AN EXPOSITORY DEVELOPMENT OF A MATHEMATICAL MODEL OF THE ELECTORAL PROCESS*

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I. INTRODUCTION

The fundamental process of politics is the aggregation of citizens' preferences into a collective—a social—choice. We develop, interpret, and explain non-technically in this expository essay the definitions, assumptions, and theorems of a mathematical model of one aggregative mechanism—the electoral process.[1] This mechanism is conceptualized here as a multidimensional model of spatial competition in which competition consists of candidates affecting turnout and the electorate's perception of each candidate's positions, and in which the social choice is a policy package which the victorious candidate advocates.

This approach, inaugurated by Downs's *An Economic Theory of Democracy*, and falling under the general rubric "spatial models of party competition," has been scrutinized, criticized, and reformulated.[2] To clarify the accomplishments of this formulation we identify and discuss in section 2 the general democratic problem of ascertaining a social preference. We review critically in section 3 the definitions and assumptions of our model. We consider in sections 4 and 5 the logic of a competitive electoral equilibrium. We assume in section 4 that the electorate's preferences can be summarized and represented by a single function; the analysis in section 5 pertains to competition between two organizational structures or two opposed ideologies (i.e., when two functions are required to summarize and represent the electorate's preference). Finally, we suggest in section 6 a conceptualization of electoral processes which facilitates extending and empirically testing our model.

II. THE DEMOCRATIC PROBLEM OF SOCIAL CHOICE

The early literature of spatial theory examines a relatively simple problem, a fundamental assumption of which is that a single dimension describes sufficiently the preferences of citizens. Assuming: (1) that candidates seek to win elections, (2) that all participants in elections (i.e., candidates and citizens) have perfect information, and (3) that the candidate can adopt any position on this single dimension; this literature seeks to ascertain the positions candidates should adopt. Finding such a position, however, requires, first, that it exists—i.e., that some position be dominant, by which

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[1] See the following: Otto A. Davis, and Melvin J. Hinich, "A Mathematical Model of Policy Formation in a Democratic Society," *Mathematical Applications in Political Science II*, J. L. Bernd, ed. (Dallas: Arnold Foundation, SMU Press, 1966); "Some Results Related to a Mathematical Model of Policy Formation in a Democratic Society," *Mathematical Applications in Political Science III*, J. L. Bernd, ed. (Charlottesville: University of Virginia Press, 1967); "On the Power and Importance of the Mean Preference in a Mathematical Model of Democratic Choice," *Public Choice*, 5 (Fall, 1968), 59–72; "Some Extensions to a Mathematical Model of Democratic Choice," forthcoming in *Social Choice*, B. Lieberman, ed. (New York: Gordon and Breach); Melvin J. Hinich and Peter C. Ordeshook, "Abstentions and Equilibrium in the Electoral Process," *Public Choice*, 7 (Fall, 1969); Social Welfare and Electoral Choice in Democratic Societies," (unpublished, Carnegie-Mellon University, 1969); Peter C. Ordeshook, "Some Extensions to a Mathematical Model of Electoral Competition, and Implications for the Theory of Responsible Parties," *Midwest Journal of Political Science*, (February 1970); *Theory of the Electoral Process* (unpublished Ph.D. dissertation, University of Rochester, 1969).

[2] Anthony Downs, *An Economic Theory of Democracy* (New York: Harper and Row, 1957). For additional theoretical developments see: Gerald Garvey: "The Theory of Party Equilibrium," this REVIEW, LX (1966), 29–38; David E. Chapman, "Models of the Working of a Two-Party Electoral System," *Papers on Non-Market Decision Making III* (Fall, 1967), and *Public Choice* (Fall, 1968)

we mean that if a candidate adopts that position then he is guaranteed at least a tie in the election and a positive plurality if his opposition selects some position other than the dominant one. Unfortunately, the existence of such positions cannot be guaranteed generally and additional assumptions are required for it to exist. Consider the following *incomplete* argument purporting to support the proposition that, if all citizens vote, the median preference of the electorate is the dominant position:

Let θ^* (see Fig. 1) represent the median position for the density of "preferences" $f(x)$; thus θ^* divides the density equally. If the first candidate selects the position $\theta_1=\theta^*$, the second candidate selects the position $\theta_2<\theta^*$, and everyone votes, the first candidate receives a positive plurality; he is preferred by all citizens to the right of θ^*, which by construction is one half of $f(x)$, and he is preferred by those citizens to the left of θ^* who are nearer to θ^* than to θ_2 (and who provide his margin of victory).

Since dominant positions exert a powerful attraction to candidates, the argument that candidates should converge to the median might appear to be trivial. However, such an argument is incomplete; it requires additional assumptions, one of the most important being that the form of each citizen's preferences is "single peaked". Specifically, the argument that all citizens to the right of θ^* prefer θ_1 with $\theta_2<\theta_1=\theta^*$, implicitly assumes the existence of a specific class of orderings of the alternatives on the horizontal axis. If preference is indicated on the vertical axis, the preference orderings in this class are represented by functions which change direction at most once from increasing to decreasing (i.e., are single peaked). If this assumption is not satisfied we may be unable to identify a dominant position so that a paradox of voting is said to exist.

The illustration of this assertion, which is a special case of Arrow's General Impossibility Theorem, is so simple that it bears repeating.[3] Consider three citizens whose preferences do not satisfy the single peakedness assumption:

Citizen	Citizen's Preference Ordering
1	A→C→B
2	C→B→A
3	B→A→C

Although each citizen has no difficulty defining a preference ordering among the alternatives—*A*, *B*, and *C*—no alternative is dominant. *B* defeats *A*, *C* defeats *B*, and *A* defeats *C*, so that the social preference ordering— $A\to C\to B\to A$ —is intransitive. Thus, if *A*, *B*, and *C* are the alternative positions for a candidate, he cannot find a strategy which guarantees him at least a tie.

The possibility that such a paradox exists poses a problem for majority decision-making. Although most standard procedures for aggregating individual preferences (e.g., voting) yield a unique social choice, if preferences are not single peaked such choices depend, for example, on the order in which the alternatives are presented.[4] Thus, if we cannot guarantee the existence of dominant positions in the context of electoral campaigns, the outcome of an election may depend on the temporal order in which candidates select their strategies.

Downs, who was perhaps the first to introduce into the contemporary political literature the problems which the paradox poses, considers only the world of one dimension. A simple example demonstrates, however, that the problems which the paradox introduces are compounded as more dimensions are considered[5]. Consider Figure 2 in which the vertical and horizontal axes index two relevant dimensions. Assume that the electorate consists of three

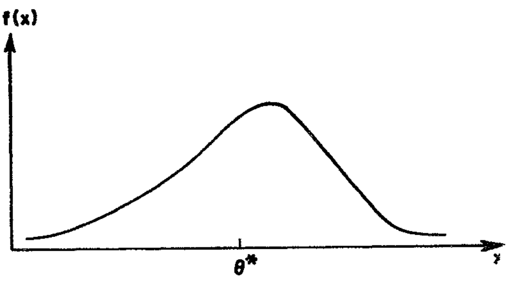

FIG. 1

[3] Kenneth J. Arrow, *Social Choice and Individual Values* (New York: Cowles Commission Monograph #12, Wiley, 1951). See also: Duncan Black, *The Theory of Committees and Elections*, (Cambridge: Cambridge University Press, 1968); and with R. A. Newing, *Committee Decisions with Complementary Valuation* (London: W. Hodge, 1951). A general exposition of the paradox and its implications is given by William H. Riker, "Voting and the Summation of Preferences: An Interpretive Bibliographical Review of Selected Developments During the Last Decade," this REVIEW, LV (December, 1961), 900–911.

[4] Duncan Black, *op. cit.*, pp. 21–25.

[5] Duncan Black and R. A. Newing, *loc. cit.*

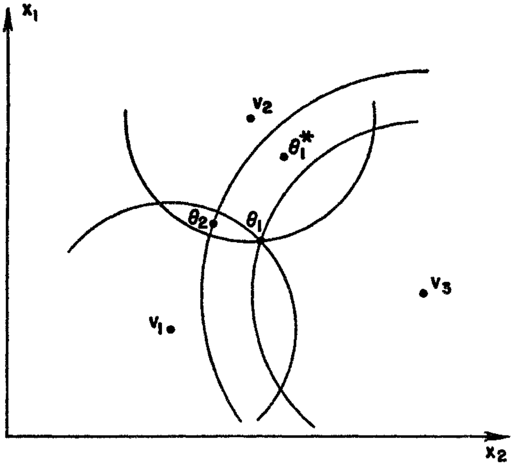

FIG. 2

voters, with their preferred positions denoted by v_1, v_2, and v_3, and that there are two candidates (who do not vote). Finally, assume that the concentric circles drawn about v_1, v_2, and v_3 represent the indifference contours of each voter's preference function. Thus, a voter is indifferent between two alternatives if they lie on the same contour, and he prefers one alternative to another if it lies on a contour closer to his preferred position—i.e., a contour of a smaller radius. Now let the first candidate adopt any strategy, say θ_1. Observe that the position θ_2 defeats θ_1 in a majority vote since it lies on indifference contours of smaller radius than θ_1 for the two voters preferring v_1 and v_2. But voters 2 and 3 prefer $\theta_1{}^*$ to θ_2 for similar reasons while θ_1 defeats $\theta_1{}^*$, etc. Obviously this cycle continues indefinitely. No dominant position exists, and the position a candidate should adopt depends on the position selected by his opponent.

Although this result may not be inordinately surprising, it demonstrates an important distinction between the unidimensional and multidimensional cases. Consider Figure 2 again but assume that citizens cannot vote on x_1 (i.e., the value of x_1 is fixed). This is equivalent to assuming that only motions on a line parallel to the x_2 axis may be considered. Preferences on this line, by construction, are single peaked so that the value of x_2 preferred by v_2 is the dominant choice. Alternatively, if x_2 is fixed, the value of x_2 preferred by v_3 is the social choice. Thus, even though a dominant position exists for each of the dimensions taken individually, the composite of these dimensions yields an intransitive social preference.

We can easily imagine such a situation whenever citizens are permitted to vote both on the amount of some service to be provided publically *and* on the fiscal institutions for funding such a service. If the electorate is provided with the opportunity to vote only for the amount of the service to be provided (with a given fiscal institution) or only for the fiscal institution (with predetermined level of public activity) an unambiguous social choice may be revealed.[6] Such choices, as the previous illustration demonstrates, are not guaranteed with the composite of these two issues.

Since such a simple example demonstrates that dominant positions, in general, do not exist for a multi-dimensional world, one wonders whether they might exist for some reasonable set of conditions. Tullock, for example, suggests that the paradox occurs with less frequency than we might otherwise anticipate from Arrow's analysis.[7] Socialization and agreement on basic normative precepts diminish the probablities of multi-peaked preferences, and certain symmetries of preference reduce the probability of a paradox occuring in a multidimensional world. Similarly, the molasses-like variability of political parameters, and the uncertainty and imperfect measuring devices of both practitioners and academics, bring into question the relevance of such precise mathematical analyses as those of Arrow and Black. Stated differently, we do not know the frequency with which the paradox occurs in reality. That the paradox *can* occur, nevertheless, raises an ominous note for democratic theory. Specifically, it decreases the probable parsimony of acceptable models. If social choices depend on the order in which motions are brought forward for a vote, or on the number of motions, or on the number of citizens voting, then those *ceteris paribus* conditions commonly scattered through academic tracts (such as this one) can be of considerable importance. We contend, therefore, that political lore, empirical generalizations, or simple graphic arguments are not satisfactory for understanding the political process. The primitive inquires of Hotelling and Smithies, and the verbal unidimensional elaborations of

[6] Such situations are examined closely by James M. Buchanan, *Public Finance in Democratic Process* (Chapel Hill: University of North Carolina Press, 1967).

