International Journal of Computational Intelligence Systems, Vol.1, No. 3 (August, 2008), 205 - 214

AN EXTENDED *K*TH-BEST APPROACH FOR REFERENTIAL-UNCOOPERATIVE BILEVEL MULTI-FOLLOWER DECISION MAKING

GUANGQUAN ZHANG, CHENGGEN SHI, JIE LU
Faculty of Information Technology, University of Technology Sydney
PO Box 123, Broadway, NSW 2007, Australia

Email: {zhangg, cshi, jielu}@it.uts.edu.au

Received: 03-01-2008
Revised: 30-06-2008

Bilevel decision techniques have been mainly developed for solving decentralized management problems with decision makers in a hierarchical organization. When multiple followers are involved in a bilevel decision problem, called a bilevel multi-follower (BLMF) decision problem, the leader's decision will be affected, not only by the reactions of these followers, but also by the relationships among these followers. The referential-uncooperative situation is one of the popular cases of BLMF decision problems where these multiple followers don't share decision variables with each other but may take others' decisions as references to their decisions. This paper presents a model for the referential-uncooperative BLMF decision problem. As the *k*th-best approach is one of the most successful approaches in dealing with normal bilevel decision problems, this paper then proposes an extended *k*th-best approach to solve the referential-uncooperative BLMF problem. Finally an example of logistics planning illustrates the application of the proposed extended *k*th-best approach.

Keywords: Bilevel programming, *k*th-best approach, Decision making, Optimization.

1. Introduction

In general, a bilevel decision problem has three important features: (1) there exists two decision units within a predominantly hierarchical structure; (2) the decision unit at the lower level executes its policies after, and in view of, a decision made at the upper level; (3) each unit independently optimizes its objective but is affected by the actions of other unit. The decision unit (decision maker) at the upper level is termed as the leader, and at the lower level, the follower.[1] The leader cannot completely control the decision made by his/her follower but is influenced by the reaction of the follower. The optimal solution of the follower allows the leader to compute his/her objective function's value. Such a decision situation has appeared in many decentralized organizations, and been mainly handled by linear bilevel programming (BLP) technique. A number of bilevel decision approaches and algorithms have been proposed to find an optimal solution for a linear bilevel decision problem, such as the Kuhn-Tucker approach,[2] branch and bound approach,[3] the *k*th-best approach,[4] and others.[1,5,6]

When a bilevel decision problem is described by a linear BLP, at least one optimal (global) solution can be attained at an extreme point of the constraint region. This result was first established by Candler and Townsley[7] with no upper-level constraints and with unique lower level solutions. Afterwards Bard[8] and Bialas and Karwan[9] proved this result under the assumption of that the constraint region is bounded. The result for the case where the upper level constraints exist was established by Savard[10] without any particular assumptions. Based on this result, Candler and Townsley[7] and Bialas and Karwan[9] proposed respectively the *k*th-best approach that computes global solutions of linear BLP problems by enumerating the extreme points of the constraint region. The *k*th-best approach has then been proven to be a valuable analysis tool with a wide range of successful applications for linear BLP.[1,7,9] Our previous work[11-13] extended the *k*th-best approach in handling a more wide range of bilevel decision problems.

In real-world bilevel decision problems, the lower level may involve multiple independent decision units, that is, multiple followers. For example, the CEO of a company is the leader and all directors of branches of this company are the followers in making a product development plan. The leader (the CEO)'s decision will be affected, not only by the reactions of the multiple followers (these directors of branches), but also by the relationships among these followers. We call such a problem a bilevel multi-follower (BLMF) decision problem. These followers may do or may don't share their decision variables, objectives or constraints. For example, those directors may have same objective of maximizing their profits in making the product development plan, but may have different constraints which are based on their individual conditions. Obviously, a BLMF decision problem occurs commonly in any organizational decision practice, and involves many different decision situations which are dependent on the relationships among the followers.

We have established a framework[14] for the BLMF decision problem, where nine main kinds of relationships amongst the followers have been identified. The uncooperative relationship, defined as the case in which there are no shared decision variables among the followers, is the most popular one of BLMF decision problems in practice. This uncooperative relationship can lead to two situations. One is that no follower take any reference from other followers' decisions, related research results have been reported in literature.[2, 15] Anotheruncooperative situation occurs when despite the followers are uncooperative in that there is no sharing of decision variables, they do, however, cross reference information by considering other followers' decision results in each of their own decision objectives and constraints. We call this case as a referential-uncooperative situation, and this paper will particularly focus on this situation. We have developed an extended branch and bound algorithm for solving this problem.[16] This paper further presents an extend *k*th-best approach to more effectively solve this problem.

This paper is organized as follows. In Section 2, a model for the referential-uncooperative situation of a linear BLMF decision problem is presented, and the definition for an optimal solutions and related theorems are given. An extended *k*th best approach for solving the referential-uncooperative BLMF decision problem is proposed in Section 3. A case-based example for the extended *k*th-best approach is illustrated in Section 4. Concluding remarks are given in Section 5.

2. A Model for the Referential-Uncooperative BLMF Decision Problems

A BLMF decision problem has been defined to have two or more followers at the low lever of the bilevel problem. Under this definition, if two followers don't have any shared decision variables, it is called an uncooperative relationship between the two followers. But if one of them has a reference to another follower's decision information in his/her objective or constraints, the two followers are defined as having a referential-uncooperative relationship. When there is a referential-uncooperative relationship in a BLMF decision model, this model is called a referential-uncooperative BLMF decision model. We present this model as follows.

