AN EXPOSITION OF THE STRUCTURE OF SOLVMANIFOLDS. PART I: ALGEBRAIC THEORY[1]

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CONTENTS

PREFACE

The modern theory of solvmanifolds began in 1949 with the beautiful and fundamental paper of A. Malcev [M, A; 1]. In the past 20 years or so the role played by algebraic group theory in establishing the basic structure theorems of solvmanifolds has become more and more apparent. In this exposition we have used algebraic groups as our basic tool in presenting a unified account of solvable Lie groups and solvmanifolds. Although there are few new theorems in this paper, many of the proofs given here will appear in print for the first time.

The general reference for algebraic groups will be A. Borel [B, A] and for Lie algebras will be N. Jacobson [J, N]. In addition we will use the following general facts and notation throughout this paper:

Let G be a connected, simply connected Lie group. We will denote its Lie algebra by $L(G)$. The group of automorphisms of G and $L(G)$ will be denoted by $A(G)$ and $A(L(G))$, respectively. Then $A(G)$ and $A(L(G))$ are canonically isomorphic. Further $A(L(G)) \subset GL(L(G))$ is an algebraic group. Thus if we talk of an automorphism θ of G being semisimple or unipotent we will mean that the image of θ in $A(L(G))$ is semisimple or unipotent. Further if $\theta \in A(G)$, by the semisimple or unipotent parts of θ, we will mean the semisimple or unipotent part of θ when viewed in $A(L(G))$ and since $A(L(G))$ is algebraic we know that the semisimple and unipotent parts of θ are also automorphisms of G. If Φ is a subgroup of an algebraic group $G \subset GL(n, R)$ by the algebraic hull of Φ, denoted by $A_h(\Phi)$, we will mean the intersection of all the algebraic subgroups of $GL(n, R)$ containing Φ. Note that $A_h(\Phi) \subset G$ and is algebraic.

CHAPTER I: ABELIAN THEORY

Although the abelian theory is by now classical, let us quickly run through it as a way of setting the stage for the generalizations to come.

Let R^n be a connected, simply connected, abelian Lie group and let H be a closed subgroup of R^n. We will also view R^n as an algebraic group of the form

$$\begin{pmatrix} 1 & 0 & . & . & . & 0 & x_1 \\ & & & & & 0 & x_2 \\ & . & & & & . & . \\ & & . & & & : & : \\ & & & . & & 0 & x_{n-1} \\ & 0 & & & 1 & x_n \\ & & & & & & 1 \end{pmatrix}$$

in $GL(n + 1, R)$. Note that viewing R^n as an algebraic group in this manner

has the important consequence that the algebraic subgroups of R^n are exactly the linear subspaces of R^n. Thus if $H \subset R^n$ we may talk either of the linear span of H or the algebraic hull of H, $A_h(H)$. We will now briefly outline how we will think about the structure of R^n/H.

Global theory. Let H_0 denote the identity component of H and let $A_h(H)$ be the linear span of H or the algebraic hull of H. Then H_0 is a connected normal subgroup of $A_h(H)$ and $A_h(H)/H$ is a compact abelian group or torus whose fundamental group is H/H_0. Further since $R^n = V \oplus A_h(H)$, where V is a vector subspace $R^n/H \approx V \times A_h(H)/H \approx V \times T$, where T is the torus $A_h(H)/H$. Thus R^n/H is compact if and only if $R^n = A_h(H)$.

Rational theory. Let $e_1, \ldots, e_n$ be a basis for R^n and let $Q^n \subset R^n$ be the subgroup of all rational linear combinations of this basis. We will call Q^n a rational form of R^n. Let G be a subgroup of Q^n. It is well known that G is a discrete subgroup of R^n if and only if G is finitely generated. Now let π be a finitely generated subgroup of Q^n and let $A_h(\pi, Q)$ be the rational subspace of Q^n spanned by π, or, equivalently, all the rational points of $A_h(\pi) \subset R^n$. Further, let W be any rational subspace of Q^n. Then

$$A_h(\pi \cap W, Q) = A_h(\pi, Q) \cap W.$$

Compact theory. Let $\pi \subset R^m$ be a discrete subgroup of R^m (or, equivalently, a finitely generated subgroup of a rational form of R^m) such that $A_h(\pi) = R^m$. Let $\pi_1 \subset R^n$ also be a discrete subgroup of R^n such that $A_h(\pi_1) = R^n$. If $\psi : \pi \to \pi_1$ is an isomorphism then ψ can be extended uniquely to an isomorphism $\psi^* : R^m \to R^n$. Hence $m = n$ and ψ^* induces a diffeomorphism of R^n/π and R^m/π_1.

<div align="center">CHAPTER II: NILPOTENT LIE GROUPS AND NILMANIFOLDS</div>

1. **What is a nilpotent Lie group?** Classically, a connected, simply connected Lie group N is said to be nilpotent if the Lie algebra of N, $L(N)$, is a nilpotent Lie algebra. This is then shown to be equivalent to requiring that N is nilpotent as an abstract group. (Recall that an abstract group is said to be nilpotent if its lower central series is finite. This amounts to requiring that if G_k is defined inductively by $G_1 = G$ and $[G_{k-1}, G] = G_k$, where $[H, K]$ is the group generated by all elements of the form $khk^{-1}h^{-1}$, $k \in K, h \in H$, then $G_k = e$ for some finite k, where e is the identity element of G.)

It is a consequence of the Birkhoff imbedding theorem, $[B, G]^3$, that we may find an isomorphism of any connected, simply connected nilpotent Lie group N into the upper triangular unipotent matrix group, $U(m)$, where

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[3] See references at end of paper.

$$U(m) = \left\{ \begin{pmatrix} 1 & & \\ & \ddots & * \\ 0 & & 1 \end{pmatrix} \right\}$$

is a subgroup of $GL(m, R)$, for some m. But any connected subgroup of $U(m)$ is easily seen to be an algebraic group. What is also important is that the algebraic subgroups of N for any imbedding in $U(m)$, any m, consist exactly of the connected subgroups of N. (Notice the similarity with R^n from Chapter I.) Thus if H is any subgroup of N we may form the algebraic hull of H, $A_h(H)$, in N, which is defined as the intersection of all the algebraic subgroups of N that contain N or equivalently the smallest connected subgroup of N containing H.

Our ability to represent N faithfully in $U(m)$ also enables us to do the following: We can find a coordinate system $X = (x_1, \ldots, x_n)$ in all of N such that multiplication in N is given by

(1) $X \cdot Y = (x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = (p_1(X, Y), \ldots, p_n(X, Y))$

where the p_i, $i = 1, \ldots, n$, are polynomials in $2n$ variables. The converse is also true and a proof can be found in [T, R ; 5]. The converse may be formally stated as follows: *Let N be a group with coordinates $X = (x_1, \ldots, x_n)$ whose law of composition satisfies equation (1), then N is a nilpotent Lie group.*

Henceforth, we will adopt either the point of view that N is an algebraic group or that the group multiplication is polynomial in talking about or working with nilpotent Lie groups. Hence, unless explicitly stated to the contrary nonabelian nilpotent Lie groups will always be connected and simply connected. We will say that a field $\mathscr{F}$ is a *field of definition* for the nilpotent Lie group N if there exists a polynomial law of composition in N all of whose coefficients are in $\mathscr{F}$ or if N can be represented as an algebraic group with $\mathscr{F}$ as field of definition. We may now talk of the points of N in any field $\mathscr{K}$ containing $\mathscr{F}$; i.e., a point of N in $\mathscr{K}$ is any n-tuple $(k_1, \ldots, k_n)$, $k_i \in \mathscr{K}$. For the rest of this paper we will call a nilpotent Lie group defined over the rational field Q a rational nilpotent Lie group. These will play a particularly important role in our theory.

2. **Global nilpotent theory.** Let H be a closed subgroup of the nilpotent Lie group N and let H_0 denote the identity component of H. If $A_h(H)$ again denotes the algebraic hull of H in N and if $A_h(H) \neq N$, it is easily seen that $N/A_h(H)$ is topologically a vector space V and that

$$N/H \approx V \times A_h(H)/H$$

where $\approx$ denotes homeomorphism.

Let us now examine the case where $H \subset N$ and $A_h(H) = N$. We begin by observing that, in this case, H_0 is a normal subgroup of N, because H_0 is normal in H and N is the algebraic hull of H. Thus we may form the nilpotent Lie group $M = N/H_0$ and consider H/H_0 as a discrete subgroup π of M. It is easily seen that $A_h(\pi) = M$. Now, by standard results in the theory of algebraic groups,

$$A_h([\pi, \pi]) = [A_h(\pi), A_h(\pi)] = [M, M].$$

A simple inductive argument now permits us to conclude that $M/\pi \approx N/H$ is compact or we have that *if H is a closed subgroup of the nilpotent Lie group N then N/H is compact if and only if $A_h(H) = N$.* Let us explicitly see that if $A_h(H) = N$, then N/H is compact. This amounts to showing that M/π is compact and by the above relation and induction we may assume that $[M, M]/[\pi, \pi]$ is compact. From this it follows that the image of π in the abelian group $M/[M, M]$ is discrete and, by an easy argument on algebraic or linear hulls, it follows that the image of π in $M/[M, M]$ is cocompact or spans the vector space. Hence M/π is a fiber bundle with compact fiber $[M, M]/[M, M] \cap \pi$ and compact base $(M/\pi[M, M])$ and so is compact.

Let us now adopt the language of calling the homogeneous space of a nilpotent Lie group a *nilmanifold*. Then we may summarize our results of this section by the statement that *a nilmanifold is the Cartesian product of a vector space and a compact nilmanifold.*

3. **Rational nilpotent Lie groups.** Let N be a nilpotent Lie group and let $(x_1, \ldots, x_n) = X$ be a coordinate system in N such that multiplication is given by polynomials defined over the rational field Q. Let $N_Q(X)$ denote the rational points of N relative to coordinate system. Now let $(y_1, \ldots, y_n) = Y$ be another coordinate system in N such that multiplication in N is given by polynomials defined over Q. Let $N_Q(Y)$ denote the rational points in N relative to this polynomial rule of multiplication. Then it may happen that $N_Q(X)$ and $N_Q(Y)$ are *not* isomorphic. We will call $N_Q(X)$ a rational form of N. Thus a nilpotent Lie group may have nonisomorphic rational forms (see [S, J] for examples). However in our discussion we will always be dealing with a fixed rational form and so we shall write N_Q rather than $N_Q(X)$. It should also be remarked that not every nilpotent Lie group has a rational form (see [M, A; 1] for example).

Let N *be a nilpotent Lie group and let N_Q be a rational form of N. Let π be a subgroup of N_Q. Then π is discrete if and only if π is finitely generated.* It is easy to see that, if π is discrete, it is finitely generated as this follows immediately from the fact that π is the fundamental group of the compact manifold $A_h(\pi)/\pi$. Hence it is the converse that needs our attention.

By standard algebraic group arguments it is easy to see that it is no

restriction to assume that $A_h(\pi) = N$. Let us now consider $\pi/[N, N] \cap \pi \subset N_Q/[N_Q, N_Q]$. It is finitely generated in the rational vector space $N_Q/[N_Q, N_Q]$ and so $\pi/[N, N] \cap \pi$ is discrete in $N/[N, N]$ and, since $A_h(\pi) = N$, $(N/[N, N])/(\pi/[N, N] \cap \pi)$ is compact. But, unlike general groups, subgroups of finitely generated nilpotent groups are finitely generated. Hence $\pi \cap [N_Q, N_Q]$ is finitely generated and so by induction $\pi \cap [N_Q, N_Q]$ is discrete in $[N, N]$. These facts combine to prove that π is discrete in N.

By an argument similar to the one above, it can be shown that if π_1 and π_2 are finitely generated subgroups of N_Q such that $A_h(\pi_1) = A_h(\pi_2)$ then π_1 and π_2 are commensurable. (Recall that two groups G_1 and G_2 are called commensurable if $G_1 \cap G_2$ is of finite index in each.)

