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AN EXPLICIT UPPER BOUND FOR THE LEAST PRIME IDEAL IN THE CHEBOTAREV DENSITY THEOREM

by Jeoung-Hwan AHN & Soun-Hi KWON (*)

Abstract. — Lagarias, Montgomery, and Odlyzko proved that there exists an effectively computable absolute constant A_1 such that for every finite extension K of $\mathbb{Q}$, every finite Galois extension L of K with Galois group G and every conjugacy class C of G, there exists a prime ideal $\mathfrak{p}$ of K which is unramified in L, for which $\left[\frac{L/K}{\mathfrak{p}}\right] = C$, for which $N_{K/\mathbb{Q}}\,\mathfrak{p}$ is a rational prime, and which satisfies $N_{K/\mathbb{Q}}\,\mathfrak{p} \leqslant 2{d_L}^{A_1}$. In this paper we show without any restriction that $N_{K/\mathbb{Q}}\,\mathfrak{p} \leqslant {d_L}^{12577}$ if $L \neq \mathbb{Q}$, using the approach developed by Lagarias, Montgomery, and Odlyzko.

Résumé. — Lagarias, Montgomery, et Odlyzko ont démontré qu'il existe une constante absolue effectivement calculable A_1 telle que pour chaque extension finie K de $\mathbb{Q}$, chaque extension galoisienne finie L de K à groupe de Galois G, et chaque classe de conjugaison C de G, il existe un idéal premier $\mathfrak{p}$ de K qui est nonramifié dans L, pour lequel $\left[\frac{L/K}{\mathfrak{p}}\right] = C$, pour lequel $N_{K/\mathbb{Q}}\,\mathfrak{p}$ est un nombre premier rationel, et qui satisfait $N_{K/\mathbb{Q}}\,\mathfrak{p} \leqslant 2{d_L}^{A_1}$. Dans cet article nous démontrons sans aucune restriction que $N_{K/\mathbb{Q}}\,\mathfrak{p} \leqslant {d_L}^{12577}$ si $L \neq \mathbb{Q}$, en suivant la méthode developpée par Lagarias, Montgomery, et Odlyzko.

1. Introduction

Let K be a finite algebraic extension of $\mathbb{Q}$, and L a finite Galois extension of K with Galois group G. Let d_L and d_K denote the absolute values of discriminants of L and K, respectively, and let $n_L = [L : \mathbb{Q}]$, $n_K = [K : \mathbb{Q}]$. To each prime ideal $\mathfrak{p}$ of K unramified in L there corresponds a certain

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conjugacy class C of G consisting of the set of Frobenius automorphisms attached to the prime ideals $\mathfrak{P}$ of L which lie over $\mathfrak{p}$. Denote this conjugacy class by the Artin symbol $\left[\frac{L/K}{\mathfrak{p}}\right]$. For a conjugacy class C of G let

$$\pi_C(x) = |\{\mathfrak{p} \mid \mathfrak{p} \text{ a prime ideal of } K, \text{ unramified in } L,\\ \left[\frac{L/K}{\mathfrak{p}}\right] = C, \text{ and } N_{K/\mathbb{Q}}\,\mathfrak{p} \leqslant x\}|.$$

The Chebotarev density theorem states that

$$\pi_C(x) \sim \frac{|C|}{|G|} Li(x)$$

as $x \to \infty$. (See [15], [53], [28], [39], and [50]. See also [47] for some extensions of Chebotarev's theorem and applications.) The error term of this theorem was estimated in [24], [41], and [59]. Lagarias, Montgomery, and Odlyzko estimated upper bound for the least prime ideal $\mathfrak{p}$ with $\left[\frac{L/K}{\mathfrak{p}}\right] = C$ under the Generalized Riemann Hypothesis (GRH), and unconditionally, in [24] and [23], respectively.

Theorem A (Lagarias and Odlyzko [24]). — *There exists an effectively computable positive absolute constant A_0 such that if the GRH holds for Dedekind zeta function of $L \neq \mathbb{Q}$, then for every conjugacy class C of G there exists an unramified prime ideal $\mathfrak{p}$ in K such that $\left[\frac{L/K}{\mathfrak{p}}\right] = C$ and*

$$N_{K/\mathbb{Q}}\,\mathfrak{p} \leqslant A_0 (\log d_L)^2.$$

Oesterlé ([41]) has stated that if GRH holds, then one may have $A_0 = 70$. Bach and Sorenson ([4]) has improved this result in two ways: If GRH holds, then for any class C of G there is a prime $\mathfrak{p}$ in K of degree 1 over $\mathbb{Q}$ with $\left[\frac{L/K}{\mathfrak{p}}\right] = C$ and $N_{K/\mathbb{Q}}\mathfrak{p} \leqslant (4 \log d_L + 2.5 n_L + 5)^2$. (See also [3], [38], and [22].) Let

$$P(C) = \left\{ \mathfrak{p} \,\middle|\, \mathfrak{p} \text{ a prime ideal of } K, \text{ unramified in } L,\\ \text{of degree one over } \mathbb{Q} \text{ and } \left[\frac{L/K}{\mathfrak{p}}\right] = C \right\}.$$

Theorem B (Lagarias, Montgomery, and Odlyzko [23]). — *There is an absolute, effectively computable constant A_1 such that for every finite extension K of $\mathbb{Q}$, every finite Galois extension L of K, and every conjugacy class C of G, there exists a prime $\mathfrak{p}$ in $P(C)$ which satisfies*

$$N_{K/\mathbb{Q}}\,\mathfrak{p} \leqslant 2{d_L}^{A_1}.$$

See also [57]. When $K=\mathbb{Q}$ and $L=\mathbb{Q}(\mathrm{e}^{2\pi i/q})$, the conjucacy classes of G correspond to the residues classes modulo q and Theorem B gives an upper bound for the least prime in an arithmetic progression ([24] and [23]). In this case Theorem B is weaker than Linnik's theorem ([29], [30], [5]). For the least prime in an arithmetic progression, see for example [7], [8], [13], [14], [17], [18], [42], [43], [55], [56], and [61]. If $K=\mathbb{Q}$, $L=\mathbb{Q}(\sqrt{D})$, and ρ is the non identity in $Gal(L/\mathbb{Q})$, Theorem B gives an upper bound for the least quadratic nonresidue module D. For this case no upper bound better than Theorem B is known ([54], [6], [24], [23], [2], [25], [26]). In this paper we compute the constant A_1.

Theorem 1.1. — *For every finite extension K of $\mathbb{Q}$, every finite Galois extension $L(\neq \mathbb{Q})$ of K with Galois group G, and every conjugacy class C of G, there exists a prime ideal $\mathfrak{p}$ in $P(C)$ which satisfies*

$$N_{K/\mathbb{Q}}\,\mathfrak{p} \leqslant {d_L}^{A_1}$$

with $A_1 = 12577$.

To compute the constant A_1 we follow the method developed by [23]. In particular, we express zero-free regions for Dedekind zeta functions, density of zeros of Dedekind zeta functions, and Deuring–Heilbronn phenomenon with explicit constants in Sections 5-7 below. Zaman showed in [63] that $N_{K/\mathbb{Q}}\,\mathfrak{p} \ll {d_L}^{40}$ for sufficiently large d_L. See also [51]. Winckler proved $A_1 = 27175010$ without any restriction in [60].

2. Outline of Lagarias–Montgomery–Odlyzko's method

Let $\Re z$ and $\Im z$ denote the real part and imaginary one of $z\in\mathbb{C}$, respectively. We review the procedure for the proof of Theorem B in [23]. Let $g\in C$ and

$$F_C(s) = -\frac{|C|}{|G|}\sum_{\psi}\overline{\psi}(g)\frac{L'}{L}(s,\psi,L/K),$$

where ψ runs over the irreducible characters of G and $L(s,\psi,L/K)$ is the Artin L-function attached to ψ. The main parts of [23] consist of estimates of inverse Mellin transforms

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} F_C(s)k(s)\,\mathrm{d}s$$

where $k(s)$ is a kernel function. The main steps of the proof of Theorem B in [23] are as follows:

(i) From the orthogonality relations for the characters ψ it follows that for $\Re s > 1$

$$F_C(s) = \sum_{\mathfrak{p}} \sum_{m=1}^{\infty} \theta(\mathfrak{p}^m)(\log N_{K/\mathbb{Q}}\mathfrak{p})(N_{K/\mathbb{Q}}\mathfrak{p})^{-ms}$$

where for prime ideals $\mathfrak{p}$ of K unramified in L

$$\theta(\mathfrak{p}^m) = \begin{cases} 1 & \text{if } \left[\frac{L/K}{\mathfrak{p}}\right]^m = C, \\ 0 & \text{otherwise,} \end{cases}$$

and $|\theta(\mathfrak{p}^m)| \leqslant 1$ if $\mathfrak{p}$ ramifies in L. So we can separate the $\mathfrak{p}^m$ with $\left[\frac{L/K}{\mathfrak{p}}\right]^m = C$ from the others. (See [24, Section 3].)

(ii) Using a method due to Deuring ([10] and [35]) $F_C(s)$ can be written as a linear combination of logarithmic derivatives of Hecke L-functions instead of Artin L-functions. Let $H =< g >$ be the cyclic subgroup generated by g, E the fixed field of H. Then

$$F_C(s) = -\frac{|C|}{|G|} \sum_{\chi} \overline{\chi}(g) \frac{L'}{L}(s, \chi, E), \tag{2.1}$$

where χ runs over the irreducible characters of H, and $L(s, \chi, E)$ is a Hecke L-function attached to field E with $\chi(\mathfrak{p}) = \chi\left(\left[\frac{L/E}{\mathfrak{p}}\right]\right)$ for all prime ideals $\mathfrak{p}$ of E unramified in L. (See [24, Section 4].) So, all the singularities of $F_C(s)$ appear at the zeros and the pole of $\zeta_L(s)$.

(iii) The kernel functions which weight prime ideals of small norm very heavily are used. Set

$$k_0(s\,; x, y) = \left(\frac{y^{s-1} - x^{s-1}}{s-1}\right)^2 \quad \text{for } y > x > 1,$$
$$k_1(s) = k_0(s; x, x^2) \quad \text{for } x \geqslant 2,$$

and

$$k_2(s) = k_2(s\,; x) = x^{s^2+s} \quad \text{for } x \geqslant 2.$$

In the case that $\zeta_L(s)$ has a real zero very close to 1 we use the kernel $k_2(s)$. Otherwise we use the kernel $k_1(s)$. The use of the kernel functions is the main innovation of [23].

(iv) For $u > 0$ we denote by $\widehat{k}(u)$ the inverse Mellin transform of the kernel function $k(s)$. Then, for $\Re s > 1$,

$$\begin{aligned} I &= \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} F_C(s)k(s)\,\mathrm{d}s \\ &= \sum_{\mathfrak{p}}\sum_{m=1}^{\infty}\theta(\mathfrak{p}^m)(\log N_{K/\mathbb{Q}}\mathfrak{p})\widehat{k}(N_{K/\mathbb{Q}}\mathfrak{p}^m), \end{aligned}$$

where the outer sum is over all prime ideals of K. An upper bound $\mathcal{E}(\log d_L)$ for

$$\left| I - \sum_{\mathfrak{p}\in P(C)}(\log N_{K/\mathbb{Q}}\mathfrak{p})\widehat{k}(N_{K/\mathbb{Q}}\mathfrak{p})\right| \leqslant \mathcal{E}(\log d_L) \tag{2.2}$$

was estimated in [23, (3.15) and (3.16)].

(v) The integral I is evaluated by contour integration:

$$\begin{aligned} I = \frac{|C|}{|G|}k(1) - \frac{|C|}{|G|}\sum_{\chi}\overline{\chi}(g)\sum_{\rho_\chi}k(\rho_\chi) \\ + \mathcal{O}\left(\frac{|C|}{|G|}n_L k(0) + \frac{|C|}{|G|}k\left(-\frac{1}{2}\right)\log d_L\right), \end{aligned}$$

where ρ_χ runs over the zeros of $L(s,\chi,E)$ in the critical strip. (See [23, Section 3].) So we get

$$\frac{|G|}{|C|}I \geqslant k(1) - \sum_{\rho}|k(\rho)| - c_6\left\{n_L k(0) + k\left(-\frac{1}{2}\right)\log d_L\right\}, \tag{2.3}$$

where ρ runs over the zeros of $\zeta_L(s)$ in the critical strip and c_6 is some constant. Note that $\zeta_L(s) = \prod_\chi L(s,\chi,E)$, where χ runs over the irreducible characters of $H = Gal(L/E)$. From (2.2) and (2.3) it follows that

$$\begin{aligned} \sum_{\mathfrak{p}\in P(C)}(\log N_{K/\mathbb{Q}}\mathfrak{p})\widehat{k}(N_{K/\mathbb{Q}}\mathfrak{p}) \geqslant \frac{|C|}{|G|}k(1) - \frac{|C|}{|G|}\sum_{\rho}|k(\rho)| \\ - c_6\frac{|C|}{|G|}\left\{n_L k(0) + k\left(-\frac{1}{2}\right)\log d_L\right\} - \mathcal{E}(\log d_L). \end{aligned} \tag{2.4}$$

(vi) The sum

$$k(1) - \sum_{\rho}|k(\rho)|$$

is estimated from below. To do this we need to know the location and the density of the zeros of $\zeta_L(s)$. If the possible exceptional zero exists, say β_0, then $k(\beta_0)$ is large. The term $k(1) - |k(\beta_0)|$

must be controlled compared to $\sum_{\rho\neq\beta_0}|k(\rho)|$. We need an enlarged zero-free region which makes possible $\sum_{\rho\neq\beta_0}|k(\rho)|$ to be small. The Deuring–Heilbronn phenomenon guarantees that the other zeros of $\zeta_L(s)$ can not be very close to 1.

(vii) We choose x of the kernel $k(s)$ in terms of d_L so that the right side of (2.4) is positive.

Then Theorem B follows. In the remaining sections of this paper we will make explicit numerically the constants intervening in the zero free regions, the density of zeros, and Deuring–Heilbronn phenomenon of $\zeta_L(s)$, and ultimately A_1.

3. Prime ideals in $P(C)$

In this section we will estimate from above

$$\left|I - \sum_{\mathfrak{p}\in P(C)} (\log N_{K/\mathbb{Q}}\mathfrak{p})\widehat{k}(N_{K/\mathbb{Q}}\mathfrak{p})\right|.$$

We will treat carefully their bounds in [23, Section 3]. We begin by recalling the inverse Mellin transform of the kernel functions. They can be easily computed. For $x \geqslant 2$ and $u > 0$ we have

$$\begin{aligned}\widehat{k_1}(u) &= \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\left\{\frac{x^{2(s-1)} - x^{s-1}}{s-1}\right\}^2 u^{-s}\,\mathrm{d}s\\ &= \begin{cases} u^{-1}\log\frac{x^4}{u} & \text{if } x^3 \leqslant u \leqslant x^4,\\ u^{-1}\log\frac{u}{x^2} & \text{if } x^2 \leqslant u \leqslant x^3,\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

and

$$\widehat{k_2}(u) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} x^{s^2+s}u^{-s}\,\mathrm{d}s = (4\pi\log x)^{-\frac{1}{2}}\exp\left\{-\frac{\left(\log\frac{u}{x}\right)^2}{4\log x}\right\},$$

where $a > -\frac{1}{2}$.

Lemma 3.1. — *Let $\sum^{\mathcal{R}}$ denote summation over the prime ideals $\mathfrak{p}$ of K that ramify in L. For $x \geqslant 2$ we have then*

$$(1)\quad \sum\nolimits^{\mathcal{R}}\sum_{m=1}^{\infty}\theta(\mathfrak{p}^m)(\log N_{K/\mathbb{Q}}\mathfrak{p})\widehat{k_1}(N_{K/\mathbb{Q}}\mathfrak{p}^m) \leqslant \frac{2\log x}{x^2}\log d_L;$$

(2) $$\sum^{\mathcal{R}} \sum_{\substack{m\geqslant 1\\ N_{K/\mathbb{Q}}\mathfrak{p}^m\leqslant x^5}} \theta(\mathfrak{p}^m)(\log N_{K/\mathbb{Q}}\mathfrak{p})\widehat{k_2}(N_{K/\mathbb{Q}}\mathfrak{p}^m) \leqslant \frac{5}{2\sqrt{\pi}\log 3}(\log x)^{\frac{1}{2}}\log d_L.$$

Proof.

(1). — Let $\mathfrak{p}$ be a prime ideal of K that is ramified in L. Note that $N_{K/\mathbb{Q}}\mathfrak{p}\geqslant 2$ and $\sum^{\mathcal{R}}\log N_{K/\mathbb{Q}}\mathfrak{p}\leqslant \log d_L$. We have

$$\begin{aligned}\sum^{\mathcal{R}}\sum_{m=1}^{\infty}\theta(\mathfrak{p}^m)(\log N_{K/\mathbb{Q}}\mathfrak{p})\widehat{k_1}(N_{K/\mathbb{Q}}\mathfrak{p}^m)&\\ \leqslant \log x\sum^{\mathcal{R}}\log N_{K/\mathbb{Q}}\mathfrak{p}&\sum_{\substack{m\geqslant 1\\ N_{K/\mathbb{Q}}\mathfrak{p}^m\geqslant x^2}}(N_{K/\mathbb{Q}}\mathfrak{p}^m)^{-1}\\ \leqslant \log x\sum^{\mathcal{R}}&\frac{\log N_{K/\mathbb{Q}}\mathfrak{p}}{N_{K/\mathbb{Q}}\mathfrak{p}^{m_{\mathfrak{p}}}}\left(\frac{1}{1-N_{K/\mathbb{Q}}\mathfrak{p}^{-1}}\right)\\ \leqslant \frac{2\log x}{x^2}&\log d_L,\end{aligned}$$

where $m_{\mathfrak{p}}=\left\lceil\frac{\log(x^2)}{\log N_{K/\mathbb{Q}}\mathfrak{p}}\right\rceil$.

