smooth one- dimensional submanifold of the the $(\lambda, y_0)$-(half)plane. The continuity of $\hat{u}_{s,-}(y;\lambda,\varepsilon)$ both as a function of $y$ and $\lambda$ implies that $\mathcal{U}_0 = \Theta_0$ is closed. Since $\frac{\partial\theta_{s,-}}{\partial y}(\lambda, y_0) = -1$ when $\lim_{\varepsilon\to 0} \theta_{s,-}(y_0;\lambda,\varepsilon) = \frac{\pi}{2}$, there is a $\eta > 0$ such that $\lim_{\varepsilon\to 0} \theta_{s,-}(y_0 - \eta;\lambda,\varepsilon) > \frac{\pi}{2}$ and $\lim_{\varepsilon\to 0} \theta_{s,-}(y_0 + \eta;\lambda,\varepsilon) < \frac{\pi}{2}$. The smoothness of $\hat{u}_{s,-}(y;\lambda,\varepsilon)$ as a function of $\lambda$ implies that these inequalities also hold for an open interval containing $\lambda$. This means that $\Theta_0$ is connected and that it does not have singular, i.e. terminal points in the interior

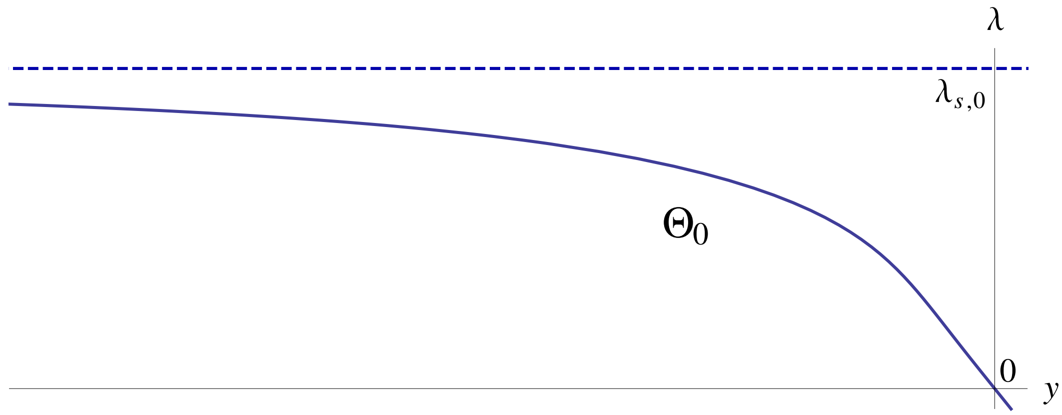



Figure 7: The zero set $\Theta_0 = \left\{(\lambda, y_0)\,|\, \lim_{\varepsilon\to 0} \theta_{s,-}(y_0,\lambda,\varepsilon) = \frac{\pi}{2}\right\}$.

of the half plane $\{(\lambda, y_0)\,|\,\lambda \geq 0\}$, except for $(0,0) \in \Theta_0$ – which also ensures that $\Theta_0$ is nonempty. We conclude that as a graph over $\lambda$, the map $\lambda \mapsto y_0(\lambda)$ defines a strictly decreasing function which has the entire negative halfline $(-\infty, 0]$ as its range, see Figure 7.
Therefore, for every $y_0 \leq 0$ there is a $\lambda \geq 0$ such that $\lim_{\varepsilon\to 0} \theta_{s,-}(y_0;\lambda,\varepsilon) = \frac{\pi}{2}$, which implies that $B_-(\lambda) = \lim_{\varepsilon\to 0} \hat{u}_{s,-}(y_*;\lambda,\varepsilon) = 0$ if we take $y_* = y_0$. □

The fact that $B_-(\lambda) \neq 0$ for $\lambda \geq 0$ only excludes *real* positive zeroes of $B_-(\lambda)$ if $y_* > 0$. In [38], we have conjectured that $B_-(\lambda) \neq 0$ for all $\lambda \in \mathbb{C}$ with $\text{Im}(\lambda) \neq 0$ for the explicit system (1.2) considered there. Even for the very simple case in which $\hat{u}_{s,-}$ can be expressed in terms of associated Legendre functions –as is the case in [38]–, there is no result in the literature about the (non-)existence of complex zeroes we are aware of. In our (numerical) investigations of $B_-(\lambda)$ we have not found any evidence of the possibility that $B_-(\lambda)$ can be zero for $\lambda \notin \mathbb{R}$.

