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An exponential formula for one-parameter semi-groups of nonlinear transformations

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For a complete normed linear space S consider a function T from $[0, \infty)$ to the set of continuous transformations from S to S which satisfies:

(1) $T(x)T(y)=T(x+y)$ if $x, y>0$,

(2) $\| T(x)p-T(x)q\| \leqq \| p-q\|$ if $x\geqq 0$, p, q are in S,

(3) if p is in S and $g_p(x)=T(x)p$ for all x in $[0, \infty)$ then g_p is continuous and $\lim_{x\to 0+} g_p(x)=p$.

If it is also specified that $T(x)$ is linear for all $x\geqq 0$, then one has a semi-group about which the following is known ([**1**] chapters 10, 11 and [3] sections 142, 143):

For all p in some dense subset of S, $g_p'(0)$ exists and if $Ap=g_p'(0)$ for all p for which $g_p'(0)$ exists, then $(I-xA)^{-1}$ exists, has domain S and is continuous for all $x\geqq 0$. Moreover, if p is in S and $x\geqq 0$,

$$(*)\qquad \lim_{n\to\infty} \| (I-(x/n)A)^{-n}p-T(x)p\| =0\,.$$

It is the purpose of this note to add to assumptions (1)-(3) a differentiability condition (which, it turns out, holds in the linear special case) which implies an "exponential formula" suggested by (*). The results of this note give a nonlinear version of the linear strong case of [**1**] (section 11.5); previous work [**2**] (section 3) of this author gave a nonlinear version of the linear uniform case of [**1**] (section 11.2).

The differentiability condition mentioned above is:

(4) there is a dense subset D of S such that if p is in D, then g_p' is continuous with domain $[0, \infty)$.

If $\delta>0$, denote $(1/\delta)[T(\delta)-I]$ by A_δ. The main result of this note follows.

THEOREM. *If* (1)-(4) *hold, p is in S and $x\geqq 0$, then*

$$\lim_{n\to\infty}\limsup_{\delta\to 0+} \| (I-(x/n)A_\delta)^{-n}p-T(x)p\| =0\,.$$

Consider first some lemmas.

LEMMA 1. *Under conditions* (1)-(3), *if $\delta, x>0$, then $(I-xA_\delta)^{-1}$ exists and has domain S.*

PROOF. Suppose w is in S. A unique point y of S is sought so that $(I-xA_\delta)y=w$, that is, $y-(x/\delta)T(\delta)y+(x/\delta)y=w$, that is, $y=[\delta/(\delta+x)]w+[x/(\delta+x)]T(\delta)y$. Define $Kz=[\delta/(\delta+x)]w+[x/(\delta+x)]T(\delta)z$ for all z in S. It is easily seen that K is a contraction mapping. Hence there is a unique y in S so that $y=Ky$. This proves the lemma.

LEMMA 2. *Under conditions* (1)-(3), *if* $\delta, x>0$, *then*

$$\|(I-xA_\delta)^{-1}u-(I-xA_\delta)^{-1}v\| \leqq \|u-v\|$$

for all u, v *in* S.

PROOF. Suppose that $(I-xA_\delta)^{-1}u=y$ and $(I-xA_\delta)^{-1}v=z$. Then $y=[\delta/(\delta+x)]u+[x/(\delta+x)]T(\delta)y$ and $z=[\delta/(\delta+x)]v+[x/(\delta+x)]T(\delta)z$ and hence, $\|y-z\| \leqq [\delta/(\delta+x)]\|u-v\|+[x/(\delta+x)]\|T(\delta)y-T(\delta)z\| \leqq [\delta/(\delta+x)]\|u-v\|+[x/(\delta+x)]\|y-z\|$. But this gives that $\|y-z\| \leqq \|u-v\|$ and hence the lemma is established.

LEMMA 3. *Under conditions* (1)-(3), *if* $\delta, x>0$, *then* $\|(I-xA_\delta)^{-1}p-T(x)p\| \leqq x\|A_xp-A_\delta T(x)p\|$ *for each* p *in* S.

PROOF.

$$\|(I-xA_\delta)^{-1}p-T(x)p\| \leqq \|p-(I-xA_\delta)T(x)p\|$$
$$=\|[T(x)-I]p-xA_\delta T(x)p\|=x\|A_xp-A_\delta T(x)p\| .$$

LEMMA 4. *Suppose that each of* L *and* M *is a continuous transformation from* S *to* S *such that if* u *and* v *are in* S, $\|Lu-Lv\| \leqq \|u-v\|$. *Then for each positive integer* n, $\|L^np-M^np\| \leqq \sum_{i=1}^{n}\|LM^{i-1}p-M^ip\|$ *for all* p *in* S.