(2). — Let $N_{\mathcal{R}}$ be the number prime ideals of K that are ramified in L/K. Note that $d_L\geqslant 3^{N_{\mathcal{R}}}$. (See [46, Chapters III and IV]).) We have

$$\begin{aligned}\sum^{\mathcal{R}}\sum_{\substack{m\geqslant 1\\ N_{K/\mathbb{Q}}\mathfrak{p}^m\leqslant x^5}}\theta(\mathfrak{p}^m)(\log N_{K/\mathbb{Q}}\mathfrak{p})\widehat{k_2}(N_{K/\mathbb{Q}}\mathfrak{p}^m)&\\ \leqslant (4\pi\log x)^{-\frac{1}{2}}\sum^{\mathcal{R}}\log N_{K/\mathbb{Q}}\mathfrak{p}&\sum_{\substack{m\geqslant 1\\ N_{K/\mathbb{Q}}\mathfrak{p}^m\leqslant x^5}}1\\ \leqslant (4\pi\log x)^{-\frac{1}{2}}\sum^{\mathcal{R}}&5\log x\\ \leqslant \frac{5}{2\sqrt{\pi}\log 3}&(\log x)^{\frac{1}{2}}\log d_L.\end{aligned}$$ □

LEMMA 3.2.

(1) *(Rosser and Schoenfeld* [44]*) For $x>1$,*

$$\pi(x)<\alpha_0\frac{x}{\log x}$$

with $\alpha_0=1.25506$, where $\pi(x)$ is the number of primes p with $p\leqslant x$.

(2) *For $x > 1$,*

$$S(x) \leqslant \frac{2\alpha_0}{\log 2}\sqrt{x},$$

where $S(x)$ is the number of prime powers p^h with $h \geqslant 2$ and $p^h \leqslant x$.

(3) *For $x \geqslant 101$*

$$\sum_{\substack{p \text{ prime} \\ p^h \geqslant x^2, h \geqslant 2}} p^{-h} \leqslant \frac{4.02\alpha_0}{x \log x}.$$

Proof.

(1). — See [44, Corollary 1].

(2). — We have

$$S(x) \leqslant \pi\left(\sqrt{x}\right)\frac{\log x}{\log 2} \leqslant \frac{2\alpha_0}{\log 2}\sqrt{x}$$

by (1).

(3). — We have

$$\sum_{\substack{p \text{ prime} \\ p^h \geqslant x^2, h \geqslant 2}} p^{-h} = \sum_{p \text{ prime}} \frac{p^{-h_p}}{1 - p^{-1}},$$

where $h_p = \max\left(\left\lceil \frac{\log(x^2)}{\log p} \right\rceil, 2\right)$ for each prime p. We observe that

$$\sum_{p \leqslant x} \frac{p^{-h_p}}{1 - p^{-1}} \leqslant \frac{2}{x^2}\pi(x) \leqslant \frac{2\alpha_0}{x \log x}.$$

For $x \geqslant 101$

$$\sum_{p > x} \frac{p^{-h_p}}{1 - p^{-1}} \leqslant \sum_{p > x} \frac{p^{-2}}{1 - p^{-1}} \leqslant \frac{x}{x-1}\sum_{p > x} p^{-2} \leqslant 1.01 \sum_{p > x} p^{-2}.$$

By using the integration by parts and (1) we estimate $\sum_{p>x} p^{-2}$ from above. Namely,

$$\begin{aligned} \sum_{p > x} p^{-2} &\leqslant \int_x^\infty \frac{1}{t^2} \mathrm{d}\pi(t) \leqslant \int_x^\infty \frac{2\pi(t)}{t^3} \mathrm{d}t \\ &\leqslant \int_x^\infty \frac{2\alpha_0}{t^2 \log t} \mathrm{d}t \leqslant \frac{2\alpha_0}{\log x} \int_x^\infty \frac{\mathrm{d}t}{t^2} = \frac{2\alpha_0}{x \log x}. \end{aligned}$$

Hence,

$$\sum_{p \text{ prime}} \frac{p^{-h_p}}{1 - p^{-1}} \leqslant \frac{4.02\alpha_0}{x \log x},$$

which yields (3). □

LEMMA 3.3. — *For $y \leqslant \infty$, let $\sum_y^{\mathcal{P}}$ denote summation over those $(\mathfrak{p}, m)$ for which $N_{K/\mathbb{Q}}\mathfrak{p}^m$ is not a rational prime and $N_{K/\mathbb{Q}}\mathfrak{p}^m \leqslant y$. Then*

(1) *for $x \geqslant 101$*

$$\sum\nolimits_{\infty}^{\mathcal{P}} \theta(\mathfrak{p}^m)(\log N_{K/\mathbb{Q}}\mathfrak{p})\widehat{k_1}(N_{K/\mathbb{Q}}\mathfrak{p}^m) \leqslant 16.08\alpha_0 n_K \frac{\log x}{x};$$

(2) *for $x \geqslant 10^{10}$*

$$\sum\nolimits_{x^5}^{\mathcal{P}} \theta(\mathfrak{p}^m)(\log N_{K/\mathbb{Q}}\mathfrak{p})\widehat{k_2}(N_{K/\mathbb{Q}}\mathfrak{p}^m) \leqslant \alpha_1 n_K x^{\frac{3}{4}} (\log x)^{\frac{3}{2}}$$

with

$$\alpha_1 = \frac{\alpha_0}{3\sqrt{\pi}\log 2}\left(\frac{15}{10^{\frac{47}{2}}\log 10} + 7 + \frac{37}{10^{\frac{5}{2}}}\right) = 2.4234\cdots.$$

Proof.

(1). — Since for a positive integer q there are at most n_K distinct prime power ideals $\mathfrak{p}^m$ with $N_{K/\mathbb{Q}}\mathfrak{p}^m = q$, it follows that

$$\begin{aligned}\sum\nolimits_{\infty}^{\mathcal{P}} \theta(\mathfrak{p}^m)(\log N_{K/\mathbb{Q}}\mathfrak{p})\widehat{k_1}(N_{K/\mathbb{Q}}\mathfrak{p}^m) &\leqslant \log x \sum\nolimits_{\infty}^{\mathcal{P}} (\log N_{K/\mathbb{Q}}\mathfrak{p})(N_{K/\mathbb{Q}}\mathfrak{p}^m)^{-1} \\ &\leqslant 4(\log x)^2 n_K \sum_{\substack{p \text{ prime} \\ x^2 \leqslant p^h \leqslant x^4, h \geqslant 2}} p^{-h}.\end{aligned}$$

Hence, by Lemma 3.2 (3) we obtain (1).

(2). — We have

$$\begin{aligned}\sum\nolimits_{x^5}^{\mathcal{P}} \theta(\mathfrak{p}^m)(\log N_{K/\mathbb{Q}}\mathfrak{p})\widehat{k_2}(N_{K/\mathbb{Q}}\mathfrak{p}^m) &\leqslant n_K \sum_{\substack{p \text{ prime} \\ p^2 \leqslant p^h \leqslant x^5}} (\log p^h)\widehat{k_2}(p^h) \\ &\leqslant n_K \int_4^{x^5} (\log u)\, \widehat{k_2}(u) \mathrm{d}S(u),\end{aligned}$$

where $S(u)$ is as Lemma 3.2 (2). According to Lemma 3.2 (2), we have

$$S(u) \leqslant \frac{2\alpha_0}{\log 2}\sqrt{u}.$$

Hence,

$$\begin{aligned}
&\int_4^{x^5} (\log u)\, \widehat{k_2}(u)\, \mathrm{d}S(u) \\
&\quad \leqslant (\log x^5)\widehat{k_2}(x^5)S(x^5) + \int_4^{x^5} \widehat{k_2}(u) \left(\frac{\log u \log \frac{u}{x}}{2\log x} - 1 \right) S(u)\, \frac{\mathrm{d}u}{u} \\
&\quad \leqslant \frac{5\alpha_0}{\sqrt{\pi}\log 2} x^{-\frac{3}{2}} (\log x)^{\frac{1}{2}} + \int_{\log\frac{4}{x}}^{4\log x} \widehat{k_2}(xe^t) \left\{ \frac{(t+\log x)t}{2\log x} \right\} S(xe^t)\, \mathrm{d}t \\
&\quad \leqslant \frac{\alpha_0}{3\sqrt{\pi}\log 2} \left(\frac{15}{x^{\frac{9}{4}}\log x} + 7 + \frac{37}{x^{\frac{1}{4}}} \right) x^{\frac{3}{4}} (\log x)^{\frac{3}{2}}. \qquad \square
\end{aligned}$$

Lemma 3.4. — *For $x \geqslant 2$, we have*

$$\sum_{\mathfrak{p}} \sum_{\substack{m\geqslant 1 \\ N_{K/\mathbb{Q}}\mathfrak{p}^m > x^5}} \theta(\mathfrak{p}^m)(\log N_{K/\mathbb{Q}}\mathfrak{p})\widehat{k_2}(N_{K/\mathbb{Q}}\mathfrak{p}^m) \leqslant \alpha_2 n_K x\, (\log x)^{\frac{1}{2}}$$

with $\alpha_2 = \frac{5}{\sqrt{\pi}}$.

Proof. — We have

$$\begin{aligned}
\sum_{\mathfrak{p}} \sum_{\substack{m\geqslant 1 \\ N_{K/\mathbb{Q}}\mathfrak{p}^m > x^5}} \theta(\mathfrak{p}^m)(\log N_{K/\mathbb{Q}}\mathfrak{p})\widehat{k_2}(N_{K/\mathbb{Q}}\mathfrak{p}^m) &\leqslant n_K \sum_{\substack{p \text{ prime} \\ p^h > x^5}} (\log p^h)\widehat{k_2}(p^h) \\
&\leqslant n_K \int_{x^5}^{\infty} (\log u)\, \widehat{k_2}(u)\, \mathrm{d}T(u),
\end{aligned}$$

where $T(u)$ is the number of prime powers p^h with $h \geqslant 1$ and $p^h \leqslant u$. Since $T(u) \leqslant u$ for $u > 0$, we have

$$\begin{aligned}
\int_{x^5}^{\infty} (\log u)\, \widehat{k_2}(u)\, \mathrm{d}T(u) &\leqslant \int_{x^5}^{\infty} \widehat{k_2}(u) \left(\frac{\log u \log \frac{u}{x}}{2\log x} - 1 \right) T(u)\, \frac{\mathrm{d}u}{u} \\
&\leqslant \int_{4\log x}^{\infty} \widehat{k_2}(xe^t) \left\{ \frac{(t+\log x)t}{2\log x} - 1 \right\} T(xe^t)\, \mathrm{d}t \\
&\leqslant \alpha_2 x (\log x)^{\frac{1}{2}}. \qquad \square
\end{aligned}$$

From Lemmas 3.1, 3.3, and 3.4 we deduce an upper bound for

$$\left| I_j - \sum_{\mathfrak{p}\in P(C)} (\log N_{K/\mathbb{Q}}\mathfrak{p})\widehat{k_j}(N_{K/\mathbb{Q}}\mathfrak{p}) \right|$$

for $j = 1, 2$ as follows.

Proposition 3.5. — *Let $k_j(s)$ be as above. Let*

$$I_j = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} F_C(s)k_j(s)\,\mathrm{d}s.$$

Assume that $L \neq \mathbb{Q}$. Then

(1) *for $x \geqslant 101$*

$$\left| I_1 - \sum_{\mathfrak{p}\in P(C)} (\log N_{K/\mathbb{Q}}\mathfrak{p})\widehat{k_1}(N_{K/\mathbb{Q}}\mathfrak{p})\right| \leqslant \frac{2\log x}{x^2}\log d_L + 16.08\alpha_0 n_K \frac{\log x}{x} \leqslant \alpha_3 \frac{\log x}{x}\log d_L \tag{3.1}$$

with

$$\alpha_3 = \frac{2}{101} + \frac{32.16\alpha_0}{\log 3} = 36.759\cdots;$$

(2) *for $x \geqslant 10^{10}$*

$$\left| I_2 - \sum_{\substack{\mathfrak{p}\in P(C)\\ N_{K/\mathbb{Q}}\mathfrak{p}\leqslant x^5}} (\log N_{K/\mathbb{Q}}\mathfrak{p})\widehat{k_2}(N_{K/\mathbb{Q}}\mathfrak{p})\right| \tag{3.2}$$

$$\leqslant \frac{5}{2\sqrt{\pi}\log 3}(\log x)^{\frac12}\log d_L + \alpha_1 n_K x^{\frac34}(\log x)^{\frac32} + \alpha_2 n_K x\,(\log x)^{\frac12}$$
$$\leqslant \alpha_4 x(\log x)^{\frac12}\log d_L$$

with

$$\alpha_4 = \frac{1}{\log 3}\left(\frac{10^{-9}}{4\sqrt{\pi}} + \frac{\alpha_1 \log 10}{5\sqrt{10}} + 2\alpha_2\right) = 5.4567\cdots.$$

Note that $d_L \geqslant 3^{n_L/2}$ for $n_L \geqslant 2$. It follows from the Hermite–Minkowski's inequality $d_L > \frac{\pi}{3}\left(\frac{3\pi}{4}\right)^{n_L-1}$ for $n_L > 1$. For $n_L = 2$, $d_L \geqslant 3$, and for $n_L \geqslant 3$, $\frac{\pi}{3}\left(\frac{3\pi}{4}\right)^{n_L-1} = \frac{4}{9}\left(\frac{3\pi}{4}\right)^{n_L} > 3^{n_L/2}$. (See also [48, p. 140] and [23, p. 291].)

4. The Contour integral

In this section we will evaluate the integrals I_1 and I_2 by contour integration. We will use $L(s,\chi)$ to denote $L(s,\chi,E)$. Let $\mathcal{F}(\chi)$ be the conductor of χ and $A(\chi) = d_E N_{E/\mathbb{Q}} \mathcal{F}(\chi)$. Let

$$\delta(\chi) = \begin{cases} 1 & \text{if } \chi \text{ is the principal character,} \\ 0 & \text{otherwise.} \end{cases}$$

We recall that for each χ there exist non-negative integers $a(\chi)$, $b(\chi)$ such that

$$a(\chi) + b(\chi) = [E : \mathbb{Q}] = n_E,$$

and such that if we define

$$\gamma_\chi(s) = \left\{\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)\right\}^{a(\chi)} \left\{\pi^{-\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right)\right\}^{b(\chi)}$$

and

$$\xi(s,\chi) = \{s(s-1)\}^{\delta(\chi)} A(\chi)^{s/2} \gamma_\chi(s) L(s,\chi),$$

then $\xi(s,\chi)$ satisfies the functional equation

$$\xi(1-s, \overline{\chi}) = W(\chi)\xi(s,\chi),$$

where $W(\chi)$ is a certain constant of absolute value 1. Furthermore, $\xi(s,\chi)$ is an entire function of order 1 and does not vanish at $s = 0$. By Hadamard product theorem we have for every $s \in \mathbb{C}$

$$-\frac{L'}{L}(s,\chi) = \frac{1}{2}\log A(\chi) + \delta(\chi)\left(\frac{1}{s} + \frac{1}{s-1}\right) + \frac{\gamma_\chi'}{\gamma_\chi}(s) - \mathcal{B}(\chi) - \sum_{\rho_\chi \in Z(\chi)} \left(\frac{1}{s-\rho_\chi} + \frac{1}{\rho_\chi}\right),$$

where $\mathcal{B}(\chi)$ is some constant and $Z(\chi)$ denotes the set of nontrivial zeros of $L(s,\chi)$. (See [48] and [24].) According to [40, (2.8)]

$$\Re\mathcal{B}(\chi) = -\sum_{\rho_\chi \in Z(\chi)} \Re\frac{1}{\rho_\chi}.$$

Hence, for every $s \in \mathbb{C}$

$$\Re\left\{-\frac{L'}{L}(s,\chi)\right\} = \frac{1}{2}\log A(\chi) + \delta(\chi)\Re\left(\frac{1}{s} + \frac{1}{s-1}\right) + \Re\frac{\gamma_\chi'}{\gamma_\chi}(s) - \sum_{\rho_\chi \in Z(\chi)} \Re\frac{1}{s-\rho_\chi}. \tag{4.1}$$

For $j = 1$, 2 we have

$$I_j = \frac{|C|}{|G|}\sum_\chi \overline{\chi}(g) J_j(\chi) \quad \text{by (2.1)},$$

where

$$J_j(\chi) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} -\frac{L'}{L}(s,\chi)k_j(s)\,\mathrm{d}s.$$

Assume that $T \geqslant 2$ does not equal the ordinate of any of the zeros of $L(s,\chi)$. Consider

$$J_j(\chi,T) = \frac{1}{2\pi i}\int_{B(T)} -\frac{L'}{L}(s,\chi)k_j(s)\,\mathrm{d}s$$

for $j = 1$, 2, where $B(T)$ is the positively oriented rectangle with vertices $2 - iT$, $2 + iT$, $-\frac{1}{2} + iT$, and $-\frac{1}{2} - iT$. By Cauchy's theorem

$$J_j(\chi,T) = \delta(\chi)k_j(1) - \{a(\chi) - \delta(\chi)\}\,k_j(0) - \sum_{\substack{\rho_\chi\in Z(\chi)\\ |\Im\rho_\chi|<T}} k_j(\rho_\chi) \tag{4.2}$$

for $j = 1$, 2.