[7] Gordon Tullock, "The General Irrelevance of the General Impossibility Theorem," *Quarterly Journal of Economics* (May, 1967). Richard G. Niemi presents an excellent formal treatment and interpretation of the probability of a paradox occurring in "Majority Decision-Making with Partial Unidimensionality," this REVIEW, LXIII (June, 1969), 488–497.

Downs, are inadequate. We also reject the argument that no generalization is possible, since such an assertion precludes all scientific inquiry. An adequate comprehension of political processes requires rigorous theory which specifies unambigously the relationships between relevant parameters. We seek, therefore, a model which promises to satisfy eventually our notions of an adequate thory (or which at least is conformable to such a theory). Given this objective, we now consider more rigorously the definitions and assumptions which constitute the foundation of our model.

III. DEFINITIONS AND ASSUMPTIONS

A theory which seeks to explain how parties and candidates do act or ought to act is predicated on the citizens' responses to the candidates' strategies. And our preoccupation with the paradox of voting in section 2 suggests that such a theory is central to a spatial analysis of the electoral process. If we assume that parties and candidates waltz annually before a blind audience—that the electorate is neither attentive nor responsive to the candidates' maneuvers—then spatial analysis is not a requisite for understanding this waltz.

We conceptualize each citizen's choices and actions as the outcome of a two-stage sequential decision process. First, we assume that the citizen evaluates both candidates' (or parties') positions in terms of his own preferences; second, that he decides whether to vote or to abstain. If he votes he supports his preferred candidate. The sequential decision process is ordered in this fashion because the model postulates that the decision concerning whether to vote or to abstain depends upon the citizen's comparative evaluation of the candidates.

Every formalization, however, reveals the ambiguities associated with one's initial conceptualization of a problem. Consider, first, the central problem of ascertaining the method citizens use to compare candidates. Downs, as we note earlier, assumes that citizens compare the candidates' ideological closeness to themselves. The inadequacy of this conceptualization is that responses to campaign issues cannot be characterized as necessarily ideological.[8] Although some elections might involve a single issue, citizens' preferences cannot be ordered unambiguously on a single continuum. Opinion cleavages demonstrate that if spatial models are to retain descriptive and predictive value, they must allow for more than one dimension of conflict and taste.

This requirement first motivated our analysis. Instead of assuming that each citizen prefers one position on a common dimension, we assume that a citizen prefers a position on each of many dimensions. We represent a preferred position by a number, x, on the scale identified with each dimension. Consequently, for the ith citizen and the kth dimension the symbol x_{ik} indicates the position that a citizen, i, most prefers with respect to the dimension, k. We represent the ith citizen's preferred positions for all n dimensions by the vector,

$$x_i = \begin{bmatrix} x_{i1} \\ x_{i2} \\ \cdot \\ \cdot \\ x_{in} \end{bmatrix} \tag{1}$$

This approach facilitates an analysis more nearly consonant with empirical evidence. For example, the complexity of modern society, the indeterminate implications of many policies, and the vagueness of political utterances guarantee the inability of even the most educated citizen to obtain a thorough knowledge and understanding of governmental policy and of the candidates' positions on issues. Thus, citizens employ criteria other than issues for evaluating candidates. The established fact that responses not related to issues (e.g., partisan identification, and candidate image) play significant, if not dominant, roles in determining electoral outcomes, however, does not vitiate the rationalistic perspective of voting behavior. Since our model is multi-dimensional, we can incorporate all criteria which we normally associate with a citizen's voting decision process—issues, style, partisan identification, and the like.[9] The assumption that candidates,

[8] V. O. Key, *Public Opinion and American Democracy* (New York: Knopf, 1963), Ch. 7; Phillip E. Converse, "The Nature of Belief Systems in Mass Publics," in David E. Apter (ed.), *Ideology and Discontent* (New York: Free Press, 1964), pp. 206–261; "The Problem of Party Distances in Models of Voting Change," in M. Kent Jennings, and L. Harmon Zeigler (eds.), *The Electoral Process* (Englewood Cliffs: Prentice-Hall, 1966), 175–207; Donald E. Stokes, "Spatial Models of Party Competition," this REVIEW, LVII (June, 1963), 368–377.

[9] The relative importance of issues, compared to image and partisan bias, as causal determinants of voting behavior remains an open question. Aggregate analyses of cross-sectional survey data demonstrate clearly the predictive dominance of partisan identification. V. O. Key, however, concludes in *The Responsible Electorate* (Cambridge: The Belknap Press of Harvard University Press,

parties, and public officials manipulate only governmental policy to win elections, therefore, is unnecessary.

Because a multi-dimensional model permits this latitude in the specification of the electorate's criteria for evaluating candidates, Downs' assumption of rational action is rendered less objectionable. Rationality in Downs' analysis ostensibly requires that each citizen has some information about the candidates' positions on issues. This information, however, is not free, and because many regard their votes as inconsequential, they avoid this cost by voting on the basis of the candidates' images or on the basis of socially determined partisan preferences.[10] And since we interpret rational choice simply as choice which conforms to the assumptions of our model, voters are rational even though the candidates' positions on issues are disregarded.[11]

Our analysis, moreover, is not sensitive to the number of relevant dimensions, or to their labels. The number and nature of issues change from election to election and it is doubtful whether anyone can successfully predict the issues that will be important in some future contest. These are parameters which must be ascertained empirically for each election.

The mathematical exercise of evaluating campaign strategies permits some ambiguity in the empirical referents for each dimension. The specification of the mathematical properties of the x_i's, though, requires precision before rigorous analysis can proceed, and it is here that we constrain the realism of our model. First, *we assume that each dimension of taste is continuous.* This requires consideration of Stokes' observation that many dimensions are discrete and some are dichotomous (which Stokes terms "valence issues").[12] Conceptually, valence issues present no serious additional problems for the analysis of voting behavior. It is difficult, however, to mix continuous and discrete dimensions in one mathematical model, so we assume continuity simply to facilitate our analysis.

Stokes's observation, nevertheless, is pertinent. If, for example, only two spatial positions (e.g., party identification in a two-party system) are available, the candidates generally are unable to vary their positions. But the candidates may employ alternative means for influencing the electorate, such as varying the saliency of their party identification by stressing their party labels differentially. Although the analysis of strategies which have no spatial location can be conducted within the framework of our model, we focus here on the analysis of spatial strategies only.

Two additional assumptions implicit in our formulation of each citizen's preferred position must also be identified. First, *we assume that citizens act as if they estimate a preferred position for every dimension.* Thus, we ignore the possibility that citizens frequently do not or cannot evaluate alternative proposals for many issues. Second, *we assume that all citizens use the same indices to measure any given policy.* Stated differently, the indices measuring the various policies are common to all voters. Thus, we fail to consider Stokes's suggestion that "we may . . . have as many perceived spaces as there are perceiving actors."[13]

Even with these assumptions we must find a more convenient summary for our information about the electorate's preference before analyzing spatial strategies. Such a summary is obtained by observing that the vestors represented by expression (1) are not simply a collection of numbers; they also define a multidimensional coordinate system. Thus, the vector x^i, which represents the ith citizen's preferred position, identifies that citizen with some point in an n-dimensional coordinate system, where the citizen's preference on the kth dimension is measured along the kth axis of the coordinate system.

Assuming now that the preferred positions of all citizens are ascertained, we estimate the probability that a citizen, selected randomly from the electorate, prefers a particular position, say x_0, by counting citizens preferring x_0 and dividing this number by the total number of citizens. When this calculation is performed for all preferences we plot a multivariate density of preferences, $f(x)$, which characterizes the population in the sense that it represents a summary statement of the preferred positions of all

1966) that policy counts heavily. Arthur S. Goldberg, moreover, demonstrates "that there is a rational component to party identification rooted in group norms" (p. 21) with the suggestion that these norms are related to issues, in "Social Determinism and Rationality as Bases of Party Identification," this REVIEW, LXIII (March, 1969), 5–25.

[10] See Goldberg, *ibid.*

[11] This interpretation of rationality is equivalent to the *as if* principle of rational behavior as presented by Milton Friedman in "The Methodology of Positive Economics," *Essays in Positive Economics* (Chicago: University of Chicago Press, 1963). See also William H. Riker, and William Zavoina, "Rational Behavior in Politics, this REVIEW, LXIV (March, 1970).

[12] *Op. cit.* For a spatial analysis of discrete dimensions see Chapman, *op. cit.*

[13] *Ibid.*

citizens. We present in Figure 3 a unidimensional example in which a citizen, selected at random from the electorate, prefers x_o with the probability $f(x_o)$.[14] Figure 4 graphs a two-dimensional example in which a citizen, selected at random from the electorate, prefers x_{o1} on the first demension and x_{o2} on the second dimension with the probability $f(x_{o1}, x_{o2})$.

The positions which a citizen prefers, however, are only a partial identification of the variables relevant for describing his calculus of voting. The act of voting implies a choice among candidates, so we require a representation of such alternatives. Additionally, since we assume that these choices involve a comparison of the alternatives and the citizen's preferred positions, we require that the representation of these alternatives conforms to expression (1). Since vectors characterize the citizen's preferences, and since a citizen's choice involves a comparison between these preferences and his perception of each candidate's position on each dimension, we assume that these positions also can be characterized by a vector. Thus, we let vector,

$$\theta_j = \begin{bmatrix} \theta_{j1} \\ \theta_{j2} \\ \cdot \\ \cdot \\ \theta_{jn} \end{bmatrix} \tag{2}$$

represent the citizen's estimate of candidate j's position on each dimension.

Although we represent the perceived position of each candidate as a vector, we cannot

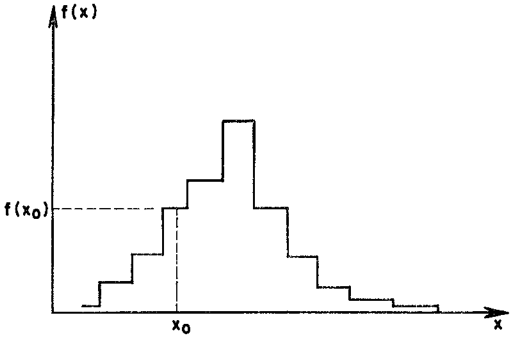

FIG. 3

[14] The densities illustrated in Figures 3 and 4 are represented as discrete although the scales are assumed to be continuous because electorates are finite populations. Nevertheless, our analysis is facilitated by assuming that $f(x)$ is continuous, which is not a serious distortion of any significance if the electorate is large. Hence, in all subsequent illustrations we represent $f(x)$ as a continuous density.

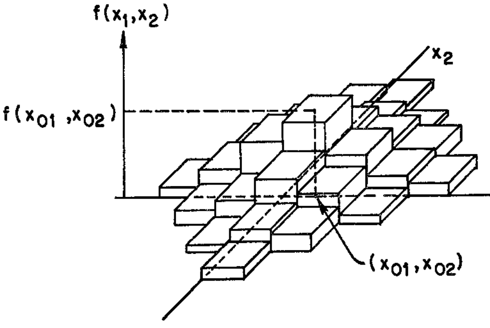

FIG. 4

assume that we know how citizens form estimates of θ_j. Downs, for example, offers several suggestions, such as estimating a candidate's strategy on the basis of past performance and probable future performance. But, like Downs, we cannot specify which suggestion is more satisfactory. Stated simply, a citizen's cognitive and evaluative processes are not sufficiently understood to permit us to identify the psychological mechanisms by which he forms estimates of θ_j.