## 5.2 The trivial eigenvalue $\lambda = 0$

While the explicit expression for $t_{s,+}$ (4.9) is in the general setting hard to analyse explicitly, it is possible to treat some specific situations in detail; in this section, we focus on the trivial eigenvalue $\lambda = 0$. From Lemmas 5.5 and 5.6 we know that $B_-(0) = 0$ if and only if $y_* = 0$ since $\frac{d}{dy}u_s^{\mathrm{s}}(y_*) = 0$ if and only if $y_* = 0$. This situation can be interpreted geometrically as a quadruple intersection of both curves $T_o$ (2.11) and $T_d$ (2.12) with $\mathcal{W}_s^{\mathrm{u}}((0,0)) \cap \mathcal{W}_s^{\mathrm{s}}((0,0))$ at $(u_M, 0)$. This implies that $p_* = 0$ (2.17) and hence $D_p(u_*) = 0$ (2.9), which in turn means that the $u$-coordinate does not make a jump (2.8). Note that this does not necessarily mean that $V$-component is identically zero, only that the $U$- and $V$-components decouple to leading order. Since $\lambda = 0$ is always a simple eigenvalue of the pulse, we can conclude the following:

**Corollary 5.8** *When $x_* = 0$, the trivial eigenvalue $\lambda = 0$ has multiplicity 2.*

Therefore, the bifurcation which changes the sign of $x_* = y_* + \mathcal{O}\left(\varepsilon^{\frac{3}{4}}\right)$, i.e. which changes the qualitative properties of the homoclinic pulse from the situation depicted in Figure 4a to Figure 4c, (further) destabilizes the pulse by sending an eigenvalue through the origin; it is highly likely that there are additional unstable eigenvalues. The fact that the trivial eigenvalue has multiplicity 2 when $x_* = 0$ can also be understood by noticing that in this case there is virtually no coupling between the slow $U$- and fast $V$-equation: the fast $V$-pulse does not have an impact on the $U$-component since $D_p(u_*)=0$. The uncoupled $U_h$- and $V_h$-components both have a zero (as well as a positive) eigenvalue, since their derivatives are a solution to their respective scalar equations.

The slow transmission function $t_{s,+}$ can be analyzed in more detail at $\lambda = 0$, yielding the following Lemma.

**Lemma 5.9** *At the trivial eigenvalue $\lambda = 0$, the slow transmission function $t_{s,+}(\lambda)$ can be expressed as*

$$t_{s,+}(0) = \frac{1}{2\sqrt{\mu}} \frac{\nu_2 D_p(u_*)}{u_{s,\infty}^2} \left\{ \mu u_* - \nu_1 F_1(u_*;0) - \frac{1}{4}\nu_2^2 D_p(u_*) \left. \frac{d}{du}\right|_{u=u_*} D_p(u) \right\}. \tag{5.12}$$

**Proof.** First, we recall Lemma 3.7: for $\lambda = 0$, we can write $v_{\text{in}}(\xi; \lambda = 0)$ as

$$v_{\text{in}}(\xi;0) = \frac{\partial}{\partial u} v_{f,h}(\xi;u)|_{u=u_*} + C\dot{v}_{f,h}(\xi;u_*)$$

where $C \in \mathbb{R}$ is a free parameter. Since $v_{f,h}(\xi,u)$ is an even function of $\xi$, the product

$$\frac{\partial F_2}{\partial V}(u_*, v_{f,h}(\xi;u_*)) \frac{\partial}{\partial \xi} v_{f,h}(\xi;u_*)$$

is odd as a function of $\xi$, hence its integral vanishes. Therefore we can write the integrand of the integral term occcuring in the expression of $t_{s,+}$ (4.9) as

$$\frac{\partial F_2}{\partial U}(u_*, v_{f,h}(\xi;u_*)) + \frac{\partial F_2}{\partial V}(u_*, v_{f,h}(\xi;u_*)) v_{\text{in}}(\xi) = \left. \frac{\mathrm{d}}{\mathrm{d}u}\right|_{u=u_*} F_2(u, v_{f,h}(\xi;u)).$$