Lemma 4.1. — *Let*

$$V_j(\chi) = \frac{1}{2\pi i}\int_{-\frac{1}{2}+i\infty}^{-\frac{1}{2}-i\infty} -\frac{L'}{L}(s,\chi)k_j(s)\,\mathrm{d}s$$

for $j = 1$, 2. Then

(1) *for $x \geqslant 101$*

$$|V_1(\chi)| \leqslant k_1\left(-\frac{1}{2}\right)\{\mu_1\log A(\chi) + n_E\nu_1\},$$

where $\mu_1 = 0.75296\cdots$ and $\nu_1 = 19.405\cdots$;

(2) *for $x \geqslant 10^{10}$*

$$|V_2(\chi)| \leqslant k_2\left(-\frac{1}{2}\right)\{\mu_2\log A(\chi) + n_E\nu_2\},$$

where $\mu_2 = 0.058787\cdots$ and $\nu_2 = 1.4793\cdots$.

Proof. — Let $s = -\frac{1}{2} + it$. By [59, Lemme 5.1]

$$\left|-\frac{L'}{L}\left(-\frac{1}{2}+it,\chi\right)\right| \leqslant \log A(\chi) + n_E v(t),$$

where

$$v(t) = \log\left(\sqrt{\frac{1}{4}+t^2}+2\right) + \frac{19683}{812}.$$

Moreover, for $x \geqslant 101$

$$\left|k_1\left(-\frac{1}{2}+it\right)\right| \leqslant \frac{x^{-3}(1+x^{-\frac{3}{2}})^2}{\frac{9}{4}+t^2}$$
$$= k_1\left(-\frac{1}{2}\right)\left(\frac{1+x^{-\frac{3}{2}}}{1-x^{-\frac{3}{2}}}\right)^2\left(\frac{9}{9+4t^2}\right)$$
$$\leqslant k_1\left(-\frac{1}{2}\right)v_1(t)$$

with $v_1(t) = \left(\frac{1+101^{-\frac{3}{2}}}{1-101^{-\frac{3}{2}}}\right)^2\left(\frac{9}{9+4t^2}\right)$ and for $x \geqslant 10^{10}$

$$\left|k_2\left(-\frac{1}{2}+it\right)\right| = x^{-\frac{1}{4}-t^2} = k_2\left(-\frac{1}{2}\right)x^{-t^2} \leqslant k_2\left(-\frac{1}{2}\right)v_2(t)$$

with $v_2(t) = 10^{-10t^2}$. Hence,

$$\left|\frac{1}{2\pi i}\int_{-\frac{1}{2}+iT}^{-\frac{1}{2}-iT} -\frac{L'}{L}(s,\chi)k_j(s)\,\mathrm{d}s\right|$$
$$\leqslant \frac{1}{\pi}k_j\left(-\frac{1}{2}\right)\int_0^T \{\log A(\chi) + n_E v(t)\}v_j(t)\,\mathrm{d}t.$$

Set

$$\mu_j = \frac{1}{\pi}\int_0^\infty v_j(t)\,\mathrm{d}t \quad \text{and} \quad \nu_j = \frac{1}{\pi}\int_0^\infty v(t)v_j(t)\,\mathrm{d}t.$$

The result follows. □

On the two segments from $2 \pm iT$ to $-\frac{1}{2} \pm iT$ we proceed with the same way as [24, Section 6]. (See [23, Section 3], [59, Section 5], and [27].) Let

$$\mathcal{H}_j(T) = \frac{1}{2\pi i}\int_{-\frac{1}{2}}^{-\frac{1}{4}}\left\{\frac{L'}{L}(\sigma+iT,\chi)k_j(\sigma+iT) - \frac{L'}{L}(\sigma-iT,\chi)k_j(\sigma-iT)\right\}\mathrm{d}\sigma$$

and

$$\mathcal{H}_j^*(T) = \frac{1}{2\pi i}\int_{-\frac{1}{4}}^{2}\left\{\frac{L'}{L}(\sigma+iT,\chi)k_j(\sigma+iT) - \frac{L'}{L}(\sigma-iT,\chi)k_j(\sigma-iT)\right\}\mathrm{d}\sigma.$$

Then

$$\mathcal{H}_j(T) + \mathcal{H}_j^*(T)$$
$$= \frac{1}{2\pi i}\left\{\int_{2+iT}^{-\frac{1}{2}+iT} -\frac{L'}{L}(s,\chi)k_j(s)\,\mathrm{d}s + \int_{-\frac{1}{2}-iT}^{2-iT} -\frac{L'}{L}(s,\chi)k_j(s)\,\mathrm{d}s\right\}.$$

Lemma 4.2. — *For $j = 1, 2$ we have*

$$\mathcal{H}_j(T) \ll |k_j(iT)|(\log A(\chi) + n_E \log T).$$

Proof. — Let $s = \sigma \pm iT$ with $-\frac{1}{2} \leqslant \sigma \leqslant -\frac{1}{4}$. Then

$$\frac{L'}{L}(s,\chi) \ll \log A(\chi) + n_E \log T$$

by [24, Lemma 6.2] and $k_j(s) \ll |k_j(iT)|$. The result follows. □

LEMMA 4.3. — *Let $-\frac{1}{4} \leqslant \sigma \leqslant 2$. Then, we have*

$$\frac{L'}{L}(\sigma \pm iT, \chi) - \sum_{\substack{\rho_\chi \in Z(\chi) \\ |\Im \rho_\chi \mp T| \leqslant 1}} \frac{1}{\sigma \pm iT - \rho_\chi} \ll \log A(\chi) + n_E \log T.$$

Proof. — See [24, Lemma 5.6]. (See also [59, Lemma 4.8].) □

Therefore, for $j = 1$, 2

$$\mathcal{H}_j^*(T) - \frac{1}{2\pi i} \int_{-\frac{1}{4}}^{2} \left\{ k_j(\sigma + iT) \sum_{\substack{\rho_\chi \in Z(\chi) \\ |\Im \rho_\chi - T| \leqslant 1}} \frac{1}{\sigma + iT - \rho_\chi} - k_j(\sigma - iT) \sum_{\substack{\rho_\chi \in Z(\chi) \\ |\Im \rho_\chi + T| \leqslant 1}} \frac{1}{\sigma - iT - \rho_\chi} \right\} \mathrm{d}\sigma$$
$$\ll |k_j(iT)|(\log A(\chi) + n_E \log T)$$

since $k_j(\sigma \pm iT) \ll |k_j(iT)|$ for $-\frac{1}{4} \leqslant \sigma \leqslant 2$.

LEMMA 4.4. — *Let $\rho_\chi \in Z(\chi)$ with $t \neq \Im \rho_\chi$. If $|t| \geqslant 2$, then*

$$\int_{-\frac{1}{4}}^{2} \frac{k_j(\sigma + it)}{\sigma + it - \rho_\chi} \,\mathrm{d}\sigma \ll |k_j(it)|$$

for $j = 1, 2$.

Proof. — Suppose first that $\Im \rho_\chi > t$. Let B_t be the positive oriented rectangle with vertices $2 + i(t-1)$, $2 + it$, $-\frac{1}{4} + it$, and $-\frac{1}{4} + i(t-1)$. By Cauchy's theorem,

$$\int_{B_t} \frac{k_j(s)}{s - \rho_\chi} \,\mathrm{d}s = 0$$

for $j = 1, 2$. However, on the three sides of the rectangle other than the segment from $-\frac{1}{4} + it$ to $2 + it$, the integrand is majorized by

$$\alpha_5 |k_j(it)|$$

for some positive constant α_5 depending on x, which proves the result for $\Im \rho_\chi > t$. A similar proof for $\Im \rho_\chi < t$ uses the rectangle with vertices $2 + it$, $2 + i(t+1)$, $-\frac{1}{4} + i(t+1)$, and $-\frac{1}{4} + it$. □

For $j = 1, 2$ we have

$$\frac{1}{2\pi i}\int_{-\frac{1}{4}}^{2}\left\{k_j(\sigma+iT)\sum_{\substack{\rho_\chi\in Z(\chi)\\ |\Im\rho_\chi - T|\leqslant 1}}\frac{1}{\sigma+iT-\rho_\chi}\right.$$
$$\left. - k_j(\sigma-iT)\sum_{\substack{\rho_\chi\in Z(\chi)\\ |\Im\rho_\chi + T|\leqslant 1}}\frac{1}{\sigma-iT-\rho_\chi}\right\}\mathrm{d}\sigma$$
$$\ll |k_j(iT)|\{n_\chi(T)+n_\chi(-T)\}$$
$$\ll |k_j(iT)|(\log A(\chi)+n_E\log T) \text{ by [24, Lemma 5.4]},$$

where $n_\chi(T)$ denotes the number of zeros $\rho_\chi \in Z(\chi)$ with $|\Im\rho_\chi - T| \leqslant 1$. We may then conclude as follows.

Lemma 4.5. — *For $j = 1, 2$ we have*

$$\mathcal{H}_j^*(T) \ll |k_j(iT)|(\log A(\chi) + n_E \log T).$$

Lemma 4.6. — *For $j = 1, 2$ we have*

$$\lim_{T\to\infty}\frac{1}{2\pi i}\left\{\int_{2+iT}^{-\frac{1}{2}+iT} -\frac{L'}{L}(s,\chi)k_j(s)\,\mathrm{d}s + \int_{-\frac{1}{2}-iT}^{2-iT} -\frac{L'}{L}(s,\chi)k_j(s)\,\mathrm{d}s\right\} = 0.$$

Proof. — By Lemmas 4.2 and 4.5

$$\mathcal{H}_j(T) + \mathcal{H}_j^*(T) \ll |k_j(iT)|\{\log A(\chi) + n_E \log T\}.$$

Since

$$|k_j(iT)| \leqslant \begin{cases} \frac{9}{4x^2(1+T^2)} & \text{if } j = 1,\\ x^{-T^2} & \text{if } j = 2,\end{cases}$$

the result follows. □

Letting $T \to \infty$ in (4.2) and combining this and Lemmas 4.6 yield

$$J_j(\chi) + V_j(\chi) = \delta(\chi)k_j(1) - \{a(\chi) - \delta(\chi)\}\,k_j(0) - \sum_{\rho_\chi\in Z(\chi)} k_j(\rho_\chi)$$

for $j = 1, 2$. Hence, we have

$$\begin{aligned}\frac{|G|}{|C|} I_j &= \sum_{\chi} \overline{\chi}(g) J_j(\chi) \\ &= k_j(1) - k_j(0) \sum_{\chi} \overline{\chi}(g) \{a(\chi) - \delta(\chi)\} - \sum_{\chi} \overline{\chi}(g) \left(\sum_{\rho_\chi \in Z(\chi)} k_j(\rho_\chi) \right) \\ &\qquad - \sum_{\chi} \overline{\chi}(g) V_j(\chi)\end{aligned}$$

for $j = 1, 2$. Note that by the conductor-discriminant formula ([46, Chapter VI, Section 3])

$$\sum_{\chi} \log A(\chi) = \log d_L.$$

We therefore conclude as follows.

Proposition 4.7. — *For $j = 1, 2$ we have*

$$\begin{aligned}\text{(4.3)} \quad \frac{|G|}{|C|} I_j \geqslant k_j(1) - \sum_{\rho \in Z(\zeta_L)} |k_j(\rho)| - \mu_j k_j \left(-\frac{1}{2}\right) \log d_L \\ - n_L \left\{ k_j(0) + \nu_j k_j \left(-\frac{1}{2}\right) \right\}\end{aligned}$$

where $Z(\zeta_L)$ denotes the set of all nontrivial zeros of $\zeta_L(s)$, μ_j and ν_j are as in Lemma 4.1.

5. Density of zeros of Dedekind zeta functions

To begin with, we recall that for every $s \in \mathbb{C}$ we have

$$\begin{aligned}\text{(5.1)} \quad \Re\left\{ -\frac{\zeta_L'}{\zeta_L}(s) \right\} = \frac{1}{2} \log d_L + \Re\left(\frac{1}{s} + \frac{1}{s-1} \right) \\ + \Re \frac{\gamma_L'}{\gamma_L}(s) - \sum_{\rho \in Z(\zeta_L)} \Re \frac{1}{s-\rho},\end{aligned}$$

where

$$\gamma_L(s) = \left\{ \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \right\}^{r_1 + r_2} \left\{ \pi^{-\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right) \right\}^{r_2},$$

r_1 and $2r_2$ are the numbers of real and complex embeddings of L. (See [24, Lemma 5.1] or [48].)

For any real number t we let

$$n_L(t) = |\{\rho = \beta + i\gamma \mid \zeta_L(\rho) = 0 \text{ with } 0 < \beta < 1 \text{ and } |\gamma - t| \leqslant 1\}|.$$

For any complex number s and positive real number $r > 0$ we let

$$n(r;s) = |\{\rho \in Z(\zeta_L) \mid |\rho - s| \leqslant r\}|.$$

From (4.1) Lagarias and Odlyzko deduced that

$$n_\chi(t) \ll \log A(\chi) + n_E \log(|t| + 2)$$

for all t. (See [24, Lemma 5.4].) In this section we will bound $n_L(t)$ and $n(r;s)$ from above using (4.1). To do this we need some lemmas.

Lemma 5.1. — *Let $s = \sigma + it$ with $\sigma > 1$. We have*

$$\sum_{\rho \in Z(\zeta_L)} \Re \frac{1}{s-\rho} \geqslant f_0(\sigma) n_L(t),$$

where

$$f_0(\sigma) = \frac{1}{2} \min\left\{\frac{\sigma - 1}{(\sigma-1)^2+1}, \frac{\sigma - \frac{1}{2}}{\left(\sigma - \frac{1}{2}\right)^2 + 1}\right\} + \frac{1}{2} \min\left\{\frac{\sigma - \frac{1}{2}}{\left(\sigma - \frac{1}{2}\right)^2 + 1}, \frac{\sigma}{\sigma^2+1}\right\}.$$

Proof. — We have

$$\sum_{\rho \in Z(\zeta_L)} \Re \frac{1}{s-\rho} \geqslant \frac{1}{2} \sum_{\substack{\beta + i\gamma \in Z(\zeta_L) \\ |t-\gamma| \leqslant 1}} \left\{\frac{\sigma - \beta}{(\sigma-\beta)^2+1} + \frac{\sigma + \beta - 1}{(\sigma+\beta-1)^2+1}\right\}$$
$$\geqslant f_0(\sigma) n_L(t).$$

□

Lemma 5.2. — *If $\Re s = \sigma > 1$, then*

$$\Re \frac{\zeta_L'}{\zeta_L}(s) \leqslant n_L f_1(\sigma),$$

where

$$f_1(\sigma) = -\frac{\zeta_{\mathbb{Q}}'}{\zeta_{\mathbb{Q}}}(\sigma).$$

Proof. — For $\Re s > 1$,

$$-\frac{\zeta_L'}{\zeta_L}(s) = \sum_{\mathfrak{P}} \frac{\log N\mathfrak{P}}{N\mathfrak{P}^s - 1} = \sum_{\mathfrak{P}} \log N\mathfrak{P} \sum_{m=1}^{\infty} N\mathfrak{P}^{-ms},$$

where $\mathfrak{P}$ runs over all prime ideals of L. Comparing $-\frac{\zeta_L'}{\zeta_L}(\sigma)$ with $-\frac{\zeta_{\mathbb{Q}}'}{\zeta_{\mathbb{Q}}}(\sigma)$ yields

$$\Re \frac{\zeta_L'}{\zeta_L}(s) \leqslant \left|-\frac{\zeta_L'}{\zeta_L}(s)\right| \leqslant -\frac{\zeta_L'}{\zeta_L}(\sigma) \leqslant n_L \left\{-\frac{\zeta_{\mathbb{Q}}'}{\zeta_{\mathbb{Q}}}(\sigma)\right\}.$$

(See [24, Lemma 3.2].) □

See also [9], [31, Lemma (*a*)], [59, Lemma 3.2], [11, p. 184], and [33, Proposition 2].

Lemma 5.3. — *Assume that* $\Re s > \frac{1}{2}$. *We have*

(1) $\Re \frac{\Gamma'}{\Gamma}(s) \leqslant \log|s| + \frac{1}{3} \leqslant \alpha_6 \log(|s|+2)$
with $\alpha_6 = 1.08$;

(2) $\Re \frac{\Gamma'}{\Gamma}(s) \geqslant \log|s| - \frac{4}{3} \geqslant \log(|s|+2) - \alpha_7$
with $\alpha_7 = \frac{4}{3} + \log 5 = 2.9427\cdots$.