But while the behavioral questions pertaining to θ_j remain unanswered, rigorous analysis can proceed only if we specify precisely the assumed mathematical properties of this vector. First, because we assume that θ_j is measured on the same dimensions as x_i, we also assume that θ_j is continuously measurable. Of greater substantive importance, however, is the additional assumption that *all citizens make identical estimates of* θ_j (thus we fail to subscript this vector with i—the citizen's index). Thus we ignore such problems as cognitive balance, imperfect information, and candidates' attempts to have different citizens believe different things about them.[15] Repeatedly, we observe citizens

[15] For a discussion of the role of cognitive balance see: Donald E. Stokes, "Some Dynamic Elements of Contests for the Presidency, this REVIEW, LX (March, 1966), 19–28; Bernard R. Berelson, Paul F. Lazarsfeld, and William N. McPhee, *Voting* (Chicago: University of Chicago Press, 1954), ch. 10; Michael J. Shapiro, "Rational Political Man: A Synthesis of Economic and Social-Psychological Perspectives," this REVIEW, LXIII (December, 1969). Cognitive balance poses a problem for our theoretical analysis, but it also reduces the validity of much cross-sectional survey research about attidudes and voting behavior. Briefly, the causal link between attitude (i.e., preference) and vote is bidirectional for many issues. Simply regressing attitude on vote does not reveal the importance of an issue for a citizen's choice—a significant

making "rational" decisions (i.e., decisions understandable to the observer) by their failure to perceive the disadvantages of an already preferred candidate and by failing to perceive the advantages of an already not preferred candidate. If, for instance, a voter favors the passage of strong civil rights measures and his preferred candidate does not, he may, nevertheless, believe this candidate favors such measures. The voter, furthermore, may guard against disruptive information by erecting a perceptual screen and filtering out dissonant messages. Party identification is known to bias citizens' perceptions of candidates' platforms, and highly salient issues often perform an equivalent function.

Such psychological possibilities identify an additional assumption of spatial analysis; *we assume that the candidates have perfect spatial mobility* (i.e., they can adopt any position in the relevant coordinate system). Perceptual distortion and imperfect information, however, frustrate a candidate's campaign objectives where, for example, citizens in a secure Democratic constituency favoring liberal labor legislation remain unconvinced that a Republican candidate is pro-labor—even though it is true. And in a multidimensional world candidates might find it impossible to alter their position on one issue without altering their positions on other issues.

A candidate, of course, prefers to have all citizens believe that he supports each and every preference (i.e., $\theta_j = x_i$ for all i)and if attainment of this ideal is impossible he should adopt the second best solution of approximating this ideal as closely as possible. Our assumptions, nevertheless, exclude this possibility, so the positions associated with a candidate cannot (except in a trivial instance) satisfy all citizens. Thus, to explain the choices citizens make when they regard neither candidate as perfectly satisfactory, we propose a measure of the citizen's evaluation of each candidate's position. Specifically, we provide for the loss in utility any citizen sustains when his preferred position is not supported by the candidates. We accomplish this end by introducing the concept of *individual loss functions*—functions which must satisfy several intuitively desirable properties before they are employed in the model.

First, if $x_i = \theta_j$ the loss citizen i sustains from candidate j's position, symbolically written $L_i(\theta_j)$, should be at some minimum value or zero, since the candidate's position and the citizen's preference are identical for all dimensions.[16] Consonant with this requirement, if $x_i \neq \theta_j$ (i.e., if at least one element of the vector x_i is not equal to the corresponding element in θ_j) the citizen should sustain a positive loss. Hence, we assume, *ceteris paribus*, that the greater the discrepancy between any element of x_i and the corresponding element in θ_j the greater is the loss citizen i associates with candidate j.

The mathematical formulation of these requirements is rendered difficult because: (1) the set of mathematical functions satisfying these two criteria is infinite, and; (2) the available empirical evidence fails to restrict this set sufficiently. The solution we propose for this problem is to conduct the analysis when only the general form of the loss function is assumed.

Consider the following expression as a potential specification for a citizen's loss function when the number of issues, n, equals 1.

$$a(x_{i1} - \theta_{j1})^2 \tag{3}$$

The term $(x_{i1} - \theta_{j1})^2$ is the squared distance be-

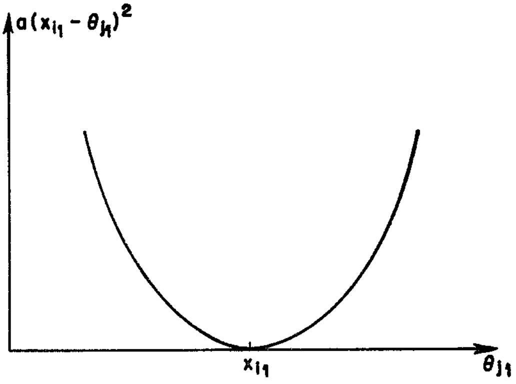

FIG. 5

regression coefficient may indicate only that the attitude has been made consistent with a predetermined preference because it is unimportant. Multiple regression analysis with many attitudinal variables, moreover, is not a satisfactory solution either. A statistically insignificant regression coefficient may indicate only that that variable is related to some other independent variable in the analysis although it may in fact be an important determinant of candidate preference. Because of such difficulties Gerald Kramer analyzes the relationship between policy preference and voting with variables which are more objectively measurable than attitudes in "An Empirical Analysis of Some Aggregative Hypotheses About U.S. Voting Behavior, 1896–1964," (unpublished, Yale University, 1968).

[16] In some of our papers individual loss functions are symbolically represented by the function $\phi(x - \theta)$ to indicate that loss is a function of the difference between x and θ.

tween citizen i's preferred position and candidate j's position. If (3) represents a citizen's loss function and if $a>0$, $a(x_{i1}-\theta_{j1})^2$ increases as x_{i1} and θ_{j1} become more disparate. (The magnitude of a is a function of the scale used to index the issue so we ignore it presently.) Furthermore, if $x_{i1}=\theta_{j1}$, $a(x_{i1}-\theta_{j1})^2=0$. Thus, expression (3) satisfies the two conditions which loss functions must satisfy. This function is illustrated in Figure 5.

Expression (3) however is not totally satisfactory since we must consider the possibility that $n>1$. Suppose, therefore, that, if $n=2$, we summed two such terms. Thus, another possible specification for the citizen's loss function is

$$a_1(x_{i1}-\theta_{j1})^2 + a_2(x_{i2}-\theta_{j2})^2 \tag{4}$$

Observe now that this expression satisfies our first condition which loss functions must satisfy; if $x_{ik}=\theta_{jk}$, for $k=1,2$, the expression reduces to zero. Thus, if this expression represents a citizen's loss function, the citizen's loss equals zero whenever $x_i=\theta_j$. This expression, moreover, satisfies our second necessary condition: if a_1, and $a_2>0$, $L_i(\theta_j)>0$ whenever $\theta_j\neq x_i$. Thus, another reasonable assumption about individual loss functions is that they are represented by expression (4). Figure 6 graphs such a function.

If the election involves more than two issues we might continue adding the necessary terms to (4). But this expression ignores one possibility—that the loss a citizen derives from a candidate's position on one issue is a function of the candidate's positions on other issues. So we add the interaction term $a_{12}(x_{i1}-\theta_{j1})(x_{i2}-\theta_{j2})$ to (4) to account for this possibility. Thus (4) becomes,

$$a_1(x_{i1}-\theta_{j1})^2 + a_2(x_{i2}-\theta_{j2})^2 + a_{12}(x_{i1}-\theta_{j1})(x_{i2}-\theta_{j2}) \tag{5}$$

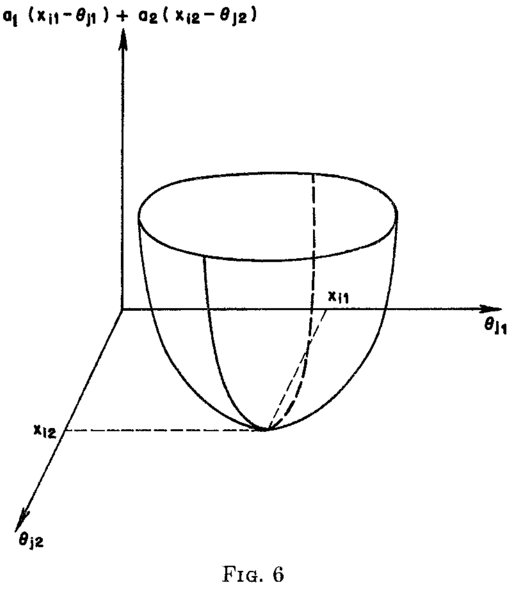

FIG. 6

If we graph expression (5) it resembles Figure 6—the graph of (4)—except that now it is rotated either to the right or left (depending on the magnitudes of a_1, a_2, and a_{12}).

For $n>2$, expression (5) is easily generalized to,[17]

$$\sum_{m=1}^{n}\sum_{k=1}^{n} a_{mk}(x_{im}-\theta_{jm})(x_{ik}-\theta_{jk}) \tag{6}$$

Thus, expression (6) represents the weighted (where the weights are the a_{km}'s) sum of squared distances (the terms for which $k=m$) plus the interaction terms between each pair of dimensions ($k\neq m$).

It remains, however, for us to interpret more precisely the weights in (6). To do so we return to the case of $n=2$—expression (5). If a_2 and a_{12} equal zero we say that only issue 1 is salient for the citizen and expression (5) reduces to (3). Consider a second example: assume that both dimensions are measured in terms of dollars spent on a program so that θ_j reads "candidate j wishes to spend θ_{j1} dollars on the first program, and θ_{j2} dollars on the second program." Assume, furthermore, that,

$$\theta_1 = \begin{bmatrix}\$1\\ \$1\end{bmatrix}, \qquad \theta_2 = \begin{bmatrix}\$3\\ 0\end{bmatrix}, \quad a_{12}=0$$

[17] In matrix notation, expression (6) becomes

$$(x_i-\theta_j)'A(x_i-\theta_j)$$

where $(x_i-\theta_j)'$ is the transpose of $(x_i-\theta_j)$, i.e.,

$$(x_i-\theta_j) = \begin{bmatrix}(x_{i1}-\theta_{j1})\\ \vdots\\ (x_{in}-\theta_{jn})\end{bmatrix},$$

$$(x_i-\theta_j)' = (x_{i1}-\theta_{i1}, \cdots, x_{in}-\theta_{jn})$$

and where A is the nxn matrix of weights, i.e.,

$$\begin{bmatrix}a_1 & a_{12}\cdots & a_{1n}\\ a_{21} & a_2 \cdot & \\ \cdot & \cdot & \\ \cdot & & \cdot\ a_n\end{bmatrix}$$

This expression is generally referred to as the quadratic form. We can guarantee that the quadratic form satisfies our requirement that it be greater than zero if $x_i\neq\theta_j$ if we assume that A is an nxn symmetric (i.e., $a_{km}=a_{mk}$) positive definite matrix. This assumption implies no substantive restrictions on our model since we simply eliminate with it citizens who do not care about any issue (i.e. $A=0$). For a discussion of quadratic forms see George Hadley, *Linear Algebra* (Reading: Addison Wesley, 1961), 251–263.

and that citizen i prefers spending no money, i.e.,

$$x_i = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Substituting these vectors into (4) (or, equivalently, into (5) since $a_{12}=0$), we get,

$$L_i(\theta_1) = a_1 + a_2$$
$$L_i(\theta_2) = 9a_1$$

Although the first candidate proposes to spend two dollars and the second candidate proposes to spend three dollars, the losses associated with each candidate are identical if $a_1+a_2=9a_1$, or, equivalently, if $a_2=8a_1$. Thus, because of the unequal weighting of the issues, the losses a citizen associates with two candidates can be equal even though the candidates adopt dissimilar programs.

