ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
*Classe di Scienze*

BJÖRN GUSTAFSSON
MIHAI PUTINAR

**An exponential transform and regularity of free boundaries in two dimensions**

*Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série*, tome 26, no 3 (1998), p. 507-543.

<http://www.numdam.org/item?id=ASNSP_1998_4_26_3_507_0>

© Scuola Normale Superiore, Pisa, 1998, tous droits réservés.

L'accès aux archives de la revue « Annali della Scuola Normale Superiore di Pisa, Classe di Scienze » (http://www.sns.it/it/edizioni/riviste/annaliscienze/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/legal.php). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
*Article numérisé dans le cadre du programme Numérisation de documents anciens mathématiques*
http://www.numdam.org/

Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)
Vol. XXVI (98), pp. 507-543

# An Exponential Transform and Regularity of Free Boundaries in Two Dimensions

BJÖRN GUSTAFSSON - MIHAI PUTINAR

*To professor Harold S. Shapiro on the occasion of his seventieth birthday and to the twenty-fifth anniversary of his quadrature domains.*

**Abstract.** We investigate the basic properties of the exponential transform $E(z, w) = \exp\left[ -\frac{1}{\pi} \int_\Omega \times \frac{dA(\zeta)}{(\zeta - z)(\bar{\zeta} - \bar{w})}\right]$ $(z, w \in \mathbb{C})$ of a domain $\Omega \subset \mathbb{C}$ and compute it in some simple cases. The main result states that if the Cauchy transform of the characteristic function of $\Omega$ has an analytic continuation from $\mathbb{C} \setminus \Omega$ across $\partial\Omega$ then the same is true for $E(z, w)$, in both variables. If $F(z, \bar{w})$ denotes this analytic-antianalytic continuation it follows that $\partial\Omega$ is contained in a real analytic set, namely the zero set of $F(z, \bar{z})$. This gives a new approach to the regularity theory for free boundaries in two dimensions.

**Mathematics Subject Classification (1991):** 32D15 (primary), 30E20, 35R35, 47B20 (secondary).

## 1. - Introduction

The exponential of the Cauchy transform of a function of a real variable appears in the canonical representation of several classes of analytic functions in the upper half-plane. One of these representations was used by R. Nevanlinna in his study of the moment problem on the line; later the same exponential expression was related to perturbation determinants of selfadjoint operators, see [K-N, Appendix]. The present paper is devoted to the analytic aspects of a two-dimensional analogue of exponentials of Cauchy transforms which have appeared in operator theory as a generalization of perturbation determinants.

A specific instance of such a generalization is a formula, due to R. W. Carey and J. D. Pincus, for the determinant of the multiplicative commutator resolvent for a pure hyponormal operator $T$ with rank one self-commutator. It reads as

Pervenuto alla Redazione il 24 febbraio 1997 e in forma definitiva il 6 ottobre 1997.

follows.

$$\begin{aligned}(1.1)\qquad &\det\left((T-w)^{*^{-1}}(T-z)(T-w)^{*}(T-z)^{-1}\right)\\ &=\exp\left[-\frac{1}{\pi}\int_{\mathbb{C}}\frac{g(\zeta)dA(\zeta)}{(\zeta-z)\left(\bar{\zeta}-\bar{w}\right)}\right],\qquad |z|,|w|>\|T\|\ .\end{aligned}$$

In this formula the function $g$ satisfies $0 \le g \le 1$ and is a kind of "spectral parameter", called the principal function, of $T$ and it characterizes $T$ up to unitary equivalence. The right member of (1.1) is what we call the exponential transform of $g$.

Quite independent of its operator theoretic importance the exponential transform turns out to be both interesting and useful from a pure function theoretic standpoint. This is basically the point of view we take in the present paper. First, in Sections 2 and 3, we derive some basic properties of the exponential transform and give a number of examples, mainly from the theory of quadrature domains.

Our main results then appear in Sections 4 and 5, and they concern analytic continuation properties of the exponential transform and applications of this to regularity questions for free boundaries. In fact, we prove that if the Cauchy transform $\widehat{\chi}_{\Omega}(z)$ of a domain $\Omega$ has an analytic continuation from outside $\Omega$ across $\partial\Omega$ then the same will be true, in both variables, for the exponential transform $E_{\Omega}(z,w)$ of $g=\chi_{\Omega}$.

If $F(z,\bar{w})$ denotes this analytic continuation ($F$ will be analytic in $z$, antianalytic in $w$) it follows, using that $E_{\Omega}(z,z) = 0$ in $\Omega$ and on (most of) $\partial\Omega$, that

$$\partial\Omega \subset \left\{z : F(z,\bar{z}) = 0\right\}.$$

Thus $\partial\Omega$ is contained in a real analytic set. This is the kind of regularity statement we obtain for boundaries $\partial\Omega$ admitting analytic continuation of the Cauchy transform. Then one can easily go further to obtain complete classification of possible singular points on $\partial\Omega$, but this we do not discuss because all this has already been done.

Indeed, there is a complete regularity theory for free boundaries in two dimensions of the type we are considering here, due to M. Sakai [Sa3], [Sa4], [Sa5]. Sakai even starts from weaker assumptions than we do, so our regularity result is just a special case of Sakai's theory. Nevertheless we think there are some good points with our approach: it is new and completely different from Sakai's approach, it is both technically and conceptually simple and it directly provides a good defining function for the free boundary.

Our proof uses a few real variable tools, namely a sequence of exhaustion functions due to L. Ahlfors and L. Bers and a lemma, due to L. Caffarelli, L. Karp, A. Margulis and H. Shahgholian, showing that $\partial\Omega$ must have area measure zero. The rest of the proof consists of complex analysis, and for this we actually give three different approaches, two function theoretic ones and one operator theoretic, the latter in Section 5. In Section 5 we also obtain a somewhat stronger version of the analytic continuation, saying that a certain resolvent extends.

ACKNOWLEDGEMENTS. Most of this paper grew out during a couple of summer weeks at KTH, and the material was then daily discussed with Peter Ebenfelt, Lavi Karp, Henrik Shahgholian, Harold S. Shapiro and occasionally others. We thank them all for their generous contribution of ideas and for letting us include some of their unpublished results here (see in particular Lemma 4.4). We also thank Siv Sandvik for typing our manuscript. The second author thanks the Department of Mathematics of KTH for its hospitality.

This work has been supported by NFR (Sweden) grants Nos. M-AA/MA 08793-310 and R-RA 08793-313 and NSF (USA) grant No. 9500954.

SOME NOTATIONS USED

$\mathbb{D}(a,r)=\{z\in\mathbb{C}:|z-a|<r\}$ (for $a\in\mathbb{C}, r\geq 0$),

$\mathbb{D}=\mathbb{D}(0,1)$,

$\mathbb{D}_{z,w}=\mathbb{D}\left(\frac{z+w}{2},\left|\frac{z-w}{2}\right|\right)$,

$\Delta=\{(z,w)\in\mathbb{C}^2:z=w\}$,

$\Delta_D=\{(z,z)\in\Delta:z\in D\}$ (if $D\subset\mathbb{C}$),

$Z$ see (2.10),

$S(z)$ see (3.4),

$D^c=\mathbb{C}\setminus D$ (if $D\subset\mathbb{C}$),

$dA$ area measure (two dimensional Lebesgue measure) in $\mathbb{C}$.

## 2. – Definition and elementary properties

DEFINITION 2.1. For $0\leq g\in L^\infty(\mathbb{C})$ the exponential transform of $g$ is defined as

$$E_g(z,w)=\exp\left[-\frac{1}{\pi}\int_{\mathbb{C}}\frac{g(\zeta)dA(\zeta)}{(\zeta-z)(\bar{\zeta}-\bar{w})}\right]\qquad(z,w\in\mathbb{C}).$$

In case $g=\chi_\Omega$ for some $\Omega\subset\mathbb{C}$ we write $E_\Omega$ in place of $E_{\chi_\Omega}$. Thus

$$E_\Omega(z,w)=\exp\left[-\frac{1}{\pi}\int_{\Omega}\frac{dA(\zeta)}{(\zeta-z)(\bar{\zeta}-\bar{w})}\right]. \tag{2.1}$$

The above definition requires some justifications. First, if $z = w$ then

$$E_g(z, z) = \exp\left[-\frac{1}{\pi}\int_{\mathbb{C}} \frac{g(\zeta)}{|\zeta - z|^2} dA(\zeta)\right],$$

where the integral is either a finite number $\geq 0$ or $+\infty$. In the latter case $E_g(z, z)$ is understood to be zero.

If $z \neq w$ then we are integrating a locally integrable function and problems can only arise for large $\zeta$.

However,

$$\mathrm{Im}\frac{g(\zeta)}{(\zeta - z)\left(\bar{\zeta} - \bar{w}\right)} = \frac{g(\zeta)\mathrm{Im}\left(\bar{z}w - \bar{\zeta}w - \zeta\bar{z}\right)}{|\zeta - z|^2|\zeta - w|^2}$$

is integrable over all $\mathbb{C}$ and

$$\mathrm{Re}\frac{g(\zeta)}{(\zeta - z)\left(\bar{\zeta} - \bar{w}\right)} \geq 0$$

for large $|\zeta|$ (see more precisely Lemma 2.4 below). Thus the negative part of $\mathrm{Re}\frac{g(\zeta)}{(\zeta-z)(\bar{\zeta}-\bar{w})}$ is integrable so that $\int_{\mathbb{C}} \mathrm{Re}\frac{g(\zeta)}{(\zeta-z)(\bar{\zeta}-\bar{w})} dA(\zeta)$ either is a finite real number or $+\infty$. In the latter case $E_g(z, w)$ is defined to be zero, otherwise it is a nonzero complex number.

In summary, $E_g(z, w)$ is a well-defined complex number for any pair $(z, w) \in \mathbb{C}^2$.

REMARK 2.2. It is easy to see that the following holds: if

$$\int_{\mathbb{C}\setminus\mathbb{D}} \frac{g(\zeta)dA(\zeta)}{|\zeta|^2} = +\infty \tag{2.2}$$

then $E_g(z, w) \equiv 0$, otherwise $E_g(z, w) \neq 0$ whenever $z \neq w$. Clearly (2.2) can occur only if $\operatorname{supp} g$ is not compact.

We recall here also the definition of the Cauchy transform: For $g \in L^\infty(\mathbb{C})$ with (say) compact support it is

$$\hat{g}(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{g(\zeta)dA(\zeta)}{\zeta - z}$$

and satisfies

$$\bar{\partial}\hat{g} = g$$

in the sense of distributions. More generally, the Cauchy transform may be defined for any distribution $\mu$ of compact support as the convolution between $\mu$ and the fundamental solution $-\frac{1}{\pi z}$ of $\bar{\partial}_z$.

In the rest of this section we go through some of the basic properties and a couple of examples of the exponential transform, usually for the case $g = \chi_\Omega$, $\Omega \subset \mathbb{C}$. Much of this material is not new (cf. in particular [MP], [P1], [P3]) and some is very elementary (like the first property below). For a characterization of which functions of two complex variables are of the form $E_g(z, w)$ for some $g$, see [P3].

a) SCALING PROPERTIES

For $0 \leq g,\ g_j \in L^{\infty}(\mathbb{C}),\ \alpha > 0$

$$E_g(z, w) = \overline{E_g(w, z)}, \tag{2.3}$$

$$E_{g_1+g_2}(z, w) = E_{g_1}(z, w) E_{g_2}(z, w), \tag{2.4}$$
$$E_{\alpha g}(z, w) = E_g(z, w)^{\alpha}.$$

EXAMPLE 2.3. We compute $E(z, w)$ for the unit disc $\mathbb{D}$ and $z, w \notin \partial\mathbb{D}$ by computing the integral

$$I = -\frac{1}{\pi} \int_{\mathbb{D}} \frac{dA(\zeta)}{(\zeta - z)(\bar{\zeta} - \bar{w})}$$

in the four components of $(\mathbb{C} \setminus \partial\mathbb{D})^2$.

For $z, w \in \overline{\mathbb{D}}^c$

$$\begin{aligned}
I &= \frac{1}{2\pi i} \int_{\mathbb{D}} \frac{d\zeta d\bar{\zeta}}{(\zeta - z)(\bar{\zeta} - \bar{w})} \\
&= \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \log(\zeta - z) \frac{d\bar{\zeta}}{\bar{\zeta} - \bar{w}} = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \log(\zeta - z) \frac{d(1/\zeta)}{1/\zeta - \bar{w}} \\
&= \log(\zeta - z)|_{1/\zeta = \bar{w}} - \log(\zeta - z)|_{1/\zeta = \infty} = \log\left(\frac{1}{\bar{w}} - z\right) - \log(-z).
\end{aligned}$$

Hence

$$E(z, w) = \frac{1/\bar{w} - z}{-z} = 1 - \frac{1}{z\bar{w}}$$

in this case.

For $z \in \overline{\mathbb{D}}^c,\ w \in \mathbb{D}$

$$\begin{aligned}
I &= \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\mathbb{D} \setminus \mathbb{D}(w, \varepsilon)} \frac{d\zeta d\bar{\zeta}}{(\zeta - z)(\bar{\zeta} - \bar{w})} \\
&= \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\partial(\mathbb{D} \setminus \mathbb{D}(w, \varepsilon))} \log(\zeta - z) \frac{d\bar{\zeta}}{\bar{\zeta} - \bar{w}} \\
&= \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \log(\zeta - z) \frac{d(1/\zeta)}{1/\zeta - \bar{w}} - \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\partial\mathbb{D}(w, \varepsilon)} \log(\zeta - z) \frac{d(\bar{\zeta} - \bar{w})}{\bar{\zeta} - \bar{w}} \\
&= -\log(-z) + \log(w - z),
\end{aligned}$$

Thus

$$E(z, w) = 1 - \frac{w}{z}.$$

The case $z \in \mathbb{D}$, $w \in \overline{\mathbb{D}}^c$ is obtained from the previous one by (2.3).