For $x \in X \subset R^n$, $y_i \in Y_i \subset R^{m_i}$, $F: X \times Y_1 \times \cdots \times Y_K \to R^1$, and $f_i : X \times Y_1 \times \ldots \times Y_K \to R^1$, $i = 1,2,\ldots,K$, a linear BLMF decision problem where $K(\geq 2)$ followers are involved and there are no shared decision variables, but shared information in objective functions and constraint functions among the followers which is defined as follows:

$$\min_{x \in X} F(x, y_1, \ldots, y_K) = cx + \sum_{s=1}^{K} d_s y_s \tag{1a}$$

$$\text{subject to } Ax + \sum_{s=1}^{K} B_s y_s \leq b \tag{1b}$$

$$\min_{y_i \in Y_i} f_i(x, y_1, \ldots, y_K) = c_i x + \sum_{s=1}^{K} e_{is} y_s \tag{1c}$$

$$\text{subject to } A_i x + \sum_{s=1}^{K} C_{is} y_s \leq b_i \tag{1d}$$

where $c \in R^n$, $c_i \in R^n$, $d_i \in R^{m_i}$, $e_{is} \in R^{m_s}$, $b \in R^p$, $b_i \in R^{q_i}$, $A \in R^{p \times n}$, $B_i \in R^{p \times m_i}$, $A_i \in R^{q_i \times n}$, $C_{is} \in R^{q_i \times m_s}$, $i, s = 1,2,\ldots,K$.

To find an optimal solution for this model (1a)-(1d), we introduce definitions of constraint region, projection of S onto the leader's decision space, feasible set for each follower, and inducible region for a linear BLMF decision problem in Definition 1.

Definition 1

(a) Constraint region of a linear BLMF decision problem:

$$S=\{(x,y_1,\ldots,y_K)\in X\times Y_1\times\ldots\times Y_k,\\ Ax+\sum_{s=1}^{K}B_s y_s\le b,\ A_i x+\sum_{s=1}^{K}C_{is}y_s\le b_i,\ i=1,2,\ldots,K\}.$$

The constraint region refers to all possible combinations of choices that the leader and followers may make.

(b) Projection of S onto the leader's decision space:

$$S(X)=\{x\in X:\exists y_i\in Y_i, Ax+\sum_{s=1}^{K}B_s y_s\le b,\\ A_i x+\sum_{s=1}^{K}C_{is}y_s\le b_i, i=1,2,\ldots,K\}$$

(c) Feasible set for each follower

$$\forall x\in S(X): S_i(x)=\{y_i\in Y_i:(x,y_1,\ldots,y_K)\in S\}.$$

The feasible region for each follower is affected by the leader's choice of x, and the allowable choices of each follower are the elements of S.

(d) Each follower's rational reaction set for $x\in S(X)$:

$$P_i(x)=\{y_i\in Y_i: y_i\in \arg\min[f_i(x,\hat{y}_i,y_j), j=1,2,\ldots,K,\\ j\ne i:\ \hat{y}_i\in S_i(x)]\}.$$

where $i=1,2,\ldots,K$,

$$\arg\min[f_i(x,\hat{y}_i,y_j):\hat{y}_i\in S_i(x)]=\{y_i\in S_i(x): f_i(x,y_1,\\ \ldots,y_K)\le f_i(x,\hat{y}_i,y_j),\ j=1,2,\ldots,K, j\ne i,\hat{y}_i\in S_i(x)\}$$

The followers observe the leader's action and simultaneously react by selecting y_i from their feasible set to minimize their objective function.

(e) Inducible region:

$$IR=\{(x,y_1,\ldots,y_K):(x,y_1,\ldots,y_K)\in S,\\ y_i\in P_i(x), i=1,2,\ldots,K\}$$

Thus the model given by expressions (1a)-(1d) can be rewritten in terms of the above notations as follows

$$\min\{F(x,y_1,\ldots,y_K):(x,y_1,\ldots,y_K)\in IR\} \qquad (2)$$

We propose the following theorem to characterize the condition under which there is an optimal solution for a referential-uncooperative linear BLMF decision problem shown in (1a)-(1d).

Theorem 1 If S is nonempty and compact, there exists an optimal solution for a linear BLMF decision problem.

Proof: Since S is nonempty, there exist a point $(x^*,y_1^*,\ldots,y_K^*)\in S$. Then, we have

$$x^*\in S(X)\ne\varnothing$$

by Definition 1(b). Consequently, we have

$$S_i(x^*)\ne\varnothing,\ i=1,2,\ldots,K$$

by Definition 1(c). Because S is compact and Definition 1(d), we have

$$P_i(x^*)=\{y_i\in Y_i: y_i\in\arg\min[f_i(x^*,\hat{y}_i,y_j), j=1,2,\ldots,K,\\ j\ne i:\hat{y}_i\in S_i(x^*)]\}=\{y_i\in Y_i: y_i\in\{y_i\in S_i(x^*): f_i(x^*,\\ y_1,\ldots,y_K)\le f_i(x^*,\hat{y}_i,y_j), j=1,2,\ldots,K, j\ne i),\\ \hat{y}_i\in S_i(x^*)\}\}\ne\varnothing$$

where $i=1,2,\ldots,K$. Hence, there exists $y_i^0\in P_i(x^*)$, $i=1,2,\ldots,K$ such that $(x^*,y_1^0,\ldots,y_K^0)\in S$. Therefore, we have:

$$IR=\{(x,y_1,\ldots,y_K):(x,y_1,\ldots,y_K)\in S,\\ y_i\in P_i(x), i=1,2,\ldots,K\}\ne\varnothing$$

by Definition 1(e). Because we are minimizing a linear function $\min_{x\in X}F(x,y_1,\ldots,y_K)=cx+\sum_{s=1}^{K}d_s y_s$ over IR, which is nonempty and bounded an optimal solution to the linear BLMF decision problem must exist. □