Proof. — For $\Re s > 0$,

$$\frac{\Gamma'}{\Gamma}(s) = \log s - \frac{1}{2s} - 2\int_0^\infty \frac{\upsilon}{(s^2+\upsilon^2)(e^{2\pi\upsilon}-1)}\,\mathrm{d}\upsilon.$$

(See [58, p. 251].) Since $|s^2+\upsilon^2| \geqslant (\Re s)^2$, we have

$$\left|\int_0^\infty \frac{\upsilon}{(s^2+\upsilon^2)(e^{2\pi\upsilon}-1)}\,\mathrm{d}\upsilon\right| \leqslant \frac{1}{(\Re s)^2}\int_0^\infty \frac{\upsilon}{e^{2\pi\upsilon}-1}\,\mathrm{d}\upsilon = \frac{1}{24(\Re s)^2}.$$

If $\Re s > \frac{1}{2}$, then

$$\Re \frac{\Gamma'}{\Gamma}(s) \leqslant \log|s| + \frac{1}{12}\frac{1}{(\Re s)^2} \leqslant \log|s| + \frac{1}{3}$$

and

$$\Re \frac{\Gamma'}{\Gamma}(s) \geqslant \log|s| - \frac{1}{2|s|} - \frac{1}{12}\frac{1}{(\Re s)^2} \geqslant \log|s| - \frac{4}{3}.$$

Set $\varphi_1(\upsilon) = \alpha_6\log(\upsilon+2) - \log\upsilon - \frac{1}{3}$ for $\upsilon > \frac{1}{2}$. Then,

$$\varphi_1'(\upsilon) = \frac{(\alpha_6-1)\upsilon - 2}{\upsilon(\upsilon+2)} \text{ and } \varphi_1(\upsilon) > \varphi_1\left(\frac{2}{\alpha_6-1}\right) > 0.$$

Hence

$$\Re \frac{\Gamma'}{\Gamma}(s) \leqslant \alpha_6 \log(|s|+2).$$

Set $\varphi_2(\upsilon) = \log\upsilon - \frac{4}{3} - \log(\upsilon+2) + \alpha_7$ for $\upsilon > \frac{1}{2}$. Then

$$\varphi_2'(\upsilon) > 0 \text{ and } \varphi_2(\upsilon) > \varphi_2\left(\frac{1}{2}\right) = 0.$$

Hence

$$\Re \frac{\Gamma'}{\Gamma}(s) \geqslant \log(|s|+2) - \alpha_7. \qquad \square$$

Lemma 5.4. — *Let $s=\sigma+it$. If $\sigma>1$, then*

$$\Re\frac{\gamma_L'}{\gamma_L}(s)\leqslant n_L\left\{f_2(\sigma)\log(|t|+2)-\frac{1}{2}\log\pi\right\},$$

where

$$f_2(\sigma)=\frac{\alpha_6}{2}\left\{\frac{\log(\sigma+5)}{\log 2}-1\right\}.$$

Proof. — By definition and Lemma 5.3 (1) we have

$$\begin{aligned}\Re\frac{\gamma_L'}{\gamma_L}(s)&=\frac{(r_1+r_2)}{2}\Re\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right)+\frac{r_2}{2}\Re\frac{\Gamma'}{\Gamma}\left(\frac{s+1}{2}\right)-\frac{n_L}{2}\log\pi\\&\leqslant\alpha_6\frac{(r_1+r_2)}{2}\log\left(\frac{|s|}{2}+2\right)+\alpha_6\frac{r_2}{2}\log\left(\frac{|s+1|}{2}+2\right)-\frac{n_L}{2}\log\pi\\&\leqslant\frac{n_L}{2}\left\{\alpha_6\log\left(\frac{|s+1|}{2}+2\right)-\log\pi\right\}.\end{aligned}$$

It is sufficient to verify that

$$\log\left(\frac{|s+1|}{2}+2\right)\leqslant\left(\frac{\log(\sigma+5)}{\log 2}-1\right)\log(|t|+2).\tag{5.2}$$

Note that $|s+1|\geqslant 2|t|$ if and only if $|t|\leqslant(\sigma+1)/\sqrt{3}$. If $|t|\geqslant(\sigma+1)/\sqrt{3}$, then (5.2) holds. We suppose now that $|t|<(\sigma+1)/\sqrt{3}$. Set $\varphi_3(v)=\varphi_5(v)/\varphi_4(v)$ with $\varphi_4(v)=v+2$ and $\varphi_5(v)=2+\sqrt{(\sigma+1)^2+v^2}/2$. Then $\varphi_3'(v)\leqslant 0$ and $\varphi_5(v)\leqslant\left(\frac{\varphi_5(0)}{\varphi_4(0)}\right)\varphi_4(v)$ for $0\leqslant v<(\sigma+1)/\sqrt{3}$. For $0\leqslant v<(\sigma+1)/\sqrt{3}$ we have then

$$\begin{aligned}\frac{\log\varphi_5(v)}{\log\varphi_4(v)}&\leqslant\frac{\log\varphi_4(v)+\log\varphi_5(0)-\log\varphi_4(0)}{\log\varphi_4(v)}\\&\leqslant\frac{\log\varphi_5(0)}{\log\varphi_4(0)}=\frac{\log(\sigma+5)}{\log 2}-1,\end{aligned}\tag{5.3}$$

which yields (5.2). □

We are now ready to bound $n_L(t)$.

Proposition 5.5. — *For all t we have*

$$n_L(t)\leqslant 1.1\log d_L+2.09\log\left\{(|t|+2)^{n_L}\right\}+0.56n_L+4.05.\tag{5.4}$$

In particular, if $L\neq\mathbb{Q}$, then

$$n_L(t)\leqslant 2.72\log\left\{d_L(|t|+2)^{n_L}\right\}.\tag{5.5}$$

Proof. — Combining (4.1), Lemmas 5.1, 5.2, 5.3, and 5.4 yields

$$f_0(\sigma)n_L(t) \leqslant \frac{1}{2}\log d_L + \frac{1}{\sigma} + \frac{1}{\sigma-1} + n_L\left\{f_2(\sigma)\log(|t|+2) - \frac{1}{2}\log\pi + f_1(\sigma)\right\}$$

for $\sigma > 1$. We write

$$n_L(t) \leqslant a_1(\sigma)\log d_L + a_2(\sigma)\log\{(|t|+2)^{n_L}\} + a_3(\sigma)n_L + a_4(\sigma) \tag{5.6}$$

for $\sigma > 1$, where

$$a_1(\sigma) = \frac{1}{2f_0(\sigma)},\ a_2(\sigma) = \frac{f_2(\sigma)}{f_0(\sigma)},\ a_3(\sigma) = \frac{1}{f_0(\sigma)}\left\{f_1(\sigma) - \frac{1}{2}\log\pi\right\},$$

and

$$a_4(\sigma) = \frac{1}{f_0(\sigma)}\left(\frac{1}{\sigma} + \frac{1}{\sigma-1}\right).$$

We choose now appropriate σ. If $\sigma = (3+\sqrt{17})/4$, then (5.6) yields (5.4). For the proof of (5.5), we choose $\sigma = 2.45$. In this case, $a_3(\sigma) < 0$ and $2a_3(\sigma) + a_4(\sigma) > 0$. Since $n_L \geqslant 2$, it follows from (5.6) that

$$\begin{aligned} n_L(t) &\leqslant a_1(\sigma)\log d_L + a_2(\sigma)\log\{(|t|+2)^{n_L}\} + 2a_3(\sigma) + a_4(\sigma) \\ &\leqslant B_1\log d_L + B_2\log\{(|t|+2)^{n_L}\}, \end{aligned}$$

where $B_1 = a_1(\sigma) + \frac{1}{\log 3}\{2a_3(\sigma) + a_4(\sigma)\} = 2.6885\cdots$ and $B_2 = a_2(\sigma) = 2.7106\cdots$. So, we obtain (5.5). □

See also [21], [52], and [59, Lemme 4.6].

Proposition 5.6. — *Let r be a positive real number.*

(1) *Assume that*

$$n_L(t) \leqslant \alpha_8\log\{d_L(|t|+2)^{n_L}\}$$

for some $\alpha_8 > 0$. Then we have

$$n(r;\sigma+it) \leqslant \alpha_8(1+r)\log\{d_L(|t|+r+2)^{n_L}\}.$$

(2) *Assume that $L \neq \mathbb{Q}$. If $\sigma \geqslant 1$ and $0 < r \leqslant 1$, then*

$$n(r;\sigma+it) \leqslant 10\left[1 + \frac{2f_2(2)}{5}r\log\{d_L(|t|+2)^{n_L}\}\right].$$

Proof. — Set

$$Z(r;s) = \{\rho\in Z(\zeta_L) \mid |\rho-s| \leqslant r\}$$
$$\text{and}\quad Z(t) = \{\beta+i\gamma\in Z(\zeta_L) \mid |\gamma-t| \leqslant 1\}.$$

Note that $n(r;s) = |Z(r;s)|$ and $n_L(t) = |Z(t)|$.

(1). — Let $t_1, t_2, \cdots, t_{1+[r]}$ be real numbers such that $t-r \leqslant t_1 < \cdots < t_{1+[r]} \leqslant t+r$ and

$$Z(r;s) \subseteq \bigcup_{i=1}^{1+[r]} Z(t_i).$$

Then

$$\begin{aligned} n(r;\sigma+it) \leqslant \sum_{i=1}^{1+[r]} n_L(t_i) &\leqslant \alpha_8 \sum_{i=1}^{1+[r]} \{\log d_L + n_L \log(|t_i|+2)\} \\ &\leqslant \alpha_8(1+r)\{\log d_L + n_L \log(|t|+r+2)\}. \end{aligned}$$

(2). — Write $z = 1+r+it$. By (4.1),

$$\sum_{\rho\in Z(\zeta_L)} \Re\frac{1}{z-\rho} = \frac{1}{2}\log d_L + \Re\frac{\gamma_L'}{\gamma_L}(z) + \Re\frac{\zeta_L'}{\zeta_L}(z) + \Re\left(\frac{1}{z}+\frac{1}{z-1}\right).$$

We now estimate $\Re\frac{\gamma_L'}{\gamma_L}(z)$ and $\Re\frac{\zeta_L'}{\zeta_L}(z)$ from above. By Lemma 5.4

$$\begin{aligned} \Re\frac{\gamma_L'}{\gamma_L}(z) &\leqslant n_L\left\{f_2(1+r)\log(|t|+2) - \frac{1}{2}\log\pi\right\} \\ &\leqslant f_2(1+r)\log\left\{(|t|+2)^{n_L}\right\}. \end{aligned}$$

It follows from [33, Proposition 2] that

$$\Re\frac{\zeta_L'}{\zeta_L}(z) \leqslant \left|\frac{\zeta_L'}{\zeta_L}(z)\right| \leqslant -\frac{\zeta_L'}{\zeta_L}(1+r) \leqslant \left(\frac{1-\frac{1}{\sqrt{5}}}{2}\right)\log d_L + \frac{1}{r}.$$

Therefore,

$$\sum_{\rho\in Z(\zeta_L)} \Re\frac{1}{z-\rho} \leqslant \left(1-\frac{1}{2\sqrt{5}}\right)\log d_L + f_2(1+r)\log\left\{(|t|+2)^{n_L}\right\} + \frac{2}{r} + \frac{1}{1+r}.$$

Moreover,

$$\sum_{\rho\in Z(\zeta_L)} \Re\frac{1}{z-\rho} \geqslant \sum_{\rho\in Z(2r;z)} \Re\frac{1}{z-\rho} \geqslant \frac{1}{4r} n(2r;z).$$

Since $Z(r;\sigma+it) \subseteq Z(r;1+it) \subseteq Z(2r;z)$ and $1-\frac{1}{2\sqrt{5}} < f_2(2)$, we have

$$\begin{aligned} &n(r;\sigma+it) \leqslant n(2r;z) \\ &\quad\leqslant 4r\left[\left(1-\frac{1}{2\sqrt{5}}\right)\log d_L + f_2(1+r)\log\left\{(|t|+2)^{n_L}\right\} + \frac{2}{r} + \frac{1}{1+r}\right] \\ &\quad\leqslant 10\left[1+\frac{2f_2(2)}{5} r\log\left\{d_L(|t|+2)^{n_L}\right\}\right]. \end{aligned}$$

□

6. Zero-free regions for Dedekind zeta functions

We abbreviate $N_{L/\mathbb{Q}}$ to N. The classical argument to obtain a zero-free region for $\zeta_L(s)$ starts from (4.1) and for $\sigma > 1$

$$\Re\left[\sum_{m=0}^{d} b_m \left\{-\frac{\zeta_L'}{\zeta_L}(\sigma + imt)\right\}\right] = \Re\sum_{m=0}^{d} b_m \sum_{\mathfrak{a}} \frac{\wedge(\mathfrak{a})}{N\mathfrak{a}^{\sigma+imt}} \geqslant 0$$

where $b_m \geqslant 0$, $\mathcal{Q}(\phi) = \sum_{m=0}^{d} b_m \cos(m\phi) \geqslant 0$, $\wedge(\mathfrak{a})$ is the generalized Von Mangoldt function, and $\mathfrak{a}$ runs over all nonzero ideals of L.

Using Stechkin's work one can reduce the constant $\frac{1}{2}$ of the term $\frac{1}{2}\log A(\chi)$ in (4.1) to $\frac{1}{2}\left(1 - \frac{1}{\sqrt{5}}\right)$, which yields larger zero-free regions for $\zeta_L(s)$. (See [49], [45], [12], [36], [14], [19], [20], [37], [34], [32], [33], and [1].) It is known that if $L \neq \mathbb{Q}$, then $\zeta_L(s)$ has at most one zero $\rho = \beta + i\gamma$ with

$$\beta > 1 - \frac{1}{2\log d_L} \quad \text{and} \quad |\gamma| < \frac{1}{2\log d_L}. \tag{6.1}$$

If this zero exists then it must be real and simple. (See [48, Lemma 3], [16, Lemma 2], and [1].) This possible zero is called the exceptional zero and denoted by ρ_0. In this section we will show the following:

Proposition 6.1. — *Assume that $L \neq \mathbb{Q}$. Let $\rho = \beta + i\gamma$ be a nontrivial zero of $\zeta_L(s)$ with $\rho \neq \rho_0$ and $\tau = |\gamma| + 2$. Then*

$$1 - \beta > (29.57\log d_L \tau^{n_L})^{-1}. \tag{6.2}$$

For the zero-free regions of $\zeta_L(s)$ see also [20, Theorem 1.1], [59, Lemme 7.1], and [62].