Using the notation introduced in (2.9), we can write the integral in (4.9) as

$$\int_{-\infty}^{\infty} \left[ \frac{\partial F_2}{\partial U}(u_*, v_{f,h}(\xi;u_*)) + \frac{\partial F_2}{\partial V}(u_*, v_{f,h}(\xi;u_*))\, v_{\text{in}}(\xi) \right] \mathrm{d}\xi =$$
$$\int_{-\infty}^{\infty} \left[ \left. \frac{\mathrm{d}}{\mathrm{d}u}\right|_{u=u_*} F_2(u, v_{f,h}(\xi;u)) \right] \mathrm{d}\xi = \left. \frac{\mathrm{d}}{\mathrm{d}u}\right|_{u=u_*} D_p(u). \tag{5.13}$$

As for the expressions $B_-$ and $B'_-$, we recall Lemma 3.2 (ii): the eigenfunction at $\lambda = 0$ for the problem $(\mathcal{L}_s(y) - \lambda)u = 0$ is (a scalar multiple of) the derivative of the function which is perturbed, in our case

$$\hat{u}_{s,-}(y_*; \lambda = 0) = C_1 \left. \frac{\mathrm{d}}{\mathrm{d}y}\right|_{y=y_*} u_s^{\mathrm{s}}(y) = C_1 \left. \frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=x_*} u_s^{\mathrm{s}}(x), \quad C_1 \in \mathbb{R}$$

where $u_s^{\mathrm{s}}(x)$ is the solution to (2.13) that spans the stable manifold $W_s^{\mathrm{s}}((0,0))$. Using (2.23), we can determine $C_1 = \frac{1}{u_{s,\infty}}$. Similarly, we can write

$$\left. \frac{\mathrm{d}}{\mathrm{d}y}\right|_{y=y_*} \hat{u}_{s,-}(y;\lambda=0) = \frac{1}{u_{s,\infty}} \left. \frac{\mathrm{d}^2}{\mathrm{d}x^2}\right|_{x=x_*} u_s^{\mathrm{s}}(x).$$

As both $B_-$ and $B'_-$ are defined as the limit of the above expressions as $\varepsilon \to 0$ (4.10), we see that

$$B_-(0) = \frac{1}{u_{s,\infty}} \left. \frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=x_*} u_s^{\mathrm{s}}(x) \quad \text{and} \quad B'_-(0) = \frac{1}{u_{s,\infty}} \left. \frac{\mathrm{d}^2}{\mathrm{d}x^2}\right|_{x=x_*} u_s^{\mathrm{s}}(x). \tag{5.14}$$

Since the flow on the stable manifold is governed by (2.13), we can write

$$u_{s,\infty} B'_{-}(0) = \mu u_* - \nu_1 F_1(u_*; 0). \tag{5.15}$$

Moreover, since the expressions for $B_-$ and $B'_-$ are evaluated at $x = x_*$, we know that at $u(x_*) = u_*$ the slow manifold intersects the touchdown curve $T_d$. Therefore, by (2.15)

$$u_{s,\infty} B_{-}(0) = -\frac{1}{2}\nu_2 D_p(u_*). \tag{5.16}$$

Substitution of (5.13), (5.15) and (5.16) in (4.9) yields the Lemma, using the fact that $u_{s,\infty} \neq 0$. □

When $y_* \neq 0$ and hence $B_-(0) \neq 0$, the trivial eigenvalue is again connected to a bifurcation of the homoclinic pulse (Remark 2.4). Comparison of the saddle-node condition (2.21) from Corollary 2.3 with the expression for $t_{s,+}(0)$ from Lemma 5.9 yields the following Corollary:

**Corollary 5.10** *Assume $B_-(0) \neq 0$. The critical eigenvalue $\lambda = 0$ has multiplicity 2 or more – or equivalently $t_{s,+}(0) = 0$ – if and only if the homoclinic orbit $\Gamma_h(\xi)$ of Theorem 2.1 undergoes a saddle node bifurcation (as described in Corollary 2.3).*

This way we may conclude that, apart from the saddle node bifurcation (Corollary 2.3) and the crossing of $x_*$ through 0 (Corollary 5.8), the homoclinic pulse $\Gamma_h$ can only lose or gain stability when a pair of complex conjugate eigenvalues –with nonzero imaginary parts– crosses the imaginary axis: the associated bifurcation is of Hopf type. In explicit settings, the bifurcation structure of these Hopf bifurcations can be analyzed in detail, see section 4 of the companion paper [38].