3. An Extended *k*th-Best Approach for the Referential-Uncooperative BLMF Decision Problems

We first give a set of related properties in this section. Based on the set of properties an extended *k*th-best approach for solving referential-uncooperative decision problems is presented.

Theorem 2 The inducible region can be written equivalently as a piecewise linear equality constraint comprised of supporting hyper planes of constraint region S.

Proof: Let us begin by writing the inducible region of Definition 1(e) explicitly as follower:

$$IR = \{(x, y_1, \ldots, y_K) : (x, y_1, \ldots, y_K) \in S,$$
$$e_{ii} y_i = \min[e_{ii}\tilde{y}_i : B_i\tilde{y}_i \le b - Ax - \sum_{s=1, s\ne i}^{K} B_s y_s,$$
$$C_{ji}\tilde{y}_i \le b_i - A_i x - \sum_{s=1, s\ne i}^{K} C_{is} y_s,$$
$$j = 1,2,\ldots,K, \tilde{y}_i \ge 0], i = 1,2,\ldots,K\}$$

Let us define:

$$b' = (b, b_1, \ldots, b_K)^T,\ A' = (A, A_1, \ldots, A_K)^T,$$
$$B_i' = (B_i, C_{1i}, \ldots, C_{Ki})^T,$$

where $i = 1,2,\ldots,K$

Now we have

$$IR = \{(x, y_1, \ldots, y_K) : (x, y_1, \ldots, y_K) \in S,$$
$$e_{ii} y_i = \min[e_{ii}\tilde{y}_i : B_i'\tilde{y}_i \le b_i' - A'x - \sum_{s=1, s\ne i}^{K} B_s' y_s, \tilde{y}_i \ge 0],$$
$$i = 1,2,\ldots,K\} \quad (3)$$

Let us define:

$$Q_i(x, y_j) = \min[e_{ii}\tilde{y}_i : B_i'\tilde{y}_i \le b_i' - A'x - \sum_{s=1, s\ne i}^{K} B_s' y_s, \tilde{y}_i \ge 0] \quad (4)$$

where $i = 1,2,\ldots,K,\ j = 1,2,\ldots K, j \ne i.$

For each value of $x \in S(X)$, the resulting feasible region to problem (1) is nonempty and compact. Thus, for Q_i, which is a linear program parameterized in x, y_j, $j = 1,2,\ldots,K$ and $j \ne i$, always has a solution. From duality theory we get

$$\max\{u(A'x + \sum_{s=1, s\ne i}^{K} B_s' y_s - b_i') : uB_i' \ge -e_{ii}, u \ge 0\} \quad (5)$$

which has the same optimal value as (4) at the solution u^*. Let $u^1, \ldots, u^s$ be a listing of all the vertices of the constraint region of (5) given by $U = \{u : uB_i' \ge -e_{ii}, u \ge 0\}$. Because we know that a solution to (5) occurs at a vertex of *U*, we get the equivalent problem:

$$\max\{u^l(A'x + \sum_{s=1, s\ne i}^{K} B_s' y_s - b_i') : u^l \in \{u^1, \ldots, u^s\}\} \quad (6)$$

which demonstrates that $Q_i(x, y_j)$ is a piecewise linear function.

Rewriting *IR* as:

$$IR = \{(x, y_1, \ldots, y_k) \in S : Q_i(x, y_j) - e_{ii} y_i = 0$$
$$i = 1,2,\ldots,K, j = 1,2,\ldots,K, j \ne i\} \quad (7)$$

yields desired result. □

Corollary 1 The problem (1) is equivalent to minimizing F over a feasible region comprised of a piecewise linear equality constraint.

Proof: By (2) and Theorem 2, we have the desired result.

Each function Q_i defined by (4) is convex and continuous. In general, because we are minimizing a linear function $F = cx + \sum_{s=1}^{K} d_s y_s$ over *IR*, and because F is bounded below S by, say, $\min\{cx + \sum_{s=1}^{K} d_s y_s : (x, y_1, \ldots, y_K) \in S\}$, the following can be concluded.

Corollary 2 A solution for the linear BLMF decision problem occurs at a vertex of *IR*.

Proof: A linear BLMF decision problem can be written as in (2). Since $F = cx + \sum_{s=1}^{K} d_s y_s$ is linear, if a solution exists, one must occur at a vertex of *IR*. □

Theorem 3 The solution $(x^*, y_1^*, \ldots, y_K^*)$ of the linear BLMF decision problem occurs at a vertex of *S*.