Let us now prove one of the fundamental existence theorems in the theory of nilmanifolds. *Let N be a nilpotent Lie group with rational form N_Q. Then there exists $\pi \subset N_Q$ such that π is a discrete subgroup of N and N/π is compact.*

Let $M_Q \subset N_Q$ be of codimension 1 or such that N_Q/M_Q is isomorphic to Q and let the real points of M_Q be M. By induction on the dimension of N there exists $\Delta \subset M_Q$ which satisfies the conclusion of our theorem relative to M. Now it is easily seen that $N_Q = Q \ltimes M_Q$ where we will use the notation $A \ltimes B$ to denote the semidirect product of A with the normal subgroup B. Now let $q \neq 0, q \in Q$, and in N_Q consider the subgroup π generated by q and Δ. Since Δ is finitely generated, so is π, and, since $A_h(\Delta) = M$, it is easily seen that $A_h(\pi) = N$. This proves our assertion.

Now let $L(N)$ denote as usual the Lie algebra to the nilpotent Lie group N and let $\exp : L(N) \to N$ denote the exponential mapping. Then by the Hausdorff-Campbell formula it follows that the exponential mapping is a homeomorphism. We will denote the inverse mapping to exp by log. A discrete subgroup $\pi \subset N$ will be called a *lattice nilpotent group* if $\log \pi$ is a subgroup of the vector space $L(N)$. Not every discrete subgroup π of N such that $A_h(\pi) = N$ is a lattice subgroup of N. For instance, if N_3 is given by

$$N_3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right\}, \qquad x, y, z \in R,$$

then

$$\pi = \left\{ \begin{pmatrix} 1 & n_1 & n_3 \\ 0 & 1 & n_2 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \qquad n_i \in Z, i = 1, 2, 3,$$

is not a lattice nilpotent group, but the group π_1 generated by π and

$$\begin{pmatrix} 1 & 0 & \frac{1}{2} \\ & 1 & 0 \\ & & 1 \end{pmatrix}$$

is a lattice nilpotent group.

The definition and the basic facts about the existence of lattice nilpotent groups are due to C. C. Moore [M, C]. Moore showed the following that will be of great importance to us in our later discussion. Let $\pi \subset N_{\mathbf{Q}} \subset N$ be discrete. Then there exists a lattice nilpotent subgroup $\Delta \supset \pi$ such that Δ/π is finite. We will not prove this. However the proof of the assertion below can be easily modified to prove this fact.

Let $(X_1, \ldots, X_n)$ be a basis of $L(N)$ such that $N_{\mathbf{Q}} = \exp(\sum q_i X_i)$, $q_i \in \mathbf{Q}$, $i = 1, \ldots, n$. Let G be a completely reducible group of automorphisms of $L(N)$ contained in $GL(n, \mathbf{Z})$ relative to the basis $(X_1, \ldots, X_n)$. Then $N_{\mathbf{Q}}$ contains a lattice nilpotent subgroup Δ such that $A_h(\Delta) = N$ and G maps Δ onto itself.

In $L(N)$ let $\mathscr{L}$ be the lattice generated by $X_1, \ldots, X_n$, i.e., $\mathscr{L} = \{\sum m_i X_i | m_i \in \mathbf{Z}, i = 1, \ldots, n\}$. Let A be the last nontrivial term of the lower central series of N and let $L(A) \cap \mathscr{L} = A_X$. Since A is a rational subgroup of $N_{\mathbf{Q}}$, A_X is a lattice subgroup of A which we will identify with $L(A)$. Now consider $M_{\mathbf{Q}} = N_{\mathbf{Q}}/A_{\mathbf{Q}}$. Since G acts completely reducible, there is a rational vector subspace W of $L(N)$ relative to $X_1, \ldots, X_n$, such that $L(N) = L(A) \oplus W$, and W is G-invariant. Clearly G maps $W \cap \mathscr{L}$ onto itself. If we identify W and $L(M)$ under the natural 1-1 mappings we have that $\mathscr{L}^* = W \cap \mathscr{L}$ is a lattice invariant under G acting as automorphisms of $M_{\mathbf{Q}}$. Hence by induction there is a lattice nilpotent subgroup Δ^* in $L(M)$ or W which is invariant under G. Now let Δ be the subgroup of N generated by $\exp(\delta)$, $\delta \in \Delta^*$. It is easy to see that δ is a finitely generated subgroup of N and for any element $\delta \in \Delta$, $\log \delta$ may be written uniquely as $a + \delta^*$, $a \in L(A)$ and $\delta^* \in \Delta^*$. Since A is central it follows that the set of all points of a that occur as the A coordinate of $\log \delta$, $\delta \in \Delta$, is a subgroup B of $A_{\mathbf{Q}}$ and is finitely generated. Hence, there exists $k \in \mathbf{Z}^+$ such that A_X/k contains B. One now verifies that the group $B\Delta$ is a lattice nilpotent subgroup of $N_{\mathbf{Q}}$ invariant under G.

There is one further concept that is important about lattice nilpotent groups that we should mention before leaving this subject. This is the concept of a *lattice hull* and is defined as follows. Let $\pi \subset N_{\mathbf{Q}} \subset N$ be discrete. Then there exists a lattice nilpotent subgroup $\Delta \supset \pi$ such that $\Delta \subset N_{\mathbf{Q}}$ and Δ/π is finite. If we let π_L denote the intersection of all the lattice nilpotent subgroups that contain π it is easy to see that π_L is the

smallest lattice nilpotent subgroup of N_Q containing π. We call π_L the *lattice hull of* π.

4. **Compact nilmanifolds.** Let π be a discrete subgroup of a nilpotent Lie group N such that π is cocompact in N or equivalently N/π is compact. Then π uniquely determines N. This statement may be made precise as follows:

Let $\pi_i \subset N_i, i = 1, 2$, *be discrete, cocompact subgroups of nilpotent Lie groups* $N_i, i = 1, 2$. *Further, let*

$$\alpha : \pi_1 \to \pi_2$$

be an isomorphism. Then α *extends uniquely to an isomorphism*

$$\alpha^* : N_1 \to N_2.$$

We will prove this theorem by induction on the dimension of N_1. Let M_1 be a connected subgroup of N_1 of codimension 1 and such that $M_1/\pi_1 \cap M_1$ is compact. (The existence of M_1 follows from the fact that the image of π_1 in $N_1/[N_1, N_1]$ is discrete and cocompact.) Let $\pi_1^* = \pi_1 \cap M_1$ and note that $\pi_1 = Z \ltimes \pi_1^*$ and $N_1 = R_1 \ltimes M_1$ with $Z_1 \subset R_1$. Now let $\pi_2 = \alpha(Z_1) \ltimes \alpha(\pi_1^*)$ and let $M_2 = A_h(\alpha(\pi_1^*)) \subset N_2$. Then, by induction, α extends uniquely to an isomorphism $\alpha_1^* : M_1 \to M_2$ since $\alpha(\pi_1^*)$ is discrete and cocompact in M_2 and dim $M_1 = $ dim $N_1 - 1$. Now it follows by a dimension argument that $N_2 = R_2 \ltimes M_2$ with $\alpha(Z_1) = Z_2 \subset R_2$. We next note that Z_1 acts by inner automorphisms of π_1 as a group of automorphisms of π_1^* and hence Z_1 may be considered to act as a group of automorphisms of M_1. Further Z_1 acts unipotently on M_1 and so the algebraic hull of Z_1 acting on M_1 is exactly the same as the action of R_1 on M_1. A similar statement holds for $\alpha(Z_1) = Z_2$ acting on M_2. Now $\alpha : Z_1 \to Z_2$ extends uniquely to $\alpha_1^* : R_1 \to R_2$. We now define $\alpha^* : R_1 \ltimes M_1 \to R_2 \ltimes M_2$ by $\alpha^*(r_1 m_1) = \alpha_1^*(r_1)\alpha_2^*(m_2)$. It is now a straightforward verification that α^* is the desired unique isomorphism extending α.

The type of argument we have just used can also be used to prove the following important fact. *Let* $\pi \subset N, N$ *a nilpotent Lie group, be discrete and cocompact. Then there exists a rational form* N_Q *such that* $\pi \subset N_Q$.

Again the proof is by induction on the dimension of N and we begin by choosing $M \subset N$ such that $M/\pi \cap M$ is compact, $N = R \ltimes M, \pi \cap R = Z$ and $\pi = Z \ltimes \pi_L^*$ where $\pi^* = \pi \cap M$. By induction there exists a rational form M_Q of M such that $M_Q \supset \pi^*$. Let π_L^* denote the lattice hull of π^* and recall that Z acts as a group of automorphisms of π_L^* and so we may view Z as a subgroup A of $GL(n, Z)$, where $n = $ dim M. Hence $A_h(A)_Q$ is a subgroup of $GL(n, Q)$ of dimension 1 and contained in $A(M)$. It is now easily seen that $N_Q = A_h(A)_Q \ltimes M_Q$ is a rational form of N.

We can repeat this type of argument once more to show that *if* π *is any*

finitely generated torsion free nilpotent group then there exists a nilpotent Lie group N such that $\pi \subset N$ and N/π is compact. The argument rests on the group theoretic fact that such groups π have a semidirect product representation of the form $\pi = \mathbf{Z} \ltimes \pi^*$, where π^* is again torsion free, finitely generated and nilpotent. Thus there exists M such that M is a nilpotent Lie group and $\pi^* \subset M$ with M/π^* compact. But then $\mathbf{Z}$ acts as a group of unipotent automorphisms of M and so we may form $N = A_h(\mathbf{Z}) \ltimes M$ and this group is easily seen to satisfy the conclusions of our theorem.

Since N and N_Q are uniquely attached to π and if π is faithfully imbedded in $U(m)$ for some m, N will be isomorphic to the algebraic hull of π in $U(m)$. We will often call N the *algebraic hull of* π. We will often denote N and N_Q by π_R and π_Q, respectively.

We come now to our final result about compact nilmanifolds. *Compact nilmanifolds with isomorphic fundamental groups are diffeomorphic.* To see this, note that if X_1 and X_2 are compact nilmanifolds with fundamental groups π_1 and π_2, respectively, then there exist nilpotent Lie groups N_1 and N_2 such that π_i is a discrete cocompact subgroup of N_i, $i = 1, 2$, and $X_1 \approx N_i/\pi_i$, $i = 1, 2$. Now let $\alpha : \pi_1 \to \pi_2$ be an isomorphism. Then α extends uniquely to an isomorphism $\alpha^* : N_1 \to N_2$ and thus induces a diffeomorphism $\alpha^{**} : N_1/\pi_1 \to N_2/\pi_2$.

CHAPTER III: SOLVABLE LIE GROUPS AND SOLVMANIFOLDS

1. **Examples.** Let N be a nilpotent Lie group and let $\mathbf{R}^s$ be the s-dimensional vector group. A (connected and simply connected) *solvable Lie group* is any Lie group R satisfying the exact sequence

$$1 \to N \to R \to \mathbf{R}^s \to 1.$$

Now let H be a closed subgroup of the solvable Lie group R. In this Chapter we will study the homogeneous spaces R/H which are called *solvmanifolds*.

Let us begin our discussion by some examples that show that many of the assertions proven for nilpotent groups and nilmanifolds are false for solvmanifolds.

EXAMPLE 1. Let C denote the complex numbers and let $R = \mathbf{R} \ltimes C^2$, where $\mathbf{R}$ acts on C^2 by

$$\text{ad}(t)(z_1, z_2) = (e^{2\pi i t} z_1, e^{2\pi i \lambda t} z_2), \qquad \lambda \text{ irrational},$$

$t \in \mathbf{R}$ and $(z_1, z_2) \in C^2$. The solvable Lie group R is *not* an algebraic group in any matrix representation because the image of R under the adjoint representation in $GL(5, \mathbf{R})$ is not a closed subset of $GL(5, \mathbf{R})$. If R were an algebraic group in any representation then the image of R under the adjoint representation would have to be an algebraic group and hence closed.

EXAMPLE 2. The group R_2 of affine transformations of the line

$$R_2 = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \right\}, \qquad a, b \in \boldsymbol{R},$$

has no discrete cocompact subgroup although in the above matrix representation it is a rational algebraic group.