PROOF.

$$\|L^np-M^np\|=\|\sum_{i=1}^{n}(L^{n-i+1}M^{i-1}p-L^{n-i}M^ip)\|$$
$$\leqq \sum_{i=1}^{n}\|L^{n-i+1}M^{i-1}p-L^{n-i}M^ip\| \leqq \sum_{i=1}^{n}\|LM^{i-1}p-M^ip\| .$$

LEMMA 5. *Under condition* (4), *suppose that* p *is in* D, R *is a bounded subinterval of* $[0,\infty)$ *and* $\varepsilon>0$. *There is a* $\delta>0$ *such that if* x, y *are in* R *and* $0<|x-y|<\delta$, *then* $\max_{w \text{ in } [x,y]}\|(x-y)^{-1}[g_p(x)-g_p(y)]-g_p'(w)\|<\varepsilon$.

PROOF. For x and y in R, $x\neq y$ and w in $[x,y]$, $g_p(x)-g_p(y)=\int_y^x g_p'$ and hence

$$\|(x-y)^{-1}[g_p(x)-g_p(y)]-g_p'(w)\|=\|(x-y)^{-1}$$
$$\int_y^x(g_p'-g_p'(w))\| \leqq \max_{c \text{ in } [x,y]}\|g_p'(c)-g_p'(w)\| .$$

The uniform continuity of g_p' on bounded intervals then gives the lemma.

PROOF OF THE THEOREM. The conclusion is obvious if $x=0$. Suppose $x>0$. Assume first that r is in D. If $\delta>0$ and n is a positive integer, then

$$\|(I-(x/n)A_\delta)^{-n}r-T(x)r\| = \|[(I-(x/n)A_\delta)^{-1}]^n r-[T(x/n)]^n r\|$$

$$\leqq \sum_{i=1}^{n} \|(I-(x/n)A_\delta)^{-1}T(x(i-1)/n)r-T(x/n)T(x(i-1)/n)r\|$$

$$\leqq \sum_{i=1}^{n} (x/n)\|A_{x/n}T(x(i-1)/n)r-A_\delta T(xi/n)r\|$$

$$=(x/n)\sum_{i=1}^{n}\|(n/x)[g_r(xi/n)-g_r(x(i-1)/n)]-(1/\delta)[g_r(\delta+xi/n)-g_r(xi/n)]\|$$

$$\leqq (x/n)\{\sum_{i=1}^{n}\|(n/x)[g_r(xi/n)-g_r(x(i-1)/n)]-g_r'(xi/n)\|$$

$$+\sum_{i=1}^{n}\|(1/\delta)[g_r(\delta+ix/n)-g_r(xi/n)]-g_r'(xi/n)\|\}.$$

Suppose now, in addition, that $\varepsilon>0$. Denote by δ' a positive number less than 1 so that if $0\leqq v, u\leqq x+1$ and $0<|u-v|<\delta'$, then $\max_{w \text{ in } [u,v]} \|(u-v)^{-1}[g_r(u)-g_r(v)]-g_r'(w)\|<\varepsilon/(4x)$. Denote by N an integer so that $x/N<\delta'$. If n is an integer greater than N and $0<\delta<\delta'$, $\|(I-(x/n)A_\delta)^{-n}r-T(x)r\| \leqq (x/n)\sum_{i=1}^{n}(2\varepsilon)/(4x)=\varepsilon/2$. From this it follows that $\limsup_{\delta\to 0+}\|(I-(x/n)A_\delta)^{-n}r-T(x)r\|<\varepsilon$.

Suppose that p is in S and $\varepsilon>0$. Since D is dense in S, there is a point r of D such that $\|p-r\|<\varepsilon/6$. By the above argument, there is an integer N and a $\delta'>0$ such that if n is an integer greater than N and $0<\delta<\delta'$, $\|(I-(x/n)A_\delta)^{-n}r-T(x)r\|<\varepsilon/6$. For δ and n chosen in such a way, $\|T(x)r-T(x)p\|<\varepsilon/6$ and $\|(I-(x/n)A_\delta)^{-n}r-(I-(x/n)A_\delta)^{-n}p\|<\varepsilon/6$ (by repeated application of Lemma 2) which gives $\|(I-(x/n)A_\delta)^{-n}p-T(x)p\|<\varepsilon/2$. From this it follows that $\limsup_{\delta\to 0+}\|(I-(x/n)A_\delta)^{-n}p-T(x)p\|<\varepsilon$ for all integers n greater than N. This proves the theorem.

In closing it is noted that conditions (1)-(3) do not imply (4). This can be seen by considering the case in which S is E_1 and

$$T(x)p=\begin{cases} p-x & \text{if } p\geqq 1 \text{ and } p-x\geqq 1,\ x\geqq 0 \\ 1 & \text{if } p\geqq 1 \text{ and } p-x<1,\ x\geqq 0 \\ p & \text{if } p<1,\ x\geqq 0. \end{cases}$$

This author considers it likely that conditions (1)-(3) do imply interesting differentiability conditions and that the conclusion to the theorem (or perhaps a stronger conclusion) can be obtained using only conditions (1)-(3) together, perhaps, with some condition much weaker than (4). Investigations into these matters may well lead to a theory of semi-groups of nonlinear transformations which parallels rather completely the well developed linear case.

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References

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