Proof: Let $(x^1, y_1^1, \ldots, y_K^1), \ldots, (x^r, y_1^r, \ldots, y_K^r)$ be the distinct vertices of *S*. Since any point in *S* can be written a convex combination of these vertices, let $(x^*, y_1^*, \ldots, y_K^*) = \sum_{j=1}^{\bar{r}} \alpha_j (x^j, y_1^j, \ldots, y_K^j)$, where $\sum_{j=1}^{\bar{r}} \alpha_j = 1$, $\alpha_j \ge 0$, $j = 1,2,\ldots,\bar{r}$ and $\bar{r} \le r$. It must be shown that $\bar{r} = 1$. To see this let us write the constraints to (1) at $(x^*, y_1^*, \ldots, y_K^*)$ in their piecewise linear form (7).

$$0 = Q_i(x, y_l^*) - e_{ii} y_i^*$$
$$i = 1,2,\ldots,K,\ l = 1,2,\ldots,K, l \ne i \quad (8)$$

Rewrite (8) as follows:

$$0 = Q_i(\sum_j \alpha_j (x^j, y_l^j)) - e_{ii}(\sum_j \alpha_j y_i^j)$$
$$\le \sum_j \alpha_i Q_i(x^j, y_l^j) - \sum_j \alpha_j e_{ii} y_i^j$$

where $i = 1,2,\ldots,K$, $l = 1,2,\ldots,K, l \ne i$

By convexity of $Q_i(x, y_l)$, we have

$$0 \le \sum_j \alpha_j (Q_i(x^j, y_l^j) - e_{ii} y_i^j)$$

where $i = 1,2,\ldots,K$, $l = 1,2,\ldots,K, l \ne i$.

But by the definition,

$$Q_i(x^j, y_l^j) = \min_{y_i \in S(x^j)} e_{ii} y_i \le e_{ii} y_i^j,$$
$$i = 1,2,\ldots,K,\ l = 1,2,\ldots,K, l \ne i.$$

Therefore, $Q_i(x^j, y_l^j) - e_{ii} y_i^j \le 0$, $j = 1,2,\ldots,\bar{r}$, $i = 1,2,\ldots,K$, $l = 1,2,\ldots,K, l \ne i$. Noting that $\alpha_j \ge 0$, $j = 1, 2,\ldots,\bar{r}$, the equality in the preceding expression must hold or else a contradiction would result in the sequence above.

Consequently, $Q_i(x^j, y_l^j) - e_{ii} y_i^j = 0$, $j = 1,2,\ldots,\bar{r}$, $i = 1,2,\ldots,K$, $l = 1,2,\ldots,K, l \ne i$. This implies that $(x^j, y_1^j, \ldots, y_K^j) \in IR$, $j = 1,2,\ldots,\bar{r}$ and $(x^*, y_1^*, \ldots, y_K^*)$ can be written as a convex combination of points in IR. Because $(x^*, y_1^*, \ldots, y_K^*)$ is a vertex of IR, a contradiction results unless $\bar{r} = 1$. □

Corollary 3 If x is an extreme point of *IR*; it is an extreme point of *S*.

Proof: Let $(x^1, y_1^1, \ldots, y_K^1), \ldots, (x^r, y_1^r, \ldots, y_K^r)$ be the distinct vertices of *S*. Since any point in *S* can be written a convex combination of these vertices, let $(x^*, y_1^*, \ldots, y_K^*) = \sum_{j=1}^{r} \alpha_j (x^j, y_1^j, \ldots, y_K^j)$, where $\sum_{j=1}^{r} \alpha_j = 1$, $\alpha_j \ge 0, j = 1,2,\ldots,\bar{r}$ and $\bar{r} \le r$. It must be shown that $\bar{r} = 1$. To see this let us write the constraints to (1) at $(x^*, y_1^*, \ldots, y_K^*)$ in their piecewise linear form (7).

$$0 = Q_i(x, y_l^*, l = 1,2,\ldots,K, l \ne i) - e_{ii} y_i^* \quad i = 1,2,\ldots,K$$

Rewrite the above formulation as follows:

$$0 = Q_i\Big(\sum_j \alpha_j (x^j, y_l^j)\Big) - e_{ii}\Big(\sum_j \alpha_j y_i^j\Big)$$
$$\le \sum_j \alpha_i Q_i(x^j, y_l^j) - \sum_j \alpha_j e_{ii} y_i^j$$

where $i = 1,2,\ldots,K$, $l = 1,2,\ldots,K, l \ne i$.

By convexity of $Q_i(x, y_l)$, we have:

$$0 \le \sum_j \alpha_j (Q_i(x^j, y_l^j) - e_{ii} y_i^j)$$

where $i = 1,2,\ldots,K, l = 1,2,\ldots,K, l \ne i$.

But by the definition,

$$Q_i(x^j, y_l^j) = \min_{y_i \in S(x^j)} e_{ii} y_i \le e_{ii} y_i^j$$
$$i = 1,2,\ldots,K,\ l = 1,2,\ldots,K, l \ne i$$

Therefore, $Q_i(x^j, y_l^j) - e_{ii} y_i^j \le 0$, $j = 1,2,\ldots,\bar{r}$, $i = 1,2,\ldots,K$, $l = 1,2,\ldots,K, l \ne i$. Noting that $\alpha_j \ge 0$, $j = 1,2,\ldots,\bar{r}$, the equality in the preceding expression must hold or else a contradiction would result in the sequence above. Consequently, $Q_i(x^j, y_l^j) - e_{ii} y_i^j = 0$, $j = 1,2\ldots,\bar{r}$, $i = 1,2,\ldots,K$, $l = 1,2,\ldots,K$, $l \ne i$. This implies that $(x^j, y_1^j, \ldots, y_K^j) \in IR$, $j = 1,2,\ldots,\bar{r}$ and $(x^*, y_1^*, \ldots, y_K^*)$ can be written as a convex combination of points in *IR*. Because $(x^*, y_1^*, \ldots, y_K^*)$ is a vertex of *IR*, a contradiction results unless $\bar{r} = 1$. This means that $(x^*, y_1^*, \ldots, y_K^*)$ is an extreme point of S. □