EXAMPLE 3. The Klein bottle is a compact solvmanifold. Once we have shown this, we will have proven that not every compact solvmanifold can be written in the form R/Γ, where Γ is a discrete subgroup of the solvable Lie group R. To see that the Klein bottle can be presented as a solvmanifold, let $R = \boldsymbol{R} \ltimes C$, where $\mathrm{ad}(t)(z) = e^{2\pi i t}z, t \in \boldsymbol{R}, z \in C$. If the elements of R are given by $(t, z), t \in \boldsymbol{R}$ and $z \in C$, let H be the closed subgroup consisting of all elements of the form $(m/2, n + iy), m, n \in \boldsymbol{Z}, y \in \boldsymbol{R}$. Then it is easily seen that R/H is the Klein bottle.

EXAMPLE 4. Let R be the universal covering group of the group of rigid motions of the plane. We may write this group as $R = \boldsymbol{R} \ltimes C$ where $\mathrm{ad}(t)z = e^{2\pi i t}z, t \in \boldsymbol{R}, z \in C$; i.e., it is the same group as in Example 3. Let Γ be the subgroup of R consisting of $(m, n_1 + in_2), m, n_1, n_2 \in \boldsymbol{Z}$. Then it is easily seen that R/Γ is topologically the three-dimensional torus. Since R is nonabelian, this shows that, for solvable Lie groups, isomorphisms of discrete cocompact subgroups may not extend to the connected groups containing them.

These examples show that the attack used in the study of nilmanifolds—a direct approach using algebraic groups—will not work for solvmanifolds. Although much work has been done in the study of solvmanifolds since the fundamental paper of Mostow in 1954 [M, G; 1], the global theory of solvmanifolds and the role played by algebraic groups in the study of solvmanifolds first became available in joint work of the author and R. Tolimieri in 1970 [A, L; 8, 13]. It was in [A, L; 8] that we first established the fundamental global result—every solvmanifold is a vector bundle over a compact solvmanifold—a result first conjectured by Mostow in his 1954 paper. Before we begin the study of solvmanifolds let us set the stage by establishing how we will think about solvable Lie groups. The basic idea we will discuss can be traced back through the works of H. C. Wang [W, H] and A. I. Malcev [M, A; 2] and much further back into the history of group theory. It is interesting to note that the construction we are going to discuss and the wreath-product of groups have the same genesis.

2. The structure of solvable Lie groups—splitting theorems. There exist two fairly recent and complete accounts of the ideas in this section in [A, L; 5] and [T, R; 1]. However, we will present an outline of this material that should help to serve as a guide through the above mentioned papers.

Let R be a solvable Lie group. Then R has a unique maximal normal

nilpotent connected subgroup which is called *the nil-radical* of R and which we will denote usually by N. Then R satisfies the exact sequence

$$1 \to N \to R \to R^s \to 1$$

and so $N \supset [R, R]$. Thus the description of R involves the "action" of R^s on N and a group extension with nonabelian kernel. It is well known that it is very difficult to work with group extensions that have nonabelian kernels. It is also clear from Chapter II that we have great control over nilmanifolds because of the algebraic group structure of nilpotent Lie groups. Hence we want to describe solvable Lie groups in terms of groups that look as much like algebraic groups as possible. In order to do this, we will replace presenting R as an exact sequence by presenting R as a subgroup of a special sort of group.

Now if $R \subset GL(n, R)$ is an algebraic solvable Lie group, then $R = T \ltimes U$, where U is a connected subgroup of $U(n)$ and T is an abelian group of semisimple matrices. (Although U is nilpotent, it is important to note that U need not be the nil-radical of R.)

At this point two options are open to us. We can use a nonintrinsic approach that makes use of the Ado imbedding theorem or an intrinsic approach that is a modification of ideas first introduced by A. I. Malcev in [M, A ; **2**]. We have chosen the second approach.

Let $L(R)$ denote as usual the Lie algebra of R and consider the adjoint representation $\text{ad}(R)$ on $L(R)$. If $A(L(R))$ denotes the algebraic group of automorphisms of $L(R)$, then $\text{ad}(R) \subset A(L(R))$ and so $A_h(\text{ad}(R))$, the algebraic hull of $\text{ad}(R)$, is contained in $A(L(R))$. Since $\text{ad}(R)$ is solvable, $A_h(\text{ad}(R))$ is a solvable algebraic group and so

$$A_h(\text{ad}(R)) = T \ltimes U$$

where T is semisimple abelian and U is unipotent. In T consider the image of $\text{ad}(R)$ and denote this by T_R. We may view T_R as a group of automorphisms of R and so we may form

$$R_s = T_R \ltimes R.$$

If R satisfies an exact sequence

$$1 \to N \to R \to R^s \to 1$$

where N is the nil-radical of R, it is easy to see that $\dim T_R = s$ and if $\sim$ denotes universal covering group then

$$\tilde{R_s} = \tilde{T_R} \ltimes R = R^s \ltimes R.$$

We have called R_s *the semisimple splitting of* R, although the name *Malcev splitting* might be also appropriate. Let us now list the basic properties of R_s.

1. T_R acts as an abelian semisimple group of automorphisms of R.
2. $R_s = M_R \rtimes T_R = R \rtimes T_R$, where M_R is the nil-radical of R_s.
3. R_s is uniquely determined by properties 1 and 2.

We will often call M_R the *nil-shadow* of R.

We may also relate $R_s^\sim$ to some exact sequences. To be precise $R_s^\sim$ satisfies the commutative diagram below where all horizontal sequences are exact and all vertical arrows are monomorphisms with the right hand arrow being an isomorphism.

$$1 \to N \to R \to R^s \to 1$$
$$\downarrow \qquad \downarrow \qquad \downarrow$$
$$1 \to M_R \to R_s^\sim \to R^s \to 1$$

The crucial property is that the bottom exact sequence splits. Thus we see the reason for the name splitting in semisimple splitting.

One of the facts that makes R_s so useful is that every automorphism θ of R gives rise to an automorphism (not unique) of R_s. The basic tool for understanding this result is the following fundamental general theorem of Mostow [M, G; 2] which we will call the "conjugacy theorem".

Let G be a Lie group with a finite number of components and Lie algebra $L(G)$. Further, let R be the radical of G and N the radical of $[G, G]$ (note, N is nil-radical of R). Let H_1 and H_2 be two closed subgroups of G each with the property that $\text{ad}(H_i)$ is a maximal subgroup of $\text{ad}(G)$ with respect to the property of semisimplicity; i.e., any invariant subspace of $L(G)$ under $\text{ad}(H_i)$ has a complementary invariant subspace under $\text{ad}(H_i), i = 1, 2$. Then there exists $n \in N$ such that $nH_1n^{-1} = H_2$.

The Mostow-conjugacy theorem enables us to prove (see [A, L; 5]) the following: *Let R be a solvable Lie group with semisimple splitting R_s and let $A(R)$ denote the group of automorphisms of R and $A(R_s, R)$ the group of automorphisms of R_s that map R onto itself. Then the restriction mapping*

$$r : A(R_s, R) \to A(R)$$

is a surjection and there is a monomorphism $c : A(R) \to A(R_s, R)$ such that $r \circ c$ is the identity on $A(R)$. Hence every group of automorphisms of R lifts to R_s, but not uniquely.

3. **A splitting theorem and a global theorem about solvmanifolds.** Now just as Malcev proved that solvable Lie groups have semisimple splittings, H. C. Wang proved that certain disconnected solvable groups have splittings. For later purposes we will state Wang's results in a slightly more general form than in his paper [W, H] but in slightly less generality than the results established by Toliemeri in [T, R; 1].

By a CN group we will mean any closed subgroup of a nilpotent Lie group. Let N be a nilpotent Lie group and let H be a closed subgroup of

N with algebraic hull $A_h(H)$. Since $A_h(H)$ is the same no matter what nil-potent Lie group we take to contain H as closed subgroup, we will adapt the notation $H_R = A_h(H)$. If H_0 is the identity component of H then H_0 is normal in H_R and H_R/H_0 contains a unique rational form, say Q, such that $Q \supset H/H_0$. Let H_Q be the pre-image of Q in H_R.

We will call a group a CS group if it is a closed subgroup of a solvable Lie group. One can demonstrate (see [A, L; **14**, Corollary 2]) that a CS group D satisfies an exact sequence

$$1 \to H \to D \to Z^s \oplus R^t \to 1$$

where H is a CN group. By the unique extension of automorphisms for CN groups, we have easily the following commutative diagram with horizontal exact sequences and vertical monomorphisms:

$$
\begin{array}{ccccccccc}
1 & \to & H & \to & D & \to & D/H & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & H_Q & \to & D_Q & \to & D/H & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & H_R & \to & D_R & \to & D/H & \to & 1
\end{array}
$$

All the statements we are about to make regarding D_R also hold for D_Q.

We will now show how to construct a semisimple splitting or simply a splitting of D_R. The method we will follow here is analogous to using the Ado theorem to prove the Malcev or semisimple splitting for solvable Lie groups. (The reader should note that in general D_R has an infinite number of components.) What H. C. Wang actually proved we will discuss in the next section and call it the abstract or the Wang splitting.

Let us begin by noting that if $D \subset R$ then $D_R \subset R$. Now *let R_s be the semisimple splitting at R and let M_R be the nil-shadow of R. Further let $R_s/M_R = T_R$ and let the image of D_R in T_R be T^*. Our main assertion is that there exists a monomorphism $\alpha: T_R \to A(R)$ such that $R_s = M_R \rtimes \alpha(T_R)$ and $\alpha(T^*)$ maps D_R onto itself.*

Since we wish to be able to use algebraic group techniques to prove our assertion, we will begin by taking a particular faithful matrix representation of R_s. To be more precise, we will use the *Birkhoff Imbedding Theorem* which states that if N is a nilpotent Lie group with automorphism group $A(N)$ then there is a faithful imbedding $B: N \to U(n)$ and $B: A(N) \to GL(n, R)$ such that $B(\theta(n)) = B(\theta)B(n)B(\theta)^{-1}$ where $\theta \in A(N)$ and $n \in N$. This representation of $N \rtimes A(N)$ has the property that if $R_s = T_R \ltimes M_R$ then not only is $B(M_R)$ unipotent, but $B(T_R)$ is semisimple.

We may now proceed to prove our assertion. Let $A = A_h(B(D_R))$. Then $[A, A]$ is the algebraic hull of $[B(D_R), B(D_R)]$. But $[B(D_R), B(D_R)] \subset B(H_R)$, which, being a connected nilpotent group, is algebraic. Hence

$[A, A] \subset B(H_R)$. Now let $A = M_A \rtimes T_A$, where M_A is unipotent and T_A is semisimple. Now let T_D^* be the image of $B(D_R)$ in T_A. Then, for $c \in B(D_R)$ and $t \in T_A$, we have

$$[t, c] = tct^{-1}c^{-1} \in B(H_R) \subset B(D_R).$$

Hence $tct^{-1} \in B(D_R)$. Now let $B(M_R) \rtimes B(T_R)$ be a semidirect product presentation of $B(R_s)$ such that

$$B(T_R) \supset T_A \cap B(T_D^*).$$

Since B is an isomorphism, this proves our assertion.

If we let T_D denote the restriction of T_D^* to D_R, we may form $(D_R)_s = D_R \rtimes T_D$. We will call $(D_R)_s$ a semisimple or Wang splitting of D_R.

We may now argue as in [A, L; 5] and show that the following four conditions are satisfied.

1. $(D_R)_s$ has a maximal nilpotent normal subgroup, which we will call the *nil-shadow* of D_R and denote by H_D. H_D is a CN group. Let the algebraic hull of H_D be denoted by M_D and be called the *real nil-shadow* of D. The group $(H_D)_Q$ will be called the *rational nil-shadow* of D.

2. $(D_R)_s = H_D \rtimes T_D$.