With this simple example we might be tempted to conclude that whenever $a_k>a_m$, issue k is "more salient" than issue m. Observe, however, that in this example we assume that the scales of both dimensions can be represented by a common measure, dollars. In general, campaign issues have no commonality of measurement and, additionally, the units of measurement for each dimension may be arbitrary. If, in the previous example, the second dimension is measured in cents, then

$$\theta_1 = \begin{bmatrix} \$1 \\ 100¢ \end{bmatrix} \qquad \theta_2 = \begin{bmatrix} \$3 \\ 0 \end{bmatrix}$$

and the necessary condition for equality of losses becomes $a_2=.0008a_1$. Thus, when both dimensions are measured in dollars equal losses requires that $a_2>a_1$, but when the second dimension is measured in cents equal losses requires that $a_2<a_1$.

This example reveals an important property of the a's—their relative magnitudes are dependent on the scales of measurement which are applied to each dimension. And since we do not know either the scales which might measure all conceivable dimensions, the relative importance citizens attach to each dimension, or the prior identification of salient issues, the proof of our theorems should not require knowing the values of the weights.[18] Thus, we attempt to avoid the criticism that it is difficult, if not impossible, to estimate these weights in any real campaign by proving theorems which are insensitive to the magnitudes of the a's.

We note, however, that in our development of a citizen's loss function, the weights are not subscripted by i; the weights are assumed to be common to all voters. This assumption, which is used in the earliest developments of our basic model, implies that the electorate is homogeneous. In other words, no restrictions are placed upon the preferred positions, so that citizens can desire widely different policies, but the assumption implies that all citizens assign the same relative weight to any dimension. Consider the issue of school desegregation. The model allows some citizens to desire segregated schools and other citizens integrated ones. The assumption that the weights are common to all citizens, however, implies that everyone assigns the same degree of importance to the issue. The model does not allow some citizens to be concerned while others do not care whether or not schools are integrated.[19]

Clearly, different citizens may not assign the same degree of importance to any issue. Yet the decision to allow individual loss functions to vary is not easily transferrable into a tractable model. Perhaps a natural method would be to assign different a's to citizens, but such a step results in a model whose complexity appears to prohibit the realization of meaningful analytical results. Accordingly, we utilize a simpler approach.

We assume that there exists some average level of concern for each issue and, if individual variations are permitted, these variations are represented as deviations from this average. *We assume, additionally, that the patterns of individual variations in level of concern do not correlate with preference.*[20] This assumption appears to conflict with the proposition that a citizen is more likely to react intensely about an issue if he prefers an extreme rather than a moderate position. If the proposi-

[18] There exists, moreover, a linear transformation on the axes so that any quadratic of the form $(x-\theta)'A(x-\theta)$ can be reduced to $(x-\theta)'(x-\theta)$ without loss of generality (i.e., A becomes the identity matrix I). Thus, without loss of generality, we can assume that

$$L_i(\theta_j) = \sum_{k=1}^{n} (x_{ik} - \theta_{jk})^2$$

so that the indifference contours for a citizen's loss function are concentric circles.

[19] The assumption of a common A matrix does not imply an interpersonal comparison of utility. It implies that, when the loss functions for citizens are ascertained independently, there exists a monotonic transformation on each loss function such that all loss functions have a common A matrix in one coordinate system.

[20] When nonvoting (which is discussed later) is caused by alienation, we must assume that variations in level of concern are *independent* of preference—a somewhat stronger assumption.

tion is correct, however, one should anticipate that relatively large and positive deviations from average concern will be associated with citizens who prefer either extreme on an issue; so there may be a zero correlation, although intensity and preference are not really independent.

If we accept expression (6) as an adequate representation of a citizen's loss function, however, we also accept the assumption of marginally increasing loss (i.e., as the citizen's preference on any dimension and a candidate's position on that dimension become more disparate, the loss which the citizen associates with this candidate increases at an increasing rate). Marginally increasing loss, however, is a hypothesis which may be disconfirmed empirically and which can be an unnecessary assumption. A more general formulation of the loss function is one which permits marginally decreasing, as well as marginally increasing, loss.[21] Such a formulation permits loss functions similar to the one illustrated in Figure 7 (and since the loss function illustrated in Figure 6 is also consistent with this assumption we employ this weaker assumption whenever possible).

Note now, from Figure 7, that this assump-

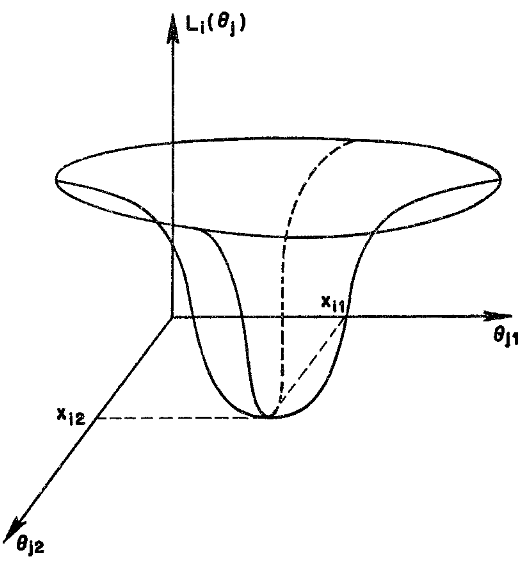

FIG 7

[21] Mathematically, we may assume that $L_i(\theta_j)$ is a function, ϕ, of the quadratic form. Thus

$$L_i(\theta_j) = \phi\ ((x_i - \theta_j)'A(x_i - \theta_j))$$

where ϕ is any monotonically increasing function of its argument. Note that if $A = I$, the citizen's indifference contours remain concentric circles under the transformation ϕ.

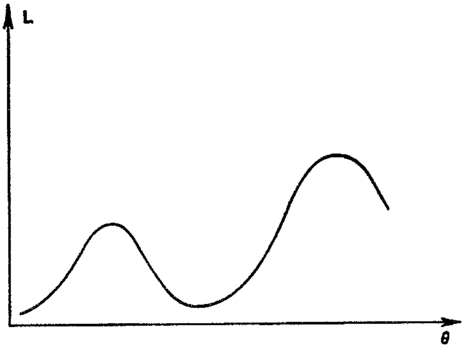

FIG. 8

tion allows variations in θ_j to have little impact on $L_i(\theta_j)$ for θ_j significantly different from x_i—the citizen can become indifferent between alternative positions if they are already quite far from his preferences. Citizens with such loss functions might, for example, be those who refuse to distinguish between two candidates if neither is regarded as satisfactory. Alternatively, citizens with loss functions such as the one illustrated in Figure 6 continue to discount heavily the candidate's movement as he proceeds farther and farther away from their preferences. But for neither situation is the citizen's loss permitted to *decrease* as θ_j and x_i become more disparate.

Thus, we come full circle to our discussion in the previous section of single peakedness and majority decision-making. Consider the following example: assume that citizens are asked to reveal their preferences for alternative tax rates. Some citizens may base their preference on the theory that "the lower the taxes the better." Others may believe that a certain amount of government activity is necessary, differing among themselves only on the amount. They prefer some intermediate tax rate. A third set of voters, however, seeks to insure the adequate financing of current programs and, consequently, favors a substantial increase in the tax rate—as opposed to an incremental increase—so that additional programs can be financed optimally. A prospective loss function for this type of citizen which fails to satisfy our assumptions is depicted in Figure 8.

From the mathematical perspective of our assumptions the occurrence of such functions offers no problem. We can satisfy our assumption about the form of the loss function if we increase the dimensionality of the analysis—by decomposing one dimension into two or more (e.g., reversing the process of factor analysis). But if loss functions on tax policy, for example, are similar to the function illustrated in Figure 8, and if our assumptions are satisfied by decomposition, what substantive political interpretation can be given to these new dimensions? We begin with a politically meaningful dimension—tax policy—which we assume has substantive meaning for both citizens and candidates. Can we assume further that the derived dimensions have substantive meaning? Are the candidates able to formulate policies on these dimensions?

Riker suggests a tentative answer to these questions:

> If the preference curves are single peaked, then . . . there exists a common qualitative dimension along which all preferences are ordered . . . the single peaked curves . . . reflect a cultural uniformity about the standard of judgment.[22]

If this "standard of judgment" is not uniform —if preferences are not single peaked—then the mathematical exercise of increasing the dimensionality of the analysis should discover the underlying multiple standards. And we assume, furthermore, that the candidates are able to formulate and manipulate policy on these standards. Consider again Figure 8. Our assumptions about loss functions might be satisfied if, for instance, we substitute two dimensions—efficiency in government spending, and government involvement (e.g., in welfare)—for the single dimension of tax policy. Thus we speak of the issues in the campaign as being efficiency and government involvement, but tax policy, per se, is not an issue.

We can now specify the citizens' rules for candidate preference. Generally, we assume that there are two candidates, and we denote the position of the first candidate as θ_1 and the position of the second candidate as θ_2. From the definitions of rationality and individual loss functions, a citizen prefers that candidate whose position yields him the smaller utility loss. Symbolically, the ith citizen prefers the first candidate if,

$$L_i(\theta_1) < L_i(\theta_2),$$

he prefers the second candidate if,

$$L_i(\theta_1) > L_i(\theta_2),$$

and he is indifferent between the candidates if,

$$L_i(\theta_1) = L_i(\theta_2).$$

We assume, moreover, that if a citizen votes, he votes for the candidate he prefers. Thus, we do not consider the possibility that citizens disguise their preferences by voting against a preferred candidate. Farquharson demonstrates that in small committees such strategic behavior can be fruitful if the paradox of voting exists or can be generated.[23] Such falsifying strategies in large electorates, however, seem worthless, since one citizen can have only an infinitesimal effect on the overall electoral preference. For large subgroups of the electorate, however, preference falsification can be rewarding, so we cannot guarantee that some citizens fail to perceive and to employ this strategy. Nevertheless, we assume that such behavior does not exist (rather than proving it does not exist or incorporating it somehow into the model).

Our assumptions concerning candidate preference and choice, however, describe only the first stage of the citizen's two stage sequential decision process. The second stage is the choice between voting for a preferred candidate or abstaining. First, some of the results reported in this essay require the assumption that all citizens vote. And for an electorate in which as many as 83 per cent of all eligible citizens vote (e.g., the 1960 Presidential election) and in which we can attribute most non-voting to habit or special circumstances, this assumption is not unduly restrictive.[24] Conversely, electoral outcomes frequently are determined by variations in turnout and other forms of participation (e.g., contributing money, ringing doorbells), and to the extent that the decision to vote or to participate otherwise is a function of the candidates' strategies, we require a theory about participation.

Presently we consider only variations in turnout, but in the final section of this essay we discuss how our analysis might be extended to include other forms of participation. The decision to vote is posited to involve a comparison of the relative expected utility from voting and not voting. Only if the expected utility of voting is greater than that of abstaining is it rational to vote. Riker and Ordeshook analyze this expected utility calculation and express

[22] *Op. cit.* p. 908. See also Niemi, *loc. cit.*, and Clyde H. Coombs, *A Theory of Data* (New York: Wiley, 1964), Cps. 5–7.

[23] *Theory of Voting* (New Haven: Yale University Press, 1969). For examples of the occurrence of paradoxes in legislatures and possible occurances of contrived paradoxes see William H. Riker, "The Paradox of Voting and Congressional Rules for Voting on Amendments," this Review, LII (1958), 349–366, and "Arrow's Theorem and Some Examples of the Paradox of Voting," in J. M. Claunch (ed.), *Mathematical Applications in Political Science* (Dallas: Arnold Foundation, SMU Press, 1965).