We use the Stechkin's work ([49]) as [36] and [20] and use the same notations as [36] and [20]. Set

$$s = \sigma + it,\ \sigma_1 = \frac{1+\sqrt{1+4\sigma^2}}{2},\ s_1 = \sigma_1 + it,\ \kappa = \frac{1}{\sqrt{5}},$$

and

$$\mathbb{F}(s,z) = \Re\left\{\frac{1}{s-z} + \frac{1}{s-(1-\overline{z})}\right\}.$$

For $\sigma > 1$

$$\Re\left\{-\frac{\zeta_L'}{\zeta_L}(s) + \kappa\frac{\zeta_L'}{\zeta_L}(s_1)\right\} = \sum_{\mathfrak{a}} \frac{\wedge(\mathfrak{a})}{N\mathfrak{a}^{\sigma}}\left(1 - \frac{\kappa}{N\mathfrak{a}^{\sigma_1-\sigma}}\right)\Re\left(N\mathfrak{a}^{-it}\right),$$

where $\mathfrak{a}$ runs over all nonzero ideals of L. Moreover, by (4.1)

$$\begin{aligned}\Re\left\{-\frac{\zeta_L'}{\zeta_L}(s)+\kappa\frac{\zeta_L'}{\zeta_L}(s_1)\right\} &= \frac{1-\kappa}{2}\log d_L + \Re\left\{\frac{\gamma_L'}{\gamma_L}(s)-\kappa\frac{\gamma_L'}{\gamma_L}(s_1)\right\}\\ &\quad + \{\mathbb{F}(s,1)-\kappa\mathbb{F}(s_1,1)\} - \sideset{}{'}\sum_{\Re\rho\geqslant\frac{1}{2}}\{\mathbb{F}(s,\rho)-\kappa\mathbb{F}(s_1,\rho)\},\end{aligned}$$

where

$$\sideset{}{'}\sum_{\Re\rho\geqslant\frac{1}{2}} = \frac{1}{2}\sum_{\substack{\rho\in Z(\zeta_L)\\ \Re\rho=\frac{1}{2}}} + \sum_{\substack{\rho\in Z(\zeta_L)\\ \frac{1}{2}<\Re\rho\leqslant 1}}.$$

Assume that $b_m\geqslant 0$ and $\mathcal{Q}(\phi)=\sum_{m=0}^{d} b_m\cos(m\phi)\geqslant 0$. Then, for $\sigma>1$

$$\begin{aligned}\sum_{m=0}^{d} b_m\Re\left\{-\frac{\zeta_L'}{\zeta_L}(\sigma+im\gamma)+\kappa\frac{\zeta_L'}{\zeta_L}(\sigma_1+im\gamma)\right\} &= \sum_{\mathfrak{a}}\frac{\wedge(\mathfrak{a})}{N\mathfrak{a}^{\sigma}}\left(1-\frac{\kappa}{N\mathfrak{a}^{\sigma_1-\sigma}}\right)\mathcal{Q}(\gamma\log N\mathfrak{a})\geqslant 0.\end{aligned}$$

So,

$$0\leqslant S_2+S_3(\sigma,\gamma)+S_4(\sigma,\gamma)-S_1(\sigma,\gamma), \tag{6.3}$$

where

$$S_1(\sigma,\gamma)=\sum_{m=0}^{d} b_m\sideset{}{'}\sum_{\Re\rho\geqslant\frac{1}{2}}\{\mathbb{F}(\sigma+im\gamma,\rho)-\kappa\mathbb{F}(\sigma_1+im\gamma,\rho)\}, \tag{6.4}$$

$$S_2=\frac{1-\kappa}{2}\mathcal{Q}(0)\log d_L, \tag{6.5}$$

$$S_3(\sigma,\gamma)=\sum_{m=0}^{d} b_m\{\mathbb{F}(\sigma+im\gamma,1)-\kappa\mathbb{F}(\sigma_1+im\gamma,1)\}, \tag{6.6}$$

and

$$S_4(\sigma,\gamma)=\sum_{m=0}^{d} b_m\Re\left\{\frac{\gamma_L'}{\gamma_L}(\sigma+im\gamma)-\kappa\frac{\gamma_L'}{\gamma_L}(\sigma_1+im\gamma)\right\}. \tag{6.7}$$

Our proof of Proposition 6.1 consists of three parts: We estimate $S_1(\sigma,\gamma)$ from below, $S_3(\sigma,\gamma)$ and $S_4(\sigma,\gamma)$ from above. Note that if ρ is a nontrivial zero with $|\gamma|<(2\log d_L)^{-1}$, then (6.2) is satisfied. So, we may assume that $\rho\in Z(\zeta_L)$ and $|\gamma|\geqslant(2\log d_L)^{-1}$. Assume that

$$1-\beta\leqslant(b\log d_L\tau^{n_L})^{-1},$$

where $b \geqslant 4$ is a constant that will be specified later. Let $\epsilon = (b \log 12)^{-1}$ and $\sigma - 1 = (b \log d_L \tau^{n_L})^{-1}$. That is, $1 - \beta \leqslant \epsilon$ and $\sigma - 1 \leqslant \epsilon$ with $\epsilon \leqslant (4 \log 12)^{-1}$.

Lemma 6.2 (Stechkin [49]). — *Let $s = \sigma + it$ with $\sigma > 1$.*

(1) *If $0 < \Re z < 1$, then*

$$\mathbb{F}(s, z) - \kappa \mathbb{F}(s_1, z) \geqslant 0.$$

(2) *If $\Im z = t$ and $\frac{1}{2} \leqslant \Re z < 1$, then*

$$\Re \frac{1}{s - 1 + \overline{z}} - \kappa \mathbb{F}(s_1, z) \geqslant 0.$$

Lemma 6.3. — *Keeping the above notation we have*

$$S_1(\sigma, \gamma) \geqslant \frac{b_1}{\sigma - \beta} - \{\mathcal{Q}(0) - b_1\}\alpha_{10} + \sum_{m \neq 1} \frac{b_m(\sigma - \beta)}{(\sigma - \beta)^2 + \{(m-1)\gamma\}^2} \tag{6.8}$$

where

$$\alpha_9 = \frac{\sqrt{5} - 1}{2} \quad \text{and} \quad \alpha_{10} = \kappa \left\{ \frac{2\epsilon}{\alpha_9^2} + \frac{\epsilon}{(\alpha_9^{-1} - \epsilon)^2} \right\} + \frac{\epsilon}{(1 - \epsilon)^2}.$$

Proof. — By Lemma 6.2 (1)

$$S_1(\sigma, \gamma) \geqslant \sum_{m=0}^{d} b_m \{\mathbb{F}(\sigma + im\gamma, \beta + i\gamma) - \kappa \mathbb{F}(\sigma_1 + im\gamma, \beta + i\gamma)\}. \tag{6.9}$$

When $m = 1$, we have

$$\mathbb{F}(\sigma + i\gamma, \beta + i\gamma) - \kappa \mathbb{F}(\sigma_1 + i\gamma, \beta + i\gamma) \geqslant \frac{1}{\sigma - \beta} \tag{6.10}$$

by Lemma 6.2 (2). When $m \neq 1$, we have

$$\begin{aligned} &\mathbb{F}(\sigma + im\gamma, \beta + i\gamma) - \kappa \mathbb{F}(\sigma_1 + im\gamma, \beta + i\gamma) \\ &\qquad = \frac{\sigma - \beta}{(\sigma - \beta)^2 + \{(m-1)\gamma\}^2} \\ &\qquad\quad - \mathcal{G}(\sigma_1 - \beta, \sigma_1 - 1 + \beta, \sigma - 1 + \beta; (m-1)\gamma), \end{aligned} \tag{6.11}$$

where

$$\mathcal{G}(\omega_1, \omega_2, \omega_3; \upsilon) = \kappa \left(\frac{\omega_1}{\omega_1^2 + \upsilon^2} + \frac{\omega_2}{\omega_2^2 + \upsilon^2} \right) - \frac{\omega_3}{\omega_3^2 + \upsilon^2}.$$

Note that

$$\begin{aligned} 0 < \sigma_1 - \beta - \alpha_9 \leqslant 2\epsilon, \quad -\epsilon \leqslant \sigma_1 - 1 + \beta - \alpha_9^{-1} \leqslant \epsilon, \\ \text{and} \quad -\epsilon \leqslant \sigma - 1 + \beta - 1 \leqslant \epsilon. \end{aligned} \tag{6.12}$$

For $u > 0$ and $u_0 > 0$

$$\left| \frac{u}{u^2 + \upsilon^2} - \frac{u_0}{u_0^2 + \upsilon^2} \right| \leqslant \frac{|u - u_0|}{\min(u, u_0)^2}. \tag{6.13}$$

(See the proof of [20, Lemma 2.2] or that of [21, Lemma 5].) Using (6.12), (6.13), and the fact that $\mathcal{G}(\alpha_9, \alpha_9^{-1}, 1; \upsilon) \leqslant 0$ for all $\upsilon \in \mathbb{R}$ ([20, Lemma 2.2 (i)] or [21, Lemma 5 (i)]) we get

$$\mathcal{G}(\sigma_1 - \beta, \sigma_1 - 1 + \beta, \sigma - 1 + \beta; (m-1)\gamma) \leqslant \alpha_{10}. \tag{6.14}$$

Substituting (6.10), (6.11), and (6.14) into (6.9) yields (6.8). □

Lemma 6.4. — *Keeping the above notation we have*

$$S_3(\sigma, \gamma) \leqslant \frac{b_0}{\sigma - 1} + b_0 f_3(1 + \epsilon) - \{\mathcal{Q}(0) - b_0\}(\mathcal{G}_0 - \alpha_{11}) + \sum_{m \neq 0} \frac{b_m(\sigma - 1)}{(\sigma - 1)^2 + (m\gamma)^2}, \tag{6.15}$$

where

$$f_3(\sigma) = \frac{1}{\sigma} - \kappa \left(\frac{1}{\sigma_1 - 1} + \frac{1}{\sigma_1} \right), \ \alpha_{11} = \kappa \left(\frac{\epsilon}{\alpha_9^2} + \frac{\epsilon}{\alpha_9^{-2}} \right) + \epsilon = (3\kappa + 1)\epsilon,$$

and $\mathcal{G}_0 = -0.121585107$.

Proof. — When $m = 0$, we have

$$\mathbb{F}(\sigma, 1) - \kappa \mathbb{F}(\sigma_1, 1) = \frac{1}{\sigma - 1} + f_3(\sigma) \leqslant \frac{1}{\sigma - 1} + f_3(1 + \epsilon) \tag{6.16}$$

since $f_3(\sigma)$ is increasing for $1 < \sigma < 1.75$. When $m \neq 0$, we have

$$\mathbb{F}(\sigma + im\gamma, 1) - \kappa \mathbb{F}(\sigma_1 + im\gamma, 1) = \frac{\sigma - 1}{(\sigma - 1)^2 + (m\gamma)^2} - \mathcal{G}(\sigma_1 - 1, \sigma_1, \sigma; m\gamma). \tag{6.17}$$

Note that $0 < \sigma_1 - 1 - \alpha_9 = \sigma_1 - \alpha_9^{-1} \leqslant \epsilon$ and $0 < \sigma - 1 \leqslant \epsilon$. Using [20, Lemma 2.2] we get

$$\mathcal{G}(\sigma_1 - 1, \sigma_1, \sigma; m\gamma) \geqslant \mathcal{G}_0 - \alpha_{11}. \tag{6.18}$$

On feeding (6.16), (6.17), and (6.18) into (6.6) we get (6.15). □

Let

$$D(m) = \begin{cases} \frac{1}{4}\{\Gamma_1(1 + \epsilon) + \Gamma_0(1 + \epsilon)\} - \frac{1 - \kappa}{2} \log \pi & \text{if } m = 0, \\ f_4(1 + \epsilon) \log m + \alpha_{12} & \text{if } m \neq 0, \end{cases}$$

where

$$\Gamma_a(s) = \frac{\Gamma'}{\Gamma}\left(\frac{s+a}{2}\right) - \kappa\frac{\Gamma'}{\Gamma}\left(\frac{s_1+a}{2}\right),\ f_4(\sigma) = \frac{\alpha_6-\kappa}{2}\left\{\frac{\log(\sigma+5)}{\log 2} - 1\right\},$$

and

$$\alpha_{12} = \frac{\kappa\alpha_7 - (1-\kappa)\log\pi}{2} = 0.34162\cdots.$$

Lemma 6.5. — *Keeping the above notation we have*

$$S_4(\sigma,\gamma) \leqslant \alpha_{13}\log\tau^{n_L} + \alpha_{14}n_L,$$

where $\alpha_{13} = \{\mathcal{Q}(0) - b_0\}f_4(1+\epsilon)$ *and* $\alpha_{14} = \sum_{m=0}^{d} b_m D(m)$.

Proof. — Since $\Gamma_0(v)$ and $\Gamma_1(v)$ are monotonically increasing and $\Gamma_1(v) > \Gamma_0(v)$ for $1 < v < 2$,

$$\begin{aligned}\Re\left\{\frac{\gamma_L'}{\gamma_L}(\sigma) - \kappa\frac{\gamma_L'}{\gamma_L}(\sigma_1)\right\} &= \frac{n_L}{2}\Gamma_0(\sigma) + \frac{r_2}{2}\{\Gamma_1(\sigma) - \Gamma_0(\sigma)\} - \frac{1-\kappa}{2}n_L\log\pi \\ &\leqslant n_L\left\{\frac{1}{4}\Gamma_1(\sigma) + \frac{1}{4}\Gamma_0(\sigma) - \frac{1-\kappa}{2}\log\pi\right\} \leqslant n_L D(0).\end{aligned}$$

Set $s = \sigma + im\gamma$ and $s_1 = \sigma_1 + im\gamma$. For $m \geqslant 1$

$$\begin{aligned}&\Re\left\{\frac{\gamma_L'}{\gamma_L}(s) - \kappa\frac{\gamma_L'}{\gamma_L}(s_1)\right\} \\ &\quad\leqslant \frac{(r_1+r_2)}{2}\left\{\alpha_6\log\left(\frac{|s|}{2}+2\right) - \kappa\log\left(\frac{|s_1|}{2}+2\right) + \kappa\alpha_7\right\} \\ &\qquad + \frac{r_2}{2}\left\{\alpha_6\log\left(\frac{|s+1|}{2}+2\right) - \kappa\log\left(\frac{|s_1+1|}{2}+2\right) + \kappa\alpha_7\right\} \\ &\qquad - \frac{1-\kappa}{2}n_L\log\pi \qquad\qquad \text{by Lemma 5.3} \\ &\quad\leqslant \frac{n_L}{2}\left\{(\alpha_6-\kappa)\log\left(\frac{|s+1|}{2}+2\right) + \kappa\alpha_7 - (1-\kappa)\log\pi\right\} \\ &\quad\leqslant n_L\{f_4(\sigma)\log(|m\gamma|+2) + \alpha_{12}\} \qquad\qquad \text{by (5.2)} \\ &\quad\leqslant n_L\{f_4(1+\epsilon)\log(|\gamma|+2) + D(m)\}.\end{aligned}$$

Hence

$$\begin{aligned}S_4(\sigma,\gamma) &\leqslant b_0 n_L D(0) + n_L\sum_{m=1}^{d} b_m\{f_4(1+\epsilon)\log\tau + D(m)\} \\ &= \alpha_{13}\log\tau^{n_L} + \alpha_{14}n_L.\end{aligned}$$

□

Now, Proposition 6.1 is ready to be proven. Combining (6.3), (6.5), Lemmas 6.3, 6.4, and 6.5 yields

$$\begin{aligned}0 \leqslant \frac{1-\kappa}{2}\mathcal{Q}(0)\log d_L + \alpha_{13}\log\tau^{n_L} + \alpha_{14}n_L + \alpha_{15} + \frac{b_0}{\sigma-1}\\ - \frac{b_1}{\sigma-\beta} + \frac{b_1(\sigma-1)}{(\sigma-1)^2+\gamma^2} - \frac{b_0(\sigma-\beta)}{(\sigma-\beta)^2+\gamma^2}\\ + \sum_{m=2}^{d} b_m\left\{\frac{(\sigma-1)}{(\sigma-1)^2+(m\gamma)^2} - \frac{(\sigma-\beta)}{(\sigma-\beta)^2+\{(m-1)\gamma\}^2}\right\},\end{aligned}$$

where $\alpha_{15} = b_0 f_3(1+\epsilon) - \{\mathcal{Q}(0)-b_0\}(\mathcal{G}_0-\alpha_{11}) + \{\mathcal{Q}(0)-b_1\}\alpha_{10}$. Since

$$\begin{aligned}\frac{b_1(\sigma-1)}{(\sigma-1)^2+\gamma^2} - \frac{b_0(\sigma-\beta)}{(\sigma-\beta)^2+\gamma^2} &\leqslant \frac{(b_1-b_0)(\sigma-1)}{(\sigma-1)^2+\gamma^2}\\ &\leqslant (b_1-b_0)\left(\frac{4b}{4+b^2}\right)\log d_L\end{aligned}$$

and for $m \geqslant 2$

$$\frac{(\sigma-1)}{(\sigma-1)^2+(m\gamma)^2} - \frac{(\sigma-\beta)}{(\sigma-\beta)^2+\{(m-1)\gamma\}^2} \leqslant 0,$$

it follows that

$$0 \leqslant \alpha_{16}\log d_L + \alpha_{13}\log\tau^{n_L} + \alpha_{14}n_L + \alpha_{15} + \frac{b_0}{\sigma-1} - \frac{b_1}{\sigma-\beta} \tag{6.19}$$

with

$$\alpha_{16} = \frac{1-\kappa}{2}\mathcal{Q}(0) + (b_1-b_0)\left(\frac{4b}{4+b^2}\right).$$

Let $0 \leqslant \delta \leqslant 1$ and $0 \leqslant \eta \leqslant 1$. Note that $d_L \geqslant 3^{n_L/2}$. Set

$$B_{11} = \alpha_{16} + \frac{2\alpha_{14}}{\log 3}\delta + \frac{\alpha_{15}}{\log 3}\eta,\ \ B_{12} = \alpha_{13} + \frac{\alpha_{14}}{\log 2}(1-\delta) + \frac{\alpha_{15}}{2\log 2}(1-\eta),$$

and

$$B_{13} = \max(B_{11}, B_{12}).$$

The inequality (6.19) is replaced by

$$0 \leqslant B_{13}\log d_L\tau^{n_L} + \frac{b_0}{\sigma-1} - \frac{b_1}{\sigma-\beta}. \tag{6.20}$$

From (6.20) it follows that

$$1-\beta \geqslant \left(\frac{b_1}{b_0 b + B_{13}} - \frac{1}{b}\right)(\log d_L\tau^{n_L})^{-1}.$$

We choose $\mathcal{Q}(\phi)$ with $b_0 < b_1$, b, δ, and η as follows:

$$\mathcal{Q}(\phi) = 4(1+\cos\phi)(0.51+\cos\phi)^2, \quad b = 8.7,\ \delta = 0.66,\ \text{and}\ \eta = 0.26,$$

and obtain (6.2).

7. The Deuring–Heilbronn phenomenon

The Deuring–Heilbronn phenomenon means that if the exceptional zero of $\zeta_L(s)$ exists then the other zeros of $\zeta_L(s)$ can not be very close to $s = 1$. In [23] Lagarias, Montgomery, and Odlyzko proved more precisely the following.