3. The set-theoretic projection $p : H_D \rtimes T_D \to H_D$ maps D_R homeomorphically onto H_D.

4. D_R and H_D generate $(D_R)_s$.

We will now use the above properties of splittings of CS groups to show that every solvmanifold is a vector bundle over a compact manifold. We will, in the next sections, introduce the abstract splitting theorem and indicate how these can be used to prove that every solvmanifold is a vector bundle over a compact solvmanifold and that compact solvmanifolds with isomorphic fundamental groups are homeomorphic.

Let R be a solvable Lie group and let D be a closed subgroup of R. Further, let $R_s = M_R \rtimes T_R$ be the semisimple splitting of R. We define the *affine action* of R on M_R as follows: Let $r \in R$ and let $r = (m, t)$, where $m \in M_R$ and $t \in T_R$. Let $m_0 \in M_R$. We define $a(g)m_0 = mt(m_0)$, where $t(m_0)$ is the image of m_0 under the automorphism determined by t. This defines R as a group of transformations acting on M_R. Note that if we denote the left multiplication by r in R by l_r and projection of R onto M_R by p, then

$$pl_r = a(r)p.$$

As an immediate corollary of this observation, we have that the solvmanifold R/D is homeomorphic to $M_R/a(D)$, where $M_R/a(D)$ denotes the quotient space of M_R under the affine action of D. This observation has the important consequence that the topological structure of R/D depends on M_R and T_D^*, but not on T_R. (This explains why nonuniqueness in the

presentation of solvmanifolds is possible as exhibited in Example 4 of §1 of this chapter.)

Now let D be a CS group and let

$$(D_R)_s = H_D \rtimes T_D$$

be the semisimple splitting of D_R. Then if D_0 denotes the identity component of D, it can be verified that the nil-shadow of D_0 in H_D, which we will denote by H_0, is normal in $(D_R)_s$. Let T_D^1 denote the image of T_D on H_D/H_0. Then we easily have that

$$((D/D_0)_R)_s = (H_D/H_0) \rtimes T_D^1 ;$$

thus we have established that if D is a CS group then the orbit of D acting affinely on its real nil-shadow M_D is compact and has fundamental group $\Gamma = D/D_0$, where Γ satisfies the exact sequence

$$1 \to \pi \to \Gamma \to Z^s \to 1$$

where π is the fundamental group of a compact nilmanifold or, equivalently, a finitely generated torsion free nilpotent group. A group Γ satisfying the above conditions will be called a FS group.

Now let D be a closed subgroup of R and let the semisimple splitting of R, $R_s = M_R \rtimes T_R$ be chosen so that if $(D_R)_s = H_D \rtimes T_D^*$ then $H_D \subset M_R$ and $T_D^* \subset T_R$. Then M_D the real nil-shadow of D_R is a connected subgroup of M_R which is invariant under $(D_R)_s$ and by the above discussion $M_D/a(D)$ is a compact manifold. Since M_R is nilpotent, we will, for the rest of this discussion, view M_R and $L(M_R)$ as identified by means of the exponential mapping. Let V be a vector subspace of M_R such that $T_D^*(V) = V$ and $M_R = M_D \oplus V$. Then one can verify that every element of M_R can be written uniquely as δv, $\delta \in M_D$ and $v \in V$. Hence if $d = (m, t) \in D$, we have

$$\mathrm{ad}(d)(\delta v) = mt(\delta)t(v) = m^1 v^1$$

where $m^1 \in M_D$ and $v^1 \in V$. Thus we see that the images of the sets δV, $\delta \in M_D$, gives a fiber bundle structure to $R/C \approx M_R/a(D)$ with $M_D/a(D)$ as base space and V as fiber with the linear action of the abelian structure group T_D^*. This proves our first structure theorem. *Every solvmanifold is a vector bundle over a compact manifold.*

4. **Abstract splitting theorems and uniqueness of compact solvmanifolds.** In §3 we established our first structure theorem for solvmanifolds. In the next two sections, we will establish two further properties: First, compact solvmanifolds are homeomorphic if and only if their fundamental groups are isomorphic; second, given any FS group it is the fundamental group of a compact solvmanifold.

Both of these results start from the Wang splitting theorem. Indeed,

the first result is an almost immediate corollary of the Wang result while the second uses the Wang splitting as a jumping off point. We will therefore begin this section with a discussion of the Wang splitting.

In §3 we saw that if D is a closed subgroup of a solvable Lie group R, or what we have been calling a CS group, then D satisfies an exact sequence

$$(1) \qquad\qquad 1 \to H \to D \to Z^s \oplus R^t \to 1$$

where H is a CN group. The converse is false. In other words there exist groups D satisfying (1) which cannot be imbedded as closed subgroups of connected, simply connected solvable Lie groups. However, H. C. Wang proved in [W, H] that if D satisfies (1) then D_R and D_Q have semi-simple splittings $(D_R)_s$ and $(D_Q)_s$ as defined for CS groups. One may then use the Birkhoff imbedding theorem to show that groups satisfying (1) have faithful matrix representations and can be imbedded in simply connected solvable Lie groups with a finite number of components. (Note that if we knew for some reason in advance that a group satisfying (1) had a faithful matrix representation, the proof of its having a Wang splitting would be a simple matter.)

Let D be a group satisfying the exact sequence (1). *Again we will state all our results for the group D_R with the understanding that exactly the same results hold for D_Q.* Let $A(D_R, H_R)$ denote all continuous automorphisms of D_R that map H_R onto itself and induce the trivial automorphism of D_R/H_R. We will say that $\theta \in A(D_R, H_R)$ is *unipotent* if θ induces a unipotent automorphism of H_R. We will call $\theta \in A(D_R, H_R)$ semisimple if

(a) there exists a cross section $f : Z^s \oplus R^t \to D_R$ such that $\theta f(z) = f(z)$, $z \in Z^s \oplus R^t$.

(b) $\theta|H_R$ is semisimple.

Wang has shown that, just as for linear transformations, every $\theta \in A(D_R, H_R)$ can be written uniquely as a product of two commuting automorphisms $s(\theta)$ and $n(\theta)$ both in $A(D_R, H_R)$ such that $s(\theta)$ is semisimple and $n(\theta)$ is unipotent.

Now let $\mathrm{ad}(D_R)$ denote the subgroup of $A(D_R, H_R)$ consisting of inner automorphisms. Let $(D_R)_w$ be the subgroup of $A(D_R, H_R)$ generated by all the semisimple and unipotent parts of $\theta \in \mathrm{ad}(D_R)$. Let B_D denote a maximal abelian subgroup of $(D_R)_w$ consisting of semisimple matrices. Then H. C. Wang has shown the following:

1. There exists a cross section $f : Z^s \oplus R^t \to D_R$ such that for each $z \in Z^s \oplus R^t$ we can find $b \in B_D$ such that $\mathrm{ad}(f(z))b^{-1}$ is unipotent.

2. The group U_D generated by the unipotent elements $\mathrm{ad}(f(z))b^{-1}$ and H_R is nilpotent and a CN group which is such that

$$1 \to H_R \to (U_D)_R \to Z^s \otimes R \oplus R^t \to 1.$$

3. B_D acts as an abelian group of semisimple automorphisms on $(U_D)_R$ which maps H_R onto itself and is trivial on $(U_D)_R/H_R$.

4. $Z^s \otimes R \oplus R^t$ contains a lattice Y such that if f is a cross section of $Z^s \oplus R^t \to D_R$ satisfying 1 then there exists a cross section $g: Y \to (U_D)_R$ such that the unipotent parts of $ad(f(y))$ equals $ad(g(y))$ restricted to H_R.

5. The group $(D_R)_w = U_D \rtimes B_D$.

Now let D be a CS group. Apparently the definition of semisimple splitting of D_R given in §3 depended on the choice of enveloping group R. We will now see that this is not the case. Let $(D_R)_s = M_D \rtimes T_D$ be a semi-simple splitting of D and let $(D_R)_w = U_D \rtimes B_D$. Then we may choose a semidirect product representation of $(D_R)_w$ such that $T_D = B_D$ as a group of automorphisms of D_R and if β assigns to each $m \in M_D$ the automorphism of D_R obtained by $mD_R m^{-1}$ then

$$\beta : M_D \to U_D$$

is an epimorphism.

Now let ξ be an automorphism of D_R. Then ξ induces an automorphism ξ^* of $(D_R)_w$ if we define ξ^* by

$$\xi^*(h) = \xi h \xi^{-1}, \qquad h \in (D_R)_w \subset A(D_R, H_R).$$

Now $\xi^*(B_D)$ acts semisimply on U_D and so by the Mostow conjugacy theorem there exists $u \in U_D$ such that

$$u B_D u^{-1} = \xi^*(B_D)$$

where u is in the range of $(B_D - I)$, where I is the identity mapping. Note that H_R maps onto the range of $(B_D - I)$. Hence there exists $n \in H_R$ such that

$$\xi B_D \xi^{-1} = n B_D n^{-1}.$$

This discussion may be formalized as follows: Let D_R be given with semisimple splitting $(D_R)_s = H_D \rtimes T_D = D_R \rtimes T_D$. Let ξ be an auto-morphism of D_R. Then
   (a) $\xi T_D \xi^{-1} = n T_D n^{-1}$ for $n \in H_R$,
   (b) $\xi^* : (D_R)_s \to (D_R)_s$ defined by

$$\xi^*(c, t) = (\xi(G), \xi t \xi^{-1})$$

is an automorphism extending ξ. This proves that groups D_R have unique semisimple splittings and nil-shadows H_D. Since D_R is a subgroup of $(D_R)_s$ that maps onto T_D by the homomorphism with kernel M_D, it follows that we may view $\tilde{T_D} = D_R/H_R$ as a group of automorphisms of H_D and form the group $(D_R)_s^{\sim} = H_D \rtimes \tilde{T_D}$.

These algebraic facts have the following immediate implications for solvmanifolds. Let R_i/D_i be solvmanifolds that are vector bundles over

the compact manifolds $M_{D_i}/a(D_i)$. Then if $D_i/(D_i)_0$, $i = 1, 2$, are isomorphic, the manifolds $M_{D_i}/a(D_i)$ are homeomorphic. This is an easy consequence of the algebraic uniqueness theorems established above and immediately implies that *compact solvmanifolds with isomorphic fundamental groups are homeomorphic.*

   5. **Existence of solvmanifolds.** We have used the Wang splitting to show that if D_1 and D_2 are CS groups such that $D_1/(D_1)_0$ is isomorphic to $D_2/(D_2)_0$ then

$$M_{D_1}/a(D_1) \approx M_{D_2}/a(D_2)$$

and the fundamental group Γ of $M_{D_1}/a(D_1)$, $\Gamma = D_1/(D_1)_0$, satisfies the exact sequence

(1) $$1 \to \pi \to \Gamma \to Z^s \to 1$$

where π is a FN group. In this section we will show that given any group Γ satisfying (1) there is a compact solvmanifold X such that $\pi_1(X) = \Gamma$. *This, combined with the results in §4, will complete the proof that any solvmanifold is a vector bundle over a compact solvmanifold.* We will not present the original proof here [A, L; **6**], but will present a new proof that is both more invariant and gets deeper into the structure of solvmanifolds. We will begin by motivating the proof and explaining where the difficulties lie. The interested reader may compare this to the results and proofs in [A, L; **16**].

   We will begin by considering the special case where π is abelian, say Z^t. Thus Δ satisfies the exact sequence

(2) $$1 \to Z^t \to \Delta \to Z^s \to 1.$$

In this special case we may proceed as follows to show why Δ is the fundamental group of a compact solvmanifold. First consider the commutative diagram, where horizontal sequences are exact and vertical arrows are monomorphisms:

(3)
$$
\begin{array}{ccccccc}
1 \to & Z^t & \to \Delta & \to Z^s & \to 1 \\
 & \downarrow & \downarrow & \downarrow \\
1 \to & Z^t \otimes R & \to \Delta_R & \to Z^s & \to 1
\end{array}
$$

Since $Z^t \otimes R = R^t$, we have a natural imbedding j of Z^s in $GL(t, R)$. What we would like to do is to complete the above diagram to the following:

(4)
$$
\begin{array}{ccccccc}
1 \to & Z^t & \to \Delta & \to & Z^s & \to 1 \\
 & \downarrow & \downarrow & & \downarrow \\
1 \to & Z^t \otimes R & \to \Delta_R & \to & Z^s & \to 1 \\
 & \downarrow & \downarrow & & \downarrow \\
1 \to & Z^t \otimes R & \to R & \to & Z^s \otimes R & \to 1
\end{array}
$$

If this could be done we would easily have that R/Δ would be a compact solvmanifold with fundamental group Δ. But the existence of the last exact sequence would imply that $j(Z^s) \subset GL(t, R)$ is contained in a connected abelian subgroup of $GL(t, R)$. Since $GL(t, R)$ is not connected, it is easy to find Δ for which R satisfying (4) does not exist.