[24] William G. Andrews, "American Voting Participation," *The Western Political Quarterly*, (December, 1966), 639–652.

the expected utility of voting hypothesis as,[25]

$$R = PB + D - C \tag{7}$$

where:

P is the citizen's subjectively estimated probability that his vote materially affects the outcome,
B is the absolute value of the subjective differential loss (or utility) between the candidates,
D is the utility a citizen derives from participating in the electoral process—termed the citizen's sense of duty,
C is the subjectively estimated cost of voting,
R is the expected utility of voting less the expected utility of not voting.

Thus, the citizen votes if and only if $R>0$, and he abstains from voting if and only if $R\leq 0$.

The earlier analysis of equation (7) demonstrates the necessity for inclusion of the PB term, and, specifically, how P might be calculated. Presently, this equation serves as an *indicator* of the relevant factors in a citizen's decision to vote. The equation, however, must be augmented by a specification of these factors' relationships to the candidates' strategies.

We consider two causes of abstention: (1) indifference and, (2) alienation.[26] First, a reasonable interpretation of B suggests that it is not independent of the candidates' strategies, and this relationship, termed *indifference*, can be represented by the variables employed in our model.[27] From the definition of B and of the loss functions, it follows that,

$$B_i = |L_i(\theta_1) - L_i(\theta_2)|$$

where $|\cdot|$ means "absolute value of."

Thus B and, therefore, R, the utility of voting, decrease as the losses associated with both candidates become less disparate. Additionally, if we assume that factors other than P, B, D, and C affect R, and that these factors have random effects on the citizen's expected utility calculations, then, with the present formulation of abstention from indifference, we can assume that *a citizen's probability of voting decreases as the difference between lossses which he associates with each candidate becomes less distinct.*

This assumption, however, appears to ignore the possible effects the candidates' strategies might have on the citizen's sense of civic duty, D, which appears to be largely the product of long-term socialization (e.g., learning or long-term reinforcement through non-voting). We consider the potential short-term effects on D, nevertheless, and assume that *a citizen's probability of voting decreases as the loss he acsociates with his preferred candidate increases and that his probability of voting increases as this loss decreases.*[28] Thus, if we say that a citizen's decision to vote is a function of alienation, we mean that in the short run he identifies a preferred candidate, and, if this candidate is not deemed to be satisfactory, the citizen abstains.

Both assumptions—alienation and indifference—have intuitive appeal and it is probable that both operate to some extent in all electorates. We consider each assumption separately, however, to provide ourselves with a tractable mathematical model. We do not consider, though, one potentially important effect on turnout which is represented in (7) by P, the subjective probability of affecting the outcome. We ignore P not because we consider its effects unimportant, but because it would be difficult to include it and because it does not affect many of our results. Generally, we wish to ascertain the dominant spatial positions in a campaign. Our analysis focuses primarily on perfect competition and on equilibrium states; under such conditions P is maximized, identical for all conditions, and can be ignored with some justification.

The list of assumptions which constitute the foundation of our model is now completed. We proceed to the specification of dominant campaign strategies. Note, however, that ascertaining dominant positions or dominant ranges of positions should not be interpreted as presuming that these are in fact the strategies candidates adopt in elections. Candidates obviously have neither the luxury of perfect spatial mobility nor the endowment of perfect information about citizens' preferences. Ascertaining the electorate's preference is one of the difficult objectives for candidates in campaigns. Thus, we assume only that on the average (or in the long run) candidates act in accordance with the model. Deviations from the predictions of our model are expected to occur. We hope, nevertheless, that the model describes and explains some fundamental forces operating in democratic electoral systems and, by a process of Darwinian selection, that these deviations occur around a mean which the model predicts. We turn now to a description of these forces

[25] "A Theory of the Calculus of Voting," this Review, LXII (March, 1968), 25–42.

[26] Our assumptions about nonvoting conform closely to the two factors Garvey (*op. cit.*) identifies.

[27] See Ordeshook, "Some Extensions . . .," *op. cit.*

[28] See Hinich and Ordeshook, "Abstentions and Equilibrium . . .," *op. cit.*

when the electorate's preference is best described by a single multivariate density.

IV. ELECTORAL STRATEGIES WITH A SINGLE DENSITY

Since dominant strategies generally do not exist in a multi-dimensional world, one objective of our analysis is to find conditions which yield dominant positions. In other words, our model should be interpreted as an attempt not only to correct the flaw of unidimensionality attributed to Downs, but also to specify conditions sufficient for majority rule. First, we consider two-candidate competition when the electorate's density of preferences is distributed unimodally. Second, the situation when the electorate's density of preferences is unknown is explored. Third, the effect of increasing the dimensionality of the election (i.e., the effect of variations in n, the number of issues) is analyzed. Finally, we consider competition when the electorate's density of preferences is bimodal.

Prior to beginning the analysis, however, we shall assume that the candidate's objective is to maximize his plurality. Although the rewards candidates seek vary (e.g., some candidates might desire idiosyncratic benefits from political activity), it is important to note that winning, at the very least, is instrumental for realizing most such goals. We assume plurality maximization, rather than vote maximization, because if winning is his criterion, a candidate must consider the votes his opponent receives as well as the votes which he receives.[29] A candidate must receive a positive plurality to win—not simply "many" votes. Even for those candidates who cannot win because of the historical prejudice of the voters of some districts (e.g., the one-party South), or those who do not seek to win (e.g., candidates who require ideological purity), the standard measure of performance is the disparity between a candidate's votes and the votes which his opponent receives. Thus, in game theoretic terms, we assume that elections are two-person zero-sum games.

It is possible to specify, with this game theoretic assumption, conditions that guarantee the dominance of a single position for *any* number of dimensions. The most prominent of these conditions is the symmetry and unimodality of the electorate's preference density. By unimodality we mean that the preference density, say $f(x)$, has a single mode (e.g., the normal density function). Symmetry implies that if, for example, the mean of $f(x)$ equals zero, the probability that a randomly selected citizen prefers the position x equals the probability that he prefers the position $-x$ (i.e., $f(x) = f(-x)$). Thus, symmetry requires that there exists for every citizen with a given preference, another citizen with a diametrically opposed preference. An example of a symmetric, unimodal density in two dimensions (Figure 9) consists of a density whose contour lines are concentric circles or ellipses.

The dominant position for these densities is μ, the vector of the means of the preferred points for each dimension. Thus, *if $f(x)$ is symmetric and unimodal, a candidate cannot be defeated if he adopts a position equal to the mean of the electorate's preference on each dimension, and this conclusion is valid if all citizens vote or if citizens abstain from voting because of alienation or indifference.*

Assuring that conditions exist in a multidimensional contest which guarantee the existence of dominant positions should not obscure the eminent restrictiveness of these conditions. It is unlikely that the electorate's preference is perfectly symmetric, even though the proof of the above theorem assumes perfect symmetry. On the other hand, symmetry and unimodality are merely sufficient conditions, and dominant positions can exist for preference densities which do not have these characteristics. Nevertheless, one should not presume the existence of dominant positions.

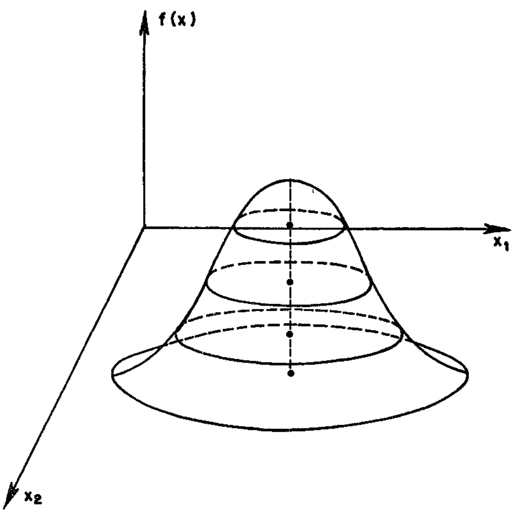

FIG. 9

[29] For an analysis of electoral strategies when vote maximization is the posited goal see Hinich and Ordeshook, "Plurality Maximization vs. Vote Maximization: A Spatial Analysis with Variable Participation," this REVIEW (forthcoming, September 1970).

This latter bleak possibility can be emphasized by returning to the simplistic Downsian world of unidimensional competition. Black demonstrates that if preferences are single peaked and if all citizens vote, a dominant position exists at the median. Furthermore, this result does not require symmetry of preferences. Note, however, that if a new feature is introduced into Black's analysis—if citizens are permitted to abstain—then this feature can preclude the existence of any dominant position under conditions equivalent to those that Black examines. Specifically, if $f(x)$ is a non-symmetric, unimodal, *and* unidimensional density, and if all citizens are assumed to vote, a dominant position will exist. On the other hand, if citizens can abstain because of indifference, then dominant positions, in general, do not exist.[30] This conclusion suggests that dominant positions are more unlikely in a multi-dimensional world, and especially one in which there is abstention, than either Arrow or Black suggest.

Let us now turn to unidimensional competition with abstention from alienation. If $f(x)$ is not symmetric, necessary and sufficient conditions for dominance are mathematically complex thus rendering difficult any substantive interpretation of the conditions. We have been unable to ascertain whether or not these conditions are satisfied for all unimodal densities of preference. These problems manifestly justify further rigorous investigation of Downs's assertion that candidates converge to a unique position when the electorate's density of preference is unimodal.

Two interesting and important observations can be culled from the conditions for dominance in unidimensional competition when $f(x)$ is unimodal and when abstention from alienation is allowed. First, if a dominant location exists, its location is, *ceteris parabus*, a function of the sensitivity of a citizen's probability of voting to variations in his preferred candidate's strategy—referred to as *the sensitivity of turnout to variations in strategy*. If this sensitivity is low the dominant position is near or at the median of $f(x)$, if sensitivity is high, the dominant position is near the mode, but if this sensitivity is at some intermediate value the dominant position is typically not near the median or the mode. Substantively, consider the logical situation of a citizen's probability of voting being inversely related to his cost of voting. Obviously, by selectively varying such costs the probabilities that certain citizens vote can be altered so as to change the policies candidates should adopt. Assume, however, that the cost of voting is varied *uniformly* throughout the electorate. Because a uniform variation increases every citizen's sensitivity (by definition) *electoral outcomes also are altered by uniform variations in the cost of voting*. Thus, such factors as the availability of polling stations, and *progressive* poll taxes affect the social choices which elections might produce.

A second observation concerns the strategy that a candidate should adopt if his opponent selects a non-optimal strategy. Consider three situations: (1) the opponent is near the dominant position; (2) the opponent is far from the dominant position; and (3) the opponent adopts some intermediate strategy. If $f(x)$ is unimodal and univariate, and if alienation causes abstention, a candidate who seeks to maximize his plurality adopts the following position for these three situations: (1) near the dominant position and closer to it than his opponent; (2) near the dominant position; and (3) near his opponent but closer to the dominant position than his opponent. Thus, if $f(x)$ is symmetric, and if we conceive of a situation in which the opponent shifts his strategy from the median to some extreme position, one can plot the candidate's plurality maximizing position against his opponent's position. Thus in Figure 10 the two axes measure the same unidimensional space but with one axis being reserved for one candidate and the other for his opponent so that the line (which does not represent a density) traces out the optimal position for the candidate if his opponent takes any given position.