### 5.3 Further instability results

The structure of $t_{s,+}(\lambda)$ at $\lambda = 0$ and near $\lambda = \lambda_{f,0}$ can be used to establish explicit conditions for the existence of real positive zeroes of $t_{s,+}(\lambda)$. Note that the line of reasoning is similar to that in [20], where the sign of the Evans function at $\lambda = 0$ and for $\lambda \to \infty$ was combined with counting arguments to establish the (non-)existence of intersections of the (real) Evans function with the positive $\lambda$-axis. Compared to [20], we have additional information about the slow component of the Evans function near its pole at $\lambda = \lambda_{f,0}$.

**Lemma 5.11** *Consider $T$ as given in (5.2). If $\nu_2 T > 0$, there exists a positive real zero of $t_{s,+}(\lambda)$; therefore, the homoclinic pulse $\Gamma_h$ unstable when $\nu_2 T > 0$.*

When $F_2$ is monotonic in $V$ and $G$ is monotonic in $U$, the coefficient $T$ is nonzero and its sign is known (see (5.2) and recall that $w_{f,0}(\xi) > 0$). In that case –which will often arise in explicit settings such as the generalized GM model– the equation $\nu_2 B_-(\lambda_{f,0}) = 0$ determines a codimension-1 instability condition, see the discussion following Corollary 5.2 on (5.4). Combining this with Lemma 5.11, we see that the homoclinic pulse $\Gamma_h$ is unstable for either $\nu_2 \in (-\infty, 0]$ or $\nu_2 \in [0, \infty)$, depending on the (fixed) sign of $T$.

**Proof.** The idea of the proof is to combine insights on the behaviour of $t_{s,+}(\lambda)$ for real $\lambda$ as $\lambda \to \infty$ with the behaviour of $t_{s,+}(\lambda)$ as $\lambda \downarrow \lambda_{f,0}$, then use the continuity of $t_{s,+}$.

Firstly, as in the proof of Lemma 3.4, define $\delta = \frac{1}{\lambda}$. For $\delta$ small enough, i.e. for $\lambda$ large enough, it can be shown analogous to the proof of Lemma 3.4 that $\hat{u}_{s,\pm}(y;\lambda,\varepsilon) = e^{\pm\sqrt{\lambda}y}(1+\mathcal{O}(\delta))$ on an $\mathcal{O}(1)$ $y$-domain $\supset \{y=0\}$, using (3.33) and (3.34). Therefore, we can approximate $B_\pm$ and $B'_\pm$ as

$$B_\pm(\lambda) = e^{\pm\sqrt{\lambda}y_*}(1+\mathcal{O}(\delta)) \text{ and } B'_\pm(\lambda) = \pm\sqrt{\lambda}\, e^{\pm\sqrt{\lambda}y_*}(1+\mathcal{O}(\delta)),$$

yielding

$$t_{s,+}(\lambda) = \frac{e^{-2\sqrt{\lambda}y_*}}{\sqrt{\lambda}} \left\{ \sqrt{\lambda} - \frac{\nu_2}{2}\int_{-\infty}^{\infty} \left[ \frac{\partial F_2}{\partial U}(u_*, v_{f,h}(\xi;u_*)) + \frac{\partial F_2}{\partial V}(u_*, v_{f,h}(\xi;u_*))\, v_{\text{in}}(\xi) \right] \mathrm{d}\xi \right\}.$$

Combining Lemma 3.4 and the elements of its proof with the expression for $v_{\text{in}}$ from Lemma 3.5, we obtain

$$v_{\text{in}}(\xi;\lambda) = \frac{1}{2\sqrt{\lambda}} \int_{-\infty}^{\infty} \frac{\partial G}{\partial U}(u_*, v_{f,h}(\tilde{\xi};u_*))\, e^{-\sqrt{\lambda}|\xi-\tilde{\xi}|} d\tilde{\xi} + \mathcal{O}(\delta).$$