Theorem 3 and Corollary 3 have provided theoretical foundation for a new algorithm used in our extended *k*th-best approach. It means that by searching extreme points on the constraint region *S*, we can efficiently find an optimal solution for a linear BLMF decision problem. The basic idea of the algorithm is that according to the objective function of the upper level, we arrange all the extreme points in S in a descending order, and select the first extreme point to check if it is on the inducible region IR. If yes, the current extreme point is the optimal solution. Otherwise, the next one will be selected and checked.

More specifically, let $(x^1, y_1^1, \ldots, y_K^1)$, …, $(x^N, y_1^N, \ldots, y_K^N)$ denote the N ordered extreme points to the linear BLMF decision problem

$$\min\{cx + \sum_{s=1}^{K} d_s y_s : (x, y_1, \ldots, y_K) \in S\} \tag{9}$$

such that:

$$cx^j + \sum_{s=1}^{K} d_s y_s^j \le cx^{j+1} + \sum_{s=1}^{K} d_s y_s^{j+1},\ j = 1,2,\ldots,N-1.$$

Let $(\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_K)$ denote the optimal solution to the following problem

$$\min\{f_i(x^j, y_1, \ldots, y_K) : y_i \in S_i(x^j),\ i = 1,2,\ldots,K\} \tag{10}$$

We only need to find the smallest j£ $j=1,2,\ldots,N$ under which $y_i^j=\tilde{y}_i$, $i=1,2,\ldots,K$.

Let us write (10) as follows:

$$\begin{aligned}&\min f_i(x,y_1,\ldots,y_K)\\&\text{subject to } y_i\in S(x)\\&x=x^j\end{aligned}$$

where $i=1,2,\ldots,K$.

We only need to find the smallest j under which $y_i^j=\tilde{y}_i$, $i=1,2,\ldots,K$.

From Definition 1(b), we rewrite (10) as follows:

$$\min f_i(x,y_1,\ldots,y_K)=c_i x+\sum_{s=1}^{K}e_{is}y_s \qquad (11a)$$

$$\text{subject to } Ax+\sum_{s=1}^{K}B_s y_s\le b \qquad (11b)$$

$$A_l x+\sum_{s=1}^{K}C_{ls}y_s\le b_l,\ l=1,2,\ldots,K \qquad (11c)$$

$$x=x^j \qquad (11d)$$

$$y_1\ge 0, y_2\ge 0,\ldots,y_K\ge 0, \qquad (11e)$$

where $i=1,2,\ldots,K$.

Solving this problem is equivalent to select one ordered extreme point $(x^j,y_1^j,\ldots,y_K^j)$ and then solve (11) to obtain the optimal solution $\tilde{y}_i$. If for all i we have $y_i^j=\tilde{y}_i$, then $(x^j,y_1^j,\ldots,y_K^j)$ is the global optimum to (1a)-(1d). Otherwise, check next extreme point.

Based on the results obtained from above procedure, an extended *k*th-best approach which can solve a referential-uncooperative BLMF decision problem is described as follows.

Step 1 Put $j\leftarrow 1$. Solve (9) with the simplex method to obtain an optimal solution $(x^1,y_1^1,\ldots,y_K^1)$. Let $W=(x^1,y_1^1,\ldots,y_K^1)$ and $T=\varnothing$. Go to Step 2.

Step 2 Solve (11) with the bounded simplex method. Let $\tilde{y}_i$ denote the optimal solution to (11). If $y_i^j=\tilde{y}_i$ for all i, $i=1,\ldots,K$, $(x^j,y_1^j,\ldots,y_K^j)$ is the global optimum to (1a)-(1d). Otherwise, go to Step 3.

Step 3 Let $W_{[j]}$ denote the set of adjacent extreme points of $(x^j,y_1^j,\ldots,y_K^j)$ such that $(x,y_1,\ldots,y_K)\in W_{[j]}$ implies $cx+\sum_{s=1}^{K}d_s y_s\le cx^j+\sum_{s=1}^{K}d_s y_s^j$. Let $T=T\cup\{(x^j,y_1^j,\ldots,y_K^j)\}$ and $W=(W\cup W_{[i]})/T$. Go to Step 4.

Step 4 Set $j\leftarrow j+1$ and choose $(x^j,y_1^j,\ldots,y_K^j)$ so that $cx^j+\sum_{s=1}^{K}d_s y_s^j=\min\{cx+\sum_{s=1}^{K}d_s y_s : (x,y_1,\ldots,y_K)\in W\}$. Go back to Step 2.

The extended *k*th-best approach is easy to be used to solve a linear referential-uncooperative BLMF decision problem.

4. An Example of Logistics Management

This section first presents a logistics planning problem modeled as a referential-uncooperative BLMF decision problem. It then shows how the proposed extended *k*th-best approach is used for solving the problem.