Observe from this illustration that the candidate adopts a strategy near the dominant position (the median, which is represented here as the origin) if his opponent is either close to or far from this position. Thus, to contradict an observation made by Tullock (in reference to situations with all citizens voting) *if alienation causes abstention, the presence of an extremist opponent should not draw the candidate far from the equilibrium point*.[31] A plurality maximizing candidate has an incentive to diverge significantly from this point only if his opponent is at some "reasonable" distance from the dominant position. This suggests an intuitively appealing strategy for candidates who do not seek to win but who simply wish to create incentives for other candidates to shift their positions—adopt a moderate as opposed to an ex-

[30] A counter example to dominance is presented in Ordeshook, "Some Extensions . . .," *op. cit.*

[31] Gordon Tullock, *Toward a Mathematics of Politics* (Ann Arbor: University of Michigan Press, 1968), p. 52.

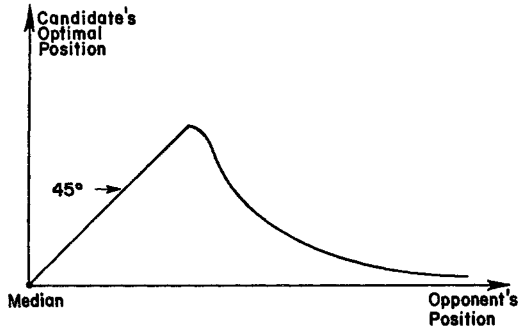

FIG. 10

treme position. Perhaps we have here an explanation for Goldwater's failure to influence Johnson's strategy of consensus.

Even with this interesting observation we must consider the situation in which no dominant position exists; specifically, one might desire to ascertain what bounds a rational candidate should place on his strategy (i.e., what region dominates alternative regions). Briefly, if everyone is assumed to vote and if little is known about the multivariate density of preference, $f(x)$, one can derive bounds on the relative distance from the mean a candidate can get before he insures that his opponent wins. We can show that *if θ_2 is farther than two standard deviations from the mean of $f(x)$ than is θ_1, then the first candidate is certain to win.* If a candidate's position is close to the mean of $f(x)$, then his opponent either should adopt a position which is also close to the mean or should face the consequence of losing the election with certainty. Alternatively, if his opponent adopts a platform distant from the mean, the candidate is afforded greater freedom in the positions he may adopt without insuring that his opponent wins.

The importance of the mean becomes more impressive if we imagine a situation in which the first candidate selects the mean as his strategy and his opponent adopts some other position. *If the number of dimensions required to describe the preferences of citizens increases, the proportion of the vote received by the first candidate increases*; *and if the number of dimensions goes to infinity, the first candidate receives all the votes.* Thus, as the number of issues increases, the strategic importance of the mean as the focal point of the candidates' strategies increases.

This result demonstrates that varying the number of relevant issues is a potentially valuable campaign strategy. Candidates in strategically advantageous positions should increase the dimensionality of the contest while candidates in disadvantageous positions should simplify the election (i.e., reduce the dimensionality) in addition to shifting to a dominant position. Note also that this result complements the intuitively satisfying notion that if a candidate is in a strategically advantageous position on a number of issues, he should attempt to increase the relative importance of these issues in the campaign.

We conclude that *as the number of issues increases, the electoral significance of candidates who advocate extreme positions decreases.* Obviously, the number of relevant issues varies from campaign to campaign. The cause of the variation is found, *inter alia,* in the exigencies of events, the candidates' focus on issues, and an increasingly pluralistic society. Perhaps as the electorate becomes more sophisticated, the number of dimensions required to represent issues increases. The civil rights "issue," for example, is no longer restricted to questions of voting and desegregation. Jobs, housing, business ownership, income distributions, and health, are now also components of this issue. Thus, assuming that responses unrelated to issues do not increase, then as a society grows more complex and the electorate more sophisticated, the chance that an extremist candidate might win is correspondingly diminished. One can also infer that the electoral fortunes of third parties are greatest when the number of issues is small, *ceteris paribus.* This inference is supported by the observation that, historically, the genesis of minor parties involves a single and dominant issue, and that any subsequent increase in the dimensionality of competition is accompanied either by a decrease in the fortunes of such parties, by their absorption by a major party, or by their replacement of a major party.

It is legitimately argued that individual voters do not perceive, and especially do not have feelings about, the entire spectrum of issues. Instead, voters are characterized as being concerned with a narrow subset of issues with the contents of the subset varying from voter to voter. Thus, farmers are supposed to care about farm price supports and those associated with the petroleum industry are supposed to be concerned with oil import quotas while the rest of us hardly even know about, and certainly are not concerned with, these issues. As the model is stated earlier, it does not include this type of phenomenon. However, with suitable assumptions, it is possible to show that our results concerning the dominanance of the mean are valid for the special case in which each voter is concerned with a single issue, but in which the issue varies from voter to voter. Specifically, if we compute the mean preference for each issue by counting only those who care about that particular issue, and if there is a symmetric preference density, then the mean vector is dominant. Also, if one candidate advocates policies which are "closer" to the vector of means than his opponent's vector, and if everyone votes, then the former candidate wins the election. Thus, our basic results obtain even if voters care only about a single issue which varies from citizen to citizen.

Thus far, however, our discussion considers competition only if $f(x)$ is either unimodal or unknown, though many electoral contests are interesting because the electorate's preference is distributed bimodally. Bimodal distributions indicate the presence of only minimal consensus, and it is competition without consensus which is of most interest for speculating about the selection of public policies by election. It is here, moreover, that we begin to uncover instances when candidates should not converge.

The results of our analysis of bimodal distributions are best expounded if we contrast these results with those achieved when preferences are distributed unimodally. In Table 1 we summarize our results.

Notice that *electoral outcomes differ between unimodal and bimodal densities only if alienation causes abstentions.* Thus, if all citizens vote or if indifference causes abstentions, the candidates should converge to the mean. One of us discusses this conclusion elsewhere within the context of the responsible parties controversy. Specifically, internal party discipline and an ability to implement programs to which the electorate has given its consent are not sufficient conditions for distinct programs.[32] This conclusion is doubtless disconcerting to proponents of a responsible two-party system. Normatively, many of us might feel that whenever preferences are bimodally distributed the two modes of opinion should be represented. Consider, as an example, a situation in which the society is governed by an omniscient and beneficent dictator faced with the task of selecting the "best" policies for his country, and where citizens' loss functions are marginally increasing (see Figure 6). The dictator should realize that in any nontrivial situation there is no possibility of satisfying everyone so he must select some scheme for evaluating the relative importance of the society's citizens—some scheme for making interpersonal comparisons of utility. Suppose that the dictator makes the judgment that such comparisons are meaningful, and decides that everyone should be weighed equally. These judgments imply that the best policies are those which minimize the total utility loss of the society. The dictator accomplishes this objective by selecting a position identical to the average desires of the population, so he selects the mean. Hence, competitive conditions which cause the two parties to converge toward the mean result in the electorial process producing the kind of result that a beneficent dictator should choose. This result, of course, although it does tend toward minimal utility losses, is quite contrary to the responsible party doctrine.

TABLE 1.—LOCATION OF DOMINANT POSITION

Distribution	All Citizens Vote	Non-Voting Because of Alienation	Non-Voting Because of Indifference
Symmetric Unimodal	Mean	Mean	Mean
Symmetric Bimodal	Mean	No General Solution	Mean*

* This result assumes that individual loss functions are quadratic, i.e., expression (6). If these loss functions are simply monotonic functions of (6), e.g., see Figure 7, then there exists no general solution. This case, however, is not yet satisfactorily analyzed.

The beneficent dictator's preference for the mean is, in fact, more pervasive than the example might suggest. Instead of weighting each citizen identically, assume that the ith voter, with the preference vector x_i, is assigned the weight $w(x_i)$. Assume also that $w(x)$ is symmetric about the mean of $f(x)$ so that $w(x) = w(-x)$ if the mean is zero. There are two general forms of the weighting function $w(x)$ which are of interest here. First, the beneficient dictator might assign more importance to those in the "middle" than to those who held extreme positions, and in this instance we say that $w(x)$ is unimodal. Second, the dictator might weight "liberals" and "conservatives" more heavily than the "moderators" and in this instance $w(x)$ is termed not unimodal. With these assumptions the dictator's preferences are presented in Table 2.

Thus, *if the citizens' loss functions are marginally increasing the dictator selects the mean for all symmetric $f(x)$ and $w(x)$.* Alternatively, if loss functions are both marginally increasing and marginally decreasing no general solution exists (unless $f(x)$ and $w(x)$ are both unimodal). The social welfare "optimality" of the mean, therefore, is sensitive to the form of the citizens' loss functions, as well as the density of citizen preferences. The point here, however, is that in a variety of situations, with a variety of ethical assumptions arbitrarily assigned, the mean appears to be a desirable point. Accordingly, contrary to the responsible parties doctrine, forces

[32] Ordeshook, *op cit.*

TABLE 2. DICTATOR'S PREFERENCE

	Loss Functions Marginally Increasing Only		Loss Functions Marginally Increasing and Marginally Decreasing	
$f(x)$	$w(x)$ Unimodal	Otherwise	$w(x)$ Unimodal	Otherwise
Symmetric Unimodal	Mean	Mean	Mean	No General Solution
Symmetric Bimodal	Mean	Mean	No General Solution	No General Solution

which cause both party platforms to converge toward the mean, rather than recognizing differences in opinion, are not necessarily "bad" and, in the majority of the above cases, are positively "good" if one is willing to accept the assumptions.

Returning to Table 1, note that no general solution exists when $f(x)$ is bimodal and alienation causes non-voting. To specify the location of the candidates' preferred positions we must consider two additional aspects of the model. These aspects are (1) restrictions on the candidates' strategies, and (2) the sensitivity of turnout to variations in strategy. First, if candidates are strategically unrestricted, then even for a single dimension, no dominant position, in general, exists. By imposing restrictions, nevertheless, equilibrium may be restored so that the candidates fail to converge and adopt strategies near the modes of $f(x)$. Specifically, if the candidates are restricted so that they cannot cross each other or cross the median, dominant strategies exist, and they can be different strategies for each candidate. And since both candidates converge to the mean if all citizens vote or if indifference causes abstentions, the only explanation for divergent positions (if candidates are afforded perfect spatial mobility) is that citizens abstain because of alienation.

This situation illustrates one of the values of formal mathematical analysis. Downs offers the intuitively satisfying but mathematically unproved proposition that whenever preferences are distributed bimodally, the forces of abstention prohibit rational candidates from converging. Our analysis demonstrates, however, that this proposition generally is false—restrictions on strategies may be necessary for equilibrium. We can also give meaningful interpretations to these restrictions. First, candidates may be committed ideologically and may be unwilling or unable to adopt platforms abridging such commitments. Second, a party's nomination is commonly a requisite for winning the general election. The candidates, therefore, may find it necessary to make public commitments in conventions or in the primaries which bind them to these policies in the general election. Third, the electorate may associate traditional policies with a candidate and with his party and, therefore, strategies may be beyond the manipulative reach of the candidates except within a limited range. Finally, by crossing either the mean or his opponent's position, a candidate can alienate citizens—political activists, opinion leaders, and interest groups—whose support is vital. In electoral politics, citizens cannot be weighted in proportion to their numerical strength. If the preferences of activists differ from the preferences of the entire electorate, a candidate's optimal strategy should not be calculated from an unweighted aggregate density of preferences.