From this, we see that $\int_{-\infty}^{\infty} \frac{\partial F_2}{\partial V}(u_*, v_{f,h}(\xi;u_*))\, v_{\text{in}}(\xi)\,\mathrm{d}\xi \to 0$ as $\lambda\to\infty$. Since $\int_{-\infty}^{\infty} \frac{\partial F_2}{\partial U}(u_*, v_{f,h}(\xi;u_*))\,\mathrm{d}\xi$ does not depend on $\lambda$, it follows that

$$t_{s,+}(\lambda) \rightsquigarrow e^{-2\lambda y_*} \quad \text{as} \quad \lambda\to\infty.$$

Secondly, we know the behaviour of $t_{s,+}$ in another limit from Lemma 5.1: recall (5.1), with $T$ as in (5.2). Now, when $\nu_2 T > 0$, then $t_{s,+}$ tends to $-\infty$ as $\lambda \downarrow \lambda_{f,0}$. Since there are no other poles of $t_{s,+}$ for $\lambda > \lambda_{f,0}$, by continuity there must be a $\lambda_* > \lambda_{f,0} > 0$ where $t_{s,+} = 0$ because $t_{s,+}$ approaches zero from above for $\lambda\to\infty$. □

Combining the statement of Corollary 5.7 with the observation that the condition $x_* < 0$ is equivalent with $\nu_2 D_p(u_*) < 0$ (combining the definition $p_* = +\frac{1}{2}\nu_2 D_p(u_*)$ with with Lemma 2.2), we see that the homoclinic pulse may only be stable when $\nu_2 D_p(u_*) > 0$. This observation can be used to obtain another instability criterion:

**Lemma 5.12** *Assume $\nu_2 D_p(u_*) > 0$, and let $R$ be defined by*

$$R = \mu u_* - \nu_1 F_1(u_*;0) - \frac{1}{4}\nu_2^2 D_p(u_*)\, \frac{d}{du}\bigg|_{u=u_*} D_p(u). \tag{5.17}$$

*If $R > 0$, the homoclinic pulse $\Gamma_h$ is unstable.*

Since $R$ is directly related to the dervative of (2.16) with respect to $u$ (see Corollary 2.3 and Lemma 5.9), it can be interpreted geometrically in the context of the existence problem as the relative slope of $T_o$ with respect to $W_s^{\mathrm{u}}((0,0))$ at their intersection $(u_*, p_*)$. In Figure 8, we have indicated the signs of $R$ related to the three possible homoclinic pulses associated to the configuration depicted in Figure 2. Lemma 5.12 directly yields the instability of the first and third intersection.

**Proof.** Since Lemma 5.11 ensures that the pulse is unstable when $\nu_2 T > 0$, we assume $\nu_2 T < 0$ without loss of generality. Using Lemma 5.9, we see that $t_{s,+}(0) = \frac{1}{2\sqrt{\mu}} \frac{\nu_2 D_p(u_*)}{u_{s,\infty}^2} R$, so $\operatorname{sgn}(t_{s,+}(0)) = \operatorname{sgn}(R)$ since $\nu_2 D_p(u_*) > 0$. Since $\nu_2 T < 0$, we can use Lemma 5.1 to conclude that $t_{s,+} \to -\infty$ as