Theorem C (Lagarias, Montgomery, Odlyzko [23]). — *There are positive, absolute, effectively computable constants c_7 and c_8 such that if $\zeta_L(s)$ has a real zero $\omega_0 > 0$ then $\zeta_L(\sigma + it) \neq 0$ for*

$$\sigma \geqslant 1 - c_8 \frac{\log\left[\frac{c_7}{(1-\omega_0)\log\{d_L(|t|+2)^{n_L}\}}\right]}{\log\{d_L(|t|+2)^{n_L}\}}$$

with the single exception $\sigma + it = \omega_0$.

See also [30]. In this section we will estimate the values of c_7 and c_8 explicitly. We will use a power sum inequality as [23]. We begin by recalling the fact that $(s-1)\zeta_L(s)$ is an entire function of order one. The Hadamard product theorem says that

$$(s-1)\zeta_L(s) = s^{r_1+r_2-1}e^{a+bs}\prod_{\omega}\left(1-\frac{s}{\omega}\right)e^{s/\omega}$$

for some constants a and b, where ω runs through all the zeros of $\zeta_L(s)$, $\omega \neq 0$, including the trivial ones, counted with multiplicity.([48]) The Euler product for $\zeta_L(s)$ gives

$$-\frac{\zeta_L'}{\zeta_L}(s) = \sum_{\mathfrak{P}}\sum_{m=1}^{\infty}(\log N\mathfrak{P})\,(N\mathfrak{P})^{-ms}$$

for $\Re s > 1$, where $\mathfrak{P}$ runs over all prime ideals of L. This series is absolutely convergent for $\Re s > 1$.

Suppose that $\zeta_L(s)$ has a real zero $\omega_0 > 0$. Differenciating $(2j-1)$ times the equality

$$\sum_{\mathfrak{P}}\sum_{m=1}^{\infty}(\log N\mathfrak{P})\,(N\mathfrak{P})^{-ms} = \frac{1}{s-1} - b - \sum_{\omega}\left(\frac{1}{s-\omega}+\frac{1}{\omega}\right) - \frac{r_1+r_2-1}{s}$$

yields that for $\Re s > 1$ and $j \geqslant 1$

$$\frac{1}{(2j-1)!}\sum_{\mathfrak{P}}\sum_{m=1}^{\infty}(\log N\mathfrak{P})(\log N\mathfrak{P}^m)^{2j-1}(N\mathfrak{P})^{-ms}$$
$$= \frac{1}{(s-1)^{2j}} - \frac{1}{(s-\omega_0)^{2j}} - \sum_{\substack{\omega\in Z(\zeta_L)\\ \omega\neq\omega_0}}\frac{1}{(s-\omega)^{2j}} - \sum_{\check{m}=0}^{\infty}\frac{\ell_{\check{m}}}{(s+\check{m})^{2j}},$$

where

$$\ell_{\check{m}} = \begin{cases} r_1+r_2-1 & \text{if } \check{m}=0,\\ r_1+r_2 & \text{if } \check{m}\neq 0 \text{ is even},\\ r_2 & \text{if } \check{m} \text{ is odd}.\end{cases}$$

If $s_0 = \sigma_0 + it_0$ with $\sigma_0 > 1$, then

$$\text{(7.1)}\quad \frac{1}{(2j-1)!}\sum_{\mathfrak{P}}\sum_{m=1}^{\infty}(\log N\mathfrak{P})(\log N\mathfrak{P}^m)^{2j-1}N\mathfrak{P}^{-m\sigma_0}\left\{1+(N\mathfrak{P}^m)^{-it_0}\right\}$$
$$+\sum_{\check{m}=2}^{\infty}\left\{\frac{\ell_{\check{m}}}{(\sigma_0+\check{m})^{2j}}+\frac{\ell_{\check{m}}}{(s_0+\check{m})^{2j}}\right\}$$
$$=\frac{1}{(\sigma_0-1)^{2j}}-\frac{1}{(\sigma_0-\omega_0)^{2j}}+\frac{1}{(s_0-1)^{2j}}-\frac{1}{(s_0-\omega_0)^{2j}}-\sum_{n=1}^{\infty}z_n^j,$$

where z_n is the series of the terms $(\sigma_0-\omega)^{-2}$ and $(s_0-\omega)^{-2}$ for all $\omega \in \{0,-1\}\cup(Z(\zeta_L)\backslash\{\omega_0\})$ such that ω is counted according to its multiplicity and $|z_n|$ is decreasing for $n \geqslant 1$. Since the real part of the left side of (7.1) is nonnegative,

$$\text{(7.2)}\quad \Re\sum_{n=1}^{\infty}z_n^j \leqslant \frac{1}{(\sigma_0-1)^{2j}}-\frac{1}{(\sigma_0-\omega_0)^{2j}}$$
$$+\Re\left[\frac{1}{\{(\sigma_0-1)+it_0\}^{2j}}-\frac{1}{\{(\sigma_0-\omega_0)+it_0\}^{2j}}\right].$$

To evaluate the constants c_7 and c_8, first, we estimate the right side of (7.2) from above.

Lemma 7.1. — *For $\sigma_0 > 1$, $j \geqslant 1$, and $0 < \upsilon \leqslant 1$ we let*

$$f_5(\sigma_0+it_0,j;\upsilon)=\Re\left[\frac{1}{\{(\sigma_0-1)+it_0\}^{2j}}-\frac{1}{\{(\sigma_0-\upsilon)+it_0\}^{2j}}\right].$$

Then

$$f_5(\sigma_0,j;\omega_0)+f_5(\sigma_0+it_0,j;\omega_0)\leqslant\frac{4j(1-\omega_0)}{(\sigma_0-1)^{2j+1}}.$$

Proof. — We have

$$f_5(\sigma_0+it_0,j;\upsilon)=2j\int_{\sigma_0-1}^{\sigma_0-\upsilon}\Re\left\{\frac{1}{(y+it_0)^{2j+1}}\right\}\mathrm{d}y\leqslant 2j\frac{1-\upsilon}{(\sigma_0-1)^{2j+1}}.$$

(See [60, (2.43)].) The result follows. □

Second, we estimate $\Re\sum_{n=1}^{\infty}z_n^j$ from below using [23, Theorem 4.2]. (See also [63, Theorem 2.3]). Set

$$\mathcal{L}=\mathcal{L}(s_0)=|z_1|^{-1}\sum_{n=1}^{\infty}|z_n|.$$

According to [23, Theorem 4.2] (see also [63, Theorem 2.3]) for any $\check{c}>12$, there exists j_0 with $1\leqslant j_0\leqslant\check{c}\mathcal{L}$ such that

$$\Re\sum_{n=1}^{\infty}z_n^{j_0}\geqslant\left(\frac{\check{c}-12}{4\check{c}}\right)|z_1|^{j_0}. \tag{7.3}$$

Now we estimate $\sum_{n=1}^{\infty}|z_n|$ from above.

Lemma 7.2. — *Let $s_0=\sigma_0+it_0$, z_n and ω_0 be as above. Then we have*

$$\sum_{n=1}^{\infty}|z_n|\leqslant B_{17}(\sigma_0)\log d_L+B_{18}(\sigma_0)\log\{(|t_0|+2)^{n_L}\}+B_{19}(\sigma_0)n_L+B_{20}(\sigma_0), \tag{7.4}$$

where $B_{17}(\sigma_0)=2a_1(\sigma_0)$, $B_{18}(\sigma_0)=a_2(\sigma_0)$, $B_{19}(\sigma_0)=a_2(\sigma_0)\log 2+2a_3(\sigma_0)+\frac{2}{\sigma_0^2}$, and $B_{20}(\sigma_0)=2a_4(\sigma_0)-\frac{2}{\sigma_0^2}$ with

$$a_1(\sigma_0)=\frac{1}{2(\sigma_0-1)},\ a_2(\sigma_0)=\frac{f_2(\sigma_0)}{\sigma_0-1},\ a_3(\sigma_0)=-\frac{\log\pi}{2(\sigma_0-1)},$$

and

$$a_4(\sigma_0)=\frac{1}{\sigma_0-1}\left(\frac{1}{\sigma_0}+\frac{1}{\sigma_0-1}\right).$$

(Here, $f_2(\sigma_0)$ is as in Section 5.)

Proof. — Note that

$$\sum_{n=1}^{\infty}|z_n|=\sum_{\substack{\omega\in Z(\zeta_L)\\ \omega\neq\omega_0}}\frac{1}{|\sigma_0-\omega|^2}+\sum_{\substack{\omega\in Z(\zeta_L)\\ \omega\neq\omega_0}}\frac{1}{|s_0-\omega|^2}+\frac{\ell_0}{|\sigma_0|^2}+\frac{\ell_0}{|s_0|^2}+\frac{\ell_1}{|\sigma_0+1|^2}+\frac{\ell_1}{|s_0+1|^2}.$$

As

$$\frac{\Re s-1}{|s-\omega|^2}\leqslant\Re\frac{1}{s-\omega}$$

for $s \in \mathbb{C}$ and $\omega \in Z(\zeta_L)$ we have

$$\sum_{\substack{\omega \in Z(\zeta_L) \\ \omega \neq \omega_0}} \frac{\Re s - 1}{|s-\omega|^2} \leqslant \sum_{\omega \in Z(\zeta_L)} \Re \frac{1}{s-\omega}$$

$$= \frac{1}{2} \log d_L + \Re\left(\frac{1}{s} + \frac{1}{s-1}\right) + \Re \frac{\gamma_L'}{\gamma_L}(s) + \Re \frac{\zeta_L'}{\zeta_L}(s). \tag{7.5}$$

Gathering together the bound in Lemma 5.4, the fact that

$$\Re\left\{\frac{\zeta_L'}{\zeta_L}(\sigma_0) + \frac{\zeta_L'}{\zeta_L}(\sigma_0 + it_0)\right\} \leqslant 0,$$

and (7.5) we get

$$\sum_{\substack{\omega \in Z(\zeta_L) \\ \omega \neq \omega_0}} \frac{1}{|\sigma_0 - \omega|^2} + \sum_{\substack{\omega \in Z(\zeta_L) \\ \omega \neq \omega_0}} \frac{1}{|s_0 - \omega|^2}$$
$$\leqslant 2a_1(\sigma_0) \log d_L + a_2(\sigma_0) \log\left\{(|t_0| + 2)^{n_L}\right\}$$
$$+ \left\{a_2(\sigma_0) \log 2 + 2a_3(\sigma_0)\right\} n_L + 2a_4(\sigma_0).$$

Moreover,

$$\frac{\ell_0}{|\sigma_0|^2} + \frac{\ell_0}{|s_0|^2} + \frac{\ell_1}{|\sigma_0 + 1|^2} + \frac{\ell_1}{|s_0 + 1|^2} \leqslant \frac{2(r_1 + r_2 - 1)}{\sigma_0^2} + \frac{2r_2}{(\sigma_0 + 1)^2}$$
$$\leqslant \frac{2}{\sigma_0^2} n_L - \frac{2}{\sigma_0^2}.$$

The result follows. □

We are now ready to prove the following.

THEOREM 7.3. — *Suppose that $L \neq \mathbb{Q}$ and $\zeta_L(s)$ has a real zero $\omega_0 > 0$. Let $\rho = \beta + i\gamma$ be a zero of $\zeta_L(s)$ with $\rho \neq \omega_0$.*

(1) *If L is not an imaginary quadratic number field, then*

$$1 - \beta \geqslant c_8 \frac{\log\left\{\frac{c_7}{(1-\omega_0) \log d_L \tau^{n_L}}\right\}}{\log d_L \tau^{n_L}}, \tag{7.6}$$

where $\tau = |\gamma| + 2$, $c_7 = 6.7934 \cdots \times 10^{-4}$, and $c_8 = 16c_7 = \frac{1}{92}$. When L is an imaginary quadratic number field, then (7.6) holds with $c_7 = 5.5803 \cdots \times 10^{-4}$ and $c_8 = 16c_7 = \frac{1}{112}$.

(2) *If ρ is a nontrivial zero of $\zeta_L(s)$, then (7.6) holds with $c_7 = 8.1168 \cdots \times 10^{-4}$ and $c_8 = 16c_7 = \frac{1}{77}$.*

Proof.

(1). — If L is not an imaginary quadratic number field, then $\zeta_L(s)$ has a zero at $s=0$ and $|z_1|^{-1} \leqslant \sigma_0^2$. Setting $t_0 = \gamma$ in (7.4) yields

$$\mathcal{L} \leqslant \sigma_0^2 \left\{B_{17}(\sigma_0)\log d_L + B_{18}(\sigma_0)\log \tau^{n_L} + B_{19}(\sigma_0)n_L + B_{20}(\sigma_0)\right\}.$$

Note that $B_{19}(\sigma_0) \geqslant 0$ for $\sigma_0 \geqslant 1.74$. For $\sigma_0 \geqslant 1.74$ and $0 \leqslant \delta,\ \eta \leqslant 1$, we let

$$B_{22}(\sigma_0,\delta,\eta) = B_{17}(\sigma_0) + \frac{2B_{19}(\sigma_0)}{\log 3}\delta + \frac{B_{20}(\sigma_0)}{\log 3}\eta,$$

$$B_{23}(\sigma_0,\delta,\eta) = B_{18}(\sigma_0) + \frac{B_{19}(\sigma_0)}{\log 2}(1-\delta) + \frac{B_{20}(\sigma_0)}{2\log 2}(1-\eta),$$

and

$$B_{24}(\sigma_0,\delta,\eta) = \max\{B_{22}(\sigma_0,\delta,\eta), B_{23}(\sigma_0,\delta,\eta)\}.$$

Then we have

$$\mathcal{L} \leqslant \sigma_0^2 B_{24}(\sigma_0,\delta,\eta)\log d_L\tau^{n_L}$$

since $d_L \geqslant 3^{n_L/2}$ and $n_L \geqslant 2$. Note that if $\rho \in Z(\zeta_L)$, then $|z_1| \geqslant |\sigma_0 + i\gamma - \rho|^{-2} = |\sigma_0 - \beta|^{-2}$ and if $\rho \notin Z(\zeta_L)$, then $\rho = \beta \leqslant 0$ and $|z_1| \geqslant |\sigma_0|^{-2} \geqslant |\sigma_0 - \beta|^{-2}$. Thus

$$|z_1| \geqslant \frac{1}{(\sigma_0-1)^2}\exp\left\{-2\left(\frac{1-\beta}{\sigma_0-1}\right)\right\}$$

and the bound (7.3) yields

$$\Re\sum_{n=1}^{\infty} z_n^{j_0} \geqslant \left(\frac{\check{c}-12}{4\check{c}}\right)\frac{1}{(\sigma_0-1)^{2j_0}}\exp\left\{-2j_0\left(\frac{1-\beta}{\sigma_0-1}\right)\right\}.$$

Combining this with (7.2) and the bound in Lemma 7.1 we have

$$\left(\frac{\check{c}-12}{4\check{c}}\right)\frac{1}{(\sigma_0-1)^{2j_0}}\exp\left\{-2j_0\left(\frac{1-\beta}{\sigma_0-1}\right)\right\} \leqslant \frac{4j_0(1-\omega_0)}{(\sigma_0-1)^{2j_0+1}}. \tag{7.7}$$

From $j_0 \leqslant \check{c}\mathcal{L} \leqslant \check{c}\sigma_0^2 B_{24}(\sigma_0,\delta,\eta)\log d_L\tau^{n_L}$ it follows that

$$1-\beta \geqslant c_8(\check{c},\sigma_0,\delta,\eta)\frac{\log\left\{\frac{c_7(\check{c},\sigma_0,\delta,\eta)}{(1-\omega_0)\log d_L\tau^{n_L}}\right\}}{\log d_L\tau^{n_L}}, \tag{7.8}$$

where $c_7(\check{c},\sigma_0,\delta,\eta) = \left(\frac{\check{c}-12}{8\check{c}}\right)c_8(\check{c},\sigma_0,\delta,\eta)$ and

$$c_8(\check{c},\sigma_0,\delta,\eta) = \frac{\sigma_0-1}{2\check{c}\sigma_0^2 B_{24}(\sigma_0,\delta,\eta)}.$$

Choosing $\check{c} = 24$, $\sigma_0 = 7.79$, $\delta = 1$, and $\eta = 1$ we get (7.6). If L is an imaginary quadratic number field, then $\zeta_L(s)$ has a zero at $s=-1$ and $|z_1|^{-1} \leqslant (\sigma_0+1)^2$. We have then

$$\mathcal{L} \leqslant (\sigma_0+1)^2 B_{24}(\sigma_0,\delta,\eta)\log d_L\tau^{n_L}$$

and $j_0 \leqslant \check{c}\mathcal{L} \leqslant \check{c}(\sigma_0+1)^2 B_{24}(\sigma_0,\delta,\eta)\log d_L \tau^{n_L}$. Moreover,

$$|z_1| \geqslant |\sigma_0-\beta|^{-2} \geqslant \frac{1}{(\sigma_0-1)^2}\exp\left\{-2\left(\frac{1-\beta}{\sigma_0-1}\right)\right\}$$

since $\zeta_L(s)$ does not have a zero at $s=0$. From (7.7) we get

$$c_8(\check{c},\sigma_0,\delta,\eta) = \frac{\sigma_0-1}{2\check{c}(\sigma_0+1)^2 B_{24}(\sigma_0,\delta,\eta)}.$$

Choosing $\check{c}=24$, $\sigma_0=12.21$, $\delta=1$, and $\eta=1$ we get the result.