For $z, w \in \mathbb{D}$, $z \neq w$ finally

$$
\begin{aligned}
I &= \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\mathbb{D} \setminus (\mathbb{D}(z,\varepsilon) \cup \mathbb{D}(w,\varepsilon))} \frac{d\zeta\, d\bar{\zeta}}{(\zeta - z)\left(\bar{\zeta} - \bar{w}\right)} \\
&= \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\partial(\mathbb{D} \setminus (\mathbb{D}(z,\varepsilon) \cup \mathbb{D}(w,\varepsilon)))} \log\left[(\zeta - z)\left(\bar{\zeta} - \bar{z}\right)\right] \frac{d\bar{\zeta}}{\bar{\zeta} - \bar{w}} \\
&= \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \log|\zeta - z|^2 \frac{d\bar{\zeta}}{\bar{\zeta} - \bar{w}} - \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\partial \mathbb{D}(z,\varepsilon)} \log|\zeta - z|^2 \frac{d\bar{\zeta}}{\bar{\zeta} - \bar{w}} \\
&\quad - \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\partial \mathbb{D}(w,\varepsilon)} \log|\zeta - z|^2 \frac{d\bar{\zeta}}{\bar{\zeta} - \bar{w}} \\
&= \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \log|1 - \bar{z}\zeta|^2 \frac{d\bar{\zeta}}{\bar{\zeta} - \bar{w}} - 0 + \log|w - z|^2 \\
&= \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \log(1 - \bar{z}\zeta) \frac{d(1/\zeta)}{1/\zeta - \bar{w}} + \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \log\left(1 - z\bar{\zeta}\right) \frac{d\bar{\zeta}}{\bar{\zeta} - \bar{w}} + \log|w - z|^2 \\
&= -\log 1 - \log(1 - z\bar{w}) + \log|w - z|^2 .
\end{aligned}
$$

Thus

$$
E(z, w) = \frac{|z - w|^2}{1 - z\bar{w}} .
$$

In summary:

$$
E(z, w) = \begin{cases}
1 - \dfrac{1}{z\bar{w}} & \left(z, w \in \overline{\mathbb{D}}^c\right), \\[2mm]
1 - \dfrac{\bar{z}}{\bar{w}} & \left(z \in \mathbb{D}, w \in \overline{\mathbb{D}}^c\right), \\[2mm]
1 - \dfrac{w}{z} & \left(z \in \overline{\mathbb{D}}^c, w \in \mathbb{D}\right), \\[2mm]
\dfrac{|z - w|^2}{1 - z\bar{w}} & (z, w \in \mathbb{D})
\end{cases}
$$

$$
= \begin{cases}
1 - \dfrac{1}{z\bar{w}} & \left(z, w \in \overline{\mathbb{D}}^c\right), \\[2mm]
-\dfrac{\bar{z} - \bar{w}}{\bar{w}} & \left(z \in \mathbb{D}, w \in \overline{\mathbb{D}}^c\right), \\[2mm]
\dfrac{z - w}{z} & \left(z \in \overline{\mathbb{D}}^c, w \in \mathbb{D}\right), \\[2mm]
\dfrac{(z - w)(\bar{z} - \bar{w})}{1 - z\bar{w}} & (z, w \in \mathbb{D}) .
\end{cases}
$$

b) BEHAVIOUR UNDER MÖBIUS TRANSFORMATIONS

Under Möbius transformations

$$f(z)=\frac{az+b}{cz+d}$$

$(ad-bc\neq 0)$ $E_\Omega$ behaves as follows. If $c=0$, then

$$E_{a\Omega+b}(az+b,aw+b)=E_\Omega(z,w). \tag{2.5}$$

If $c\neq 0$, then

$$E_{f(\Omega)}\big(f(z),f(w)\big)E_\Omega(z,-d/c)E_\Omega(-d/c,w)=E_\Omega(z,w)E_\Omega(-d/c,-d/c) \tag{2.6}$$

(of interest only if $-d/c\notin\Omega$) and

$$E_\Omega(z,w)E_{f(\Omega)^c}\big(f(z),f(w)\big)=C|f(z)-f(w)|^2E_\Omega(z,-d/c)E_\Omega(-d/c,w). \tag{2.7}$$

Here $C\geq 0$ is a constant depending on $\Omega$ and $f$. If $-d/c\in\Omega$ then $C$ is strictly positive.

PROOF. We assume that $E_\Omega\not\equiv 0$ since otherwise all statements are trivial. Making the change of variable $\zeta=f(\eta)$ in the integral in (2.1) and using the identity

$$\frac{f'(\eta)}{f(\eta)-f(z)}=\frac{1}{\eta-z}-\frac{c}{c\eta+d}$$

one obtains

$$\begin{aligned}E_{f(\Omega)}(f(z),f(w))&=\exp\left[-\frac{1}{\pi}\int_\Omega\frac{f'(\eta)\overline{f'(\eta)}dA(\eta)}{(f(\eta)-f(z))\,\overline{(f(\eta)-f(w))}}\right]\\&=\exp\left[-\frac{1}{\pi}\int_\Omega\left(\frac{1}{\eta-z}-\frac{c}{c\eta+d}\right)\left(\frac{1}{\bar\eta-\bar w}-\frac{\bar c}{\overline{c\eta+d}}\right)dA(\eta)\right].\end{aligned}$$

This equals $E_\Omega(z,w)$ if $c=0$ (proving (2.5)) and equals

$$\frac{E_\Omega(z,w)E_\Omega(-d/c,-d/c)}{E_\Omega(z,-d/c)E_\Omega(-d/c,w)}$$

if $c\neq 0$, at least if all the integrals appearing are absolutely convergent, i.e. if all $E_\Omega$'s occuring are nonzero. If some $E_\Omega$ vanishes it is easy to see that (2.6) still holds.

For $-d/c\in\Omega$, i.e. for $f(\Omega)$ containing a neighbourhood of infinity, equation (2.6) is not of much use because both members then are identically zero.

However, applying (2.6) to $\Omega \setminus \mathbb{D}(-d/c, r)$, where $\mathbb{D}(-d/c, r) \subset \Omega$, and then multiplying both members by $E_{f(\Omega)^c}(f(z), f(w))E_{\mathbb{D}(-d/c,r)}(z, w)$ gives

$$\begin{aligned}
&E_{f(\Omega)^c}\big(f(z), f(w)\big)E_{\mathbb{D}(-d/c,r)}(z, w)E_{f(\Omega\setminus\mathbb{D}(-d/c))}\big(f(z), f(w)\big)\\
&\cdot E_{\Omega\setminus\mathbb{D}(-d/c,r)}(z, -d/c)E_{\Omega\setminus\mathbb{D}(-d/c,r)}(-d/c, w)\\
=&E_{f(\Omega)^c}\big(f(z), f(w)\big)E_{\mathbb{D}(-d/c,r)}(z, w)\\
&\cdot E_{\Omega\setminus\mathbb{D}(-d/c,r)}(z, w)E_{\Omega\setminus\mathbb{D}(-d/c,r)}(-d/c, -d/c).
\end{aligned}$$

Since

$$f\big(\Omega \setminus \mathbb{D}(-d/c, r)\big) = f(\Omega) \setminus f\big(\mathbb{D}(-d/c, r)\big) = f(\Omega) \cap \mathbb{D}\left(\frac{a}{c}, \left|\frac{ad-bc}{c^2 r}\right|\right),$$

we find

$$\begin{aligned}
&E_{\mathbb{D}(-d/c,r)}(z, w)E_{\mathbb{D}\left(\frac{a}{c}, \left|\frac{ad-bc}{c^2 r}\right|\right)}\big(f(z), f(w)\big)\\
&\cdot E_{\Omega\setminus\mathbb{D}(-d/c,r)}(z, -d/c)E_{\Omega\setminus\mathbb{D}(-d/c,r)}(-d/c, w)\\
=&E_{f(\Omega)^c}\big(f(z), f(w)\big)E_{\Omega}(z, w)E_{\Omega\setminus\mathbb{D}(-d/c,r)}(-d/c, -d/c).
\end{aligned}$$

Let now $r \to 0$. It is easily verified then that, for $z, w \neq -d/c$,

$$\begin{aligned}
E_{\mathbb{D}(-d/c,r)}(z, w) &\to 1,\\
E_{\Omega\setminus\mathbb{D}(-d/c,r)}(z, -d/c) &\to E_{\Omega}(z, -d/c),\\
E_{\Omega\setminus\mathbb{D}(-d/c,r)}(-d/c, w) &\to E_{\Omega}(-d/c, w),\\
E_{\mathbb{D}\left(\frac{a}{c}, \left|\frac{ad-bc}{c^2 r}\right|\right)}\big(f(z), f(w)\big) &= C_1 r^2 |f(z) - f(w)|^2 + \mathcal{O}(r^3),\\
E_{\Omega\setminus\mathbb{D}(-d/c,r)}(-d/c, -d/c) &= C_2 r^2
\end{aligned}$$

for suitable constants $C_1, C_2 > 0$. Putting the pieces together we obtain the desired formula (2.7) in the case that $-d/c \in \Omega$.

For $-d/c \notin \overline{\Omega}$ (2.7) holds with $C = 0$ ($E_{f(\Omega)^c} \equiv 0$ since $f(\Omega)^c$ contains a neighbourhood of infinity). If $-d/c \in \partial\Omega$ then $f(\Omega)^c$ still is unbounded and typically $E_{f(\Omega)^c} \equiv 0$, so that (2.7) holds with $C = 0$. In the exceptional cases that $E_{f(\Omega)^c} \not\equiv 0$ (for $-d/c \in \partial\Omega$) small modifications of the proof show that (2.7) still holds (with $C > 0$). □

c) Behaviour at infinity

We assume that $0 \leq g \in L^\infty(\mathbb{C})$ has compact support so that $E_g$ is analytic-antianalytic in a neighbourhood of infinity. Expanding the integral in the definition of $E_g(z, w)$ in power series in $1/z$ and/or $1/\bar{w}$ gives

$$E_g(z, w) = 1 - \frac{1}{\pi z \bar{w}} \int_{\mathbb{C}} g\, dA + \mathcal{O}\left(\left(\frac{1}{|z|} + \frac{1}{|w|}\right)^3\right)$$

as $|z|, |w| \to \infty$;

$$E_g(z, w) = 1 - \frac{1}{\bar{w}}\,\hat{g}(z) + \mathcal{O}\left(\frac{1}{|w|^2}\right)$$

as $|w| \to \infty$, with $z \in \mathbb{C}$ fixed.

Thus the Cauchy transform is a derivative at infinity in one of the variables of the exponential transform.

d) Estimates

We only treat the case $g = \chi_\Omega$. If $z = w$ then the exponent in (2.1) is nonpositive, hence

$$0 \le E_\Omega(z, z) \le 1.$$

If $z \neq w$, let $\mathbb{D}_{z,w} = \mathbb{D}\left(\frac{z+w}{2}, \left|\frac{z-w}{2}\right|\right)$ be the disc with diameter $|z-w|$ and $z$, $w \in \partial\mathbb{D}_{z,w}$. Since $\mathrm{Re}(\bar{\zeta} - \bar{z})(\zeta - w)$ has the interpretation of being the scalar product between $\zeta - z$ and $\zeta - w$ considered as vectors we have

$$\mathrm{Re}(\bar{\zeta} - \bar{z})(\zeta - w) = \begin{cases} < 0 & \text{if } \zeta \in \mathbb{D}_{z,w}, \\ = 0 & \text{if } \zeta \in \partial\mathbb{D}_{z,w}, \\ > 0 & \text{if } \zeta \notin \overline{\mathbb{D}}_{z,w}. \end{cases}$$

Thus

$$\begin{aligned} |E_\Omega(z, w)| &= \exp\left[-\frac{1}{\pi}\int_\Omega \frac{\mathrm{Re}\,(\bar{\zeta} - \bar{z})\,(\zeta - w)}{|\zeta - z|^2|\zeta - w|^2}\,dA(\zeta)\right] \\ &\le \exp\left[-\frac{1}{\pi}\int_{\mathbb{D}_{z,w}} \frac{\mathrm{Re}\,(\bar{\zeta} - \bar{z})\,(\zeta - w)}{|\zeta - z|^2|\zeta - w|^2}\,dA(\zeta)\right] \\ &= |E_{\mathbb{D}_{z,w}}(z, w)|, \end{aligned}$$

with equality if and only if $\Omega = \mathbb{D}_{z,w}$ up to null-sets. By (2.5) $E_{\mathbb{D}_{z,w}}(z, w)$ is an absolute constant, in fact $E_{\mathbb{D}_{z,w}}(z, w) = 2$ by Example 2.3.

More specifically we may decompose

$$\begin{aligned} E_\Omega(z, w) &= \exp\left[-\frac{1}{\pi}\int_{\Omega\cap\mathbb{D}_{z,w}} \frac{\mathrm{Re}\,(\bar{\zeta} - \bar{z})\,(\zeta - w)}{|\zeta - z|^2|\zeta - w|^2}\,dA(\zeta)\right] \\ &\quad\cdot \exp\left[-\frac{1}{\pi}\int_{\Omega\setminus\mathbb{D}_{z,w}} \frac{\mathrm{Re}\,(\bar{\zeta} - \bar{z})\,(\zeta - w)}{|\zeta - z|^2|\zeta - w|^2}\,dA(\zeta)\right] \\ &\quad\cdot \exp\left[-\frac{i}{\pi}\int_\Omega \frac{\mathrm{Im}\,(\bar{\zeta} - \bar{z})\,(\zeta - w)}{|\zeta - z|^2|\zeta - w|^2}\,dA(\zeta)\right] \\ &= E_1 \cdot E_2 \cdot E_3. \end{aligned}$$

The integrands in the $E_j$ are:
for $E_1$: $\leq 0$ and integrable,
for $E_2$: $\geq 0$ but possibly nonintegrable,
for $E_3$: integrable. Thus

$$\begin{aligned} 1 \leq E_1 \leq 2, \\ 0 \leq E_2 \leq 1, \\ |E_3| = 1. \end{aligned}$$

In summary:

LEMMA 2.4. *For any $\Omega \subset \mathbb{C}$, $z, w \in \mathbb{C}$*

$$|E_\Omega(z, w)| \leq 2$$

*with equality if and only if $\Omega$ is a disc (up to null-sets) and $z, w$ diametrically opposite points on the boundary. Furthermore, for fixed $z, w \in \mathbb{C}$ the set function $\Omega \mapsto |E_\Omega(z, w)|$ is increasing as a function of $\Omega \cap \mathbb{D}_{z,w}$ and decreasing as a function of $\Omega \setminus \mathbb{D}_{z,w}$. Finally,*

$$\begin{aligned} 1 \leq |E_\Omega(z, w)| \leq 2 \quad &\textit{if} \quad \Omega \subset \mathbb{D}_{z,w}, \\ 0 \leq |E_\Omega(z, w)| \leq 1 \quad &\textit{if} \quad \Omega \subset \mathbb{D}^c_{z,w}. \end{aligned}$$

For $z \in \Omega$, let $r = \text{dist}\,(z, \Omega^c)$. If $w \in \mathbb{D}(z, r)$ then $\mathbb{D}_{z,w} \subset \mathbb{D}(z, r) \subset \Omega$. Hence the second part of Lemma 2.4 shows that

$$|E_\Omega(z, w)| \leq |E_{\mathbb{D}(z,r)}(z, w)| = \frac{|z - w|^2}{\text{dist}\,(z, \Omega^c)^2}, \tag{2.8}$$

where we used also Example 2.3 and (2.5). If $w \notin \mathbb{D}(z, r)$ then the right member in (2.8) is $\geq 1$. Therefore we have in any case, using also Lemma 2.4,

$$|E_\Omega(z, w)| \leq \frac{2|z - w|^2}{\text{dist}\,(z, \Omega^c)^2}$$

for $z \in \Omega$. By symmetry we get:

LEMMA 2.5. *For any $z, w \in \Omega$,*

$$\begin{aligned} |E_\Omega(z, w)| &\leq \frac{2|z - w|^2}{\left[\max\left\{\text{dist}\,(z, \Omega^c)\,, \text{dist}\,(w, \Omega^c)\,\right\}\right]^2} \\ &\leq \frac{2|z - w|^2}{\text{dist}\,(z, \Omega^c)\,\text{dist}\,(w, \Omega^c)}. \end{aligned} \tag{2.9}$$

e) CONTINUITY PROPERTIES

The following example shows that $E(z, w)$ need not be continuous on $\mathbb{C} \times \mathbb{C}$.