A logistics chain often involves a series of units such as supplier and distributor. All the units involved in the chain are interrelated in a way that a decision made at one unit affects the performance of next unit(s). In the meantime, when one unit tries to optimize its objective, it may need to consider the objective of next unit, and its decision will be affected by the next unit's reaction as well. Both supplier and distributor, two important units in a logistics chain, have their own objectives such as to maximize their benefits and minimize their costs; constraints such as time, locations and facilities; and variables such as prices. For each of possible decision made by the supplier, the distributor finds a way to optimize his/her objective value. The optimal solution of the distributor allows the supplier to compute his/her objective function's value. As the main purpose of making a logistics plan is to optimize the supplier's objective function's value, the supplier is the leader, and the distributor is the follower in the case.

We assume that there are two kinds of distributors A and B in this case. They have their own decision variables, objectives and constraints. But they have cross reference of information by considering other followers' decision results in each of their own decision objective and constraint. For example, distributor A considers the price of transportation of distributor B. We therefore establish a referential-cooperative BLMF model for this problem.

For $x\in X\subset R^n$ the supplier's (leader's) decision variable, $y\in Y\subset R^m$ the distributor A's (follower A's) decision variable, $z \in Z \subset R^m$ the distributor B's (follower B's) decision variable, $F : X \times Y \to R^1$ the supplier's objective function, and $f_1 : X \times Y \to R^1$ and $f_2 : X \times Y \to R^1$ the distributor A's and distributor B's objective functions respectively. In order to easily show the use for the proposed *k*th-best approach, the logistics planning problem is simplified into $X = \{x \ge 0\}$, $Y = \{y \ge 0\}$, $Z = \{z \ge 0\}$ with $x \in R^1$, $y \in R^1$, $z \in R^1$. The supplier's objective is to minimize, over the set *X*, the total transportation cost of the system described by min$F(x,y,z)$. The distributor A seeks to minimize his/her transportation time delay described by $\min f_1(x,y,z)$ over the set *Y*, and the distributor B by $\min f_2(x,y,z)$ over the set *Z*. Although the two kinds of distributors have different decision variables, decision objective and constraints, but each of them takes other's decision variable into their objective and constraints as references. This is a typical referential-uncooperative BLMF decision problem. The problem's model is presented as follows:

$$\begin{aligned}
&\min_{x \in X} F(x,y,z) = -x + 2y + 3z \\
&\text{subject to } x \ge 1 \\
&\qquad z \le 1 \\
&\qquad \min_{y \in Y} f_1(x,y,z) = x - y + z \\
&\qquad \text{subject to } x + y + z \ge 1 \\
&\qquad\qquad y \le 1 \\
&\qquad \min_{z \in Z} f_2(x,y,z) = x + y - z \\
&\qquad \text{subject to } x + y + z \le 8 \\
&\qquad\qquad x \le 2 .
\end{aligned}$$

According to the extended *k*th-best approach, this model can be rewritten in the format of (9) as follows:

$$\begin{aligned}
&\min F(x,y,z) = -x + 2y + 3z \\
&\text{subject to } x \ge 1 \\
&\qquad z \le 1 \\
&\qquad x + y + z \ge 1 \\
&\qquad y \le 1 \\
&\qquad x + y + z \le 8 \\
&\qquad x \le 2 \\
&\qquad x \ge 0 \\
&\qquad y \ge 0 \\
&\qquad z \ge 0 .
\end{aligned}$$

Now we go through this extended *k*th-best approach from Step 1 to Step 4.

In Step 1, set $j = 1$, and solve above problem with the simplex method to obtain an optimal solution $(x_{[1]}, y_{[1]}, z_{[1]}) = (2,0,0)$. Let $W = \{(2,0,0)\}$ and $T = \phi$. Go to Step 2.

In the Loop 1:

Setting $i \leftarrow 1$ and by (11), we have:

$$\begin{aligned}
&\min f_1(x,y,z) = x - y + z \\
&\text{subject to } x \ge 1 \\
&\qquad z \le 1 \\
&\qquad x + y + z \ge 1 \\
&\qquad y \le 1 \\
&\qquad x + y + z \le 8 \\
&\qquad x \le 2 \\
&\qquad x = 2 \\
&\qquad y \ge 0 \\
&\qquad z \ge 0 .
\end{aligned}$$

Using the bounded simplex method, we have $\tilde{y}_j = 1$. Because of $\tilde{y}_j \ne y_{[j]}$, we go to Step 3 and then have $W_{[j]} = \{(1,0,0),(2,1,0),(2,0,1)\}$, $T = \{(2,0,0)\}$ and $W = \{(1,0,0), (2,1,0),(2,0,1)\}$. We then go to Step 4. Update $j = 2$, and choose $(x_{[j]}, y_{[j]}, z_{[j]}) = (1,0,0)$, go back to Step 2.

In the Loop 2:

Setting $i \leftarrow 1$ and by (11), we have

$$\begin{aligned}
&\min f_1(x,y,z) = x - y + z \\
&\text{subject to } x \ge 1 \\
&\qquad z \le 1 \\
&\qquad x + y + z \ge 1 \\
&\qquad y \le 1 \\
&\qquad x + y + z \le 8 \\
&\qquad x \le 2 \\
&\qquad x = 1 \\
&\qquad y \ge 0 \\
&\qquad z \ge 0 .
\end{aligned}$$

Same as loop1, by using the bounded simplex method, we have $\tilde{y}_j = 1$. Because of $\tilde{y}_j \ne y_{[j]}$, we go to Step 3, and obtain:

$$\begin{aligned}
&W_{[j]} = \{(2,0,0),(1,1,0),(1,0,1)\} \\
&T = \{(2,0,0),(1,0,0)\} \\
&W = \{(2,1,0),(2,0,1),(1,1,0),(1,0,1)\}
\end{aligned}$$

Then go to Step 4. Update $j = 3$, and choose $(x_{[j]}, y_{[j]}, z_{[j]}) = (2,1,0)$, then go to Step 2 again.