(2). — We consider $\sum_{n=1}^{\infty}\widehat{z}_n^j$ (instead of $\sum_{n=1}^{\infty} z_n^j$ in (7.2)), where $\widehat{z}_n$ is the series of terms $(\sigma_0-\omega)^{-2}$ and $(\sigma_0+it_0-\omega)^{-2}$ for all $\omega \in Z(\zeta_L)\setminus\{\omega_0\}$ such that ω is counted according to its multiplicity and $|\widehat{z}_n|$ is decreasing for $n \geqslant 1$. Since

$$\Re\sum_{n=1}^{\infty}\widehat{z}_n^j + \Re\left\{\frac{\ell_0}{\sigma_0^{2j}} + \frac{\ell_0}{(\sigma_0+it_0)^{2j}} + \frac{\ell_1}{(\sigma_0+1)^{2j}} + \frac{\ell_1}{(\sigma_0+it_0+1)^{2j}}\right\} = \Re\sum_{n=1}^{\infty} z_n^j$$

and

$$\Re\left\{\frac{1}{(\sigma_0-\omega)^{2j}} + \frac{1}{(\sigma_0+it_0-\omega)^{2j}}\right\} \geqslant 0 \quad \text{for } \omega = 0,-1,$$

$$\Re\sum_{n=1}^{\infty}\widehat{z}_n^j \leqslant \frac{1}{(\sigma_0-1)^{2j}} - \frac{1}{(\sigma_0-\omega_0)^{2j}} + \Re\left[\frac{1}{\{(\sigma_0-1)+it_0\}^{2j}} - \frac{1}{\{(\sigma_0-\omega_0)+it_0\}^{2j}}\right]. \tag{7.9}$$

We use the power-sum inequality in [23, Theorem 4.2] for $\sum_{n=1}^{\infty}\widehat{z}_n^j$. Set $\widehat{\mathcal{L}} = |\widehat{z}_1|^{-1}\sum_{n=1}^{\infty}|\widehat{z}_n|$. For any $\check{c} > 12$, there exists $\widehat{j}_0$ with $1 \leqslant \widehat{j}_0 \leqslant \check{c}\widehat{\mathcal{L}}$ such that

$$\Re\sum_{n=1}^{\infty}\widehat{z}_n^{\widehat{j}_0} \geqslant \left(\frac{\check{c}-12}{4\check{c}}\right)|\widehat{z}_1|^{\widehat{j}_0}. \tag{7.10}$$

If $\rho \in Z(\zeta_L)$, then $1-\overline{\rho} \in Z(\zeta_L)$. Set $t_0=\gamma$. Then

$$|\widehat{z}_1|^{-1} \leqslant \min\{(\sigma_0-\beta)^2, (\sigma_0-1+\beta)^2\} \leqslant \left(\sigma_0-\frac{1}{2}\right)^2.$$

Then we have

$$\widehat{\mathcal{L}} \leqslant \left(\sigma_0-\frac{1}{2}\right)^2\left\{B_{17}(\sigma_0)\log d_L + B_{18}(\sigma_0)\log\tau^{n_L} + \widehat{B}_{19}(\sigma_0)n_L + \widehat{B}_{20}(\sigma_0)\right\},$$

where $\widehat{B}_{19}(\sigma_0) = a_2(\sigma_0)\log 2 + 2a_3(\sigma_0)$ and $\widehat{B}_{20}(\sigma_0) = 2a_4(\sigma_0)$. Note that $\widehat{B}_{19}(\sigma_0) \leqslant 0$ and $2\widehat{B}_{19}(\sigma_0) + \widehat{B}_{20}(\sigma_0) \geqslant 0$ for $1 < \sigma_0 \leqslant 11.66$. So, for $1 < \sigma_0 \leqslant 11.66$

$$\widehat{\mathcal{L}} \leqslant \left(\sigma_0 - \frac{1}{2}\right)^2 \Big\{B_{17}(\sigma_0)\log d_L + B_{18}(\sigma_0)\log \tau^{n_L} + 2\widehat{B}_{19}(\sigma_0) + \widehat{B}_{20}(\sigma_0)\Big\}.$$

For $1 < \sigma_0 \leqslant 11.66$ and $0 \leqslant \eta \leqslant 1$, we let

$$B_{25}(\sigma_0,\eta) = B_{17}(\sigma_0) + \frac{2\widehat{B}_{19}(\sigma_0) + \widehat{B}_{20}(\sigma_0)}{\log 3}\eta,$$
$$B_{26}(\sigma_0,\eta) = B_{18}(\sigma_0) + \frac{2\widehat{B}_{19}(\sigma_0) + \widehat{B}_{20}(\sigma_0)}{2\log 2}(1-\eta),$$

and

$$B_{27}(\sigma_0,\eta) = \max\{B_{25}(\sigma_0,\eta), B_{26}(\sigma_0,\eta)\}.$$

Then we have

$$\widehat{\mathcal{L}} \leqslant \left(\sigma_0 - \frac{1}{2}\right)^2 B_{27}(\sigma_0,\eta)\log d_L\tau^{n_L}.$$

Note that $d_L \geqslant 3^{n_L/2}$. Since

$$|z_1| \geqslant |\sigma_0 + i\gamma - \rho|^{-2} \geqslant \frac{1}{(\sigma_0-1)^2}\exp\left\{-2\left(\frac{1-\beta}{\sigma_0-1}\right)\right\},$$

the bound (7.10) yields

$$\Re\sum_{n=1}^{\infty} \widehat{z_n^{j_0}} \geqslant \left(\frac{\check{c}-12}{4\check{c}}\right)\frac{1}{(\sigma_0-1)^{2\widehat{j_0}}}\exp\left\{-2\widehat{j_0}\left(\frac{1-\beta}{\sigma_0-1}\right)\right\}.$$

Combining this with (7.9) and the bound in Lemma 7.1 we have

$$\left(\frac{\check{c}-12}{4\check{c}}\right)\frac{1}{(\sigma_0-1)^{2\widehat{j_0}}}\exp\left\{-2\widehat{j_0}\left(\frac{1-\beta}{\sigma_0-1}\right)\right\} \leqslant \frac{4\widehat{j_0}(1-\omega_0)}{(\sigma_0-1)^{2\widehat{j_0}+1}}.$$

From $\widehat{j_0} \leqslant \check{c}\mathcal{L} \leqslant \check{c}\left(\sigma_0 - \frac{1}{2}\right)^2 B_{27}(\sigma_0,\eta)\log d_L\tau^{n_L}$ it follows that

$$c_8(\check{c},\sigma_0,\eta) = \frac{\sigma_0 - 1}{2\check{c}\left(\sigma_0-\frac{1}{2}\right)^2 B_{27}(\sigma_0,\eta)}.$$

Choosing $\check{c} = 24$, $\sigma_0 = 5.42$, and $\eta = 1$ we get the result. □

Remark. — To get an upper bound for $\mathcal{L}$ the zero-density estimate for the number of zeros of $\zeta_L(s)$ was used in [23]:

$$\begin{aligned}\mathcal{L} &\ll (2-\beta)^2 \sum_{\omega} \left(\frac{1}{|2-\omega|^2} + \frac{1}{|2+i\gamma-\omega|} \right) \\ &\ll \int_0^\infty \frac{1}{u^2+1}\,\mathrm{d}n(u) + \int_0^\infty \frac{1}{u^2+1}\,\mathrm{d}n(u+\tau) \\ &\ll \log d_L \tau^{n_L},\end{aligned}$$

where ω runs through all the zeros of $\zeta_L(s)$ including the trivial ones. (See [23, (5.6)].) However we used

$$\sum_{\rho\in Z(\zeta_L)} \frac{\sigma-1}{|s-\rho|^2} \leqslant \sum_{\rho\in Z(\zeta_L)} \Re \frac{1}{s-\rho}$$

for $\Re s = \sigma > 1$ and (5.1). (See (7.5) above.)

Corollary 7.4. — *Assume that $L \neq \mathbb{Q}$. Then for any real zero $\omega_0 > 0$ of $\zeta_L(s)$ we have*

$$1-\omega_0 \geqslant d_L^{-c_{10}} \tag{7.11}$$

with $c_{10} = 114.72\cdots$.

Proof. — When L is not an imaginary quadratic number fields, we let $\check{c} = 12.1$, $\sigma_0 = 7.79$, $\delta = 1$, and $\eta = 1$. The inequality (7.8) yields

$$1-\beta \geqslant c_8 \frac{\log c_7 + \log(1-\omega_0)^{-1} - \log\log d_L \tau^{n_L}}{\log d_L \tau^{n_L}} \tag{7.12}$$

for any zero $\beta + i\gamma \neq \omega_0$ of $\zeta_L(s)$, where $c_7 = 2.2434\cdots \times 10^{-5}$ and $c_8 = 2.1716\cdots \times 10^{-2}$. Set $1-\omega_0 = d_L^{-c}$. Since $\zeta_L(s)$ always has a trivial zero at $s=0$ and $d_L \geqslant 3^{n_L/2}$, we have

$$\begin{aligned}1 &\geqslant c_8 \left\{ \frac{\log c_7 + c\log d_L}{\left(1+\frac{2\log 2}{\log 3}\right)\log d_L} - \frac{\log\log d_L 2^{n_L}}{\log d_L 2^{n_L}} \right\} \\ &\geqslant c_8 \left\{ \left(1+\frac{2\log 2}{\log 3}\right)^{-1} \left(\frac{\log c_7}{\log d_L} + c\right) - \frac{1}{e} \right\}. \end{aligned} \tag{7.13}$$

Note that $\frac{\log x}{x} \leqslant \frac{1}{e}$ for $x > 0$. Then (7.13) yields

$$c \leqslant \left(\frac{1}{c_8} + \frac{1}{e}\right)\left(1+\frac{2\log 2}{\log 3}\right) - \frac{\log c_7}{\log 3} = 114.72\cdots.$$

When L is an imaginary quadratic number field, it is known that $\zeta_L(\sigma) \neq 0$ for $\sigma \geqslant 1 - \left(\frac{\pi}{6}\sqrt{d_L}\right)^{-1}$. (See [48, proof of Lemma 11].) The result follows. □

Remarks.

(1) For the zero-free regions for $\zeta_L(s)$ see also [48].
(2) In [63], Zaman proved that, for d_L sufficiently large, $1-\omega_0 \gg d_L^{-21.3}$.

8. Proof of Theorem 1.1

Theorem 1.1 is ready to be proven. We will choose appropriate kernel functions $k(s)$ and estimate

$$k(1) - \sum_{\rho \in Z(\zeta_L)} |k(\rho)|$$

from below. From now on we denote by β_0 the exceptional zero of $\zeta_L(s)$ if it exists, and $\beta_0 = 1 - (2\log d_L)^{-1}$ otherwise. Our proof is divided into a sequence of lemmas.

LEMMA 8.1. — *We have*

$$k_1(1) - k_1(\beta_0) \geqslant \frac{9}{10}(\log x)^2 \min\{1, (1-\beta_0)\log x\} \tag{8.1}$$

and

$$k_2(1) - k_2(\beta_0) \geqslant \frac{9}{10}x^2 \min\{1, (1-\beta_0)\log x\}. \tag{8.2}$$

Proof. — We have

$$\begin{aligned} k_1(1) - k_1(\beta_0) &= (\log x)^2 - \left(\frac{x^{(\beta_0-1)} - x^{2(\beta_0-1)}}{1-\beta_0}\right)^2 \\ &= (\log x)^2 \varphi_6((1-\beta_0)\log x), \end{aligned}$$

where

$$\varphi_6(v) = 1 - \left(\frac{e^{-v} - e^{-2v}}{v}\right)^2.$$

It is easily verified that

$$\varphi_6(v) \geqslant \begin{cases} \varphi_6(1)v & \text{for } 0 < v \leqslant 1, \\ \varphi_6(1) & \text{for } v \geqslant 1 \end{cases}$$

with $\varphi_6(1) = 0.94592\cdots$. Hence $\varphi_6(v) \geqslant \varphi_6(1)\min\{1, v\}$, which yields (8.1). We have

$$k_2(1) - k_2(\beta_0) = x^2(1 - x^{(\beta_0-1)(\beta_0+2)}) \geqslant x^2\varphi_7((1-\beta_0)\log x),$$

where $\varphi_7(v) = 1 - e^{-\frac{5}{2}v}$. It is easy to see that

$$\varphi_7(v) \geqslant \begin{cases} \varphi_7(1)v & \text{for } 0 < v \leqslant 1, \\ \varphi_7(1) & \text{for } v \geqslant 1 \end{cases}$$

with $\varphi_7(1) = 0.91791\cdots$. Hence $\varphi_7(v) \geqslant \varphi_7(1)\min\{1,v\}$, which yields (8.2). ☐

In the following c_7 and c_8 are as in Theorem 7.3 (2).

Lemma 8.2. — *Suppose that $\beta_0 \leqslant 1 - {c_7}^2(\log d_L 3^{n_L})^{-2}$. We use the kernel function $k_1(s)$ and obtain*

$$\sum_{\substack{\rho\in Z(\zeta_L)\\ \rho\neq\beta_0}} |k_1(\rho)| \leqslant c_{13}\log d_L + c_{14}(\log d_L)^2\{(1-\beta_0)\log d_L\}^{2c_{12}\frac{\log x}{\log d_L}},$$

where $c_{12} = 6.8610\cdots\times 10^{-4}$, $c_{13} = 124.14\cdots$, and $c_{14} = 1.7700\cdots\times 10^8$.

Proof. — Write

$$\sum_{\substack{\rho\neq\beta_0\\ \rho\in Z(\zeta_L)}} |k_1(\rho)| = \sum_{|\rho-1|>1} |k_1(\rho)| + \sum_{|\rho-1|\leqslant 1} |k_1(\rho)|,$$

where $\sum_{|\rho-1|>1}$ (resp. $\sum_{|\rho-1|\leqslant 1}$) denotes that we sum over $\rho = \beta + i\gamma$ such that $\rho \in Z(\zeta_L)$ with $\rho \neq \beta_0$ and $|\rho - 1| > 1$ (resp. $|\rho-1| \leqslant 1$). Since

$$|k_1(\rho)| = \left|\frac{x^{2(\rho-1)} - x^{\rho-1}}{\rho - 1}\right|^2 \leqslant \frac{4x^{-2(1-\beta)}}{|\rho-1|^2},$$

it follows that

$$\begin{aligned}\sum_{|\rho-1|>1} |k_1(\rho)| &\leqslant 4\int_1^\infty \frac{1}{r^2}\,\mathrm{d}n(r;1)\\ &\leqslant 21.76\int_1^\infty \frac{(1+r)\{\log d_L + n_L\log(r+2)\}}{r^3}\,\mathrm{d}r\\ &\qquad\qquad \text{(by (5.5) and Proposition 5.6 (1))}\\ &\leqslant c_{13}\log d_L\end{aligned}$$

where $c_{13} = 21.76\left(\frac{3}{2} + \frac{2+15\log 3}{4\log 3}\right) = 124.14\cdots$. For the sum $\sum_{|\rho-1|\leqslant 1}|k_1(\rho)|$ we consider two cases separately.

(i) If an exceptional zero β_0 exists with $1-\beta_0 \leqslant \left(\frac{c_7}{3}\right)^2(\log d_L)^{-1}$, then

$$\frac{c_7}{(1-\beta_0)\log d_L \tau^{n_L}} \geqslant \frac{c_7}{3(1-\beta_0)\log d_L} \geqslant \{(1-\beta_0)\log d_L\}^{-\frac{1}{2}}.$$

Hence, by Theorem 7.3 (2)

$$1-\beta \geqslant c_8\frac{\log\{(1-\beta_0)\log d_L\}^{-\frac{1}{2}}}{\log d_L\tau^{n_L}} \geqslant c_{11}\frac{\log\{(1-\beta_0)\log d_L\}^{-1}}{\log d_L}$$

with $c_{11} = \frac{c_8}{6} = \frac{1}{462}$.