EXAMPLE 2.6. For $\alpha > 0$, set

$$\Omega_\alpha = \{x + iy \in \mathbb{C} : 0 < x < 1, |y| < x^\alpha\}.$$

Thus $\Omega_\alpha$ is a domain in the right half-plane with $0 \in \partial\Omega_\alpha$ a $C^1$ smooth boundary point if $0 < \alpha < 1$, a Lipschitz corner if $\alpha = 1$ and an outward cusp if $\alpha > 1$. Let $E = E_{\Omega_\alpha}$. We compute

$$\begin{aligned} E(0,0) &= \exp\left[-\frac{1}{\pi}\int_{\Omega_\alpha}\frac{dA(\zeta)}{|\zeta|^2}\right] \\ &= \exp\left[-\frac{2}{\pi}\int_0^1\int_0^{x^\alpha}\frac{dy}{x^2+y^2}\,dx\right] \\ &= \exp\left[-\frac{2}{\pi}\int_0^1\frac{\arctan x^{\alpha-1}}{x}\,dx\right], \end{aligned}$$

which is $= 0$ if $0 < \alpha \le 1$ but $> 0$ if $\alpha > 1$.

Since $E(z,z) = 0$ for all $z \in \Omega_\alpha$ it follows that $E$ is not continuous at (0,0) if $\alpha > 1$.

Now for a general domain $\Omega \subset \mathbb{C}$ set

$$Z = \left\{z \in \partial\Omega : \int_{\Omega\cap\mathbb{D}(z,1)}\frac{dA(\zeta)}{|\zeta - z|^2} < \infty\right\}. \tag{2.10}$$

If $E_\Omega \not\equiv 0$ this means

$$Z = \{z \in \partial\Omega : E_\Omega(z,z) > 0\}. \tag{2.11}$$

Clearly $Z$ is an exceptional subset of $\partial\Omega$. It is empty if $\partial\Omega$ is smooth, or even Lipschitz. Some simple observations are

(i) $\Omega$ has vanishing Lebesgue density at each point of $z \in Z$. (Proof: if $z \in Z$ then $0 \le \frac{1}{\varepsilon^2}\int_{\Omega\cap\mathbb{D}(z,\varepsilon)} dA \le \int_{\Omega\cap\mathbb{D}(z,\varepsilon)}\frac{dA(\zeta)}{|\zeta-z|^2} \to 0$ as $\varepsilon \to 0$.)

(ii) $\partial\Omega \setminus Z$ is always dense in $\partial\Omega$. (Proof: if $z \in \partial\Omega$, take $z_n \in \Omega$, $z_n \to z$; then any closest neighbours $\zeta_n$ of $z_n$ on $\partial\Omega$ satisfy $\zeta_n \to z_n$ and $\zeta_n \in \partial\Omega\setminus Z$, the latter in view of (i) and the fact that $\partial\Omega$ satisfies an inner ball condition at $\zeta_n$.)

(iii) If $E_\Omega \not\equiv 0$ then $E_\Omega$ is always discontinuous at each point of $\Delta_Z$ (as in Example 2.6).

PROPOSITION 2.7. *For any open set $\Omega \subset \mathbb{C}$, $E_\Omega(z,w)$ is continuous on $\mathbb{C}^2 \setminus \Delta_Z$. On $\Delta_Z$, $E_\Omega$ is still continuous in each variable separately when the other variable is kept fixed.*

NOTE: Thus $E_\Omega$ is continuous in all $\mathbb{C}\times\mathbb{C}$ if and only if $E_\Omega \equiv 0$ or $Z=\varnothing$.

PROOF. Since $E_\Omega$ clearly is continuous if $E\equiv 0$ we may exclude this case from discussion. Then, by Remark 2.2, $E_\Omega \neq 0$ off the diagonal. When $E_\Omega(z,w)\neq 0$ continuity of $E_\Omega$ at $(z,w)$ is the same thing as continuity at $(z,w)$ of $I(z,w)=\int_\Omega \frac{dA(\zeta)}{(\zeta-z)(\zeta-w)}$, which is immediately verified when $z\neq w$, and also when $z=w\in\overline{\Omega}^c$. For the first statement of the proposition it therefore only remains to consider points $(z,z)$ with $z\in\Omega\cup(\partial\Omega\setminus Z)$, i.e. exactly those points at which $E_\Omega(z,z)=0$.

Let $(z_n,w_n)\to(z,z)$ and let $\mathbb{D}_n=\mathbb{D}\left(\frac{z_n+w_n}{2},|\frac{z_n-w_n}{2}|\right)$. Then (cf. the proof of Lemma 2.4)

$$\operatorname{Re}\frac{(\bar\zeta-\bar z_n)(\zeta-w_n)}{|\zeta-z_n|^2|\zeta-w_n|^2}\chi_{\Omega\setminus\mathbb{D}_n}(\zeta)$$

is a sequence of nonnegative functions converging pointwise almost everywhere to $\chi_\Omega(\zeta)/|\zeta-z|^2$. Therefore by assumption and Fatou's lemma

$$\begin{aligned}+\infty&=\int_\Omega\frac{dA(\zeta)}{|\zeta-z|^2}\\ &\le\lim_{n\to\infty}\int_{\Omega\setminus\mathbb{D}_n}\frac{\operatorname{Re}(\bar\zeta-\bar z_n)(\zeta-w_n)}{|\zeta-z_n|^2|\zeta-w_n|^2}dA(\zeta).\end{aligned}\tag{2.12}$$

The integrals in the right member of (2.12), which hence tend to $+\infty$, are parts of the integrals in the exponents of $E(z_n,w_n)$. The remaining parts of the exponents give uniformly bounded contributions to $E(z_n,w_n)$ as shown in the proof of Lemma 2.4. (In the notation of that proof $E_2\to 0$, while $1\le E_1\le 2$, $|E_3|=1$.) Thus $E(z_n,w_n)\to 0=E(z,z)$ completing the proof of the first statement of the proposition.

The separate continuity on $\Delta_Z$ is a previously known fact [Cl1] [MP, p. 258] but let us still give a short proof. We need to prove that $E(z_n,z)\to E(z,z)$ as $z_n\to z$ for $z\in Z$. This is equivalent to proving that

$$\int_{\Omega\cap\mathbb{D}(z,1)}\frac{dA(\zeta)}{(\zeta-z_n)(\bar\zeta-\bar z)}\longrightarrow\int_{\Omega\cap\mathbb{D}(z,1)}\frac{dA(\zeta)}{|\zeta-z|^2}$$

as $z_n\to z$, where by assumption the right member is finite. For notational convenience we assume that $z=0$, $\Omega\subset\mathbb{D}(0,1)$.

So let $z_n\to 0$ and let $0<\varepsilon<1$. Since $|\zeta|>(1-\varepsilon)|z_n|$ for $\zeta\in\mathbb{D}(z_n,\varepsilon|z_n|)$ we have

$$\begin{aligned}\left|\int_\Omega\frac{dA(\zeta)}{(\zeta-z_n)\bar\zeta}-\int_\Omega\frac{dA(\zeta)}{|\zeta|^2}\right|&=\left|\int_\Omega\frac{z_n}{\zeta-z_n}\frac{dA(\zeta)}{|\zeta|^2}\right|\\ &\le\int_{\mathbb{D}(z_n,\varepsilon|z_n|)}\left|\frac{z_n}{\zeta-z_n}\right|\frac{dA(\zeta)}{|\zeta|^2}+\int_{\Omega\setminus\mathbb{D}(z_n,\varepsilon|z_n|)}\left|\frac{z_n}{\zeta-z_n}\right|\frac{dA(\zeta)}{|\zeta|^2}\\ &\le\frac{1}{(1-\varepsilon)^2}\frac{1}{|z_n|}\int_{\mathbb{D}(z_n,\varepsilon|z_n|)}\frac{dA(\zeta)}{|\zeta-z_n|}+\int_\Omega\min\left\{\left|\frac{z_n}{\zeta-z_n}\right|,\frac{1}{\varepsilon}\right\}\frac{dA(\zeta)}{|\zeta|^2}.\end{aligned}$$

Here the first term equals $2\pi\varepsilon/(1-\varepsilon)^2$, hence can be made arbitrarily small by taking $\varepsilon$ small enough, and for any fixed $\varepsilon$ the second term tends to zero as $z_n \to 0$ by Lebesgue's dominated convergence theorem.

This finishes the proof of Proposition 2.7. □

f) Derivatives and general structure of $E(z, w)$

Since $E$ is a bounded function it is also a distribution in $\mathbb{C}^2$. We shall compute some of the classical and distributional derivatives of $E = E_\Omega$, $\Omega \subset \mathbb{C}$ open. We exclude the case $E \equiv 0$ from discussion.

It is clear that $\bar\partial_z E = 0$ for $z \in \overline{\Omega}^c$ and $\partial_w E = 0$ for $w \in \overline{\Omega}^c$, i.e. $E(z, w)$ is analytic in $z$, antianalytic in $w$ when the variable in question is outside $\overline{\Omega}$. One trick one may use for computing, e.g., derivatives $\bar\partial_z$ inside $\Omega$ is to excise a small disc $D \subset \Omega$ containing $z$ and write (by (2.4))

$$E_\Omega(z, w) = E_D(z, w) \cdot E_{\Omega\setminus D}(z, w).$$

Since the second factor is analytic in $z$, $\bar\partial_z$ will only act on the first factor, for which we have explicit expressions by Example 2.3.

If $D = \mathbb{D}(a, r)$ and also $w$ is in $D$ this gives

$$\begin{aligned}\bar\partial_z E_\Omega(z, w) &= \bar\partial_z \left( \frac{(z-w)(\bar z - \bar w)}{r^2 - (z-a)(\bar w - \bar a)} \right) E_{\Omega\setminus D}(z, w)\\ &= \frac{z-w}{r^2-(z-a)(\bar w - \bar a)} E_{\Omega\setminus D}(z, w)\\ &= \frac{E_\Omega(z, w)}{\bar z - \bar w}.\end{aligned}$$

The same expression is obtained for $w \notin D$.

Thus (using also (2.3))

$$\begin{aligned}\bar\partial_z E(z, w) &= \frac{E(z, w)}{\bar z - \bar w} \quad \text{in} \quad \Omega \times \mathbb{C},\\ \partial_w E(z, w) &= -\frac{E(z, w)}{z - w} \quad \text{in} \quad \mathbb{C} \times \Omega.\end{aligned} \tag{2.13}$$

Handling higher derivatives in the same way gives

$$\begin{aligned}\bar\partial_z^2 E(z, w) &= 0 && \text{in} \quad (\mathbb{C} \setminus \partial\Omega) \times \mathbb{C},\\ \partial_w^2 E(z, w) &= 0 && \text{in} \quad \mathbb{C} \times (\mathbb{C} \setminus \partial\Omega),\\ \bar\partial_z \partial_w E(z, w) &= -\frac{E(z, w)}{(z-w)(\bar z - \bar w)} && \text{in} \quad \Omega \times \Omega.\end{aligned} \tag{2.14}$$

It also follows that $E_\Omega(z, w)$ has the same type of structure in the components of $(\mathbb{C} \setminus \partial\Omega)^2$ as $E_{\mathbb{D}}$ has in $(\mathbb{C} \setminus \partial\mathbb{D})^2$, namely

$$
E(z,w) = \begin{cases} F(z,\bar{w}) & \text{in } \overline{\Omega}^c \times \overline{\Omega}^c, \\ (\bar{z}-\bar{w})\, G(z,\bar{w}) & \text{in } \Omega \times \overline{\Omega}^c, \\ -(z-w)\overline{G(w,\bar{z})} & \text{in } \overline{\Omega}^c \times \Omega, \\ (z-w)(\bar{z}-\bar{w})\, H(z,\bar{w}) & \text{in } \Omega \times \Omega, \end{cases} \tag{2.15}
$$

where $F$, $G$, $H$ are functions which are analytic in their arguments. Comparison with (2.13), (2.14) gives that

$$
\begin{aligned}
& F(z,\bar{w}) = E(z,w) && \text{in } \overline{\Omega}^c \times \overline{\Omega}^c, \\
& G(z,\bar{w}) = \bar{\partial}_z E(z,w) && \text{in } \Omega \times \overline{\Omega}^c, && (2.16) \\
& H(z,\bar{w}) = -\bar{\partial}_z \partial_w E(z,w) && \text{in } \Omega \times \Omega. && (2.17)
\end{aligned}
$$

Since $E(z,z) > 0$ for $z \in \overline{\Omega}^c$ we have $F(z,\bar{z}) > 0$ for $z \in \overline{\Omega}^c$, and it also follows that

$$
H(z,\bar{z}) > 0 \qquad \text{for} \qquad z \in \Omega. \tag{2.18}
$$

Indeed, for $z, w \in D \subset \Omega$, $z \neq w$ ($D$ a disc as above) we have

$$
\frac{E_\Omega(z,w)}{|z-w|^2} = \frac{E_D(z,w)}{|z-w|^2} \cdot E_{\Omega \setminus D}(z,w)
$$

so (2.18) follows from the corresponding statement for the disc $D$ along with (2.15) and by letting $w \to z$.

Next we take contributions across $\partial\Omega$ into account by computing derivatives in the distributional sense in all $\mathbb{C}^2$. For computing the distributional derivative $\bar{\partial}_z E$ we are allowed to use the chain rule ([E-G, Section 4.22]) and the fact that the exponent in $E(z,w)$ as a function of $z$ is the Cauchy-transform of the function $\zeta \mapsto (\bar{\zeta} - \bar{w})^{-1} \chi_\Omega(\zeta)$, which is in $L^p_{\text{loc}}(\mathbb{C})$ for all $p < 2$. This gives

$$
\bar{\partial}_z E(z,w) = E(z,w) \frac{\chi_\Omega(z)}{\bar{z}-\bar{w}} \tag{2.19}
$$

in all $\mathbb{C}^2$. Similarly,

$$
\partial_w E(z,w) = E(z,w) \frac{\chi_\Omega(w)}{w-z}.
$$

Note that the right members are locally integrable functions. Thus there are no distributional contributions on $\partial\Omega$ for the first order derivatives.