However we know that if we consider $Z^t \otimes C$, C the complex numbers, instead of $Z^t \otimes R$, we have $j(Z^s) \subset GL(t, C)$ and now it is well known that we can always extend the homomorphism j from Z^s to $Z^s \otimes R$. One can now show, and we will see how in a few moments, to complete the commutative diagram below with the usual conventions in vertical and horizontal mappings.

(5)
$$
\begin{array}{ccccccc}
1 \to & Z^t & \to \Delta \to & Z^s & \to 1 \\
& \downarrow & \downarrow & \downarrow & \\
1 \to & Z^t \otimes C & \to \Delta_C \to & Z^s & \to 1 \\
& \downarrow & \downarrow & \downarrow & \\
1 \to & Z^t \otimes C & \to R \to & Z^s \otimes R & \to 1
\end{array}
$$

Unfortunately, R/Δ is not compact. This difficulty can be overcome by letting D be the subgroup of R generated by the image of Δ and the subgroup $\mathscr{I}$ of $Z^c \otimes C$ consisting of the purely imaginary vectors. It is easily checked that D is closed in R and that $\pi_1(R/D) = D/D_0 = D/\mathscr{I} = \Delta$. Thus we see how to go about solving our problem in this special case.

Let us now see what happens when we try to do this in the general case of the exact sequence (1). Let N be the algebraic hull of π and let N_C be the complex points of N. Then we may try to form (5) with $Z^t \otimes C$ replaced by N_C. Unfortunately this attack does not work. One reason is that the subset of N_C with purely imaginary coordinates does not form a subgroup of N_C when N is not abelian as the simplest examples will show.

Until now we have glossed over the problem of extending the cocycles that are implicit in the construction of diagrams (4) and (5). The reason for this was twofold: first to isolate the automorphism extension problem and second because the Wang splitting will enable us to overcome this difficulty. Let us now see how this is done.

Consider $(\Delta_R)_s^\sim = Z^s \ltimes H_\Delta$ and let M_Δ be the algebraic hull of H_Δ. Then M_Δ satisfies the exact sequence

$$1 \to Z^s \otimes R \to M_\Delta \to Z^t \otimes R \to 1.$$

Further we may form $(M_\Delta)_C$ by requiring it to satisfy the above exact sequence with $Z \otimes R$ replaced by $Z \otimes C$.

Now let $j(Z^t) \subset GL(s, C)$ and let $j^*(Z^t \otimes R)$ be an extension of j. Let A be the algebraic hull of $j^*(Z^t \otimes R)$ and let $A = T_A \oplus U_A$ where U_A is the unipotent part and T_A the semisimple part of A. Let $u': A \to U_A$ have

kernel T_A and $s': A \to T_A$ have kernel U_A. Then form $u:(Z^t \otimes R)$ $\to GL(s, C)$ and $s: Z^t \otimes R \to GL(s, C)$ by $u = u' \circ j^*$ and $s = s' \circ j^*$. It should be remarked that u is independent of how j is extended to j^*, but s is not. It is also easily seen how to form $(Z^t \otimes R)s^* \ltimes (M_\Delta)_C$, where $(Z^t \otimes R)s^*$ means that $Z^t \otimes R$ acts semisimply on $(M_\Delta)_C$ so as to agree with s when restricted to $Z^s \otimes C$ and to act trivially on $(M_\Delta)_C/Z^s \otimes C$. Further the following commutative diagram exists where δ is the mapping defined by $\delta(z) = s^*(z \otimes 1) \oplus (z \otimes 1)$, $z \in Z^t$, and all vertical arrows are monomorphisms:

$$
\begin{array}{ccccccc}
1 \to Z^s \otimes C \to & & \Delta_C & & \to & Z^t & \to 1 \\
\downarrow & & \downarrow & & & \downarrow & \\
1 \to Z^s \otimes C \to (Z^t \otimes R)s^* \ltimes (M_\Delta)_C & \xrightarrow{\delta} & (Z^t \otimes R) \oplus (Z^t \otimes R) \to 1
\end{array}
$$

If we let $R = \rho^{-1}(z \otimes r \oplus z \otimes r)$, $z \in Z^t$, $r \in R$, it is easily seen that $R \subset (Z^t \otimes R)s^* \ltimes (M_\Delta)_C$ satisfies (5).

Let us now go through the general arguments. We begin our discussion with the introduction of the concept of the free nilpotent Lie group of step k generated by a vector space V. This will be our replacement for $Z^s \otimes C$ in the above argument. We will begin by recalling some of the basic properties of the free Lie algebra generated by $F(V)$ a vector space V.

1. Let $\alpha: V \to L$ be a linear mapping of V into a Lie algebra L. Then α extends uniquely to an algebra homomorphism $\alpha^*: F(V) \to L$.

2. Let G be a group of automorphisms of V. Then G extends uniquely to a group of automorphisms of $F(V)$.

3. Let G be a group of automorphisms of V and assume that $W \subset V$ is a linear subspace that is G-invariant. Let $\mathscr{I}(W)$ be the ideal in $F(V)$ generated by W. Then $\mathscr{I}(W)$ is G-invariant.

4. Let $F_k(V)$ be the kth term of the lower central series of $F(V)$. We call $F(V)/F_{k+1}(V) = F^k(V)$ the k step free nilpotent Lie algebra generated by V. If N is a k step nilpotent Lie algebra and $\alpha: F(V) \to N$ is a homomorphism then α factors through $F^k(V)$.

Henceforth we will identify a nilpotent Lie algebra with the corresponding Lie group by means of the exponential mapping implicitly whenever this is convenient.

Before we proceed to a proof of our main assertion, let us pause to consider the following problem: Let M be a connected simply connected nilpotent Lie group and let $\rho: R^s \to A(M)$ be a homomorphism whose image is semisimple. Let $G = (R^s)\rho \ltimes M$, where $(R^s)\rho$ means that R^s acts on M as the image of the mapping ρ.

Problem 1. Give a necessary and sufficient condition for there to exist a solvable Lie group R such that $R_s^\sim = G$.

Clearly if $M \rtimes (R^s)\rho$ is to be the $R_s^\sim$ for some R, there must exist a

connected subgroup $N \subset M$, such that $N \supset [M, M]$ and
1. N is $\rho(R^s)$ invariant and $\rho(R^s)$ acts trivially on M/N,
2. $\dim M - \dim N = s$.

Now assume that G satisfies (1) and (2). Then $G/N = M/N \oplus R^s$. Let $\alpha: M/N \to R^s$ be a linear isomorphism and let $\Delta = \{(x, \alpha x) | x \in MN\}$. If $\mu: G \to G/N$ is the canonical map with kernel N, let $\mu^{-1}(\Delta) = R$. Then it is trivial to verify that $R_s = M \rtimes (R^s)\rho$.

We now want to look at a slight variant of this problem. Consider a group

$$1 \to H \to D \to Z^s \to 1$$

where H is a CN group. Let $(D_R)_s^{\sim} = H_D \rtimes Z^s$ is the Wang splitting and let M_D be the algebraic hull of H_D. Assume $(D_R)_s^{\sim} \subset M_D \rtimes (R^s)\rho$, where $(R^s)\rho$ acts semisimply on M_D.

Problem 2. Can we find $R \subset M_D \rtimes (R^s)\rho$ such that $R \supset D$ and $R_s^{\sim} = M_D \rtimes (R^s)\rho$.

The answer is of course yes and is proven by combining the last two arguments.

Now let Γ satisfy the exact sequence

(1)　　　　　　　　　　$1 \to \pi \to \Gamma \to Z^s \to 1$

and let $(\Gamma_R)_s^{\sim} = H_\Gamma \rtimes Z^s$ have the property that Z^s acting on M_Γ extends to an action of $Z^s \otimes R$ acting on M_Γ and preserving eigenvalue one subspaces. Then the group R constructed as the solution to Problem 2 above has the property that $R \supset \Gamma$ and R/Γ is compact. Thus we have proven the following theorem.

A necessary and sufficient condition that group Γ satisfying (1) be a discrete cocompact subgroup of a connected simply connected solvable Lie group is that if M_Γ is the algebraic hull of the nil-shadow of Γ and Z^s is the semisimple part of Γ then the action of Z^s in M_Γ can be extended to $Z^s \otimes R$ and preserves the eigenvalue one subspaces.

This shows the central role played by the automorphism extension problem for compact solvmanifolds.

We are now in a position to combine the free nilpotent Lie groups with the above discussion to characterize the fundamental groups of compact solvmanifolds.

Let Γ as usual satisfy (1) and let $(\Gamma_R)_s^{\sim} = H_\Gamma \rtimes Z^s$ and let M_Γ be the algebraic hull of H_Γ. Viewing M_Γ as a vector space, assume it is a subspace of a vector space W such that Z^s extends semisimply to W with the property that this action of Z^s in W can be extended to $Z^s \otimes R$. (Note we can always take $W = M_\Gamma \otimes C$.) Let M_Γ be k-step nilpotent and form $F^k(W)$. Choose a linear map $W \to M_\Gamma$ whose kernel K is $(Z^s \otimes 1)$-invariant. Then we may view $Z^s \otimes R$ acting on $F^k(W)$ and so we may form $Z^s \otimes R \rtimes F^k(W)$

and let $\mathscr{I}(K)$ be the ideal in $F^k(W)$ generated by K. Then let $\mathscr{K}:F^k(W)$ $\to M_\Gamma$ be the homomorphism with kernel $\mathscr{I}(K)$. One checks that $\mathscr{K}$ extends to a homomorphism $\mathscr{K}^*:Z^s \otimes 1 \ltimes F^k(W) \to Z^s \ltimes M_\Gamma$. Now $\Gamma \subset Z^s \ltimes M_\Gamma$ and so we may consider $\mathscr{K}^{*-1}(\Gamma) \subset Z^s \otimes R \ltimes F^k(W)$. By the solution of Problem 2 there exists $\mathscr{R} \supset \mathscr{K}^{*-1}(\Gamma)$ and such that $\mathscr{R} \subset Z^s \otimes R \ltimes F^k(W)$ with the full group as its semisimple splitting. Clearly $\mathscr{R}/\mathscr{K}^{*-1}(\Gamma)$ is compact and $\mathscr{K}^{*-1}(\Gamma)/\mathscr{K}^{*-1}(\Gamma)_0 \approx \Gamma$ and this proves our assertion that any group Γ satisfying (1) is the fundamental group of a *compact* solvmanifold.

6. Presentations of compact solvmanifolds. Let R be a connected, simply connected solvable Lie group and let D be a closed subgroup of R such that R/D is compact. If D has no connected subgroup normal in R, we will say that R/D is a *presentation* of $X = R/D$, X a compact solvmanifold. It is clear that the concept of presentation has been introduced to avoid an annoying but trivial complication. For if $H \subset D$ is connected and normal in R then

$$(R/H)/(D/H) \approx X$$

and so we may replace R by R/H when writing the homogeneous space R/D.

With this definition out of the way, we may state the two problems that we will study in this section.

Problem 1. Find a necessary and sufficient condition for a solvable Lie group to have a discrete subgroup Γ such that R/Γ is compact.

Problem 2. Given a compact solvmanifold X find all pairs (R_α, D_α) such that R_α/D_α is a presentation for X.