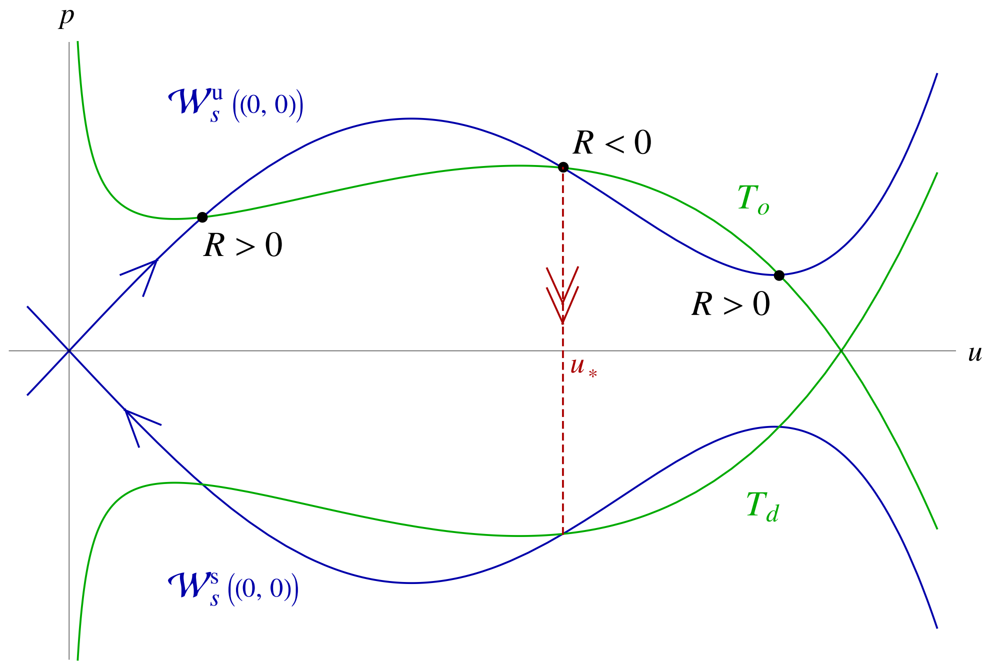
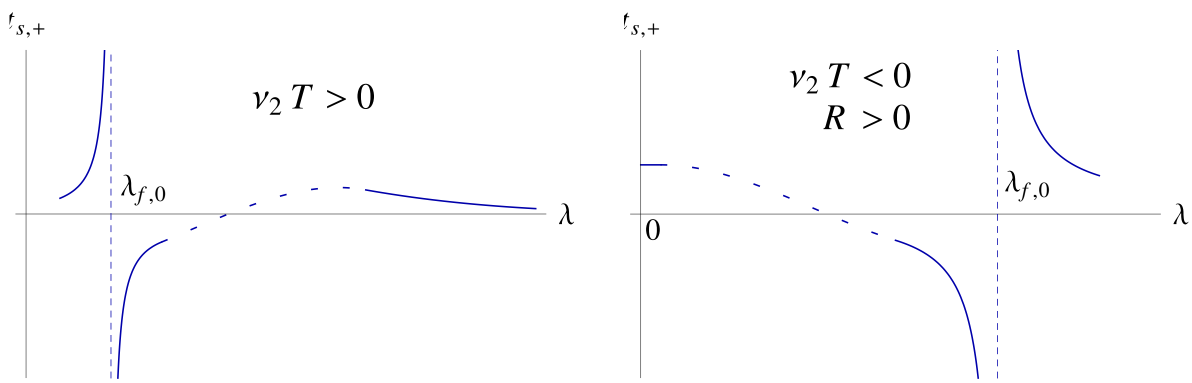



Figure 8: The coefficient $R$ (5.17) interpreted geometrically as the relative slope of $T_o$ with respect to $\mathcal{W}_s^{\mathrm{u}}((0,0))$ at their intersection point. Only the homoclinic pulse associated to the second intersection can be stable (Lemma 5.12).




(a) (b)

Figure 9: An illustration of the proof of Lemma 5.11 (9a) and Lemma 5.12 (9b). The (singular) behaviour of $t_{s,+}$ near $\lambda = \lambda_{f,0}$ is determined by the sign of $\nu_2 T$ (Lemma 5.1). In (a), this leads to at least one root of $t_{s,+}$ to the right of $\lambda_{f,0} > 0$ since $t_{s,+} \rightsquigarrow e^{-2y_*\lambda}$ as $\lambda \to \infty$. In (b), $t_{s,+}$ has to cross the horizontal axis in the interval $\lambda \in (0, \lambda_{f,0})$ at least once if $R > 0$.

$\lambda \uparrow \lambda_{f,0}$. If $R > 0$, i.e. $t_{s,+}(0) > 0$, it follows that there is a $\lambda_0 \in (0, \lambda_{f,0})$ for which $t_{s,+}(\lambda_0) = 0$ since $t_{s,+}(\lambda)$ is continuous for $\lambda \in [0, \lambda_{f,0})$. Since $\lambda_0 > 0$ is a positive zero of $t_{s,+}(\lambda)$, the homoclinic pulse $\Gamma_h$ is unstable. □

We refer to Figure 9 for an illustration of the necessary existence of unstable eigenvalues in the case $\nu_2 T > 0$ (Lemma 5.11) and the case $R > 0$, $\nu_2 T < 0$ (Lemma 5.12). Note that $R < 0$ for the only existing pulse in the explicit model (1.2), see Figure 2.3 (a) in [38].