For the computation of $\bar{\partial}_z^2 E$ we assume that $\partial\Omega$ is smooth. In this case we have, for any test function $\varphi \in \mathcal{D}(\mathbb{C}^2)$ and using (2.19), (2.15).

$$\begin{aligned}
\langle \bar{\partial}_z^2 E, \varphi\rangle &= \left\langle \bar{\partial}_z \left(\frac{E(z,w)\chi_\Omega(z)}{\bar{z}-\bar{w}}\right), \varphi\right\rangle \\
&= -\int_{\mathbb{C}\setminus\partial\Omega}\int_\Omega \frac{E(z,w)}{\bar{z}-\bar{w}}\,\bar{\partial}_z\varphi(z,w)dA(z)dA(w) \\
&= -\int_{\mathbb{C}\setminus\partial\Omega}\int_\Omega \bar{\partial}_z\left(\frac{E(z,w)\varphi(z,w)}{\bar{z}-\bar{w}}\right) dA(z)dA(w) \\
&= -\frac{1}{2i}\int_{\mathbb{C}\setminus\partial\Omega}\int_{\partial\Omega} \frac{E(z,w)\varphi(z,w)}{\bar{z}-\bar{w}}\, dz\, dA(w).
\end{aligned}$$

Here $\langle f, \varphi\rangle$ denotes the action $f(\varphi)$ of a distribution $f$ on a test function $\varphi$.

Let $n$ denote the outward unit normal vector on $\partial\Omega$ considered as a complex valued function on $\partial\Omega$. Then $dz = inds$ where $ds$ is arc-length measure on $\partial\Omega$. We conclude that $\bar{\partial}_z^2 E$ equals the complex measure

$$\bar{\partial}_z^2 E(z,w) = -\frac{1}{2}\frac{E(z,w)}{\bar{z}-\bar{w}}\, n(z)ds(z)dA(w).$$

Note that $\frac{E(z,w)}{\bar{z}-\bar{w}}\, n(z) \in L^1(ds\otimes dA)$.

$\partial_w^2 E$ is computed similarly. For $\bar{\partial}_z\partial_w E$ we have

$$\begin{aligned}
\langle \bar{\partial}_z\partial_w E, \varphi\rangle &= \left\langle \partial_w \left(\frac{E(z,w)\,\chi_\Omega(z)}{\bar{z}-\bar{w}}\right), \varphi\right\rangle \\
&= -\int_\Omega\int_{\mathbb{C}} \frac{E(z,w)}{\bar{z}-\bar{w}}\,\partial_w\,\varphi(z,w)\,dA(w)dA(z) \\
&= -\int_\Omega\left[\lim_{\varepsilon\to 0}\int_{\mathbb{C}\setminus\mathbb{D}(z,\varepsilon)} \frac{E(z,w)}{\bar{z}-\bar{w}}\,\partial_w\,\varphi(z,w)\,dA(w)\right] dA(z) \\
&= -\frac{1}{2i}\int_\Omega\left[\lim_{\varepsilon\to 0}\int_{\partial\mathbb{D}(z,\varepsilon)} \frac{E(z,w)\varphi(z,w)}{\bar{z}-\bar{w}}\, d\bar{w}\right] dA(z) \\
&\quad + \int_\Omega\left[\lim_{\varepsilon\to 0}\int_{\mathbb{C}\setminus\mathbb{D}(z,\varepsilon)} \partial_w\left(\frac{E(z,w)}{\bar{z}-\bar{w}}\right)\,\varphi(z,w)\,dA(w)\right] dA(z) \\
&= -\pi\int_\Omega E(z,z)\,\varphi\,(z,z)dA(z) \\
&\quad - \int_\Omega\left[\lim_{\varepsilon\to 0}\int_{\mathbb{C}\setminus\mathbb{D}(z,\varepsilon)} \frac{E(z,w)}{(z-w)\,(\bar{z}-\bar{w})}\chi_\Omega\,(w)\varphi(z,w)dA(w)\right] dA(z) \\
&= -\int_\Omega\left[\int_\Omega \frac{E(z,w)}{|z-w|^2}\varphi(z,w)dA(w)\right] dA(z),
\end{aligned}$$

where we used $E(z,z) = 0$, $z \in \Omega$, continuity of $E$ at these points and the estimate (2.9). Thus at least in the sense of iterated integration, as written above, we have

$$\overline{\partial}_z \partial_w E(z, w) = -\frac{E(z, w)}{|z - w|^2} \chi_\Omega(z) \chi_\Omega(w). \tag{2.20}$$

We do not know whether the right member here always is a locally integrable function.

One way to get a statement which only involves absolutely integrable functions is to replace, in the above computation, occurrences of factors $\chi_\Omega(z)$ (or integrations over $\Omega$ with respect to $z$) by functions $\psi_n(z)$ which are compactly supported in $\Omega$ and satisfy $0 \leq \psi_n \leq \chi_\Omega$, $\psi_n \to \chi_\Omega$ pointwise as $n \to \infty$. This gives the statement that

$$-\frac{E(z, w)}{|z - w|^2} \psi_n(z) \chi_\Omega(w) \rightharpoonup \overline{\partial}_z \partial_w E(z, w) \tag{2.21}$$

in the sense of distributions as $n \to \infty$. Here the left members certainly are in $L^1_{\mathrm{loc}}(\mathbb{C}^2)$ because of Lemma 2.5.

## 3. – Further examples

a) Classical quadrature domains

A (classical, bounded) quadrature domain is a bounded domain $\Omega \subset \mathbb{C}$ with the property that there exists a distribution $\mu$ with support in a finite number of points in $\Omega$ such that

$$\int_\Omega \varphi dA = \langle \mu, \varphi \rangle \tag{3.1}$$

for every integrable analytic function $\varphi$ in $\Omega$.

The basic example is any disc $\Omega = \mathbb{D}(a, r)$, in which case $\langle \mu, \varphi \rangle = \pi r^2 \varphi(a)$. In general the action of $\mu$ on $\varphi$ analytic is of the form

$$\langle \mu, \varphi \rangle = \sum_{k=1}^{m} \sum_{j=0}^{n_k - 1} c_{kj} \varphi^{(j)}(z_k)$$

for suitable $z_1, \ldots, z_m \subset \mathbb{C}$, $n_k \geq 1$, $c_{kj} \in \mathbb{C}$. Set then (assuming $c_{k,n_k-1} \neq 0$) $n = \sum_{k=1}^m n_k$ and

$$P(z) = (z - z_1)^{n_1} \cdot \ldots \cdot (z - z_m)^{n_m}.$$

It is known [Ah-Sh], [Gu1], [Sh2] that, with $\Omega, \mu$ as above, $\partial\Omega$ is an algebraic curve, more precisely that

$$\Omega = \left\{ z \in \mathbb{C} : Q(z, \bar{z}) < 0 \right\} \cup (\text{a finite set}), \tag{3.2}$$

where $Q(z, w)$ is an irreducible polynomial of the form

$$Q(z, w) = \sum_{k,j=0}^{n} a_{kj} z^k w^j$$

with $a_{kj} = \bar{a}_{jk}$ and $a_{nn} = 1$. For $\Omega = \mathbb{D}(0, 1)$ we have $P(z) = z$, $Q(z, w) = zw - 1$ for example.

The "finite set" in (3.2) is a subset of the set of isolated points in $\{z \in \mathbb{C} : Q(z, \bar{z}) = 0\}$ and this subset can be chosen arbitrarily.

Clearly $\Omega$ determines $\mu$, $P$ and $Q$ as above uniquely when $\Omega$ is a quadrature domain ($\mu$ is determined only as an analytic functional). Conversely, $Q$ determines $\Omega$ up to the finite set in (3.2).

Taking $\varphi(\zeta) = \frac{1}{\zeta - z}$ for $z \in \Omega^c$ in (3.1) gives that

$$\widehat{\chi}_\Omega = \hat{\mu} \quad \text{on} \quad \Omega^c, \tag{3.3}$$

and by an approximation argument (using [Be]) this equation is in fact found to be equivalent to $\Omega$ being a quadrature domain for $\mu$ [Ah-Sh], [Sa1], [Sh2]. Note that $\hat{\mu}$ here is a rational function vanishing at infinity. Since any such rational function is the Cauchy transform of some distribution $\mu$ as above it follows that $\Omega$ is a quadrature domain if and only if $\widehat{\chi}_\Omega$ coincides on $\Omega^c$ with a rational function.

It is convenient to set

$$S(z) = \bar{z} - \widehat{\chi}_\Omega(z) + \hat{\mu}(z)$$

for $z \in \overline{\Omega}$ when (3.3) holds. Then $S(z)$ is meromorphic in $\Omega$ and satisfies

$$S(z) = \bar{z} \quad \text{on} \quad \partial\Omega, \tag{3.4}$$

thus $S(z)$ is the *Schwarz function* for $\partial\Omega$ [D], [Sh2]. Since $Q(z, \bar{z}) = 0$ on $\partial\Omega$ it follows from (3.4) and analyticity of $S(z)$ that

$$Q\big(z, S(z)\big) \equiv 0 \quad (z \in \Omega). \tag{3.5}$$

Hence $S(z)$ is the algebraic function defined by $Q(z, w) = 0$.

The exponential transform for a quadrature domain with data as above was computed in [P1] to be

$$E_\Omega(z, w) = \frac{Q(z, \bar{w})}{P(z)\overline{P(w)}} \tag{3.6}$$

for $z, w \in \overline{\Omega}^c$. Moreover, using operator theory a strong converse of this result was obtained, namely saying that whenever an exponential transform $E_g(z, w)$ with $0 \leq g \leq 1$ is of the form of the right member in (3.6) for large $|z|, |w|$, then $g = \chi_\Omega$ a.e. for a quadrature domain $\Omega$ as above.

Having the Schwarz function at hand it is easy, in view of continuity (Proposition 2.7) and general structure (2.15) results, to compute $E$ everywhere within the realm of $S(z)$ once it is known in $\overline{\Omega}^c \times \overline{\Omega}^c$. When, for example, $z$ is moved from $\overline{\Omega}^c$ to $\Omega$ then $E(z, w)$ should by (2.15) acquire a factor $\bar{z} - \bar{w}$. In order for the transition to be continuous one has to compensate by a factor $S(z) - \bar{w}$ in the denominator. Thus $E(z, w)$ acquires the factor $(\bar{z} - \bar{w}) / (S(z) - \bar{w})$ compared to its natural analytic continuation given (in the case of quadrature domains) by the right member of (3.6). (These matters will be discussed in more depth in Section 4.)

In this way we obtain the following formulas, complementing (3.6).

$$E(z, w) = \frac{(\bar{z} - \bar{w})\, Q(z, \bar{w})}{\big(S(z) - \bar{w}\big) P(z) \overline{P(w)}} \quad \text{in} \quad \Omega \times \overline{\Omega}^c, \tag{3.7}$$

$$E(z, w) = \frac{(z - w) Q(z, \bar{w})}{\big(z - \overline{S(w)}\big)\, P(z) \overline{P(w)}} \quad \text{in} \quad \overline{\Omega}^c \times \Omega, \tag{3.8}$$

$$E(z, w) = \frac{(z - w)(\bar{z} - \bar{w})\, Q(z, \bar{w})}{\big(z - \overline{S(w)}\big)\big(S(z) - \bar{w}\big) P(z) \overline{P(w)}} \quad \text{in} \quad \Omega \times \Omega. \tag{3.9}$$

EXAMPLE 3.1. For the functional

$$\langle \mu, \varphi \rangle = \pi r^2 \big(\varphi(-1) + \varphi(1)\big),$$

where $r > 1$, there turns out to be an essentially unique $\Omega$, and the corresponding data $P$ and $Q$ are

$$\begin{aligned} P(z) &= (z - 1)(z + 1), \\ Q(z, w) &= z^2 w^2 - z^2 - w^2 - 2r^2 z w \end{aligned} \tag{3.10}$$

(see [Sh2]). Thus

$$E_\Omega(z, w) = \frac{z^2 \bar{w}^2 - z^2 - \bar{w}^2 - 2r^2 z \bar{w}}{\big(z^2 - 1\big)\big(\bar{w}^2 - 1\big)}$$

for $z, w \in \overline{\Omega}^c$. For $z, w$ in the remaining parts of $\mathbb{C}$, $E_\Omega(z, w)$ is obtained from (3.7)-(3.9) inserting the Schwarz function (determined by (3.5), (3.10))

$$S(z) = \frac{z}{z^2 - 1}\left(r^2 + \sqrt{r^4 + z^2 - 1}\right). \tag{3.11}$$

The "finite set" in (3.2) either is empty or consists of just $z = 0$ in the present case.

EXAMPLE 3.2. For

$$\langle \mu, \varphi \rangle = 6\pi \varphi(-1) - 4\pi \varphi'(-1)$$

there also is a unique quadrature domain $\Omega$ [Ah-Sh]. This is given by the data

$$\begin{aligned} Q(z,w) &= z^2w^2 + 2(zw^2 + z^2w) + z^2 + w^2 - 2zw, \\ P(z) &= (z+1)^2. \end{aligned} \tag{3.12}$$

Thus

$$E_\Omega(z,w) = \frac{z^2\bar{w}^2 + 2\left(z\bar{w}^2 + z^2\bar{w}\right) + z^2 + \bar{w}^2 - 2z\bar{w}}{(z+1)^2\,(\bar{w}+1)^2}$$

for $z, w \in \overline{\Omega}^c$.

We remark that $\partial\Omega$ is a cardioid and that it has an inward cusp at $z = 0$. (Since $Q(z,\bar{z})$ vanishes to the second order at $z = 0$, like in the previous example, this point is a singular point of $\{z \in \mathbb{C} : Q(z,\bar{z}) = 0\}$.)

The Schwarz function is

$$S(z) = \frac{z}{(z+1)^2}\left(-z + 1 + \sqrt{-4z}\right). \tag{3.13}$$

b) Complements of unbounded quadrature domains

There are also unbounded domains $\Omega$ satisfying identities of the kind (3.1) for all integrable analytic functions in $\Omega$ ("unbounded quadrature domains"). There are even examples with $\mu = 0$ ("null quadrature domains"), e.g. half-planes, exterior of ellipses and exterior of parabolas [Sa2], [Sh1]. It can be shown that (2.2) necessarily holds for an unbounded quadrature domain [Ah-Sh], [Sa1, Section 11], so the exponential kernel will vanish identically.

However, the complement of an unbounded quadrature domain often has a nontrivial exponential kernel, which we now proceed to compute.

It is known [Sh1], [Sh2], [Sa4] that the class of those unbounded quadrature domains which are not dense in the Riemann sphere coincides with the class of images of bounded quadrature domains $\Omega$ under Möbius transformations with the pole on $\overline{\Omega}$. It is enough to consider just inversions $f(z) = 1/z$ and domains $\Omega$ with $0 \in \overline{\Omega}$.