In the Loop3:

Setting $i \leftarrow 1$ and we have by (11):

$$\begin{aligned} \min f_1(x,y,z) &= x - y + z \\ \text{subject to } & x \geq 1 \\ & z \leq 1 \\ & x + y + z \geq 1 \\ & y \leq 1 \\ & x + y + z \leq 8 \\ & x \leq 2 \\ & x = 2 \\ & y \geq 0 \\ & z \geq 0. \end{aligned}$$

Through using the bounded simplex method, we obtain $\tilde{y}_j = 1$ and $\tilde{y}_j = y_{[j]}$. This is a different situation from last loop. We thus set $i \leftarrow i+1$ and have a new expression of distributor's function f_2 by (11):

$$\begin{aligned} \min f_2(x,y,z) &= x + y - z \\ \text{subject to } & x \geq 1 \\ & z \leq 1 \\ & x + y + z \geq 1 \\ & y \leq 1 \\ & x + y + z \leq 8 \\ & x \leq 2 \\ & x = 2 \\ & y \geq 0 \\ & z \geq 0. \end{aligned}$$

Same as before, by using the bounded simplex method again, we have $\tilde{z}_j = 1$. Because $\tilde{z}_j \neq z_{[j]}$, we go to Step 3, and have:

$$W_{[j]} = \{(2,0,0),(1,1,0),(2,1,1)\}$$
$$T = \{(2,0,0),(1,0,0),(2,1,0)\}$$
$$W = \{(2,0,1),(1,1,0),(1,0,1),(2,1,1)\}.$$

We then go to Step 4. Updating $j = 4$ and choosing $(x_{[j]}, y_{[j]}, z_{[j]}) = (2,0,1)$, then we go back to Step 2.

In Loop 4:

Setting $i \leftarrow 1$ and we have by (11):

$$\begin{aligned} \min f_1(x,y,z) &= x - y + z \\ \text{subject to } & x \geq 1 \\ & z \leq 1 \\ & x + y + z \geq 1 \\ & y \leq 1 \\ & x + y + z \leq 8 \\ & x \leq 2 \\ & x = 2 \\ & y \geq 0 \\ & z \geq 0 \end{aligned}$$

Through using the bounded simplex method, we obtain $\tilde{y}_j = 1$ and $\tilde{y}_j \neq y_{[j]}$. We go to Step 3, have:

$$W_{[j]} = \{(2,0,0),(1,0,1),(2,1,1)\}$$
$$T = \{(2,0,0),(1,0,0),(2,1,0),(2,0,1)\}$$
$$W = \{(1,1,0),(1,0,1),(2,1,1)\}$$

We then go to Step 4. Updating $j = 5$ we get $(x_{[j]}, y_{[j]}, z_{[j]}) = (1,1,0)$

In Loop 5:

Setting $i \leftarrow 1$ and we have by (11):

$$\begin{aligned} \min f_1(x,y,z) &= x - y + z \\ \text{subject to } & x \geq 1 \\ & z \leq 1 \\ & x + y + z \geq 1 \\ & y \leq 1 \\ & x + y + z \leq 8 \\ & x \leq 2 \\ & x = 1 \\ & y \geq 0 \\ & z \geq 0 \end{aligned}$$

Through using the bounded simplex method, we obtain $\tilde{y}_j = 1$ and $\tilde{y}_j = y_{[j]}$. We set $i = i + 1$, have:

$$\begin{aligned} \min f_2(x,y,z) &= x + y - z \\ \text{subject to } & x \geq 1 \\ & z \leq 1 \\ & x + y + z \geq 1 \\ & y \leq 1 \\ & x + y + z \leq 8 \\ & x \leq 2 \\ & x = 1 \\ & y \geq 0 \\ & z \geq 0 \end{aligned}$$

We have: $\tilde{z}_j = 1$, $\tilde{z}_j \neq z_{[j]}$, go to Step 3, we have:

$$W_{[j]} = \{(1,0,0),(1,1,1),(2,1,0)\}$$
$$T = \{(2,0,0),(1,0,0),(2,1,0),(2,0,1),(1,1,0\}$$
$$W = \{(1,0,1),(2,1,1),(1,1,1)\}$$

We then go to Step 4. Updating $j=5$ we get $(x_{[j]}, y_{[j]}, z_{[j]}) = (1,0,1)$.