(ii) If $1-\beta_0 > \left(\frac{c_7}{3}\right)^2 (\log d_L)^{-1}$, then by (6.2)

$$1-\beta \geqslant (29.57 \log d_L \tau^{n_L})^{-1} \geqslant (88.71 \log d_L)^{-1}.$$

Set $c_{12} = \left\{177.42 \log\left(\frac{3}{c_7}\right)\right\}^{-1} = 6.8610\cdots \times 10^{-4}$. Then

$$(88.71)^{-1} = 2c_{12} \log\left(\frac{3}{c_7}\right) > c_{12} \log\{(1-\beta_0)\log d_L\}^{-1}$$

and

$$1-\beta > c_{12} \frac{\log\{(1-\beta_0)\log d_L\}^{-1}}{\log d_L}.$$

As $c_{11} > c_{12}$ we have

$$1-\beta > c_{12} \frac{\log\{(1-\beta_0)\log d_L\}^{-1}}{\log d_L}$$

in all cases. Let

$$B = c_{12} \frac{\log\{(1-\beta_0)\log d_L\}^{-1}}{\log d_L}.$$

Then

$$|k_1(\rho)| \leqslant \frac{4x^{2(\beta-1)}}{|\rho-1|^2} \leqslant \frac{4x^{-2B}}{|\rho-1|^2}.$$

By Proposition 5.6 (2),

$$\begin{aligned}\sum_{|\rho-1|\leqslant 1} |k_1(\rho)| &\leqslant 4x^{-2B} \int_B^1 \frac{1}{r^2}\, \mathrm{d}n(r;1) \\ &\leqslant 4x^{-2B} \left\{ n(1;1) + 20 \int_B^1 \frac{1+\frac{2f_2(2)}{5}\left(1+\frac{2\log 2}{\log 3}\right) r \log d_L}{r^3}\, \mathrm{d}r \right\} \\ &\qquad \text{(by Proposition 5.6 (2))} \\ &\leqslant 40x^{-2B} \left\{ B^{-2} + \frac{4f_2(2)}{5}\left(1+\frac{2\log 2}{\log 3}\right) B^{-1} \log d_L \right. \\ &\qquad \left. - \frac{2f_2(2)}{5}\left(1+\frac{2\log 2}{\log 3}\right) \log d_L \right\} \\ &\leqslant c_{14} (\log d_L)^2 \{(1-\beta_0)\log d_L\}^{2c_{12}\frac{\log x}{\log d_L}}\end{aligned}$$

where

$$c_{14} = \frac{40}{c_{12}\log 2} \left\{ \frac{1}{c_{12}\log 2} + \frac{4f_2(2)}{5}\left(1+\frac{2\log 2}{\log 3}\right) \right\} = 1.7700\cdots \times 10^8.$$

For the last inequality we used (6.1), which yields

$$B = c_{12} \frac{\log\{(1-\beta_0)\log d_L\}^{-1}}{\log d_L} \geqslant \frac{c_{12}\log 2}{\log d_L}. \qquad \square$$

We have therefore

$$k_1(1) - \sum_{\rho\in Z(\zeta_L)} |k_1(\rho)| \geqslant \frac{9}{10}(\log x)^2 \min\{1,(1-\beta_0)\log x\} - c_{13}\log d_L - c_{14}(\log d_L)^2\left\{(1-\beta_0)\log d_L\right\}^{2c_{12}\frac{\log x}{\log d_L}}. \tag{8.3}$$

Note that for $x \geqslant 101$

$$\begin{aligned}\mu_1 k_1\left(-\frac{1}{2}\right)\log d_L + n_L\left\{k_1(0)+\nu_1 k_1\left(-\frac{1}{2}\right)\right\} &\leqslant \left\{\frac{2}{\log 3}\left(x^{-2}-x^{-1}\right)^2 + \frac{4}{9}\left(\mu_1+\frac{2}{\log 3}\nu_1\right)\left(x^{-3}-x^{-3/2}\right)^2\right\}\log d_L\\ &\leqslant \left\{\frac{2}{\log 3}x^{-2}+\frac{4}{9}\left(\mu_1+\frac{2}{\log 3}\nu_1\right)x^{-3}\right\}\log d_L\\ &\leqslant c_{15}x^{-2}\log d_L,\end{aligned} \tag{8.4}$$

where

$$c_{15} = \frac{2}{\log 3} + \frac{4}{909}\left(\mu_1 + \frac{2}{\log 3}\nu_1\right) = 1.9792\cdots.$$

Gathering together the bounds (3.1), (4.3), (8.3), and (8.4) we conclude the following:

Lemma 8.3. — *Suppose that $\beta_0 \leqslant 1 - {c_7}^2(\log d_L 3^{n_L})^{-2}$. We have then*

$$\begin{aligned}\frac{|G|}{|C|}\sum_{\mathfrak{p}\in P(C)}(\log N_{K/\mathbb{Q}}\mathfrak{p})\widehat{k_1}(N_{K/\mathbb{Q}}\mathfrak{p}) &\geqslant \frac{9}{10}(\log x)^2\min\{1,(1-\beta_0)\log x\} - c_{13}\log d_L\\ &\quad - c_{14}(\log d_L)^2\left\{(1-\beta_0)\log d_L\right\}^{2c_{12}\frac{\log x}{\log d_L}} - c_{15}x^{-2}\log d_L\\ &\quad - \alpha_3\frac{|G|}{|C|}\frac{\log x}{x}\log d_L.\end{aligned} \tag{8.5}$$

Lemma 8.4. — *Suppose that $\beta_0 \leqslant 1 - {c_7}^2(\log d_L 3^{n_L})^{-2}$. For $\log x = c_{16}\log d_L$ with $c_{16} = 3144.25$, we have*

$$\sum_{\mathfrak{p}\in P(C)}(\log N_{K/\mathbb{Q}}\mathfrak{p})\widehat{k_1}(N_{K/\mathbb{Q}}\mathfrak{p}) > 0.$$

In particular, there is a prime $\mathfrak{p}\in P(C)$ with $N_{K/\mathbb{Q}}\mathfrak{p} \leqslant x^4 = d_L^{4c_{16}}$.

Proof. — Let $\log x = c_{16} \log d_L$.

(i) Suppose that $1 \leqslant c_{16}(1-\beta_0)\log d_L$. (8.5) and (6.1) yield

$$(\log d_L)^{-2}\frac{|G|}{|C|}\sum_{\mathfrak{p}\in P(C)}(\log N_{K/\mathbb{Q}}\mathfrak{p})\widehat{k_1}(N_{K/\mathbb{Q}}\mathfrak{p}) \geqslant \left\{\frac{9}{10}c_{16}^2 - c_{14}\left(\frac{1}{2}\right)^{2c_{12}c_{16}}\right\} - \epsilon_1,$$

where

$$\epsilon_1 = \frac{c_{13}}{\log d_L} + \frac{c_{15}}{d_L^{2c_{16}}\log d_L} + \frac{2\alpha_3 c_{16}\log d_L}{d_L^{c_{16}}\log 3}.$$

(Note that $\frac{|G|}{|C|} \leqslant |G| = \frac{n_L}{n_K} \leqslant n_L \leqslant \frac{2}{\log 3}\log d_L$.) For $c_{16} = 3144.25$, we have

$$\frac{9}{10}c_{16}^2 > c_{14}\left(\frac{1}{2}\right)^{2c_{12}c_{16}} + \epsilon_1.$$

(ii) Suppose that $1 \geqslant c_{16}(1-\beta_0)\log d_L$. Since $1-\beta_0 \geqslant c_7^2(\log d_L 3^{n_L})^{-2} \geqslant \left(\frac{c_7}{3}\right)^2(\log d_L)^{-2}$, (8.5) and (6.1) yield

$$\begin{aligned}\{(1-\beta_0)\log d_L\}^{-1}&(\log d_L)^{-2}\frac{|G|}{|C|}\sum_{\mathfrak{p}\in P(C)}(\log N_{K/\mathbb{Q}}\mathfrak{p})\widehat{k_1}(N_{K/\mathbb{Q}}\mathfrak{p})\\ &\geqslant \frac{9}{10}c_{16}^3 - c_{14}\{(1-\beta_0)\log d_L\}^{2c_{12}c_{16}-1} - \frac{c_{13}}{(1-\beta_0)(\log d_L)^2}\\ &\quad - \frac{c_{15}}{d_L^{2c_{16}}(1-\beta_0)(\log d_L)^2} - \frac{2\alpha_3 c_{16}}{d_L^{c_{16}}(1-\beta_0)\log 3}\\ &\geqslant \frac{9}{10}c_{16}^3 - c_{14}\left(\frac{1}{2}\right)^{2c_{12}c_{16}-1} - c_{13}\left(\frac{3}{c_7}\right)^2 - \epsilon_2,\end{aligned}$$

where

$$\epsilon_2 = \left(\frac{3}{c_7}\right)^2\left\{\frac{c_{15}}{d_L^{2c_{16}}} + \frac{2\alpha_3 c_{16}}{\log 3}\frac{(\log d_L)^2}{d_L^{c_{16}}}\right\}.$$

For $c_{16} = 1261$, we have

$$\frac{9}{10}c_{16}^3 > c_{14}\left(\frac{1}{2}\right)^{2c_{12}c_{16}-1} + c_{13}\left(\frac{3}{c_7}\right)^2 + \epsilon_2.$$

The result follows. □

Lemma 8.5. — *Suppose that* $1-\beta_0 \leqslant {c_7}^2(\log d_L 3^{n_L})^{-2}$. *We have then*

$$\frac{|G|}{|C|} \sum_{\substack{\mathfrak{p}\in P(C) \\ N_{K/\mathbb{Q}}\mathfrak{p}\leqslant x^5}} (\log N_{K/\mathbb{Q}}\mathfrak{p})\widehat{k_2}(N_{K/\mathbb{Q}}\mathfrak{p}) \tag{8.6}$$

$$\geqslant \frac{9}{10}x^2\min\{1,(1-\beta_0)\log x\} - c_{20}x\log d_L - c_{21}x^2(1-\beta_0)^{2c_{19}\frac{\log x}{\log d_L}}\log d_L - c'_{15}\log d_L - \alpha_4\frac{|G|}{|C|}x(\log x)^{\frac{1}{2}}\log d_L,$$

where $c_{20} = 19.16\cdots$, $c_{21} = 6.1522\cdots$, $c_{19} = \frac{c_8}{6} = \frac{1}{462}$, *and* $c'_{15} = 1.8291\cdots$.

Proof. — For $\rho=\beta+i\gamma \in Z(\zeta_L)$ with $|\gamma|\leqslant 1$ we have by Theorem 7.3 (2)

$$1-\beta \geqslant c_8\frac{\log\left\{\frac{c_7}{(1-\beta_0)\log d_L 3^{n_L}}\right\}}{\log d_L 3^{n_L}} \geqslant c_{19}\frac{\log(1-\beta_0)^{-1}}{\log d_L}$$

with $c_{19} = \frac{c_8}{6} = \frac{1}{462}$. Since

$$|k_2(\rho)| \leqslant x^{\beta^2+\beta} \leqslant x^{2-2(1-\beta)} \leqslant x^2(1-\beta_0)^{2c_{19}\frac{\log x}{\log d_L}},$$

$$\sum_{|\gamma|\leqslant 1}|k_2(\rho)| \leqslant x^2(1-\beta_0)^{2c_{19}\frac{\log x}{\log d_L}}\sum_{|\gamma|\leqslant 1}1 \leqslant c_{21}x^2(1-\beta_0)^{2c_{19}\frac{\log x}{\log d_L}}\log d_L \quad \text{by (5.5)}$$

with $c_{21} = 2.72\left(1+\frac{2\log 2}{\log 3}\right) = 6.1522\cdots$. For zeros $\rho=\beta+i\gamma$ with $|\gamma|>1$ and $x\geqslant 10^{10}$ we have

$$\begin{aligned}\sum_{|\gamma|>1}|k_2(\rho)| &\leqslant x^2\sum_{m=1}^{\infty}\{n_L(2m)+n_L(-2m)\}x^{-(2m-1)^2} \\ &\leqslant 5.44x^2\sum_{m=1}^{\infty}\{\log d_L + n_L\log(2m+2)\}x^{-(2m-1)^2} \quad \text{by (5.5)} \\ &\leqslant c_{20}x\log d_L,\end{aligned}$$

where

$$c_{20} = 5.44\sum_{m=1}^{\infty}\left\{1+\frac{2}{\log 3}\log(2m+2)\right\}10^{-40m^2+40m} = 19.16\cdots.$$

It follows that for $x \geqslant 10^{10}$

$$(8.7)\quad k_2(1) - \sum_{\rho} |k_2(\rho)| \geqslant \frac{9}{10} x^2 \min\{1, (1-\beta_0)\log x\} \\ - c_{21} x^2 (1-\beta_0)^{2c_{19}\frac{\log x}{\log d_L}} \log d_L - c_{20} x \log d_L.$$

Note that for $x \geqslant 10^{10}$

$$(8.8)\quad \mu_2 k_2\left(-\frac{1}{2}\right) \log d_L + n_L \left\{k_2(0) + \nu_2 k_2\left(-\frac{1}{2}\right)\right\} \\ \leqslant \left\{\frac{2}{\log 3} + \left(\mu_2 + \frac{2}{\log 3}\nu_2\right) x^{-\frac{1}{4}}\right\} \log d_L \\ \leqslant c_{15}' \log d_L,$$

where

$$c_{15}' = \frac{2}{\log 3} + \left(\mu_2 + \frac{2}{\log 3}\nu_2\right) 10^{-\frac{5}{2}} = 1.8291\cdots.$$

Combining (3.2), (4.3), (8.7), and (8.8) yields (8.6). □

LEMMA 8.6. — *Suppose that $1 - \beta_0 \leqslant {c_7}^2(\log d_L 3^{n_L})^{-2}$. If $x = d_L^{c_{23}}$ with $c_{23} = 179$, then*

$$\sum_{\substack{\mathfrak{p}\in P(C)\\ N_{K/\mathbb{Q}}\mathfrak{p}\leqslant x^5}} (\log N_{K/\mathbb{Q}}\mathfrak{p}) \widehat{k_2}(N_{K/\mathbb{Q}}\mathfrak{p}) > 0.$$

In particular, there is a prime $\mathfrak{p} \in P(C)$ with $N_{K/\mathbb{Q}}\mathfrak{p} \leqslant x^5 = d_L^{5c_{23}}$.

Proof. — Let $x = d_L^{c_{23}}$. Then (8.6) becomes

$$\frac{|G|}{|C|} \sum_{\substack{\mathfrak{p}\in P(C)\\ N_{K/\mathbb{Q}}\mathfrak{p}\leqslant x^5}} (\log N_{K/\mathbb{Q}}\mathfrak{p}) \widehat{k_2}(N_{K/\mathbb{Q}}\mathfrak{p}) \\ \geqslant \frac{9}{10} d_L^{2c_{23}} \min\{1, c_{23}(1-\beta_0)\log d_L\} - c_{20} d_L^{c_{23}} \log d_L \\ - c_{21} d_L^{2c_{23}} (1-\beta_0)^{2c_{19}c_{23}} \log d_L - c_{15}' \log d_L - \frac{2\alpha_4 c_{23}^{\frac{1}{2}}}{\log 3} d_L^{c_{23}} (\log d_L)^{\frac{5}{2}}.$$

When $1 \leqslant c_{23}(1-\beta_0)\log d_L$, we have

$$d_L^{-2c_{23}} \frac{|G|}{|C|} \sum_{\substack{\mathfrak{p}\in P(C)\\ N_{K/\mathbb{Q}}\mathfrak{p}\leqslant x^5}} (\log N_{K/\mathbb{Q}}\mathfrak{p})\widehat{k_2}(N_{K/\mathbb{Q}}\mathfrak{p})$$
$$\geqslant \frac{9}{10} - c_{21}\{c_7^2(\log d_L)^{-2}\}^{2c_{19}c_{23}}\log d_L - \epsilon_3$$
$$= \frac{9}{10} - c_{21}c_7^{4c_{19}c_{23}}(\log d_L)^{1-4c_{19}c_{23}} - \epsilon_3,$$

where

$$\epsilon_3 = c_{20}\frac{\log d_L}{d_L^{c_{23}}} + c'_{15}\frac{\log d_L}{d_L^{2c_{23}}} + \frac{2\alpha_4 c_{23}^{\frac{1}{2}}}{\log 3}\frac{(\log d_L)^{\frac{5}{2}}}{d_L^{c_{23}}}.$$

If $c_{23} = (4c_{19})^{-1} = 114.76\cdots$, then

$$\frac{9}{10} > c_{21}c_7 + \epsilon_3.$$

When $1 \geqslant c_{23}(1-\beta_0)\log d_L$, using Corollary 7.4 we have

$$d_L^{-2c_{23}}\{(1-\beta_0)\log d_L\}^{-1}\frac{|G|}{|C|} \sum_{\substack{\mathfrak{p}\in P(C)\\ N_{K/\mathbb{Q}}\mathfrak{p}\leqslant x^5}} (\log N_{K/\mathbb{Q}}\mathfrak{p})\widehat{k_2}(N_{K/\mathbb{Q}}\mathfrak{p})$$
$$\geqslant \frac{9}{10}c_{23} - \frac{c_{20}}{d_L^{c_{23}}(1-\beta_0)} - c_{21}(1-\beta_0)^{2c_{19}c_{23}-1} - \frac{c'_{15}}{d_L^{2c_{23}}(1-\beta_0)}$$
$$- \frac{2\alpha_4 c_{23}^{\frac{1}{2}}}{\log 3}\frac{(\log d_L)^{\frac{3}{2}}}{d_L^{c_{23}}(1-\beta_0)}$$
$$\geqslant \frac{9}{10}c_{23} - \epsilon_4,$$

where

$$\epsilon_4 = \frac{c_{20}}{d_L^{c_{23}-c_{10}}} + c_{21}c_7^{4c_{19}c_{23}-2}(\log d_L)^{2-4c_{19}c_{23}}$$
$$+ \frac{c'_{15}}{d_L^{2c_{23}-c_{10}}} + \frac{2\alpha_4 c_{23}^{\frac{1}{2}}}{\log 3}\frac{(\log d_L)^{\frac{3}{2}}}{d_L^{c_{23}-c_{10}}}.$$

If $c_{23} = 179$, then

$$\frac{9}{10}c_{23} > \epsilon_4.$$

The result follows. □

Lemma 8.4 and 8.6 yield Theorem 1.1.

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