So let $\Omega$ be a bounded quadrature domain with $0 \in \overline{\Omega}$ and data $P$, $Q$ as in the beginning of this section and let $D = \overline{f(\Omega)}^c$, $f(z) = 1/z$. Then (2.7), (3.6) give, for $z, w \in D$ (i.e., $1/z, 1/w \in \overline{\Omega}^c$)

$$\frac{Q\left(\dfrac{1}{z}, \dfrac{1}{\bar{w}}\right)}{P\left(\dfrac{1}{z}\right)\overline{P\left(\dfrac{1}{w}\right)}} \cdot E_D(z,w) = C|z-w|^2 E_\Omega\left(\frac{1}{z}, 0\right) E_\Omega\left(0, \frac{1}{w}\right).$$

For $\zeta \in \overline{\Omega}^c$, $\eta \in \overline{\Omega}$ we have

$$E_\Omega(\zeta,\eta) = \frac{\zeta - \eta}{\zeta - \overline{S(\eta)}} \cdot \frac{Q(\zeta,\bar{\eta})}{P(\zeta)\overline{P(\eta)}}$$

(cf. (3.8)), hence

$$E_\Omega\left(\frac{1}{z},0\right)=\frac{1}{1-z\overline{S(0)}}\,\frac{Q\left(\frac{1}{z},0\right)}{P\left(\frac{1}{z}\right)\overline{P(0)}}.$$

Similarly,

$$E_\Omega\left(0,\frac{1}{w}\right)=\frac{1}{1-\bar{w}S(0)}\,\frac{Q\left(0,\frac{1}{\bar{w}}\right)}{P(0)\overline{P\left(\frac{1}{w}\right)}}.$$

Thus we find (incorporating $|P(0)|^2$ into $C$)

$$E_D(z,w)=\frac{C|z-w|^2Q\left(\frac{1}{z},0\right)Q\left(0,\frac{1}{\bar{w}}\right)}{\left(1-z\overline{S(0)}\right)\left(1-\bar{w}S(0)\right)Q\left(\frac{1}{z},\frac{1}{\bar{w}}\right)} \tag{3.14}$$

for $z, w \in D$.

EXAMPLE 3.3. The ellipse.

With $\Omega$ as in Example 3.1 and $f$ a Möbius transformation with the pole at the origin $D=\overline{f(\Omega)^c}$ will be an ellipse, and all (genuine) ellipses can be obtained that way. Taking for example $f(z)=1/z$ we get

$$\begin{aligned} D&=\left\{z\in\mathbb{C}:2r^2z\bar{z}+z^2+\bar{z}^2<1\right\}\\ &=\left\{x+iy\in\mathbb{C}:2(r^2+1)x^2+2(r^2-1)y^2<1\right\}.\end{aligned}$$

Since $S(0)=0$ by (3.11) ($S(z)$ the Schwarz function of $\Omega$) equation (3.14) together with (3.10) gives

$$E_D(z,w)=\frac{C|z-w|^2}{1-z^2-\bar{w}^2-2r^2z\bar{w}} \tag{3.15}$$

for $z, w \in D$.

EXAMPLE 3.4. The parabola.

In similarity with the previous example, inversion of the quadrature domain $\Omega$ in Example 3.2 at the cusp point on $\partial\Omega$ gives the exterior of a parabola. Indeed, with $f(z)=1/z$ and $D=\overline{f(\Omega)}^c$ we have

$$D=\left\{x+iy\in\mathbb{C}:y^2<x+\frac{1}{4}\right\}.$$

The present example is a case in which formula (2.7) gives a nontrivial result (i.e. $C > 0$) despite the fact that the pole of the Möbius transformation is on $\partial\Omega$. This is due to the fact that $D$ is so small at infinity that $E_D \not\equiv 0$. Thus (3.14) applies again (with $C > 0$) and gives, by (3.12), (3.13),

$$E_D(z, w) = \frac{C|z - w|^2}{1 + 2(z + \bar{w}) + z^2 + \bar{w}^2 - 2z\bar{w}} \tag{3.16}$$

for $z, w \in D$.

The results of Examples 3.3 and 3.4 can be expressed by saying that for $D$ the interior of an ellipse or parabola

$$E_D(z, w) = \frac{|z - w|^2}{q(z, \bar{w})} \qquad (z, w \in D), \tag{3.17}$$

where $q$ is a quadratic polynomial defining $\partial D$ ($D = \{z \in \mathbb{C} : q(z, \bar{z}) > 0\}$), here normalized so that the constant $C$ in (3.15), (3.16) disappears. The formula (3.17) also holds for $D$ an infinite strip, in which case $q$ is a reducible quadratic polynomial. For any other $D$ with $\partial D$ an algebraic curve of degree two or less $E_D \equiv 0$ because of the size of $D$ at infinity.

## 4. – Analytic continuation properties and regularity of free boundaries

Let $\Omega \subset \mathbb{C}$ be a bounded domain and recall that the Cauchy transform $\widehat{\chi}_\Omega(z)$ of $\Omega$ is analytic in the exterior $\overline{\Omega}^c$. In this section we shall study the following type of question: Assume that $\widehat{\chi}_\Omega|_{\Omega^c}$ has an analytic continuation across $\partial\Omega$ into $\Omega$, does then also $E_\Omega|_{\Omega^c\times\Omega^c}$ admit an analytic continuation across $\partial\Omega$ in one or both variables?

The answer to this question is in fact known to be "yes": By the work of M. Sakai [Sa3], [Sa4] $\widehat{\chi}_\Omega$ extends analytically if and only if $\partial\Omega$ is analytic with certain types of singularities allowed and for $\partial\Omega$ analytic the analytic continuation of $E$ has been proved by one of the authors [P3], at least when no singularities are present.

The classification by Sakai of boundaries $\partial\Omega$ admitting analytic continuation of $\widehat{\chi}_\Omega$ is quite long and technical. Here we shall reverse the order of reasoning and first give a direct and relatively simple proof of the fact that analytic continuation of $\widehat{\chi}_\Omega|_{\Omega^c}$ to say $\mathbb{C} \setminus K$, $K \subset \Omega$ compact, implies analytic continuation of $E|_{\Omega^c\times\Omega^c}$ to $(\mathbb{C} \setminus K)^2$. Then it follows from this fact that $\partial\Omega$ is analytic, more precisely that $\partial\Omega \subset \{z \in \mathbb{C} \setminus K : F(z, \bar{z}) = 0\}$ where $F$ is the analytic continuation of $E$.

In analogy with Section 3 we have that $\widehat{\chi}_\Omega|_{\Omega^c}$ has an analytic continuation across $\partial\Omega$ if and only if $\Omega$ is a kind of "quadrature domain" ("quadrature domain in the wide sense" in the terminology of [Sh1], [Sh2]), namely that there exists a distribution $\mu$ with compact support in $\Omega$ such that

$$\int_\Omega \varphi dA = \langle \mu, \varphi \rangle \tag{4.1}$$

for every integrable analytic function $\varphi$ in $\Omega$. Indeed, like in Section 3, (4.1) is equivalent to

$$\widehat{\chi}_\Omega = \hat{\mu} \text{ on } \Omega^c, \tag{4.2}$$

and $\hat{\mu}$ is analytic outside supp $\mu$.

In the case of classical quadrature domains (those in Section 3) there is a beautiful argument of D. Aharonov-H.S. Shapiro [Ah-Sh] showing that, without any apriori regularity assumption whatsoever, the boundary of such a domain $\Omega$ must be a subset of an algebraic curve. Once one has this global grip on $\partial\Omega$ the more detailed analysis of it (e.g. investigation of possible singular points) is relatively easy.

One main point with our theorem below is that we do the same thing for the general quadrature domains as Aharonov-Shapiro do for the classical ones: we show that the boundary $\partial\Omega$ of such a domain must be contained in the zero set of a certain real analytic function, namely the analytic continuation of the exponential transform from outside. In the special case of classical quadrature domains this function agrees (apart from some harmless one-variable factors) with the Aharonov-Shapiro polynomial. Thus we feel that it is kind of a "canonical" description of $\partial\Omega$ that we obtain.

As a special case our result applies to many two-dimensional free boundary problems in mathematical physics, e.g. the obstacle problem (with analytic data), the dam problem, Hele-Shaw flow moving boundary problems etc. Indeed, in these problems one usually has more information about $\partial\Omega$ to start with, namely (translated to our context) in addition to our assumption (4.2) one has $u = 0$ on $\Omega^c$, $u \geq 0$ in $\Omega \setminus K$ where $u$ is a certain real valued potential of $\widehat{\chi}_\Omega - \hat{\mu}$ (so that $\partial u = \chi_\Omega - \mu$). See [Ro], [Sa5] for details.

Note, however, that our regularity result is not new. Sakai [Sa3], [Sa4] starts from even weaker assumptions than we do. We essentially need a Schwarz function $S(z)$ satisfying the estimate (4.8) below whereas Sakai merely assumes continuity of $S(z)$. Also, we do not pursue the detailed analysis of possible singularities of $\partial\Omega$ since this was done already in [Sa3], [Sa4].

THEOREM 4.1. *Let $\Omega \subset \mathbb{C}$ be a bounded open set and assume that there exists a compact subset $K \subset \Omega$ and an analytic function $f$ in $K^c = \mathbb{C} \setminus K$ such that*

$$\widehat{\chi}_\Omega(z) = f(z) \quad \textit{for every} \quad z \in \Omega^c. \tag{4.3}$$

*Then there exists an analytic-antianalytic function $F(z, \bar{w})$ defined for $(z, w) \in K^c \times K^c$ such that*

$$E(z, w) = F(z, \bar{w}) \quad \textit{for} \quad z, w \in \Omega^c. \tag{4.4}$$

*Moreover,*

$$\partial\Omega \subset \{z \in K^c : F(z,\bar z) = 0\}. \tag{4.5}$$

*More precisely, setting* $A = \{z \in \Omega \setminus K : \widehat{\chi}_\Omega(z) = f(z)\}$ *we have* $F(z,\bar z) > 0$ *for* $z \in \overline{\Omega}^c$, $F(z,\bar z) = 0$ *for* $z \in \partial\Omega \cup A$ *and* $F(z,\bar z) < 0$ *for* $z \in \Omega \setminus (K \cup A)$.

Like in Section 3, given the extension (4.3) we define the *Schwarz function* [D], [Sh2] $S(z)$ of $\partial\Omega$ to be

$$S(z) = \bar z - \widehat{\chi}_\Omega(z) + f(z), \quad z \in K^c.$$

Then

$$S(z) = \bar z \quad \text{for} \quad z \in \Omega^c, \tag{4.6}$$

$$S(z) \text{ is analytic in } \Omega \setminus K. \tag{4.7}$$

Moreover, $S$ is continuous. In fact, it is known that the Cauchy-transform satisfies an estimate

$$|\widehat{\chi}_\Omega(z_1) - \widehat{\chi}_\Omega(z_2)| \le C|z_1 - z_2| \log \frac{1}{|z_1 - z_2|}$$

if $|z_1 - z_2| \le e^{-2}$, hence the same is true for $S(z)$. In particular,

$$|S(z) - \bar z| \le C\, d(z) \log \frac{1}{d(z)} \tag{4.8}$$

where $d(z) = \min(e^{-2}, \operatorname{dist}(z, \Omega^c))$. Finally note that

$$\bar\partial S(z) = 1 - \chi_\Omega(z) \qquad (z \in K^c)$$

in the sense of distributions.

Next, following Ahlfors [Af] and Bers [Be] we choose a nondecreasing function $\psi \in C^\infty(\mathbb{R})$ satisfying $0 \le \psi \le 1$, $\psi(t) = 0$ for $t \le 1$, $\psi(t) = 1$ for $t \ge 2$ and define cut-off functions by

$$\psi_n(z) = \psi\left(\frac{n}{\log\log \dfrac{1}{d(z)}}\right) \tag{4.9}$$

for $z \in \Omega$, $\psi_n(z) = 0$ for $z \in \Omega^c$ ($n = 1, 2, \dots$). Then each $\psi_n$ is Lipschitz continuous, has compact support in $\Omega$ and $0 \le \psi_n \nearrow \chi_\Omega$ pointwise as $n \to \infty$. Moreover, we have a good bound on the gradient of $\psi_n$:

$$\left|\bar\partial\psi_n(z)\right| \le \frac{C}{n d(z) \log \dfrac{1}{d(z)}}. \tag{4.10}$$

Combining (4.10) with (4.8) gives the important estimate

$$\left\|(S(z) - \bar z)\,\bar\partial\psi_n(z)\right\|_\infty \le \frac{C}{n}. \tag{4.11}$$

This enables us, e.g., to use a kind of Stokes' formula without having any a priori knowledge about the smoothness of $\partial\Omega$ (besides the property (4.3)). Indeed we have:

LEMMA 4.2. *Let $\Omega \subset \mathbb{C}$ satisfy the hypotheses of Theorem 4.1 and let $\Phi(z, w)$ be a function which is analytic and Lipschitz continuous in an open set in $\mathbb{C}^2$ containing $\{(z, \bar{z}) : z \in \overline{\Omega}\} \cup \{(z, S(z)) : z \in \Omega \setminus K\}$. Then*

$$\int_{\Omega} \bar{\partial}_z \Phi(z, \bar{z})\, d\bar{z}\, dz = \int_{\partial D} \Phi\big(z, S(z)\big) dz \tag{4.12}$$

*for any open set $D$, $K \subset D \subset \Omega$, with smooth boundary ("smooth" means here smooth enough for Stokes' formula).*

NOTE: $\partial D$ should be thought of as a freely moving contour just inside $\partial\Omega$ and then assuming smoothness of $\partial D$ is of course harmless (as long as $D \subset\subset \Omega$).

