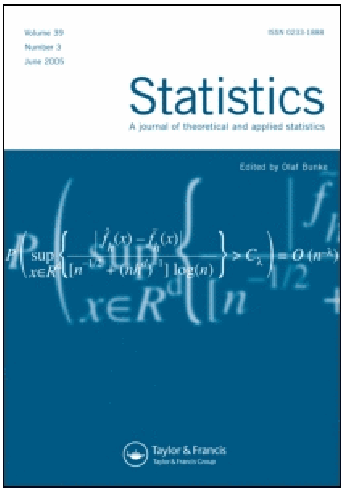

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An extended Lomax distribution

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A new five-parameter continuous distribution, the so-called McDonald Lomax distribution, that extends the Lomax distribution and some other distributions is proposed and studied. The model has as special sub-models new four- and three-parameter distributions. Various structural properties of the new distribution are derived, including expansions for the density function, explicit expressions for the moments, generating and quantile functions, mean deviations and Rényi entropy. The score function is derived and the estimation is performed by maximum likelihood. We also obtain the observed information matrix. An application illustrates the usefulness of the proposed model.

Keywords: Lomax distribution; McDonald distribution; maximum-likelihood estimation; mean deviation; Pareto type II distribution

1. Introduction

The statistics literature is filled with hundreds of continuous univariate distributions (see, e.g. [1,2]). Numerous classical distributions have been extensively used over the past decades for modelling data in several areas such as engineering, actuarial, environmental and medical sciences, biological studies, demography, economics, finance and insurance. However, in many applied areas such as lifetime analysis, finance and insurance, there is a clear need for extended forms of these distributions. For any continuous baseline cumulative distribution function (cdf) $G(x)$, the cumulative function $F(x)$ of the McDonald-G ('Mc-G' for short) distribution is defined by

$$F(x) = I_{G(x)^c}(ac^{-1}, b) = \frac{1}{B(ac^{-1}, b)} \int_0^{G(x)^c} \omega^{a/c-1} (1-\omega)^{b-1}\, \mathrm{d}\omega, \qquad (1)$$

where $a > 0$, $b > 0$ and $c > 0$ are additional shape parameters that aim to introduce skewness and to provide greater flexibility of its tails, $I_y(a, b) = B_y(a, b)/B(a, b)$ is the incomplete beta function ratio, $B_y(a, b) = \int_0^y \omega^{a-1}(1-\omega)^{b-1}\, \mathrm{d}\omega$ is the incomplete beta function, $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the beta function and $\Gamma(\alpha) = \int_0^\infty w^{\alpha-1}\, \mathrm{e}^{-w}\, \mathrm{d}w$ is the gamma function.

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Application of $X = G^{-1}(V^{1/c})$ to a beta random variable V with positive parameters a/c and b yields X with cumulative function (1).

The probability density function (pdf) and hazard rate function associated with Equation (1) are given by

$$f(x) = \frac{c}{B(ac^{-1}, b)} g(x) G(x)^{a-1}\{1 - G(x)^c\}^{b-1} \tag{2}$$

and

$$r(x) = \frac{c g(x) G(x)^{a-1}\{1 - G(x)^c\}^{b-1}}{B(ac^{-1}, b)\{1 - I_{G(x)^c}(ac^{-1}, b)\}},$$

respectively. One major benefit of the Mc-G distribution is its ability to fit skewed data that cannot be properly fitted by existing distributions.

The class of distributions (2) includes two important special sub-classes: the beta generalized distributions [3] for $c = 1$ and the Kumaraswamy generalized distributions [4] for $a = c$. It follows from Equation (2) that the Mc-G distribution with baseline $G(x)$ is the beta generalized distribution with baseline $G(x)^c$. This simple transformation may facilitate the computation of several of its properties. Furthermore, when $G(x) = x$, we obtain the classical beta ($c = 1$) and Kumaraswamy ($a = c$) distributions. The distribution of Kumaraswamy [5] is commonly termed the 'minimax' distribution. It is also called the generalized beta distribution of the first kind (or beta type I). Jones [6] advocates its tractability, especially in simulations, because its quantile function takes a simple form and because of its pedagogical appeal relative to the classical beta distribution.