This last cause of strategy restrictions also offers an explanation of why candidates, who are afforded perfect spatial mobility, might not seek to converge when the mass electorate's preference is unimodally distributed. If the preferences of activists—citizens who are credited with disproportionately greater weights—are bimodally distributed, an electorate in which citizens are weighted in proportion to their potential contribution to a campaign, from a strategic perspective, may be equivalent to an unweighted electorate whose density of preferences is bimodal.[33] This idea is consistent with the observation that in competitive districts the legislative acts of Congressmen diverge more frequently from their constituencies' preferences than in less competitive districts.[34]

[33] See, for example, Herbert McCosky, Paul J. Hoffman, and Rosemary O'Hara, "Issue Conflict and Concensus Among Party Leaders and Followers," this REVIEW, LIV (June 1960), 406–427, and; Samuel J. Eldersveld, *Political Parties* (Chicago: Rand McNally, 1964), ch. 8.

[34] Warren E. Miller, "Majority Rule and the Representative System of Government," in E.

It may be that competition forces Congressmen to assign disproportionate weights to activists, thereby affecting a bimodal distribution (even when the electorate's preference density may be unimodal). Thus, if candidates diverge and if preferences are arrayed unimodally, this should not be interpreted as a refutation of spatial models, but as an indication that the components of competition are far more complex than such simple minded tests.

Earlier we note that a second important consideration in the discussion of bimodal densities is the sensitivity of turnout to variations in strategy. Specifically, if this sensitivity is sufficiently low, the candidates converge to the mean. Thus, we find a second important qualification for Downs's analysis of such distributions of preference. *Candidates diverge when preferences are bimodally distributed and when alienation causes abstentions only if the sensitivity of turnout to variations in strategy is sufficiently great. Bimodal distributions and abstentions caused by alienation, then, are not sufficient conditions for non-convergence.* "Sufficiently great" is imprecise and most likely it must remain so. The incentives for convergence or divergence are sensitive to so many parameters of $f(x)$ and of turnout that generalization appears impossible. (It is possible, nevertheless, to ascertain dominant positions in specific instances.) We can generalize only by stating that, *as the sensitivity of turnout increases, the incentives for non-convergence increase if $f(x)$ is bimodal.*

V. ELECTORAL STRATEGIES WITH TWO DENSITIES

Having sketched some theoretical results for competition between two candidates in an electorate which is characterized by a single density of preferences, we now consider electorates characterized by two densities. Specifically, assume that the sum of two symmetric unimodal densities characterizes the electorate's preferences. This permits us to consider the phenomenon of political parties. We cannot analyze, of course, the detailed quality and variety of parties. It is possible to consider in these developments only some salient characteristics. Our approach to party politics assumes that a multivariate, symmetric, unimodal density characterizes the desires of the party membership and two (possibly overlapping) densities characterize an electorate of two competing parties. Additionally, assume: (1) all citizens vote; (2) candidates are first selected in primaries; (3) in primary elections citizens can only vote in their party's primary; and (4) in the general election citizens select the candidate whose strategy yields them the smallest loss without regard to party.

The term "without regard to party" may be viewed as a most unsatisfactory assumption. Previously, we indicate that a multidimensional model is valuable because partisan identification is admissible as an additional dimension of taste. However, it is now desirable to assume that partisan identification can be ignored—although this assumption is clearly contradicted by empirical fact. A solution to this *contretemps* is available. First, since the model permits as many dimensions of taste as necessary, one might assert that a substantial basis for party identification is found in these dimensions. Thus, if a sufficient number of dimensions is provided, we can minimize the distortion afforded by ignoring party identification. Alternatively, we may retain partisan identification as a dimension, say the nth, without affecting our definition of parties. We can analyze strategies, then, on the first n-1 dimensions and take cognizance of the citizen's bias inherent in the nth dimension.

The idea now is to utilize previous results to analyze the relationship between victory in the general election and the preferences of party identifiers. Thus, if candidacies are determined by primaries, a candidate is nominated whose position is identical with the mean vector of the preferred positions of the members of his party. Symbolically,

$$\theta_1 = \mu_1; \qquad \theta_2 = \mu_2$$

where μ_1 and μ_2 are the means of the first and second parties' densities of preference respectively.

Imagine the means of the two densities being pulled apart or moved away from each other. In American politics, for example, the Democratic and Republican densities "overlap." However, as the means are moved further apart by shifting the distributions, this overlap diminishes until it vanishes. An illustrative, one-dimensional situation is set forth in Figure 11 in which no overlap exists. Obviously, since citizens select the candidate whose strategy is nearer their preferred position, at some point during the shifting all or nearly all citizens prefer the candidate of their party. In this instance the majority party always wins. The minority cannot exert an influence upon the formation of policy and is totally ignored. While

Allardt, and Y. Littunen (eds.), *Cleavages, Ideologies, and Party Systems: Contributions to Comparative Political Sociology* (Helsinki: Transactions of the Westermarck Society, 1964), 343–376.

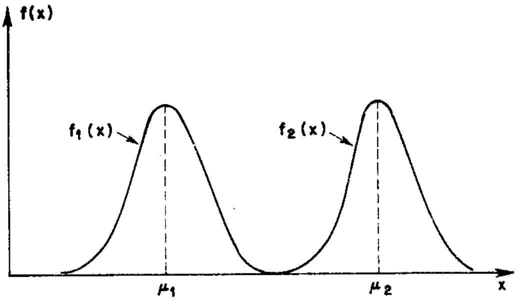

FIG. 11

the formal proof of this argument is of the limiting variety, the argument need not be stated here in such a form. The inference derived from the theorem is that, *ceteris paribus, the greater the discrepancy between μ_1 and μ_2, the less a minority party should favor government by simple majority rule.*

Among the *ceteris paribus* conditions, however, are two important parameters of $f_1(x)$ and $f_2(x)$—their variances. Considering the usual case in which overlap exists between the two densities, and assuming that the densities are normal or multivariate normal, it is possible to obtain some mathematically rigorous results. Assume that the preferences of the second party are more dispersed (in the sense that its members represent a "wider range" of opinion) than are the preferences of the first party. Two such distributions are represented for the unidimensional case in Figure 12.

Under these circumstances a necessary condition for the second candidate to win is that it be the majority party. Thus, a "dispersed" minority party, one which encompasses a much wider range of opinion than the majority party, cannot receive a positive plurality. An obvious corollary to this theorem is that *minority parties can win elections. Their chances increase, furthermore, as the range of opinion of their membership diminishes, ceteris paribus, relative to the dispersion of the opposition.* In the situation depicted in Figure 12 the first party, assumed to be a minority, might win, since the variance of $f_1(x)$ around μ_1 is considerably less than the variance of $f_2(x)$ around μ_2. The discrepancy implies that the first party attracts more votes from the opposition than it loses from defections of its own membership. This point is intuitively satisfying and is useful for interpreting the efforts of minority parties to enforce a singleness of purpose and ideology in countries holding meaningful elections.

The preceding analysis assumes that each candidate adopts the mean vector of his party as his position in the general election. Obviously parties should not seek to constrain their nominees to strategies reflecting solely the preferences of party activists. Otherwise, movement toward the mean of the opposition party, which might increase a candidate's probability of winning, is prohibited. This reasoning suggests an interesting conflict. The minority party has the most to gain by permitting its candidates to diverge from the party's preference, but it must work hard toward retaining its singleness of purpose. The internal tension of minority parties, therefore, is the tempering of ideological purity with the necessity of nominating viable candidates. The majority party, on the other hand has less to gain by permitting its candidates to diverge from the preferences of the party, but it is less concerned with its ideological purity. The internal tension of majority parties, therefore, is the necessity for selecting among an abundance of viable candidates on the basis of some criterion other than ideology.

VI. FROM THEORY TO THE REAL WORLD

Our definition of party structure undoubtedly abstracts many interesting and important distinctions between minority and majority parties, as well as the pervasive conflicts within such organizations. This is, of course, a feature of all theories. Abstractness is not an evil nor can it be avoided in the development of any science. The relevant question is whether we have deduced an empirically valid and meaningful situation or whether we have provided only an insight into the logical equivalences of a mathematical structure which bears little relevance to actual campaigns. Political scientists should not be concerned *per se* with insights into a mathematical structure. They *should* be concerned with the relevance of such a structure to exceedingly complex processes. Hence, the correspondences between the real world and our model require identification, and it is these correspondences which should be considered.

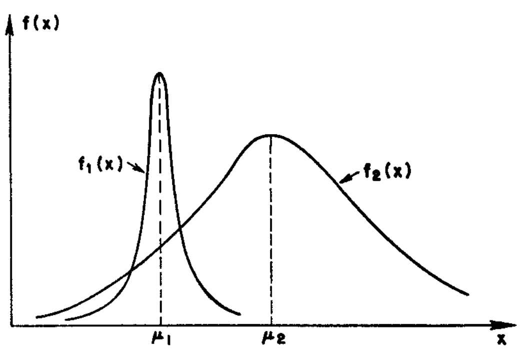

FIG. 12

Arrow's General Impossibility Theorem is an example of the properties of an abstract analysis and its relation to the real world. In section 2 the paradox implied by this theorem is illustrated with a relatively simple situation consisting of three citizens, with unambiguously identified preference orderings over three alternatives, and a predetermined decision rule. One concludes from that illustration that the possibility of intransitive social preference is a pervasive feature of all collective decision-making situations. More importantly, Arrow's analysis consists of ascertaining the logical consequences of certain assumptions—*assumptions which do not begin to encompass the complexity of social processes.* We regard his General Impossibility Theorem as a relevant consideration in all such processes, though, because we believe that he abstracts from these processes certain fundamental characteristics. Our model, like Arrow's, should not be interpreted as a *description* of the electoral process, but as an *abstraction* of characteristics which seem fundamental and pervasive in electoral processes.

Despite such conditional statements, some scholars discount a model's value if its assumptions seem naive and unrealistic or if the opportunities for empirical analysis and further development are obscure. These conclusions about a model's assumptions, however, often are the result of two factors which, if recognized, can render the assumptions more palatable. First, there may be an unintentional resistance either to conceptualizing (perhaps diverse) empirical phenomena in terms of a model's parameters or to reinterpreting these parameters. Second, an assumption's suitability may be disputed because of a confusion between the properties of an adequate theory and the properties of a description of reality in terms of the theory.

As an example of the first factor, consider Downs's assumption that competition consists of parties presenting alternative ideological positions to the electorate. Obviously, a party is a complex and heterogeneous organization, and no single point on a scale is an adequate description of its campaign behavior. Thus, Downs abstracts from his analysis a pervasive and important feature of elections. If this assumption seems to be essential for spatial analysis, and if weakening it is a formidable task, we might reject spatial analysis. But if we represent reality as competition between candidates (individuals) and change Downs's label from "party" to "candidate" we increase somewhat the promise of his approach. A party can be interpreted then as a density of individual preferences which constrain the positions candidates adopt in the general elections.[35] Hence, a seemingly naive assumption becomes less objectionable by a simple reinterpretation of its content.

As a second example of the first factor, consider our multidimensional model. A common criticism of spatial analysis is that many citizens do not evaluate candidates on the basis of "issues", since voters' preferences are explained by partisan bias or the candidates' images. Some scholars conclude, therefore, that spatial models of competition are wholly inappropriate for understanding elections. Nevertheless, one can argue that partisan identification or candidate "image" can be conceptualized not simply as biases or new parameters but as additional preference dimensions (i.e., elements in each citizen's x vector). Thus, while additional measurement problems require consideration, we can at least reinterpret our data so that no new theoretical variables are required.

As a final example of the first factor, consider the observation that many citizens do more than simply vote in an election—many people contribute time and financial resources to one candidate or to the other. Much of a candidate's energies, moreover, are directed towards such citizens because their support is worth more than an equal number of citizens who contribute only their vote. Thus, spatial models might be construed to be inappropriate for understanding this vital aspect of elections. If we interpret voting as *only* one kind of political participation, however, and if we assume, by an appropriate redefinition of the terms in equation (7), that R is the utility a citizen derives from participating in some specified manner, we may interpret our results as the strategies candidates should adopt if they seek to maximize their plurality of any measure of participation. Thus, if we employ the assumption that variations in alternative forms of political participation can be explained by an equivalent calculus we can extend our analysis to the competition for these forms of support.