PROOF. By Lipschitz continuity and (4.11) we have

$$\left\|\bar{\partial}\psi_n(z)\, (\Phi(z, \bar{z}) - \Phi(z, S(z)))\right\|_\infty \le C \left\|\bar{\partial}\psi_n(z)\, (\bar{z} - S(z))\right\|_\infty \le \frac{C}{n}.$$

Thus, noting that $\partial(\Omega \setminus D) \cap \Omega$ is smooth and equals $\partial D \cap \Omega$ but has opposite orientation, we obtain

$$\begin{aligned} 0 &= \lim_{n\to\infty} \int_{\Omega\setminus D} \bar{\partial}\psi_n(z)\big(\Phi(z, \bar{z}) - \Phi\big(z, S(z)\big)\big) d\bar{z}\, dz \\ &= \lim_{n\to\infty} \int_{\partial(\Omega\setminus D)\cap\Omega} \psi_n(z)\big(\Phi(z, \bar{z}) - \Phi\big(z, S(z)\big)\big) dz \\ &\quad - \lim_{n\to\infty} \int_{\Omega\setminus D} \psi_n(z) \bar{\partial}_z \Phi(z, \bar{z})\, d\bar{z}\, dz \\ &= -\int_{\partial D\cap\Omega} \Phi(z, \bar{z})\, dz + \int_{\partial D\cap\Omega} \Phi\big(z, S(z)\big) dz - \int_{\Omega\setminus D} \bar{\partial}_z \Phi(z, \bar{z}) d\bar{z}\, dz. \end{aligned}$$

By the usual Stokes' theorem applied to the smooth domain $D$

$$\begin{aligned} \int_D \bar{\partial}_z \Phi(z, \bar{z}) d\bar{z}\, dz &= \int_{\partial D} \Phi(z, \bar{z}) dz \\ &= \int_{\partial D\cap\Omega} \Phi(z, \bar{z}) dz + \int_{\partial D\setminus\Omega} \Phi\big(z, S(z)\big) dz \end{aligned}$$

(since $S(z) = \bar{z}$ on $\partial D \setminus \Omega \subset \partial\Omega$). Combining the two formulas gives (4.12). □

REMARK 4.3. As is clear from the proof, what we really used about $\widehat{\chi}_\Omega$ is that there exists a continuous function $S(z)$ defined and satisfying the estimate (4.8) in $\overline{\Omega}\setminus D$, satisfying $S(z) = \bar{z}$ for $z \in \partial\Omega$ and being analytic in $\Omega \setminus \overline{D}$.

It is allowed in the lemma that part of $\partial D$ agrees with $\partial\Omega$ but then this part of $\partial\Omega$ must be smooth. Without that smoothness the result of the first computation of the proof is still valid. This gives a more local version of the lemma which applies e.g. to $D = \Omega \setminus \overline{\mathbb{D}(z_0, r)}$, $z_0 \in \partial\Omega$, $r > 0$, whenever there exists a Schwarz function as above in $\Omega \cap \overline{\mathbb{D}(z_0, r)}$.

The following crucial lemma is due to L. Caffarelli, L. Karp, A. Margulis and H. Shahgholian. For completeness we include here most of the proof. The main idea is found in [K-Mg]. It is based on an estimate by Caffarelli [Ca] which has been generalized to our situation by Shahgholian (unpublished). The last statement (b) is due to Karp (unpublished). The lemma generalizes in a natural way to higher dimensions.

LEMMA 4.4 (Caffarelli-Karp-Margulis-Shahgholian). *If $\Omega$ satisfies the hypotheses in Theorem* 4.1 *then*

(a) *$\partial\Omega$ has area measure zero,*

(b) $Z = \varnothing$

(*see* (2.10) *for the notation*).

PROOF. By Lebesgue's theorem, almost every point of $\partial\Omega \subset \Omega^c$ is a density point of $\Omega^c$. Moreover, $\widehat{\chi}_\Omega$ is differentiable at almost every point [Cd-Z], [Re]. For the first statement of the lemma it is therefore enough to prove that the set of points $z \in \partial\Omega$ at which both of

(i) $\Omega^c$ has density one at $z$,

(ii) $\widehat{\chi}_\Omega$ is differentiable at $z$

hold has measure zero. We shall prove that this set is actually empty.

From this it will also follow that $Z$ is empty, because (i) and (ii) do hold for every $z \in Z$. For (i) this was remarked after (2.11) and for (ii) we have, using the same arguments as in the proof of Proposition 2.7 (the final estimates) or directly invoking e.g. [St, p. 243-246], that

$$\widehat{\chi}_\Omega(w) - \widehat{\chi}_\Omega(z) = (w - z)\left[-\frac{1}{\pi}\int_\Omega \frac{dA(\zeta)}{(\zeta - z)^2}\right] + o(|w - z|) \text{ as } w \to z.$$

We thank L. Karp and the referee for useful comments in this context.

Now, to prove that (i) and (ii) are contradictory, let $z \in \partial\Omega$ be a point satisfying (i) and (ii). We may assume that $z = 0$. By (ii)

$$S(z) = \bar{z} - \widehat{\chi}_\Omega(z) + f(z)$$

is differentiable at $z = 0$, i.e.

$$S(z) = \partial S(0)z + \bar{\partial} S(0)\bar{z} + o(|z|)$$

($|z| \to 0$). Using (i) and that $\widehat{\chi}_\Omega = f$ on $\Omega^c$ it follows that $\partial S(0) = 0$, $\bar{\partial} S(0) = 1$, hence that

$$S(z) - \bar{z} = o(|z|). \tag{4.13}$$

Next, fix any small $r > 0$, let $z_n \in \Omega \cap \mathbb{D}(0, r)$, $z_n \to 0$ $(n \to \infty)$ and consider the functions

$$u_n(z) = |S(z) - \bar{z}|^2 - |z - z_n|^2$$

for $z \in \Omega \cap \mathbb{D}(0,r)$. We have $\bar{\partial}\partial u_n(z) = |S'(z)|^2 \geq 0$, hence each $u_n$ is subharmonic. Since $u_n(z) = -|z-z_n|^2 < 0$ on $\partial\Omega \cap \mathbb{D}(0,r)$, $u_n(z_n) = |S(z_n) - \bar{z}_n|^2 \geq 0$ it follows that $u_n \geq 0$ somewhere on $\partial\mathbb{D}(0,r) \cap \Omega$.

Thus

$$|S(\zeta_n) - \bar{\zeta}_n| \geq |\zeta_n - z_n|$$

for some $\zeta_n \in \partial\mathbb{D}(0,r) \cap \Omega$, and since $z_n \to 0$, $|\zeta_n| = r > 0$ this contradicts (4.13). Hence the lemma is proved. □

As a corollary of Proposition 2.7 and b) of Lemma 4.4 we have

COROLLARY 4.5. *$E_\Omega$ is continuous everywhere and $E_\Omega(z,z) = 0$ for all $z \in \partial\Omega$ when $\Omega$ satisfies the hypotheses of Theorem* 4.1.

PROOF OF THEOREM 4.1. We shall prove that there exists an analytic-antianalytic function $F(z,\bar{w})$ in $K^c \times K^c$ satisfying

$$F(z,\bar{w}) = \frac{z - \overline{S(w)}}{z - w} \cdot \frac{S(z) - \bar{w}}{\bar{z} - \bar{w}} \cdot E(z,w) \tag{4.14}$$

for $(z,w) \in (K^c \times K^c) \setminus \Delta_\Omega$ and satisfying

$$F(z,\bar{w}) = \left(z - \overline{S(w)}\right)\left(S(z) - \bar{w}\right) \cdot \frac{E(z,w)}{|z-w|^2} \tag{4.15}$$

for $(z,w) \in (K^c \times K^c) \setminus \Delta_{\Omega^c}$.

In (4.14) it is understood that $\frac{z-\overline{S(w)}}{z-w} = 1$ whenever $w \in \Omega^c$, $\frac{S(z)-\bar{w}}{\bar{z}-\bar{w}} = 1$ whenever $z \in \Omega^c$ (i.e., even if $z = w$) and in (4.15) it is understood that, in $\Omega \times \Omega$, $E(z,w)/|z-w|^2$ is the analytic-antianalytic function $H(z,\bar{w})$ appearing in (2.15). Note that at each point of $(K^c)^2$ at least one of (4.14) and (4.15) is in force. Indeed, both make plain sense, and say the same thing, in $(K^c)^2 \setminus \Delta$. At each point of $\Delta_{K^c}$ exactly one of (4.14) and (4.15) makes sense as above.

Note also that (4.14) in particular says that $F = E$ on $\Omega^c \times \Omega^c$ so $F$ will indeed be the required analytic-antianalytic extension of $E$. Since, by Corollary 4.5, $E = 0$ on $\Delta_{\partial\Omega}$ also (4.5) will follow. The last statement of the theorem follows by setting $z = w$ in (4.15) and using (2.15), (2.18). Thus the theorem is proved once $F$ as above is constructed.

Next we remark that it is enough to prove (4.14), (4.15) for the set int $\overline{\Omega}$ instead of $\Omega$.

Indeed, the difference set $B = (\operatorname{int}\overline{\Omega}) \setminus \Omega \subset \partial\Omega$ has measure zero by Lemma 4.4 so $E$ does not recognize the difference (and neither does $F$). If equations (4.14), (4.15) hold for int $\overline{\Omega}$ they also hold for $\Omega$ (and conversely) because the change just means that the set of points $\Delta_B$ is moved from the domain of validity of (4.15) to that of (4.14), and since $S(z) = \bar{z}$ and $E(z,z) = 0$ for $z \in B$ any equation (4.14), (4.15) which is valid for $\Delta_B$ will say the same thing, namely that $F = 0$ on $\Delta_B$.

Thus, renaming $\Omega$ to be int $\overline{\Omega}$ we may assume that

$$\Omega = \operatorname{int} \overline{\Omega}, \tag{4.16}$$

i.e., that each point of $\partial\Omega$ is accessible from $\overline{\Omega}^c$ ($\partial\Omega = \partial(\overline{\Omega}^c)$).

We will present two methods of constructing $F$ satisfying (4.14), (4.15). The first method is to write the integral in the definition of $E$ as a boundary integral, then perform the analytic continuation of $E$ by deforming the contour of integration and finally check that (4.14), (4.15) hold for the result $F$ of the analytic continuation.

The second method is to define $F$ by (4.14), (4.15) and then check that $F$ is analytic-antianalytic.

Method 1.

We start by taking a fixed $w \in \overline{\Omega}^c$ and choosing an open set $D = D_w$ with $K \subset D \subset\subset \Omega$, $\partial D$ smooth, such that

$$\left| \frac{S(\zeta) - \bar{\zeta}}{\bar{\zeta} - \bar{w}} \right| < \frac{1}{2} \quad \text{for} \quad \zeta \in \Omega \setminus D.$$

This is possible by continuity of $S(\zeta)$ and since $|\zeta - w| \geq c > 0$. It follows that

$$\log\left[(\zeta - w)\left(S(\zeta) - \bar{w}\right)\right] = \log\left(|\zeta - w|^2 \left(1 + \frac{S(\zeta) - \bar{\zeta}}{\bar{\zeta} - \bar{w}}\right)\right)$$

is single-valued analytic in $\Omega \setminus \overline{D}$ (we choose the natural branch of the logarithm, for which the argument is in the interval $(-\pi/2, \pi/2)$).

For any $z \in \overline{\Omega}^c$, now apply Lemma 4.2 with

$$\Phi\left(\zeta, \bar{\zeta}\right) = \frac{\log|\zeta - w|^2}{\zeta - z} = \frac{\log\left[(\zeta - w)\left(\bar{\zeta} - \bar{w}\right)\right]}{\zeta - z}.$$

Since

$$\Phi(\zeta, S(\zeta)) = \frac{\log\left[(\zeta - w)\left(S(\zeta) - \bar{w}\right)\right]}{\zeta - z},$$

$$\bar{\partial}_\zeta \Phi\left(\zeta, \bar{\zeta}\right) = \frac{1}{(\zeta - z)\left(\bar{\zeta} - \bar{w}\right)}$$

this gives

$$\begin{aligned} E_\Omega(z, w) &= \exp\left[-\frac{1}{2\pi i} \int_\Omega \frac{d\bar{\zeta} d\zeta}{\left(\bar{\zeta} - \bar{w}\right)(\zeta - z)}\right] \\ &= \exp\left[-\frac{1}{2\pi i} \int_{\partial D} \log\left[\left(S(\zeta) - \bar{w}\right)(\zeta - w)\right] \frac{d\zeta}{\zeta - z}\right]. \end{aligned}$$

As a function of $z$, the right member above is analytic in all $\overline{D}^c$. This means that we have already produced a little bit of analytic continuation of $E$.

Next we want to continue the so obtained analytic function further, to $\mathbb{C} \setminus K$.

The integral in the last expression above defines one analytic function in $D$ and one other in $\mathbb{C} \setminus \overline{D}$, say

$$\frac{1}{2\pi i} \int_{\partial D} \log\left[(S(\zeta) - \bar{w})(\zeta - w)\right] \frac{d\zeta}{\zeta - z} = \begin{cases} h^e(z), & z \in \mathbb{C} \setminus \overline{D}, \\ h^i(z), & z \in D. \end{cases}$$

Since $\partial D$ is smooth the two functions $h^e$ and $h^i$ extend continuously up to $\partial D$ and satisfy the jump condition

$$h^i(z) - h^e(z) = \log\left[(S(z) - \bar{w})(z - w)\right]$$

for $z \in \partial D$.

Since

$$E_\Omega(z, w) = \exp\left[ - h^e(z)\right] \qquad \left(z \in \mathbb{C} \setminus \overline{\Omega}\right)$$

by the previous computation it follows that the function

$$\begin{aligned} &\exp\left[\log\left((S(z) - \bar{w})(z - w)\right) - h^i(z)\right] \\ &= (S(z) - \bar{w})\,(z - w) \exp\left[-\frac{1}{2\pi i} \int_{\partial D} \log\left(S(\zeta) - \bar{w})(\zeta - w\right) \frac{d\zeta}{\zeta - z}\right], \end{aligned}$$

which is analytic in $D \setminus K$, provides the further analytic extension of $E_\Omega(z, w)$ down to $K$. Finally one notices that, for $z \in D \setminus K$, the path of integration $\partial D$ in the last integral can be moved back to $\partial\Omega$ again. Thus the analytic extension of $E_\Omega(z, w)$ with respect to $z$ is given by

$$F(z, \bar{w}) = (S(z) - \bar{w})\,(z - w) \exp\left[-\frac{1}{2\pi i} \int_{\sim\partial\Omega} \log\left[(S(\zeta) - \bar{w})(\zeta - w)\right] \frac{d\zeta}{\zeta - z}\right]$$

for $z \in \Omega \setminus K$, $w \in \overline{\Omega}^c$ and where $\sim \partial\Omega$ denotes any path $\partial D$, with $\{z\} \cup K \subset D \subset\subset \Omega$.

Next we want to express $F$ directly in terms of $E$. For $z \in \Omega \setminus K$, $w \in \overline{\Omega}^c$ choose $\mathbb{D}(z, r) \subset\subset \Omega \setminus K$. Since $E_{\mathbb{D}(z,r)}(z, w) = 1$ (Example 2.3) we have, using Lemma 4.2 again

$$\begin{aligned} E_\Omega(z, w) &= E_{\Omega\setminus\mathbb{D}(z,r)}(z, w) E_{\mathbb{D}(z,r)}(z, w) \\ &= \exp\left[-\frac{1}{2\pi i} \int_{\Omega\setminus\mathbb{D}(z,r)} \frac{d\bar{\zeta} d\zeta}{\left(\bar{\zeta} - \bar{w}\right)(\zeta - z)}\right] \\ &= \exp\left[-\frac{1}{2\pi i} \int_{\sim\partial\Omega} \log\left[(S(\zeta) - \bar{w})(\zeta - w)\right] \frac{d\zeta}{\zeta - z}\right] \\ &\quad \cdot \exp\left[\frac{1}{2\pi i} \int_{\partial\mathbb{D}(z,r)} \log\left[(\bar{\zeta} - \bar{w})\,(\zeta - w)\right] \frac{d\zeta}{\zeta - z}\right] \\ &= \exp\left[-\frac{1}{2\pi i} \int_{\sim\partial\Omega} \log\left[(S(\zeta) - \bar{w})(\zeta - w)\right] \frac{d\zeta}{\zeta - z}\right] \cdot |z - w|^2. \end{aligned}$$

Thus

$$F(z,\bar{w}) = \big(S(z)-\bar{w}\big)(z-w)\,\frac{E(z,w)}{|z-w|^2} = \frac{S(z)-\bar{w}}{\bar{z}-\bar{w}}\,E(z,w) \tag{4.17}$$

for $z \in \Omega$, $w \in \overline{\Omega}^c$.