Combining Corollary 5.7, Lemma 5.11 and Lemma 5.12, we may conclude:

**Theorem 5.13** *Let $\Gamma_h$ be a homoclinic pulse whose existence is established by Theorem 2.1. $\Gamma_h$ can only be stable if $\nu_2 D_p(u_*) > 0$, $\nu_2 T < 0$ and $R < 0$, where $D_p(u_*)$, $T$ and $R$ are explicitly computable expressions given in* (2.9), (5.2) *and* (5.17).

Finally, we formulate another instability result that is again based on the fact that we know $t_{s,+}$ has a pole at $\lambda = \lambda_{f,0}$.

**Lemma 5.14** *Assume $\nu_2 \neq 0$ and $B_-(\lambda_{f,0}) \neq 0$. Let $S$ be defined by*

$$S = \frac{2}{\nu_2} \frac{B'_-(\lambda_{f,0})}{B_-(\lambda_{f,0})} + \int_{-\infty}^{\infty} \frac{\partial F_2}{\partial U}(u_*, v_{f,h}(\xi; u_*)) d\xi. \tag{5.18}$$

*If $|S|$ is large enough, then $t_{s,+}(\lambda)$ has a zero near $\lambda_{f,0}$, rendering the homoclinic pulse unstable.*

**Proof.** Take the interval $I(\lambda_{f,0}, \delta)$ to be a (symmetric) $\delta$-neighbourhood of $\lambda_{f,0}$ in $\mathbb{R}$ with $0 < \delta \ll 1$ small enough. We can rewrite the equation $t_{s,+}(\lambda) = 0$ using (4.9) as

$$\int_{-\infty}^{\infty} \frac{\partial F_2}{\partial V}(u_*, v_{f,h}(\xi; u_*))\, v_{\text{in}}(\xi) \mathrm{d}\xi = -\frac{2}{\nu_2} \frac{B'_-}{B_-} - \int_{-\infty}^{\infty} \frac{\partial F_2}{\partial U}(u_*, v_{f,h}(\xi; u_*)) \mathrm{d}\xi \tag{5.19}$$

From Corollary 3.6 we know that the lefthand side of (5.19) behaves as $\frac{1}{\lambda - \lambda_{f,0}}$ in $I(\lambda_{f,0}, \delta)$, while the righthand side of (5.19), given by $S$ to leading order in $\delta$, is continuous –and to leading order constant– in $\lambda$ on the same interval. Therefore, a solution to (5.19) in the interval $I(\lambda_{f,0}, \delta)$ can be found if $|S|$ is large enough; see also Figure 10. □

This Lemma can be used to clarify the scaling of the $F_2$ term in (1.1) / (1.7), as argued in the introductory section 1:

**Corollary 5.15** *When $\nu_2$ is small enough, in particular when $\nu_2 = \mathcal{O}(\varepsilon)$, the homoclinic pulse is unstable.*

## 6 Discussion

The existence and stability theory for localised homoclinic pulses in the general setting of equation (1.1) presented in this paper can be seen as the first fundamental step in the analysis of the dynamics of interacting localised structures. Based on this work, some next steps can now be taken. Several of these steps have already been made in the context of GS/GM-type models – see [2, 8, 9, 18, 22, 23, 24, 30, 33, 41] and the references therein. The present paper and its companion [38] show that there will be fundamental analytical challenges in further developing the theory in the general setting of (1.1). Moreover, it is clear that the 'slow nonlinearity' of (1.1) will generate pulse dynamics that is much richer than that of 'slowly linear' models – see Remark 1.1 for the case of one localised homoclinic pulse.