Both of these problems have been studied in some detail. The first by G. D. Mostow in [M, G; 3] and the second by R. Johnson in [J, R]. However our treatment of both problems is different than the above treatments. Indeed, to Problem 2 we will present a complete solution, up to isomorphism, which except for the torus Johnson failed to achieve. However he did show that there exist presentations (S_α, C_α) such that dim $S_\alpha \to \infty$ as $\alpha \to \infty$.

Let R be a solvable Lie group with semisimple splitting $R_s^\sim = M_R \rtimes R_R$. Necessary and sufficient conditions for R to contain a discrete cocompact subgroup are the following:

(1) M_s has a rational form M_Q such that if N is the nil-radical N of R then $M_Q \cap N$ is a rational form of N.

(2) Consider $M_R/N \oplus R^s$ which contains R/N as a subgroup. Let $p_1:M_R/N \oplus R^s \to M_R/N$ and $p_2:M_R/N \oplus R^s \to R^s$. Then $p_1|R/N = q_1$ and $p_2|R/N = q_2$ are linear isometries. Let $q_2q_1^{-1}(M_Q/N)$ be $\Phi \subset R_s$. Then Φ contains a lattice subgroup L such that R_s/L is compact and L acts as integer

unimodular matrices on a basis of M_Q.

The necessity follows easily by replacing a discrete subgroup by a lattice nilpotent subgroup. The sufficiency follows from the fact proven about lattice nilpotent subgroups invariant under groups of integer unimodular matrices of Chapter II, §3 and the diagonal arguments of the previous section.

Our solution to Problem 2 is also very simple once we are in possession of the Mostow structure theorem for presentations of compact solvmanifolds. We will state it and use it now although we will delay its proof to Chapter IV, §3. The structure theorem says the following: Let R be a solvable Lie group and let D be a closed subgroup such that R/D is a presentation. Let N be the nil-radical of R. Then D satisfies

$$1 \to N \cap D \to D \to \mathbf{Z}^s \to 1$$

where N is the algebraic hull of $N \cap D$ and $\mathbf{Z}^s$ is a lattice in R/N.

If we form $(D_R)_s^{\sim} = H_D \ltimes \mathbf{Z}^s$ and M_D is the algebraic hull of H_D then $R_s^{\sim} = M_D \ltimes (\mathbf{Z}^s \otimes \mathbf{R})\rho$ where as usual $(\mathbf{Z}^s \otimes \mathbf{R})\rho$ denotes an action of $\mathbf{Z}^s \otimes \mathbf{R}$ on M_D. Now let D_0 be the identity compact of D then D_0 is a normal subgroup of M_D and H_D and if $\Gamma = D/D_0$, then

$$(\Gamma_R)_s^{\sim} = H_D/D_0 \rtimes \mathbf{Z}^s$$

and M_D/D_0 is the algebraic hull of H_D/D_0.

Now let M_D be the k-step nilpotent and let $\sigma: F^k(M_D) \to M_D$ be the canonical mapping obtained from the identity mapping. Then since $(\mathbf{Z}^s \otimes \mathbf{R})\rho$ operates on M_D we may extend σ to a homomorphism

$$\sigma: (\mathbf{Z}^s \otimes \mathbf{R})\rho \ltimes F^k(M_D) \to (\mathbf{Z}^s \otimes \mathbf{R})\rho \ltimes M_D$$

and let $\sigma^{-1}(R) = \mathscr{R}$.

Now since $\mathbf{Z}^s \otimes 1$ operates semisimply in M_D leaving D_0 invariant we have

$$(\mathbf{Z}^s \otimes 1)\rho \ltimes F^k(M_D) \xrightarrow{\mu} M_D/D_0 \rtimes \mathbf{Z}^s$$

is well defined. Now $\Gamma \subset M_D/D_0 \rtimes \mathbf{Z}^s$ and $\mu^{-1}(\Gamma)$ is cocompact in $\mathscr{R}$. It is straightforward to now verify that R is obtained from $\mathscr{R}$ by dividing out by the maximal connected subgroup of $\mu^{-1}(\Gamma)$ that is normal in $\mathscr{R}$. *This shows that the construction given in the previous section can be made into a generic construction for all presentations of a compact solvmanifold by dividing $\mathscr{R}$ by the maximal connected subgroup of $\mathscr{X}^{*-1}(\Gamma)$ that is normal in $\mathscr{R}$.*

Thus we see that the presentation problem for a compact solvmanifold reduces to an extension problem and an isomorphism problem.

7. Compact solvmanifolds with discrete isotropy groups. Let Γ be a

discrete subgroup of a connected nilpotent group N such that N/Γ is compact. Then, by the results of Chapter II, we saw that N is unique. But, in Example 4 of Chapter III, §1, we saw that if $\tilde{R}$ denotes the universal covering group of the group of rigid motions of the plane, then $\tilde{R}$ operates transitively and with discrete isotropy group on the three-dimensional torus. Hence a torus may be written in more than one way in the form R/Γ where R is a solvable Lie group and Γ is a discrete subgroup. The basic question of this section is given two presentations R_1/Γ and R_2/Γ of a compact solvmanifold with discrete isotropy group Γ, how much structure do R_1 and R_2 have in common? (It will be clear from the discrete case what will be true for the closed subgroup case so we will not present the details.)

This problem has been studied in [A, L; **3, 12, 10**] and [T, R; **2**]. We will use the semisimple splitting, in the spirit of [T, R; **2**], to unify these ideas and present a true uniqueness theorem that has already been used to study the automorphism groups of polycyclic groups [A, L; **14**].

Let $X = R_i/\Gamma$, $i = 1, 2$, and let us write X in the affine presentation $X = M/a(\Gamma)$, where M is the algebraic hull of the nil-shadow of the Wang splitting of Γ_R. Further $a(\Gamma)$ is a faithful representation of Γ in $A(M) \ltimes M$. Now we know from Chapter III, §4 that $(R_i)_s = T_i \ltimes M$, $i = 1, 2$, where $T_i \subset A(M)$ is completely reducible and abelian.

Now let $\alpha: A(M) \ltimes M \to A(M)$ be the homomorphism with kernel M and let A^0 denote the identity component of the algebraic hull of $\alpha \circ a(\Gamma)$. If Γ satisfies

$$1 \to \pi \to \Gamma \to Z^s \to 1$$

where π is nilpotent then $\pi_R \subset M$. Further A^0 acts trivially on M/π_R since Γ does. Now $\alpha(T_i)$ acts trivially on $q(\Gamma)$ and so $\alpha(T_i)$ acts trivially on A^0. It follows easily from this that $\alpha(T_i)A^0 = C_i \oplus A^0$, where C_i is a compact abelian subgroup of $A(M)$.

Now let $\Gamma_0 \subset \Gamma$ be the subgroup of Γ which is the pre-image of the group $\alpha(\Gamma) \cap A^0$. In [A, L; **10, 14**], I defined a subgroup of finite index of Γ_0 and called such groups "algebraic strongly torsion free S groups of type T" and "predivisible" polycyclic groups in [A, L; **10**] and [A, L; **14**] respectively. For the rest of this paper we will call Γ_0 "*the predivisible subgroup of Γ*" and if Γ is its own predivisible subgroup we will call Γ a predivisible polycyclic group.

Let Γ be a predivisible polycyclic group.

Since A^0 is an abelian connected Lie group, A^0 has a unique maximal compact subgroup C and so we may form A^0/C and consider the image $\alpha \circ a(\Gamma)$ in A^0/C. Let W be the linear hull of the image of $\alpha \circ a(\Gamma)$ in A^0/C and let B be the pre-image of W in A^0. Let $D^\#(\Gamma) = M_\Gamma \rtimes B$. Now let $\pi \supset [\Gamma, \Gamma]$ and be such that π is nilpotent and satisfies the exact sequence

$$1 \to \pi \to \Gamma \to Z^s \to 1.$$

Let π_R be the algebraic hull of π. Then π_R is normal in $D^{\#}(\Gamma)$ and $D^{\#}(\Gamma)/\pi_R$ is abelian. Thus we have a well-defined isomorphism

$$\gamma: \Gamma/\pi \to M_\Gamma \rtimes B/\pi_R$$

which, since the right side is a connected abelian group, can be extended to a continuous isomorphism

$$\gamma^*: \Gamma/\pi \otimes R \to M_\Gamma \rtimes B/\pi_R.$$

Although $\gamma^*(\Gamma/\pi \otimes R) = D^*$ is not unique, the group D^*C contains $\gamma^*(\Gamma/\pi \otimes R)$ for all possible choices of extensions. Let $\eta : M_\Gamma \rtimes B \to M_\Gamma \rtimes B/\pi_R$ and let $\eta^{-1}(D^*) = R(\Gamma)$ and $\eta^*(D^*C) = D(\Gamma)$. Clearly

1. $D(\Gamma) \supset \Gamma$ as discrete cocompact subgroup and $D(\Gamma)/\Gamma$ is compact. Although $D(\Gamma)$ is a solvable Lie group, it is *not simply connected in general*.

2. $R(\Gamma) \supset \Gamma$ and $R(\Gamma)/\Gamma$ is compact. Although $R(\Gamma)$ is simply connected, *it is not unique*.

3. $D(\Gamma) = C \ltimes R(\Gamma)$ where C is a compact abelian group.

The groups $D(\Gamma)$ were introduced and studied in [A, L; **14**]. From the point of view adopted here Theorems 1 and 2 of that paper become trivial consequences of the definitions and may be stated as follows: Let Γ be a predivisible group and let $D(\Gamma)$ be defined as above and called the *split-Lie hull of* Γ.

1. Every automorphism of Γ uniquely extends to an automorphism of $D(\Gamma)$.

2. Let G be a group that contains Γ as a normal subgroup of finite index and let $F = G/\Gamma$. Then if the group $GD(\Gamma)$ is defined by the commutative diagram

$$
\begin{array}{ccccccccc}
1 & \to & \Gamma & \to & G & \to & F & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & D(\Gamma) & \to & GD(\Gamma) & \to & F & \to & 1
\end{array}
$$

the bottom exact sequence splits.

The groups $R(\Gamma)$ were first introduced in [A, L; **10**] where their existence and basic properties were first established. Again using the point of view adopted here Theorems 1 and 2 of [A, L; **10**] become trivial consequences of the definition.

1. $R(\Gamma)$ contains Γ as a discrete cocompact subgroup.

2. Let θ be an automorphism of Γ_R. Then if θ is trivial in Γ_R/π_R, θ is uniquely extendable to an automorphism of $R(\Gamma)$.

Until now we have not stated results using exact sequences. Such results are of course implicit in what we have already proven and we will

now make them explicit. The result below generalizes the results in [A, L; 3, 16].

Let R_1 and R_2 be connected, simply connected solvable Lie groups with nil-radicals N_1 and N_2 respectively. Further, let Γ_i be discrete co-compact subgroups of R_i, $i = 1, 2$, and let α be an isomorphism of Γ_1 and Γ_2 with the following properties:

1. α induces an isomorphism α_1 of $\Gamma_1 \cap N_1$ onto $\Gamma_2 \cap N_2$ and so α uniquely extends to an isomorphism α_1^* of N_1 onto N_2.

2. Let $V_i = R_i/N_i$, $i = 1, 2$, and let α induce the isomorphism α_2 of $\Gamma_1/\Gamma_1 \cap N_1$ and $\Gamma_2/\Gamma_2 \cap N_2$. Then α_2 can be extended uniquely to give an isomorphism α_2^* of V_1 onto V_2. Further, we assume that

$$
\begin{array}{ccc}
V_1 & \to & A(N_1)/\mathscr{I}(N_1) \\
\alpha_2^* \downarrow & & \downarrow \alpha_1^{**} \\
V_2 & \to & A(N_2)/\mathscr{I}(N_2)
\end{array}
$$

where the horizontal arrows are induced by coset actions is commutative where $\mathscr{I}(\ )$ is the group of inner automorphisms of the group in the bracket and α_1^{**} is determined from α_1^*.

Under all the above conditions R_1 is isomorphic to R_2.