Next one observes that all the above works in the same way also if $w \in \Omega \backslash K$. It is just to replace $\Omega$ by $\Omega \setminus \overline{\mathbb{D}(w,r)}$ ($\mathbb{D}(w,r) \subset\subset \Omega \setminus K$) and use that $E_{D(w,r)}(z,w) = 1$ for $z \in \mathbb{D}(w,r)^c$. The conclusion is that for fixed $w \in \Omega$, the analytic continuation of $z \mapsto E(z,w)$ from $\overline{\Omega}^c$ to $(K \cup \{w\})^c$ is still given by the right member of (4.17).

Similarly, for fixed $z \in (K \cup \partial\Omega)^c$ the antianalytic continuation of $w \mapsto E(z,w)$ from $\overline{\Omega}^c$ to $(K \cup \{z\})^c$ is given by

$$\big(z-\overline{S(w)}\big)\,(\bar{z}-\bar{w})\,\frac{E(z,w)}{|z-w|^2} = \frac{z-\overline{S(w)}}{z-w}\,E(z,w). \tag{4.18}$$

Combining the above two analytic continuations it follows that there exists an analytic-antianalytic function $F$ in $(K^c \times K^c) \setminus (\Delta_{\overline{\Omega}} \cup (\partial\Omega \times \partial\Omega))$ which agrees with $E$ in $\overline{\Omega}^c \times \overline{\Omega}^c$ and which satisfies (4.14), (4.15) in $(K^c \times K^c) \setminus (\Delta \cup (\partial\Omega \times \partial\Omega))$. Indeed, the joint analytic continuation from $\overline{\Omega}^c \times \overline{\Omega}^c$ to a point $(z,w)$ with say $z \notin \partial\Omega$ and $z \neq w$ is achieved by first moving $z$ to the right place by (4.17) and then moving $w$ according to (4.18). The latter means that the expression (4.18) replaces $E$ in (4.17) when $w$ moves into $\Omega$.

Since the functions $E(z,w)/|z-w|^2$ and $(S(z)-w)(z-\overline{S(w)})$ are analytic-antianalytic in all $(\Omega \setminus K) \times (\Omega \setminus K)$ (2.15) we can add $\Delta_{\Omega\backslash K}$ to the domain of validity of (4.15) by taking the right member of (4.15) as the definition of $F$ on $\Delta_{\Omega\backslash K}$. By this $F$ is defined and analytic-antianalytic, and (4.14), (4.15) hold, in $(K^c \times K^c) \setminus (\partial\Omega \times \partial\Omega)$.

The remaining continuation of $F$ to all $K^c \times K^c$ is now automatic. Indeed, let $z_0 \in \partial\Omega$, $w_0 \in \partial\Omega$, let $r > 0$ denote the distance between $\partial\Omega$ and $K$ and let $0 < \varepsilon < 1/2$. Then for each $z_1 \in \mathbb{D}(z_0, \varepsilon r) \setminus \partial\Omega$, $w_1 \in \mathbb{D}(w_0, \varepsilon r) \setminus \partial\Omega$ the power series of $F(z,\bar{w})$ at $(z_1, w_1)$ has radius of convergence at least $(1-\varepsilon)r$ in each variable separately when the other variable is kept fixed.

By log-convexity of joint domain of convergence [Hm, Thm 2.4.3] the full power series of $F(z,w)$ at $(z_1,w_1)$ converges in all $D = \mathbb{D}(z_1, \sqrt{\varepsilon}\sqrt{1-\varepsilon r}) \times \mathbb{D}\left(w_1, \sqrt{\varepsilon}\ \sqrt{1-\varepsilon r}\right)$. Since $\varepsilon < 1/2$ this domain contains $(z_0, w_0)$. There will be no problems of multivaluedness for the analytic continuation because the initial domain of analyticity $D \setminus (\partial\Omega \times \partial\Omega)$ is connected and the extension to $D$ is defined by a single power series.

The above shows that $F(z,\bar{w})$ has an analytic-antianalytic extension to all $K^c \times K^c$. We must still check that (4.14), (4.15) are valid on $\partial\Omega \times \partial\Omega$, i.e. that $F = E$ on $\partial\Omega \times \partial\Omega$. But this holds in $\overline{\Omega}^c \times \overline{\Omega}^c$, both $F$ and $E$ are continuous and $\partial\Omega \times \partial\Omega \subset \partial(\overline{\Omega}^c \times \overline{\Omega}^c)$ by (4.16). Thus (4.14), (4.15) hold as stated and by this the construction of $F$ by Method 1 as finished.

METHOD 2.

Define $F(z, \bar{w})$ at each point $(z, w) \in K^c \times K^c$ by (4.14), (4.15). We shall prove that $F$ is analytic-antianalytic. For this, in the virtue of Hartog's theorem [Hm, Thm 2.2.8], it is enough to prove that $F$ is analytic (antianalytic) in $z$ (resp. $w$) when the other variable is held fixed and by symmetry it is enough to consider one of the cases, say to prove analyticity in $z$ when $w$ is held fixed.

So let $w \in K^c$ be fixed. The factor

$$\frac{z - \overline{S(w)}}{z - w} = 1 + \frac{w - \overline{S(w)}}{z - w}$$

is clearly analytic in $z$, except possibly for a simple pole at $z = w$ in the case that $w \in \Omega$. Therefore it is enough to show that the function $G(z) = G_w(z)$ defined in $K^c$ by

$$G(z) = \left(1 + \frac{S(z) - \bar{z}}{\bar{z} - \bar{w}}\right) E(z, w) = (z - w)\big(S(z) - \bar{w}\big) \cdot \frac{E(z, w)}{|z - w|^2} \tag{4.19}$$

is analytic and, in case $w \in \Omega$, has a zero at $z = w$. As to the interpretation of (4.19) on the diagonal similar rules as for (4.14), (4.15) apply: if $z = w \in \Omega^c$ the first expression shall be used, saying that $G = E$ then, if $z = w \in \Omega$ the second expression shall be used.

The analyticity of $G$ in $\overline{\Omega}^c$ and in $\Omega \setminus K$, and that it vanishes at $w$ if $w \in \Omega$, is immediately clear from the general structure (2.15) of $E(z, w)$. Problems can arise only on $\partial\Omega$.

Since $G$ is locally bounded except for the factor $1/(\bar{z} - \bar{w})$, it is locally integrable, in particular $G$ is a distribution in $K^c$. We shall compute $\bar{\partial} G$ in the sense of distributions. Let $\{\psi_n\}$ be the Ahlfors-Bers cut-off functions (4.9). By $G \in L^1_{\text{loc}}$, $S(z) - \bar{z} = 0$ on $\Omega^c$ we have, as $n \to \infty$,

$$E(z, w) + \psi_n(z)\big(S(z) - \bar{z}\big)\frac{E(z, w)}{\bar{z} - \bar{w}} \rightharpoonup E(z, w) + \big(S(z) - \bar{z}\big)\frac{E(z, w)}{\bar{z} - \bar{w}} = G(z)$$

in the sense of distributions (use e.g. dominated convergence on the above expressions action on test functions). It follows that

$$\bar{\partial}_z E(z, w) + \bar{\partial}_z \left[\psi_n(z)\big(S(z) - \bar{z}\big)\, \frac{E(z, w)}{\bar{z} - \bar{w}}\right] \rightharpoonup \bar{\partial}_z G(z) \tag{4.20}$$

in the sense of distributions.

Due to the factor $\psi_n(z)$ everything in the second term in (4.20) takes place inside a compact subset of $\Omega$, where $S(z) - \bar{z}$ is smooth and $E(z, w)/\,(\bar{z} - \bar{w})$ is analytic in $z$ by (2.15). Therefore we are allowed to use the product rule for the derivative, giving that

$$\bar{\partial}_z \left[\psi_n(z)\big(S(z) - \bar{z}\big)\frac{E(z, w)}{\bar{z} - \bar{w}}\right] = \bar{\partial}_z \psi_n(z)\big(S(z) - \bar{z}\big)\frac{E(z, w)}{\bar{z} - \bar{w}} - \psi_n(z)\frac{E(z, w)}{\bar{z} - \bar{w}}.$$

In view of the estimate (4.11) and the integrability of $E(z, w)/(\bar{z} - \bar{w})$ the first term above tends to zero (in the sense of distributions) as $n \to \infty$ and the second term tends to $-\chi_\Omega(z)E(z, w)/(\bar{z} - \bar{w})$. Taking the first term in (4.20) into account and using (2.19) we therefore conclude from (4.20) that

$$\bar{\partial} G = 0.$$

Thus $G(z)$ is analytic in $K^c$ as a distribution, i.e. there exists an analytic function in $K^c$ which differs from $G$ at most on a nullset. The final observation now is that this nullset has to be empty. Indeed, we claim that if we change the definition of $G$ on any nonempty nullset the result will be a discontinuous function.

For this we have to use the assumption $\Omega = \operatorname{int} \overline{\Omega}$ and the fact that $E$ is continuous everywhere. By definition (4.19) $G$ is continuous in $\Omega$ and equals $E(\cdot, w)$ on $\Omega^c$. By the above remarks the restriction of $G$ to $\Omega^c$ therefore is continuous and each point of $\Omega^c$ is in the closure of the open subset $\overline{\Omega}^c$. It follows that if we change the definition of $G$ at any point $z_0$, in $\Omega \setminus K$ or in $\Omega^c$, but leave it unchanged almost everywhere it will become discontinuous at $z_0$.

Thus we conclude that $G$ already from the beginning was analytic as a function, not only as a distribution. This finishes the Method 2 proof of Theorem 1. □

REMARK 4.6. By small modifications of the proof one arrives at a local version of Theorem 4.1 saying the following: if $\Omega, U \subset \mathbb{C}$ are open (think of $U$ as a neighbourhood of some point on $\partial\Omega$) and there exists $f$ analytic in $U$ such that $\widehat{\chi}_\Omega = f$ in $U \setminus \Omega$ then there exists $F(z, \bar{w})$ analytic-antianalytic in $U \times U$ such that $E = F$ in $(U \setminus \Omega)^2$. Moreover, $\partial\Omega \cap U \subset \{z \in U : F(z, \bar{z}) = 0\}$.

REMAK 4.7. The role of part a) of Lemma 4.4 in the proof of Theorem 4.1 is that it permits us to work with int $\overline{\Omega}$ instead of $\Omega$. Part b) of the lemma is really not needed, it is enough to use that $\partial\Omega \setminus Z$ is dense in $\partial\Omega$ (see before Proposition 2.7). If one is willing to assume in advance that $\Omega = \operatorname{int} \overline{\Omega}$ then Lemma 4.4 can therefore be dispensed with and the proof of Theorem 4.1 becomes shorter. Moreover, assumption (4.3) may in this case be relaxed to

$$\widehat{\chi}_\Omega = f \text{ in } \overline{\Omega}^c \tag{4.21}$$

because (4.3) will follow from (4.21) by continuity.

## 5. – Analytic continuation of resolvents of hyponormal operators

Let $T \in \mathcal{L}(H)$ be the unique irreducible hyponormal operator with rank-one self-commutator (i.e., $[T^*, T] = \xi \otimes \xi$ for some nonzero $\xi \in H$) and with principal function equal to the characteristic function $\chi_\Omega$ of a bounded domain $\Omega$. Here $H$ is a separable complex Hilbert space with inner product denoted $\langle \cdot, \cdot \rangle$.

It is well known that a generalized weakly continuous resolvent $(T^* - \bar{z})^{-1}\xi$ exists for every $z \in \mathbb{C}$, see [MP, Ch. XI]. Moreover, a fundamental identity established by K. Clancey [Cl1] asserts that

$$E_\Omega(z, w) = 1 - \left\langle \left(T^* - \bar{w}\right)^{-1}\xi, \left(T^* - \bar{z}\right)^{-1}\xi \right\rangle, \quad (z, w \in \mathbb{C}).$$

Starting with the same assumption as in Section 4 (namely that the Cauchy transform $\widehat{\chi}_\Omega$ extends analytically across the boundary of $\Omega$) we will prove that the resolvent $(T^* - \bar{z})^{-1}\xi$ itself extends antianalytically across $\partial\Omega$. This will provide another proof of the main result in Section 4 and a little bit more. In the present section we will for simplicity assume that int $\overline{\Omega} = \Omega$ (cf. Remark 4.7).

Assume that $\widehat{\chi}_\Omega(z) = f(z)$, $z \notin \overline{\Omega}$, where $f$ is analytic in $\mathbb{C} \setminus K$ and define as above the corresponding Schwarz function

$$S(z) = \bar{z} - \widehat{\chi}_\Omega(z) + f(z), \quad z \in \mathbb{C} \setminus K.$$

Let $\psi_n$ be the Ahlfors-Bers exhaustion functions (4.9) satisfying $\psi_n \nearrow \chi_\Omega$ and (4.11).