Reconceptualization also increases opportunities for testing the model. It may be difficult to measure adequately many parameters of a citizen's calculus in mass electorates; factors which we abstract out of the real world (e.g., uncertainty) may confound testability. Our sources of data, though, need not be confined to mass electorates. If we allow the

[35] See Peter H. Aranson and Peter C. Ordeshook, "Spatial Strategies for Sequential Elections," (forthcoming); and R. G. Niemi and H. F. Weisberg, *Probability Models in Political Science.*

generalization concerning participation, for example, we can focus on citizens who might contribute money. And if the model supplies a satisfactory explanation for such people's actions then we have some confidence that the model is useful for discussing other forms of participation. Citizens comprising this restricted data base, furthermore, should have more information concerning the candidates' strategies, and their own preferences. Some of our ideal-type assumptions thereby are renered more consonant with our view of reality.

Nevertheless, reconceptualization cannot account for other phenomena, such as the degree of uncertainty citizens associate with a candidate's strategy. Obviously, citizens associate some uncertainty with each element of θ_j, and candidates manipulate the stochastic properties of these elements as a strategic alternative to varying spatial location. Because we explicitly assume that θ_j is deterministic, we cannot consider such strategies presently. A fundamental variation in our assumption is necessary, then, if we wish to incorporate uncertainty into the analysis.

Reconceptualization, moreover, cannot diminish the difference in complexity between our theorems about elections and the real elections themselves. What is a candidate's optimal strategy, for example, when he and his opponent seek *both* votes and finances, when the subset of voter and financial contributors overlap, when the concerns of both subsets overlap, and when the candidates eventually hope to convert dollars into votes? If one assumes that competition occurs between candidates and not parties, as another example, what is the proper role of parties and party structures in a spatial model? Thus, even reconceptualization cannot now include many important elements of elections such as uncertainty, cognitive dissonance, misperception, and the strategies of varying uncertainty and issue saliency. Therefore, one must evaluate the adequacy of spatial models from the perspective of these omissions.

This evaluation entails clarification of a confusion which plagues political research, and which is the second factor accounting for the charge of naivete or unrealism of assumptions. Political scientists commonly confuse the desirable properties of a deductive theory with the properties of an adequate description of reality in terms of such a theory.[36] Theory construction consists of formulating general sentences about reality. The general sentences of one or more theories may be applied to particular classes of real world situations. Thus, science proposes no general "theory of falling feathers," for example, which attempts to predict every twist and turn of a feather's flight. One the other hand, one might attempt to fully explain and predict a feather's path if the environmental conditions are well specified, and if sufficient computer resources are available; and such an effort would undoubtedly utilize existing theory. Nevertheless, the fact that parsimonious propositions about falling feathers cannot be constructed is not interpreted as an inadequacy of Newtonian physics.

For identical reasons, political scientists should not expect theories from which we deduce directly all relevant or interesting aspects of reality. Instead we must differentiate between the process of constructing theories and that of applying them. This distinction necessarily entails differentiating between those facets of reality which we do not conceptualize as elements of our theory and those elements which are simple complex combinations of laws we understand theoretically.

Consider, for example, our assumption that all citizens weight the issues in an identical fashion (or our weaker assumption that these weights are distributed independently of preference). Obviously, this assumption is not satisfied generally. Hence, the critic might reject our analysis or demand that such an assumption be removed. If we remove it, however, no general sentences may be forthcoming —the situation appears to be entirely too unstructured for the construction of law-like generalizations, although our perception may or may not be correct. We can suggest, however, a strategy for empirical research about the positions which candidates are likely to adopt: (1) decompose the electorate into subgroups such that for each subgroup one reasonably might anticipate compliance with the independence assumption (e.g., Pool *et al's* categories);[37] (2) assuming that, for each subgroup, preferences are distributed symmetrically and unimodally, ascertain the mean

[36] For our use of the word *theory* see Carl G. Hempel, *Philosophy of Natural Science* (Englewood Cliffs: Prentice-Hall, 1966). See also, Otto A. Davis, "Notes on Strategy and Methodology for a Scientific Political Science," J. Bernd (ed.), *Mathematical Applications in Political Science, IV* (Charlottesville: University of Virginia Press, 1969).

[37] Ithiel de Sola Pool, Robert P. Abelson, and Samuel Popkin, *Candidates, Issues, and Strategies* (Cambridge: M.I.T. Press, 1964).

preference vector of each such group; (3) for each candidate, ascertain those subgroups he is unlikely or unwilling to satisfy under any circumstances; (4) for each candidate, ascertain those subgroups he is likely to satisfy under any circumstance, and; (5) for the remaining (i.e., pivotal) subgroups find some strategy vector which comes closest to the aggregate mean preferences of these groups. This latter step, admittedly, is ambiguous, and it suggests employing devices such as ascertaining an optimal strategy by trial and error in simulated campaigns.

A second example in which we might combine productively techniques of application such as simulation with abstract theoretical principles concerns the potential conflict between the policy preferences of activists and those of voters.[38] By activists, we mean those citizens who, in addition to their vote, contribute valuable resources, such as finances or a party's nomination, to a candidate. Candidates must accord these citizens additional consideration when formulating strategies. Generally, however, the means of the preference densities of the activists and of the mass electorate do not coincide, so the candidate somehow must compromise his strategies. He might, for instance, attempt simply to assign a weight to each citizen's preference on the basis of the citizen's value, ascertain the mean preference of the weighted population, and adopt this mean. But such *ad hoc* procedures require additional justification. A citizen's value is likely to be a function of the particular resource he contributes, the amount he contributes, and the candidate's opportunities for utilizing this resource—all of which may be functions of the candidate's present strategy which, in turn, is a function of the citizen's value and the weight the candidate assigns him. Within such cyclical relationships we can suggest a few factors candidates must consider (and which we must consider when testing spatial models):

1. the resources various groups of activists can contribute,
2. the preference density of each group and of the mass electorate,
3. the patterns of issue saliency within each group and within the mass electorate,
4. the opportunities for converting each resource into votes.
5. the tradeoffs between resources necessitated by conflicting policy preferences.

This list illustrates only some of the complexity of electoral processes. A deductive approach may be suited to analyzing abstractly the opportunities for converting resources such as finances into resources such as votes and the tradeoffs between resource procurement necessitated by conflicting policy preferences.[39] Candidates, moreover, probably employ simplifying decision rules. But elections are far more complex than falling feathers, so political scientists must be cognizant of the distinction between the processes of constructing theories and those of applying them in particular instances. The important problem is determining what aspects of electoral behavior are amenable to parsimonious deductive examination and what aspects are not susceptible to the development of law-like propositions.

Definitive answers to such questions, of course, are difficult to ascertain, and the absence of adequate research about even a few of

[38] Some practitioners of simulation methods might object to our removing simulations from the class of deductive scientific theories. We agree with Hayward R. Alker's observation that the logical operations of computer simulations are deductive ("Computer Simulation, Conceptual Frameworks and Coalition Behavior," in S. Groennings, et. al. (eds.), *The Study of Coalition Behavior*, forthcoming). But Alker's assertion constitutes a serious confusion of the use of deduction as a method of science with the notion of a deductive theory. In the first usage a deduction is the process of inference from general statements to concrete instances. Thus, one infers that, if all α are β, then a particular α is a β. In the second usage, deduction is the process of inference from general sentence to general sentence. Thus, one infers that if all α are β and if all β are α, then α is equivalent to β. The first kind of deduction is used in fitting models to reality, and, where analysis is complex, is the proper function of stimulations; the second constitutes finding necessary and sufficient relations (i.e., cause) and is the proper domain of abstract mathematics. See, for example, Kenneth Waltz, "Realities, Assumptions, and Simulations," in William D. Coplin (ed.), *Simulation in the Study of Politics* (Chicago: Markham, 1968), and Charles A. Powell's review of Coplin's book, this REVIEW, LXIII (September 1969), p. 937. Perhaps the most fruitful attempt at applying in concert simulation and the generalizations of game theory and coalition theory, and the one which comports with our understanding of the proper uses of simulation—is presented in Coplin's volume by Howard Rosenthal in "Voting and Coalition Models in Election Simulations."

[39] See Gerald Kramer, "A Decision-Theoretic Analysis of a Problem in Political Campaigning," in J. L. Bernd (ed.), *Mathematical Applications in Political Science II* (Dallas: Arnold Foundation, SMU Press, 1966).

the factors we illustrate lessens the value of speculation. Nevertheless, we propose to identify some areas in which a deductive approach (theory *qua* science) might best be applied and some in which a less deductive approach might be more suitable. First, it seems reasonable to suppose that uncertainty can be introduced into the model if we let either the citizens' preference vectors, x, or the candidates' strategy vectors, θ_1 and θ_2, be random variables.[40] Thus, we might examine the strategy of varying the uncertainty associated with a candidate's position and thereby consider situations in which an incumbent's position is known (because his position is the policies he supports while in office) and his opponent's position is a matter for speculation. Similarly, we should be able to construct general propositions concerning the strategy of varying the relative saliencies of issues, and to contrast the efficacy of each strategy—varying uncertainty, saliency, and spatial location.

A citizen's cognitive processes, however, are undoubtedly less amenable to such aggregate analysis, and approaches similar to those which McPhee suggests may be more appropriate.[41] We might incorporate the effects of cognitive dissonance in a deductive analysis, for example, by assuming that the candidates' strategies are restricted and that these restrictions are mathematical functions of issue saliency. As with our assumption concerning a common pattern of saliency, however, an analysis of cognitive dissonance and its effects in heterogeneous electorates may not be susceptible to the development of general law-like propositions.

The inherent limitations of theories also should be kept in mind when analyzing situations in which no dominant position exists. We note previously that dominant positions do not exist for most preference densities—even for idealized situations. So we propose in section 4 a general bound on acceptable strategies for any density, and our analysis of symmetric densities suggests that strategies near or at the median are powerful attractions for candidates in less restricted circumstances. The strategies candidates really do adopt if no dominant position exists, however, are at present a function of factors which are not included in our model. It is reasonable to suppose that some of these factors, such as restrictions on spatial location and the candidates' reaction paths, can be incorporated rigorously into the model. Other factors, such as miscalculation, probably must remain external.

Obviously one can imagine many additional extensions and inherent limitations of our model. We offer these suggestions as an alternative to a banal call for further research and the observation that people develop theory slowly and incrementally. Instead, we identify some extensions which we are currently researching, and we offer some notes of caution to others who seek to develop deductive political theories. These notes of caution are relevant also to those who might attempt to test some of our conclusions or who might attempt naively to draw inferences from the model about reality. One cannot, for example, conclude that two candidates should converge whenever polls reveal that preferences are distributed unimodally. The underlying distribution of activists' preferences may be bimodal and the support of these activists may be essential. If this is the case, it would be unwise for either candidate to set his position equal to the median because to do so would be to alienate this support. Additional complications are easily imagined. Hence, those who would test and develop a theory—as well as those who would criticize it—must recognize the distinction between features of politics which are expressions of general theoretical propositions and those which are complex combinations of these propositions.

[40] For a unidimensional spatial analysis of uncertainty see Kenneth Shepsle, "Essays on Risky Choice in Electoral Competition," (unpublished Doctoral dissertation, University of Rochester, 1970).

[41] William N. McPhee, *Formal Theories of Mass Behavior* (New York: The Free Press, 1963), p. 40.