We may see the assertion is true by noting that α induces an isomorphism of $(R_1)_s$ onto $(R_2)_s$. The rest is then trivial.

CHAPTER IV: DISCRETE SEMISIMPLE SPLITTING AND APPLICATIONS

1. **Introduction.** Let me begin by explaining the reason for the length and detail of the next section. When I first proved the Philip Hall conjecture [A, L; 15], H. C. Wang found a way of shortening the proof. The shortened version was the one that was published in [A, L; 7]. The published version does not show how the Hall conjecture can be derived from a discrete semisimple splitting theorem. Instead, one directly proves the Hall conjecture. Later it was stated that one can derive the discrete semisimple splitting theorem from the Hall conjecture. Since it is my belief that the discrete semisimple splitting is the key tool, I will reproduce here a refined version of my original proof. This starts by directly proving the existence of a discrete splitting theorem and deduces from it the Hall conjecture. We also use the discrete splitting theorem to prove Mostow's structure theorem for presentations of compact solvmanifolds.

Before plunging ahead with the details of this discussion let us pause to resumé some facts and notations that have already been established.

Let N be a connected, simply connected nilpotent Lie group. A closed subgroup of N will be called a CN group. We know from Chapter II that a CN group Δ can be characterized as a torsion free nilpotent topological group whose identity component Δ_0 is a connected, simply connected Lie group and such that Δ/Δ_0 is finitely generated. Further, if Δ is a CN

group, there exists a unique nilpotent Lie group Δ_R (Δ_Q) such that $\Delta_R \supset \Delta$ and Δ_R/Δ is compact (Δ_Q/Δ_0 is a rational nilpotent Lie group and the real completion of Δ_Q is Δ_R). If Δ is a CN group with trivial identity component we will call Δ an FN group.

We have seen in Chapter III that Γ is the fundamental group of a compact solvmanifold if and only if Γ satisfies an exact sequence

(1) $$1 \to \Delta \to \Gamma \to Z^s \to 1$$

where Δ is an FN group. Fundamental groups of compact solvmanifolds will be called FS groups. If Δ in (1) is a CN group we will call Γ a CS group. If Γ is a CS group satisfying (1), there exists a unique group Γ_Q (Γ_R) satisfying the commutative diagram

$$\begin{array}{ccccccc}
1 \to & \Delta & \to & \Gamma & \to & Z^s & \to 1 \\
 & \downarrow & & \downarrow & & \downarrow & \\
1 \to & \Delta_Q & \to & \Gamma_Q & \to & Z^s & \to 1
\end{array}$$

(Γ_R is defined as above with Δ_R replacing Δ_Q).

2. **Discrete semisimple splittings.** Let us begin by formally stating the main existence theorem.

Let Γ be a CS group. Then there exists a closed subgroup Δ^* of Δ_Q such that

(a) $\Delta^* \supset \Delta$ and Δ^*/Δ is finite,

(b) Δ^* is normalized by Γ.

Further if we let $\Gamma^* = \Gamma\Delta^*$ and let $(\Gamma_R)_s = T_\Gamma \ltimes H_\Gamma$, then we may choose $n \in H_\Gamma$ such that $nT_\Gamma n^{-1} = T$ has the property that

(c) Γ^* and Δ^* are invariant under T.

Given a CS group Γ the group $T \times \Gamma^*$ whose existence we have just asserted is called a discrete semisimple splitting of Γ.

We will only prove the existence of a discrete semisimple splitting of Γ under the restriction that Γ has trivial identity component. The proof in the general case goes through in essentially the same way or may be reduced to the special case treated here.

Let us begin by remarking on the following: Let Γ satisfy

$$1 \to \Delta \to \Gamma \to Z^s \to 1$$

where Δ is an FN group. Let Δ_* be the lattice hull of Δ. Then there exists a unique group Γ_* satisfying the following commutative diagram:

$$\begin{array}{ccccccc}
1 \to & \Delta & \to & \Gamma & \to & Z^s & \to 1 \\
 & \downarrow & & \downarrow & & \downarrow & \\
1 \to & \Delta_* & \to & \Gamma_* & \to & Z^s & \to 1
\end{array}$$

Further any automorphism of Γ is uniquely extendable to Γ_*. We will call Γ_* a *lattice polycyclic group*.

We next need the following fairly straight forward result. *Let*

$$1 \to \Delta \to \Gamma \to Z^s \to 1$$

be a lattice polycyclic group and let $(\Gamma_R)_s^{\sim} = H_\Gamma \rtimes Z^s$. *Let* $e_1, \ldots, e_r$ *be a basis of* $L(\Delta_R)$ *that generates* $\log(\Delta)$ *as a* Z *module. Then using the cross section* f *and the mapping* g *introduced in Chapter* III, §4, *we have* $\mathrm{ad}(g(y))$, $y \in Z^s$, *is rational relative to* $e_1, \ldots, e_r$ *and* $[g(y_1), g(y_2)] \in \Delta$ *for* $y_1, y_2 \in Z^s$.

This assertion may be verified as follows. Since $\log(\Delta)$ is a lattice in $L(\Delta_R)$, and $\mathrm{ad}(f(g))$ preserves $\log(\Delta)$, $g \in Z^s$, we have $\mathrm{ad}(f(g))$ is integer unimodular and so its semisimple and unipotent parts are rational. But $\mathrm{ad}(g(y))$, $y \in Z^s$, is exactly the unipotent part of $\mathrm{ad}(f(y))$.

To see that $[g(y_1), g(y_2)] \in \Delta$ we just have to see that $[f(y_1), f(y_2)] \in \Delta$ and that $\mathrm{ad}(f(y_1)) = b_i \, \mathrm{ad}(g(y_i))$, b_i semisimple and acting trivially on $g(y_i)$ all $y_i \in Z^s$.

Let us now state the discrete splitting theorem we will actually prove and give a proof for it.

Let Γ be a lattice polycyclic group and let $(\Gamma_R)_s^{\sim} = H_\Gamma \times Z^s$. Then H_Γ contains a discrete cocompact subgroup D such that

    (a) D is a lattice nilpotent group,

    (b) $D \cap \Delta_R \supset \Delta$,

    (c) D contains the set $g(Z^s)$ in the above notation,

    (d) Z^s acting on H_Γ maps D onto itself.

The existence of D satisfying (a), (b), and (c) follows easily from the fact that $(H_\Gamma)_R$ is rational nilpotent relative to the appropriate basis of $L((H_\Gamma)_R)$. It is thus condition (d) that is the essential condition. We will prove this by a slight modification of the argument given in Chapter II, §3. Let $M = (H_\Gamma)_R$ and let M_k be the last nontrivial subgroup in the lower central series of M. Then $H_\Gamma/M_k \cap H_\Gamma \rtimes Z^s$ is the Wang splitting of $\Gamma/M_{k \cap \Gamma}$. Hence by induction we may find D' in M/M_k satisfying the conditions of our theorem. Now Z^s acting on M_k is integer unimodular relative to $\Delta \cap M_k$. But then the construction of lattice nilpotent subgroups discussed in Chapter II, §3 assures us that the group generated by a pre-image of D' in M and $(D \cap M_k)/e$, some $e \in Z^+$, will be a lattice nilpotent subgroup of M invariant under Z^s. This proves our assertion.

Given a group Γ the group $Z^s \ltimes D$ constructed above will be called a discrete splitting of Γ. It is easy to see that the intersection of all D's satisfying the conclusions of the above theorem also satisfies the theorem. Henceforth we will restrict ourselves to this minimal D and call $Z^s \ltimes D$ *the* discrete semisimple splitting of Γ. It is easily verified that every automorphism of Γ extends uniquely to its discrete semisimple splitting.

3. **The P. Hall conjecture and Mostow's structure theorem for solv-manifolds.** In this section we will prove the P. Hall conjecture and then

the Mostow structure theorem for presentations of solvmanifolds as corollaries of the discrete semisimple splitting theorem. In Chapter III, §3 we discussed the Birkhoff imbedding theorem [B, G]. We will need the following slight refinement of the results stated in Chapter III, §3. Let L be a nilpotent Lie algebra and let $e_1, \ldots, e_r$ be a basis of L with integer constants of structure. Then the Birkhoff imbedding of L, $B(L)$, will have the property that $B(e_i)$, $i = 1, \ldots, r$, are integer nilpotent matrices. Further if A is an automorphism of L which is integer unimodular relative to $e_1, \ldots, e_r$, then $B(A)$ is an integer unimodular matrix.

The following is the key step in proving the P. Hall conjecture from one point of view.

Let D be a lattice nilpotent polycyclic group and let K be a group of automorphisms of D. Then there exists a subgroup $D^{\#}$ of finite index in D such that D is invariant under K and $K \ltimes D^{\#}$ has a faithful representation in $GL(n, Z)$.

Let $e_1, \ldots, e_r$ be a basis of $L(D_R)$ be such that the Z-module generated by this basis is $\log(D)$. Since the constants of structure of $L(D_R)$ are rational relative to $e_1, \ldots, e_r$, there exists an integer d_1 such that $L(D_R)$ has integer constants of structure relative to $d_1 e_1, \ldots, d_1 e_r$. Further, K acting on $L(D_R)$ is still integer unimodular relative to the basis $d_1 e_1, \ldots, d_1 e_r$. Hence the Birkhoff imbedding B of $L(D_R)$ maps $d_1 \log(D)$ into nilpotent integer matrices and $B(K)$ is a group of integer unimodular matrices. Let Exp denote matrix exponentiation. Then it is easy to see that $\text{Exp}(d_1 \log(D)) \cap GL(n, Z) = D^{+}$ is commensurable with D and K invariant. Thus clearly $D \cap GL(n, Z) = D^{\#}$ has the same property.

P. Hall conjectured that every polycyclic group G has a faithful representation in $GL(n, Z)$. We are now in a position to prove this conjecture.

We first note that, using induced representations, it is enough to prove that a group commensurable with G satisfies the Hall conjecture to conclude the Hall conjecture is true in general. It is a fact, essentially proven by Malcev [M, A; 3], that any polycyclic group is commensurable with a lattice polycyclic group Γ. Imbed Γ in its discrete semisimple splitting $Z^s \ltimes D$. Then by the above $Z^s \ltimes D$ has a commensurable subgroup imbeddable in $GL(n, Z)$. This proves that Γ has a commensurable subgroup imbeddable in $GL(n, Z)$ and thus we have proven the P. Hall conjecture.

We now turn to the proofs of two theorems of G. D. Mostow. The first theorem may be stated as follows:

Let R be a connected, simply connected solvable Lie group and let H be a closed subgroup of R such that R/H is a presentation of a compact solvmanifold. If N is the nil-radical of R then HN/N is a discrete subgroup of R/N.

Let us first reformulate this assertion using the language of semisimple splittings. Since R/H is compact, we have by the Mostow conjecture that the real points of the nil-shadow of the semisimple splitting of H_R is the nil-shadow of R. Let us introduce the notation $(R)_s = M_R \rtimes T_R$ and $H_R = M_H \rtimes T_H$ with $M_H \subset M_R$ and $T_H \subset T_R$. It is now trivial to verify that the following holds: HN/N is a discrete subgroup of R/N if and only if T_H is a discrete subgroup of the group $A(M_R)$. Now let M_0 denote the nil-shadow of H_0. Let us now check that if R/H is a presentation of a compact solvmanifold, then the natural map $q : T_H \to A(M_R/M_0)$ has trivial kernel.

To see this, let $t \in T_H$ be in the kernel of q. Then the range X of $(t - I)$, where I is the identity on M_R, is such that $X \cap M_0 = X'$ is nontrivial. Since T_R is abelian X and M_0 are invariant subspaces of M_R under T_R. Hence the ideal $I(X')$ generated by X' in M_R is in M_0 and invariant under T_R. This means that $I(X')$ is an ideal in R_s and certainly in R. Since R/H is a presentation, this implies that $X' = 0$ and the kernel of q is trivial.

Now we can prove the Mostow structure theorem. The proof is that since $q(T_H)$ preserves a discrete cocompact subgroup of M_R/M, $q(T_H)$ is a discrete subgroup of $A(M_R/M_0)$ and hence certainly of $A(M_R)$.