LEMMA 5.1. *The following limit exists in* $\mathcal{D}'(\mathbb{C})\widehat{\otimes}H$:

$$U(z) = \lim_{n\to\infty} \psi_n(z)\left(\overline{S(z)} - z\right)\partial \left(T^* - \bar{z}\right)^{-1}\xi.$$

PROOF. Let $\varphi \in \mathcal{D}(\mathbb{C})$ and let $n \in \mathbb{N}$ be fixed. Then

$$\int_{\mathbb{C}} \psi_n(z)\left(\overline{S(z)} - z\right) \partial \left(T^* - \bar{z}\right)^{-1}\xi\, \varphi(z) dA(z)$$

$$\stackrel{\text{def}}{=} -\int_{\mathbb{C}} \left(T^* - \bar{z}\right)^{-1}\xi\, \partial \left[\psi_n(z)\, \left(\overline{S(z)} - z\right)\, \varphi(z)\right] dA(z)$$

$$= -\int_{\mathbb{C}} \left(T^* - \bar{z}\right)^{-1}\xi\, \partial\psi_n(z)\, \left(\overline{S(z)} - z\right)\varphi(z)\, dA(z)$$

$$+ \int_{\mathbb{C}} \left(T^* - \bar{z}\right)^{-1}\xi\, \psi_n(z)\, \varphi(z)\, dA(z)$$

$$- \int_{\mathbb{C}} \left(T^* - \bar{z}\right)^{-1}\xi\, \psi_n(z)\, \left(\overline{S(z)} - z\right) \partial\varphi(z)\, dA(z).$$

By passing to the limit we find

$$\int_{\mathbb{C}} U(z)\, \varphi(z)\, dA(z) = \int_\Omega \left(T^* - \bar{z}\right)^{-1}\xi\, \varphi(z)\, dA(z) - \int_\Omega \left(T^* - \bar{z}\right)^{-1}\xi\, \left(\overline{S(z)} - z\right) \partial\varphi(z)\, dA(z).$$

Recall that $\|(T^* - \bar{z})^{-1}\xi\| \leq 1$, $z \in \mathbb{C}$, see [MP]. □

We consider the closed linear span

$$H_1 = \vee_{w \notin \overline{\Omega}} \left(T^* - \bar{w}\right)^{-1} \xi$$

and the restriction $R^* = T^* |_{H_1}$ (of $T^*$ to this invariant subspace). Then $R \in \mathcal{L}(H_1)$ and

$$\left(R^* - \bar{w}\right)^{-1} \xi = \left(T^* - \bar{w}\right)^{-1} \xi, \quad w \notin \overline{\Omega}.$$

In particular $\sigma(R) \subset \overline{\Omega} = \sigma(T)$. Let $P$ denote the orthogonal projection of $H$ onto $H_1$.

PROPOSITION 5.2. *The distribution*

$$V(z) = P\left(T^* - \bar{z}\right)^{-1} \xi + PU(z) \in \mathcal{D}'(\mathbb{C}) \widehat{\otimes} H_1$$

*is antianalytic in* $\mathbb{C} \setminus K$.

PROOF. Let $w \notin \overline{\Omega}$, and consider the scalar valued distribution

$$\begin{aligned}
\left\langle V(z), \left(T^* - \bar{w}\right)^{-1} \xi\right\rangle &= \left\langle P\left(T^* - \bar{z}\right)^{-1} \xi + PU(z), \left(T^* - \bar{w}\right)^{-1} \xi\right\rangle \\
&= \left\langle \left(T^* - \bar{z}\right)^{-1} \xi + U(z), \left(T^* - \bar{w}\right)^{-1} \xi\right\rangle \\
&= \left\langle \left(T^* - \bar{z}\right)^{-1} \xi, \left(T^* - \bar{w}\right)^{-1} \xi\right\rangle \\
&\quad + \lim_{n \to \infty} \psi_n(z) \left(\overline{S(z)} - z\right) \partial_z \left\langle \left(T^* - \bar{z}\right)^{-1} \xi, \left(T^* - \bar{w}\right)^{-1} \xi\right\rangle \\
&= 1 - E(w, z) - \lim_{n \to \infty} \psi_n(z) \left(\overline{S(z)} - z\right) \partial_z E(w, z) \\
&= 1 - E(w, z) - \chi_\Omega(z) \left(\overline{S(z)} - z\right) \partial_z E(w, z) \\
&= 1 - \overline{G_w(z)},
\end{aligned}$$

$G_w(z)$ defined by (4.19). But according to Section 4, $G_w(z)$ is analytic for $z \notin K$. Hence the distribution $\partial V(z)$ satisfies

$$\left\langle \partial V(z), \left(T^* - \bar{w}\right)^{-1} \xi\right\rangle = 0, \quad w \notin \overline{\Omega}, z \notin K.$$

Since the space $H_1$ is generated by the vectors $(T^* - \bar{w})^{-1} \xi$, $w \notin \overline{\Omega}$, it follows that $\partial V(z) = 0$ for $z \notin K$. □

COROLLARY 5.3. *The Cauchy transform $\widehat{\chi}_\Omega(z)$ extends analytically from $\overline{\Omega}^c$ to $K^c$ if and only if the resolvent $(T^* - \bar{z})^{-1}\xi$ extends antianalytically from $\overline{\Omega}^c$ to $K^c$.*

PROOF. One implication follows from Proposition 5.2. Indeed, the function

$$\left(T^* - \bar{z}\right) V(z), \quad z \notin K$$

is antianalytic, and it satisfies for $z \notin \overline{\Omega}$

$$\left(T^* - \bar{z}\right) V(z) = \left(T^* - \bar{z}\right) P \left(T^* - \bar{z}\right)^{-1} \xi = \left(T^* - \bar{z}\right) \left(T^* - \bar{z}\right)^{-1} \xi = \xi,$$

whence $V(z)$ is an antianalytic extension to $K^c$ of the resolvent $(T^* - \bar{z})^{-1}\xi$.
Conversely, suppose that the antianalytic extension $V(z)$ of the resolvent $(T^* - \bar{z})^{-1}\xi$ exists for $z \notin K$. Then, for $z \notin \overline{\Omega}$, it is known as a consequence of Helton-Howe trace formula, see [MP], that

$$\langle \xi, V(z) \rangle = \left\langle \xi, \left(T^* - \bar{z}\right)^{-1} \xi \right\rangle = \frac{1}{\pi} \int_\Omega \frac{dA(\zeta)}{\zeta - z},$$

whence the analytic extension of $\widehat{\chi}_\Omega(z)$ is given by the function $\langle \xi, V(z) \rangle$. □

THEOREM 5.4. *Assume that the Cauchy transform of the bounded domain $\Omega$ analytically extends from $\overline{\Omega}^c$ to $K^c$ ($K \subset\subset \Omega$). Then the exponential kernel $E_\Omega(z, w)$ extends analytically from $\overline{\Omega}^c \times \overline{\Omega}^c$ to $K^c \times K^c$ and its extension is given by the formula*

$$F(z, \bar{w}) = 1 - \langle V(w), V(z) \rangle$$

(*where $V$ is the function defined in Proposition* 5.2).
*Moreover, $F(z, \bar{z}) > 0$ for $z \notin \overline{\Omega}$ and $F(z, \bar{z}) \leq 0$ for $z \in \Omega \setminus K$.*

PROOF. Only the last statement needs a proof. Since $(T^* - \bar{z}) V(z) = \xi$ for $z \notin K$, it follows that

$$\|V(z)\| \geq \left\| \left(T^* - \bar{z}\right)^{-1} \xi \right\|, \quad z \notin K,$$

because $(T^* - \bar{z})^{-1}\xi = x$ is by definition the vector of minimal norm satisfying the equation $(T^* - \bar{z})x = \xi$. In addition, we know that $\|(T^* - \bar{z})^{-1}\xi\| < 1$ for $z \notin \overline{\Omega}$ and $\|(T^* - \bar{z})^{-1}\xi\| = 1$ for $z \in \Omega$. Moreover, $V(z) = (T^* - \bar{z})^{-1}\xi$, $z \notin \overline{\Omega}$, and the statement follows. □

COROLLARY 5.5. *Let $F(z, \bar{w})$ be the analytic extension of the exponential kernel $E_\Omega(z, w)$. Then the function $1 - F(z, \bar{w})$ is positive semidefinite.*

PROPOSITION 5.6.

$$\left\|\partial_z \left(T^* - \bar{z}\right)^{-1} \xi\right\| \le \frac{\sqrt{2}}{\operatorname{dist}(z, \partial\Omega)}, \quad z \in \Omega.$$

PROOF. Since

$$\bar{\partial}_z \partial_w E(z, w) = \left\langle \partial_w \left(T^* - \bar{w}\right)^{-1} \xi, \partial_z \left(T^* - \bar{z}\right)^{-1} \xi \right\rangle$$

for every pair $(z, w) \in \Omega \times \Omega$, the estimate follows from (2.9), (2.14). □

The following semidefinite scalar product on $\mathcal{D}(\mathbb{C})$ appears in a functional model which diagonalizes the operator $T^*$, see [MP]

$$\langle\langle \varphi, \psi \rangle\rangle \overset{\text{def}}{=} \frac{1}{\pi^2} \int_{\mathbb{C}\times\mathbb{C}} \left(1 - E(z, w)\right) \partial_w \varphi(w)\, \bar{\partial}_z \overline{\psi(z)}\, dA(z) dA(w).$$

A simple application of Stokes' theorem leads to the following result.

LEMMA 5.7. *Let $\Omega \subset \mathbb{C}$ be a bounded domain with piecewise smooth boundary and let $\varphi, \psi \in \mathcal{D}(\mathbb{C})$, $\chi_n \in \mathcal{D}(\Omega)$, $\chi_n \nearrow \chi_\Omega$. Then*

$$\begin{aligned} -\pi^2 \langle\langle \varphi, \psi \rangle\rangle &= \lim_{n\to\infty} \int_{\Omega\times\Omega} \bar{\partial}_z \partial_w\, E(z, w) \chi_n(z)\, \chi_n(w)\, \varphi(w) \overline{\psi(z)} dA(z)\, dA(w) \\ &= -\lim_{n\to\infty} \int_{\Omega\times\Omega} \frac{E(z, w)}{|z - w|^2}\, \chi_n(z)\, \chi_n(w)\, \varphi(w)\, \overline{\psi(z)}\, dA(z)\, dA(w). \end{aligned}$$

We omit the details. We finally remark that the operator $T$ is rationally cyclic on the completion of $\mathcal{D}(\mathbb{C})$ with respect to the above scalar product, see [P2].

## REFERENCES


[Af] L. V. AHLFORS, *Finitely generated Kleinian groups*, Amer. J. Math. **86** (1964), 413-429.

[Ah-Sh] D. AHARONOV – H. S. SHAPIRO, *Domains in which analytic functions satisfy quadrature identities*, J. Analyse Math. **30** (1976), 39-73.

[Be] L. BERS, *An approximation theorem*, J. Anal. Math. **14** (1965), 1-4.

[Br] L. BROWN, *The determinant invariant for operators with trace class self-commutator*, In: "Proc. Conference on Operator Theory", Lecture Notes in Math. Vol. 345, Springer, Berlin, 1973.

[Ca] L. CAFFARELLI, *Compactness methods in free boundary problems*, Comm. Partial Differential Equations **5** (1980), 427-448.

[Cd-Z] A. P. CALDERON – A. ZYGMUND, *Local properties of solutions of elliptic partial differential equations*, Studia Math. **20** (1961), 171-225.

[C-Pi] R. W. CAREY – J. D. PINCUS, *An exponential formula for determining functions*, Indiana Univ. Math. J. **23** (1974), 1031-1042.

[Cl1] K. CLANCEY, *A kernel for operators with one-dimensional self-commutator*, Integral Equations Operator Theory **7** (1984), 441-458.

[Cl2] K. CLANCEY, *Hilbert space operators with one-dimensional self-commutator*, J. Operator Theory **13** (1985), 265-289.

[E-G] L. C. EVANS – R. F. GARIEPY, Measure Theory and Fine Properties of Functions, CRC Press, Boca Raton, 1992.

[D] P. J. DAVIS, The Schwarz Function and its Applications, Carus Mathematical Monographs, Math. Assoc. Amer., 1974.

[Gu1] B. GUSTAFSSON, *Quadrature identities and the Schottky double*, Acta Appl. Math. **1** (1983), 209-240.

[Gu2] B. GUSTAFSSON, *Singular and special points on quadrature domains from an algebraic geometric point of view*, J. Anal. Math. **51** (1988), 91-117.

[He-Ho] J. W. HELTON – R. HOWE, *Traces of commutators of integral operators*, Acta Math. **135** (1975), 271-305.

[Hm] L. HÖRMANDER, An Introduction to Complex Analysis in Several Variables, North-Holland, Amsterdam, 1989.

[K-Mg] L. KARP – A. MARGULIS, *On the Newtonian potential theory for unbounded sources and its applications to free boundary problems*, J. Anal. Math. **70** (1996), 1-63.

[K-N] M. G. KREIN – A. A. NUDELMAN, The Markov moment problem and extremal problems, Transl. Math. Monographs, vol. 50, Amer. Math. Soc., Providence, R.I., 1977.

[MP] M. MARTIN – M. PUTINAR, Lectures on Hyponormal Operators, Birkhäuser, Basel, 1989.

[Pi] J. D. PINCUS, *Commutators and systems of singular integral equations I*, Acta Math. **121** (1968), 219-249.

[Pi-Rv] J. D. PINCUS – J. ROVNYAK, *A representation for determining functions*, Proc. Amer. Math. Soc. **22** (1969), 498-502.

[P1] M. PUTINAR, *Extremal solutions of the two-dimensional L-problem of moments*, J. Funct. Anal. **136** (1996), 331-364.

[P2] M. PUTINAR, *Linear analysis of quadrature domains*, Ark. Mat. **33** (1995), 357-376.

[P3] M. PUTINAR, *Extremal solutions of the two-dimensional L-problem of moments II*, J. Approx. Theory (to appear).

[Re] YU G. RESHETNYAK, *Generalized derivatives and differentiability almost everywhere*, Mat. Sb. **75** (1968), 323-334 (translated from Russian).

[Ro] J. F. RODRIGUES, Obstacle Problems in Mathematical Physics, North-Holland, Amsterdam, 1987.

[Sa1] M. SAKAI, Quadrature Domains, Lecture Notes in Math. Vol. no **934**, Springer-Verlag, Berlin-Heidelberg, 1982.

[Sa2] M. SAKAI, *Null quadrature domains*, J. Anal. Math. **40** (1981), 144-154.

[Sa3] M. SAKAI, *Regularity of boundary having a Schwarz function*, Acta Math. **166** (1991), 263-297.

[Sa4] M. SAKAI, *Regularity of boundaries of quadrature domains in two dimensions*, SIAM J. Math. Anal. **24** (1994), 341-364.

[Sa5] M. SAKAI, *Regularity of free boundaries in two dimensions*, Ann. Scuola Norm Sup. Pisa Cl. Sci. (4) **20** (1993), 323-339.

[Sh1] H. S. SHAPIRO, *Unbounded quadrature domains*, In: "Complex Analysis I, Proceedings, University of Maryland 1985-86", C.A. Berenstein (ed.), Lecture Notes in Mat., Vol. 1275, Springer-Verlag, 1987, 287-331.

[Sh2] H. S. SHAPIRO, The Schwarz function and Its Generalization to Higher Dimensions, Wiley, New York, University of Arkansas Lecture Notes, 1992.

[St] E. M. STEIN, Singular Integrals and Differentiability Properties of functions, Princeton University Press, Princeton, 1970.

[X] D. XIA, *On the kernels associated with a class of hyponormal operators*, Integral Equations Operator Theory **6** (1983), 444-452.

Department of Mathematics
Royal Institute of Technology
S-100 44 Stockholm, Sweden
gbjorn@math.kth.se

Department of Mathematics
University of California
Santa Barbara, CA 93106, USA
mputinar@math.ucsb.edu