In Loop 6:

Setting $i \leftarrow 1$ and we have by (11):

$$\begin{aligned} \min f_1(x,y,z) &= x - y + z \\ \text{subject to } & x \geq 1 \\ & z \leq 1 \\ & x+y+z \geq 1 \\ & y \leq 1 \\ & x+y+z \leq 8 \\ & x \leq 2 \\ & x=1 \\ & y \geq 0 \\ & z \geq 0 \end{aligned}$$

Through using the bounded simplex method, we obtain $\tilde{y}_j = 1$ and $\tilde{y}_j \neq y_{[j]}$. We go to Step 3, have:

$$W_{[j]} = \{(1,0,0),(1,1,1),(2,0,1)\}$$
$$T = \{(2,0,0),(1,0,0),(2,1,0),(2,0,1),(1,1,0),(1,0,1)\}$$
$$W = \{(2,1,1),(1,1,1)\}$$

We then go to Step 7. Updating $j=5$ we get $(x_{[j]}, y_{[j]}, z_{[j]}) = (2,1,1)$

In Loop 7:

Setting $i \leftarrow 1$ and we have by (11):

$$\begin{aligned} \min f_1(x,y,z) &= x - y + z \\ \text{subject to } & x \geq 1 \\ & z \leq 1 \\ & x+y+z \geq 1 \\ & y \leq 1 \\ & x+y+z \leq 8 \\ & x \leq 2 \\ & x=2 \\ & y \geq 0 \\ & z \geq 0 \end{aligned}$$

Through using the bounded simplex method, we obtain $\tilde{y}_j = 1$ and $\tilde{y}_j = y_{[j]}$. We set $i = i + 1$, have:

$$\begin{aligned} \min f_2(x,y,z) &= x - y + z \\ \text{subject to } & x \geq 1 \\ & z \leq 1 \\ & x+y+z \geq 1 \\ & y \leq 1 \\ & x+y+z \leq 8 \\ & x \leq 2 \\ & x = 2 \\ & y \geq 0 \\ & z \geq 0 \end{aligned}$$

We have: $\tilde{z}_j = 1$, $\tilde{z}_j = z_{[j]}$. Go to Step 4, we have: $(x_{[j]}, y_{[j]}, z_{[j]}) = (2,1,1)$.

It has been found that from loop 7 that the optimal solution of the referential-uncooperative BLMF problem occurs at the point $(x^*, y^*, z^*) = (2,1,1)$ with the leader's objective value $F^* = 3$, and two followers' objective values $f_1^* = 2$ and $f_2^* = 2$ respectively.

5. Conclusions and Further Study

A referential-uncooperative BLMF decision problem occurs commonly in management and planning of many organizations. For solving such a BLMF decision problem, this paper extended the *k*th-best approach from dealing with simple one-leader-and-one-follower situation to complex referential-uncooperative multiple followers' situation. This paper further illustrated the details of the proposed approach by an example of logistic planning problems. Initial experiment results showed that this extended *k*th-best approach can effectively solve the proposed BLMF decision problem. Some practical use of this extended approach will be considered as our future research task for BLMF decision making.

Acknowledgements

The work presented in this paper was supported by Australian Research Council (ARC) under discovery grant DP0557154.

References

1. J. Bard, *Practical bilevel optimization: algorithms and applications* (Kluwer Academic Publishers, USA, 1998).
2. J. Bard and J. Falk, An explicit solution to the multi-level programming problem, *Computers and Operations Research*, 9(1982) 77-100.
3. J. Bard and J. Moore, A branch and bound algorithm for the bilevel programming problem, *SIAM Journal of Scientific and Statistical Computing,* 11 (1990) 281-292.
4. W. Bialas and M. Karwan, On two-level optimization, *IEEE Tran. Automatic Control*, AC-27 (1982) 211-214.
5. P. Hansen, B. Jaumard, and G. Savard, *New branch-and-bound rules for linear bilevel programming,* Journal on Scientific and Statistical Computing **13** (1992) 1194-1217.

6. D. White and G. Anandalingam, A penalty function approach for solving bi-level linear programs, *Journal of Global Optimization* **3** (1993) 397-419.
7. W. Candler and R. Townsley, A linear two-level programming problem, *Computers and Operations Research* **9** (1982) 59-76.
8. J. Bard, An investigation of the linear three level programming problems, *IEEE Transactions on Systems, Man, and Cybernetics* **14** (1984) 711-717.
9. W. Bialas and M. Karwan, Two-level linear programming, *Management Science* **30** (1984) 1004-1020.
10. G. Savard, *Contributions á la programmation mathématique á deux niveaux*, PhD thesis (Université de Montréal, École Polytechnique, 1989).
11. C. Shi, J, Lu and G. Zhang, An extended *K*th-best approach for linear bilevel programming, *Applied Mathematics and Computation* **164** (2005) 843-855.
12. C, Shi, G. Zhang and J. Lu, On the definition of linear bilevel programming solution, *Applied Mathematics and Computation* **160** (2005) 169-176.
13. C. Shi, G. Zhang and J. Lu, A *K*th-bast approach for linear bilevel multi-follower programming, *Journal of Global Optimization* **33** (4) (2005) 563-578
14. J. Lu, C. Shi and G. Zhang, On bilevel multi-follower decision-making: general framework and solutions, *Information Science* **176** (2006) 1607-1627
15. J. Lu, C. Shi, G. Zhang and D. Ruan, Multi-follower Linear Bilevel Programming: Model and Kuhn-Tucker Approach, Applied Computational Intelligence, in *Proceedings of the IADIS International Conference in Applied Computing* (Portugal, Algarve, 2005), pp. 81-88.
16. J. Lu, C. Shi, G. Zhang and D. Ruan, An extended branch and bound algorithm for bilevel multi-follower decision making in a referential-uncooperative situation, *International Journal of Information Technology and Decision Making* **6** (2) (2007) 371-388