Our final assertion of this section may be stated as follows: Let R be a connected, simply connected solvable Lie group with discrete cocompact subgroup Γ. Then there exists a faithful representation ζ of G in $GL(n, R)$ such that $\zeta(G) \cap GL(n, Z)$ is commensurable with Γ. To see this let $R_s = M_R \rtimes T_R$ be the semisimple splitting of R and let $\Gamma_s = \Delta_\Gamma \rtimes T_\Gamma$ be the discrete semisimple splitting of Γ with $\Delta_\Gamma \subset M_R$, $T_\Gamma \subset T_R$. Now $(\Delta_\Gamma)_R = M_R$. Hence the Birkhoff imbedding imbeds R_s and Γ_s in $GL(n, R)$ with $\Gamma_s \cap GL(n, Z)$ commensurable with Γ_s. This proves our assertion.

4. **Solvable Lie groups of exponential type.** There are two main classes of solvable Lie groups that have been studied. These are the exponential groups (groups of type E) and the type R solvable groups (groups of type R). We will discuss groups of type E in this section and type R groups in the next section.

A connected, simply connected solvable Lie group R is said to be of *exponential type* if the exponential map of $L(R)$ to R is a homeomorphism. The simplest solvable Lie group that is not of type E is the universal covering group of the group of rigid motions of the euclidean plane, $\tilde{\mathcal{R}}$. This may be seen by noting that $\tilde{\mathcal{R}}$ has a matrix representation

$$
\begin{pmatrix}
\cos 2\pi t & \sin 2\pi t & 0 & x \\
-\sin 2\pi t & \cos 2\pi t & 0 & y \\
0 & 0 & 1 & t \\
0 & 0 & 0 & 1
\end{pmatrix},
$$

$x, y, t \in R$. It is easy to check that the element $x = \frac{1}{2}, y = 0, t = 1$ has no square root in $\tilde{\mathscr{R}}$.

Groups of type E have been characterized of J. Dixmier [D, J] and M. Saito [S, M]. Clearly if R is type E both any connected subgroup of R is type E and any simply connected homomorphic image of R is type E. (Note the first assertion follows from the semisimple splitting of R and the fact that nilpotent groups obviously have this property.) Hence it is clear that R is not of type E if it has a connected subgroup H such that H has a homomorphism onto $\tilde{\mathscr{R}}$. The next theorems show that this condition is also necessary and give a simple criterion for R not to be of type E.

Let R be a connected, simply connected solvable Lie group with semisimple splitting $R_s = M_R \rtimes T_R$. Then there exists $t \in T_R$ which has an eigenvalue θ such that $\theta \neq 1$ but $|\theta| = 1$ if and only if R has a connected subgroup H which has a homomorphism onto $\tilde{\mathscr{R}}$.

Let us begin by assuming that $H \subset R$ exists such that $\gamma : H \to \tilde{\mathscr{R}}$ is a surjection. Let $H_s = M_H \rtimes T_H$ be a semisimple splitting so chosen that $M_H \subset M_R$ and $T_H \subset T_R$. Let the kernel of γ be K. Note that $K \cap M_H$ is normal in H and so invariant under T_H. Hence the action of T_H on the complementary invariant subspace in M_H to $K \cap M_H$ is the same as the semisimple part of $\mathrm{ad}(\tilde{\mathscr{R}})$. Thus T_H has an element t with an eigenvalue θ such that $|\theta| = 1$ but $\theta \neq 1$.

Conversely, let $t \in T_H$ act on M_H with an eigenvalue θ such that $|\theta| = 1$ but $\theta \neq 1$. Let $(M_H)_i, i = 1, \ldots, k$, be the lower central series of M_H. Then there is a largest index j such that $t|(M_H)_j$ has eigenvalue θ with $|\theta| = 1$ and $\theta \neq 1$. Then notice that T_H is connected and hence there are $t_0 \in T_H$ arbitrarily close to the identity with the same property for its eigenvalues. Hence there is in R a one parameter group $p(s)$ such that the semisimple part of $p(s)$ contains $t_0 \in T_H$. This is because the exponential map is a homeomorphism in some neighborhood of the identity.

Now form the subgroup $p(t) \ltimes (M_H)_j \subset R$. It is easily seen that $p(t) \ltimes (M_H)_j$ has a homomorphism onto $\tilde{\mathscr{R}}$. Indeed in $L((M_H)_j)$ choose any subspace W containing $L((M_H)_{j+1})$ and of codimension 2 in $L((M_H)_j)$ with the property that it is $p(t)$ invariant and $\mathrm{ad}(p(t))$ has eigenvalue of absolute value 1, but not constantly 1, on the complement of W. Clearly, $\exp(W)$ is a normal subgroup of $p(t) \ltimes (M_H)_j$ with quotient group $\tilde{\mathscr{R}}$.

We will now give a criterion that tells when a solvable Lie group is of type E.

Let R be a connected, simply connected solvable Lie group with semisimple splitting $R_s = M_R \rtimes T_R$. Then R is type E if and only if T_R has no eigenvalues such that $|\theta| = 1$ but $\theta \neq 1$.

By our previous discussion if T_R has an eigenvalue θ such that $|\theta| = 1$ but $\theta \neq 1$, then R has a subgroup H with a surjection onto $\tilde{\mathscr{R}}$ and so R

is not type E. Hence it only remains to verify the converse.

Now let T_R have no eigenvalue θ such that $|\theta| = 1$ but $\theta \neq 1$. Then it is easily verified that T_R is simply connected and if $t_0 \in T_R$ with eigenvector v with eigenvalue 1 and $t(s) \subset T_R$ is *the* one parameter group through t_0 then $t(s)(v) = v$ for all s. Now let $r \in R$ and let $R_s = M_R \rtimes T_R$ be chosen so that $r = mt$ with $m \in M_R$, $t \in T_R$ and $mt = tm$. Let $t(s)$ be the one parameter group in T_R such that $t(1) = t$ and $m(s)$ the one parameter group in M_R such that $m(1) = m$. Then it is easily verified that $m(s)t(s) = t(s)m(s)$ and so is a one parameter group and also that $m(s)t(s) \subset R$.

In the course of the above discussion we showed that if R is of type E, T_R is simply connected. The converse to this assertion is easily seen to be false.

Our discussion above has four important but easily verified consequences (in the order presented) by us. Let R be a connected, simply connected group of exponential type and let Γ be a discrete subgroup of R.

(1) Let $h(\Gamma)$ be the intersection of all the connected subgroups of R which contain Γ. Then R/Γ is compact if and only if $h(\Gamma) = R$.

(2) Let $r \in R$, let $p(s)$ be the one parameter group such that $p(1) = r$ and let X be the tangent space to $p(s)$ at $s = 0$. Then $[X, L(R)] = L([p(s), R]) = \text{range}(\text{ad}(r) - I)L(R)$.

(3) Let G be a connected subgroup of R such that $G/\Gamma \cap G$ is compact. Then $[L(G), L(R)] = L[G, R] = \text{span} \bigcup \{(\text{ad}(\gamma_i) - I)L(R)\}$ where $\gamma_i \in \Gamma \cap G$ and the one parameter groups through γ_i have tangent vectors that span $L(G)$.

(4) Let V be a connected abelian subgroup of R such that $\Gamma \cap V = L$ is cocompact in V. Then $\Gamma \cap [R, V]$ is compact in $[R, V]$. (Note in proving this last assertion that, for $\gamma \in \Gamma, (\text{ad}(\gamma) - I)L$ is a discrete cocompact subgroup of $(\text{ad}(\gamma) - I)V$.)

We will now discuss the result we have been aiming towards in this discussion. This result was first presented in [A, L; 17, Appendix to VII]. On rereading the proof it seems confusing to me and not all assertions are correct. However the argument in broad outline seems to me to be about the same as presented here.

Let R be an exponential solvable Lie group and let Γ be a discrete cocompact subgroup. Let G be a connected normal subgroup of R such that $\Gamma \cap G$ is cocompact in G. Then $[R, G]/\Gamma \cap [R, G]$ is compact.

Let $[G, G] = N$ and let M be the nil-radical of G. Then $\Gamma \cap M$ is cocompact in M and hence $\Gamma \cap [M, M]$ is cocompact in $[M, M]$. Since $[M, M]$ is characteristic in M it is normal in R and so, by induction, it is easily seen we may reduce to the case where $[M, M] = e$.

We may now use assertion 4 above to conclude that $\Gamma \cap [G, G]$ is cocompact in $[G, G]$ and so we may again assume that $[G, G] = e$. But

then G is abelian and applying 4 again completes the argument.

If we define

$$R^\infty = \lim_{k \to \infty} [\underbrace{R \cdots [R, R] \cdots]}_{k}$$

we have by simple induction that $\Gamma \cap R^\infty$ is cocompact in R^∞.

5. **Type R solvmanifolds and their generalizations.** Let us begin by defining type R solvable groups. Let R be a connected, simply connected, solvable Lie group with semisimple splitting $R_s = M_R \rtimes T_R$. Then R is said to be of type R if all the eigenvalues of T_R have absolute value 1. (*Note.* A solvable Lie group is of type R and type E if and only if it is nilpotent.) Let us begin this section with a proof of a theorem on the structure of presentations of compact solvmanifolds R/H where R is a type R solvable Lie group. (See Theorem 4.2 of [A, L; 4].)

Since T_R is a subgroup of the orthogonal group if R is type R its closure T_R^* is compact. Let $R_s^* = M_R \rtimes T_R^*$. If R/H is the presentation of a compact solvmanifold then $T_R = T_R^*$ and $H/H \cap M_R$ is finite.

We may see this as follows. Since R/H is compact and $R_s^* = R \rtimes C$, where C is a compact group, R_s^*/H is compact. Let R^* be the universal covering group of R_s^* and let H^* be the pre-image of H in R^*. Then $R^* = V \ltimes M_R$, where V is the vector space covering group of T_R^*. Now the nil-radical of R^* is easily seen to be M_R. Hence, by the Mostow structure theorem, $M_R/H^* \cap M_R$ is compact and the identity component of H^*, H_0^*, is contained in $H^* \cap M_R$. Further, the image of H^* in $R^*/M_R = V$ is a lattice and so discrete. Hence $H/H \cap M_R$ must be finite. Hence $H^* \cap M_R$ is cocompact in M_R and the theorem is now trivial to verify.

It now follows easily that if R/H is the presentation of a compact solvmanifold with R of type R then R/H is finite covered by a nil-manifold of the form $M_R/H^\#$ where M_R is the nil-shadow of R and $H^\#$ is of finite index in H.

These results may be looked at from a different point of view that leads to the relation between our results and some classical results of Bieberbach and Zassenhaus. Let M be a connected, simply connected nilpotent Lie group and C a compact abelian group of automorphisms of M. Form $M \rtimes C$ and let Γ be a discrete subgroup of $M \rtimes C$. Then our results above assure us that $M/M \cap \Gamma$ is compact and $\Gamma/M \cap \Gamma$ is finite. It is reasonable to expect this result to be true if we drop the hypothesis that C is abelian and only require that C be a compact group of automorphisms of M. This was verified in [A, L; 2] and a simple proof of the classical theorems of Bieberbach (the special case where M is abelian) was given in [A, L; 9].

The main tool theorem in [A, L; 2] is the following:

Let R be a connected, simply connected solvable Lie group and let C be a compact group of automorphism of R. Further, let Γ be a discrete subgroup of $R \rtimes C$ and let $\overline{\Gamma R}$ denote the closure of the group generated by the elements of Γ and R. Then the identity component of $\overline{\Gamma R}$ is solvable.

In [A, L; 1], we re-examined the above tool theorem to try and remove the hypothesis that C is compact. We then obtained the following result.

Let G be a Lie group with radical R. Further let Γ be a discrete subgroup of G. Then the identity component of $\overline{\Gamma R}$ is solvable.

We will content ourselves with the statements of these results and how they relate to the theory of solvmanifolds. We do not know how to improve on the proofs in the original articles and so will refer the interested reader to these papers.

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