Equation (2) will be most tractable when the functions $G(x)$ and $g(x)$ have simple analytical expressions. The major benefit of Equation (2) is its ability of fitting skewed data that cannot be properly fitted by existing distributions. The cumulative function (1) can also be expressed in terms of the hypergeometric function as

$$F(x) = \frac{cG(x)^a}{aB(ac^{-1}, b)} {}_2F_1(ac^{-1}, 1 - b; ac^{-1} + 1; G(x)^a),$$

where

$${}_2F_1(p, q; r; x) = \frac{\Gamma(r)}{\Gamma(p)\Gamma(q)} \sum_{j=0}^{\infty} \frac{\Gamma(p + j)\Gamma(q + j)}{\Gamma(r + j)} \frac{x^j}{j!}.$$

Thus, for any parent $G(x)$, the properties of $F(x)$ could, in principle, be obtained from the well-established properties of the hypergeometric function (see [7]). The hypergeometric function can be computed, for example, using the MATHEMATICA software. For example, ${}_2F_1(p, q; r; x)$ is obtained from MATHEMATICA as `HypergeometricPFQ[{p, q}, {r}, x]`.

The Lomax distribution (also known as the Pareto distribution of second kind) with parameters $\alpha > 0$ and $\beta > 0$, say Lomax(α, β), is given by the cdf

$$G(x; \alpha, \beta) = 1 - \left(\frac{\beta}{\beta + x}\right)^{\alpha}, \quad x > 0, \tag{3}$$

where α and β are the shape and scale parameters, respectively. The pdf corresponding to Equation (3) is

$$g(x; \alpha, \beta) = \frac{\alpha\beta^{\alpha}}{(\beta + x)^{\alpha+1}}, \quad x > 0. \tag{4}$$

Equation (4) is a special form of Pearson type VI distribution.

Let $Y \sim$ Lomax(α, β). The sth moment of Y can be written as [8]

$$\mathrm{E}(Y^s) = \frac{\beta\Gamma(\alpha - s)\Gamma(s + 1)}{\Gamma(\alpha)}, \quad s < \alpha. \tag{5}$$

From Equation (5), if $\alpha > 2$, simple calculations yield $\mathrm{E}(Y) = \beta/(\alpha - 1)$ and $\mathrm{Var}(Y) = \alpha\beta^2/\{(\alpha - 1)^2(\alpha - 2)\}$.

After the work of Lomax [9], who used the distribution (3) to analyse business failure data, the Lomax distribution has been studied by several authors. Balkema and de Haan [10] showed that the cdf (3) arises as a limit distribution of residual lifetime at great age. According to Arnold [11], the Lomax distribution is well adapted for modelling reliability problems, since many of its properties are interpretable in that context and could be an alternative to the well-known distributions used in reliability. Ahsanullah [12] and Balakrishnan and Ahsanullah [13] considered some distributional properties and moments of the Lomax distribution, respectively. This distribution was used for modelling size spectra data in aquatic ecology by Vidondo *et al.* [14]. Childs *et al.* [15] considered order statistics from non-identical right-truncated Lomax distributions and provided applications for this situation. Al-Awadhi and Ghitany [16] used the Lomax distribution as a mixing distribution for the Poisson parameter and obtained the discrete Poisson–Lomax distribution. Howlader and Hossain [17] investigated the Bayesian estimation of the Lomax survival function. Abd-Ellah [18] obtained the Bayesian prediction bounds for certain Lomax order statistics. Nadarajah [19] derived several properties of the logarithm of the Lomax random variable as, for example, the moment-generating function (mgf), expectation, variance and skewness. More recently, Abd-Elfattah *et al.* [20] derived the non-Bayesian and Bayesian estimators of the sample size in the case of type I censored samples from the Lomax distribution. Ghitany *et al.* [21] investigated the properties of a new parametric distribution generated by Marshall and Olkin [22] and extended the family of distributions applied to the Lomax model. Hassan and Al-Ghamdi [23] determined the optimal times of changing stress level for simple stress plans under a cumulative exposure model using the Lomax distribution. For multivariate extensions of the Lomax distribution, the reader is referred to Nayak [8], Roy and Gupta [24], Petropoulos and Kourouklis [25] and Nadarajah [26].