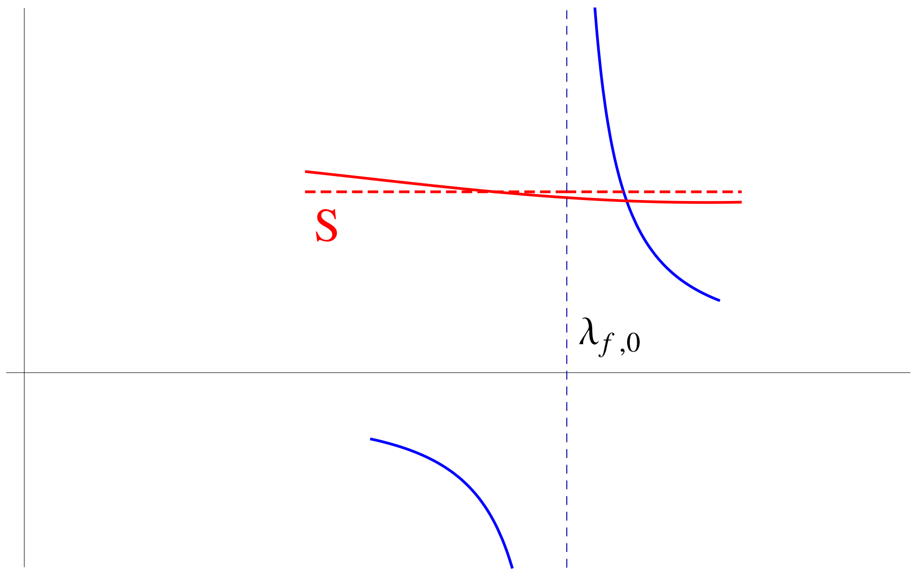



Figure 10: The statement of Lemma 5.14 graphically explained. The lefthand side of (5.19), which is singular at $\lambda = \lambda_{f,0}$, is indicated in blue, the righthand side of (5.19) is indicated in red and its approximation $S$ (5.18) by the dashed line.

A first next step –one that in fact largely inspired the present work– is the stability analysis of localised spatially periodic patterns to systems of type (1.1) on bounded and/or unbounded domains. Based on [30], it was found in the recent work [9] that the nature of the destabilization of spatially periodic multi-pulse patterns with long wavelength is quite complex. It is shown in [9] in the context of GM-type models that such patterns can be destabilized by two distinct types of Hopf bifurcations: one in which the destabilization makes the pulses of the periodic pattern oscillate exactly in phase with their neighbouring pulses, and one in which each destabilized pulse starts to oscillate exactly out of phase with its neighbours. Moreover, on the unbounded domain $x \in \mathbb{R}$, the character of the destabilization alternates countably many times between these two types of Hopf bifurcation as the wavelength of the underlying pattern grows, i.e. as the spatially periodic pattern approaches the homoclinic limit. This so-called 'Hopf dance' has also been found numerically by AUTO-simulations in generalized Gray-Scott models – models that even include nonlinear diffusion in the slow $U$-component [9, 35]. The analysis of [9] clearly shows that the Hopf dance, and especially the associated higher order 'belly dance', has its origins in the 'slowly linear' character of GM/GS-type models. It can be expected that the destabilization of long wavelength periodic patterns in system (1.1) has an even richer structure. This is the subject of work in progress.

Already in the case of GS/GM-type models, interacting pulses may exhibit complicated, even chaotic, behaviour [28, 29]. However, in the parameter regimes in which the pulse dynamics can be studied in full analytical detail –i.e. the regime in which pulse self-replication does not occur– the pulse interactions are of a much more simple nature, see [2, 8, 22, 23, 24, 33] and the references therein. Nevertheless, the semi-strong pulse dynamics exhibited by GS/GM-type models are much richer than in the weakly interacting case. Weak pulse interactions are only driven by exponentially small tail interactions [10, 31, 32]. The semi-strong GS/GM-type dynamics are largely determined by the slow $U$-component that does not approach its background state in between the fast $V$-pulses. However, in the GS/GM-type models studied in the literature, the slow $U$-dynamics are linear, and –exactly as in the stability analysis for homoclinic pulses– this linearity plays a crucial role in the analysis. In the general system (1.1), also the slow $U$-dynamics between localised $V$-pulses will be nonlinear. In combination with the observations of [38] –especially the possibility of stably oscillating pulses (Remark 1.1)– this implies that even in the semi-strong regime, the pulse dynamics generated by systems of the type (1.1) will be much more rich and complex than encountered so far in the literature.

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