In this article, we introduce a new distribution, the so-called McDonald Lomax (McLomax) distribution with density function obtained from Equation (2) by taking $G(x)$ and $g(x)$ to be the cdf and pdf of the Lomax(α, β) distribution, respectively. We adopt a different approach to much of the literature so far: rather than considering the classical beta generator [3] or the Kumaraswamy generator [4] applied to a baseline distribution, we propose the use of a more flexible McDonald generator applied to the Lomax distribution. We obtain some mathematical properties of this distribution and discuss maximum-likelihood estimation of its parameters. The rest of the article is outlined as follows. In Section 2, we introduce the McLomax distribution and provide plots of the density and hazard rate functions. In Section 3, we demonstrate that the McLomax density function can be expressed as an infinite linear combination of Lomax density functions. A general expression for the moments is provided in Section 4. Two representations for the mgf are derived in Section 5. In Section 6, we give an expression for the quantile function. The Rényi entropy is determined in Section 7. The mean deviations are calculated in Section 8. The maximum-likelihood estimation is addressed in Section 9. An empirical application is presented and discussed in Section 10. Finally, concluding remarks are given in Section 11.

2. McLomax distribution

We consider the generalization of the Lomax distribution by inserting Equations (3) and (4) in Equation (2). To avoid non-identifiability problems, we allow b to vary on $[1, \infty)$ only. We then write $\eta = b - 1$, which varies on $[0, \infty)$. The McLomax density function with five parameters α, β, a, η and c, denoted by McLomax$(\alpha, \beta, a, \eta, c)$, is expressed as (for $x > 0$)

$$f(x) = \frac{c\alpha\beta^{\alpha}(\beta + x)^{-(\alpha+1)}}{B(ac^{-1}, \eta + 1)} \left\{1 - \left(\frac{\beta}{\beta + x}\right)^{\alpha}\right\}^{a-1} \left[1 - \left\{1 - \left(\frac{\beta}{\beta + x}\right)^{\alpha}\right\}^{c}\right]^{\eta}. \qquad (6)$$

Evidently, the density function (6) does not involve any complicated function, and it includes several distributions as special sub-models not previously considered in the literature. In fact, the Lomax distribution (with parameters α and β) is clearly a basic exemplar for $a = c = 1$ and $\eta = 0$. The beta Lomax (BLomax) and Kumaraswamy Lomax (KwLomax) distributions are new models which arise for $c = 1$ and $a = c$, respectively. For $\eta = 0$ and $c = 1$, it leads to a new distribution referred to as the exponentiated Lomax (ELomax) distribution that extends the exponentiated standard Lomax (ESLomax) distribution [27] for $\beta = 1$. The McLomax distribution can also be applied in engineering as the Lomax distribution [11] and can be used to model reliability and survival problems. The McLomax distribution allows for greater flexibility of its tails and can be widely applied in many areas.

The cdf and hazard rate function corresponding to Equation (6) (for $x > 0$) are given by

$$F(x) = I_{\{1-\beta^\alpha(\beta+x)^{-\alpha}\}^c}(ac^{-1}, \eta + 1) \tag{7}$$

and

$$r(x) = \frac{c\alpha\beta^\alpha(\beta + x)^{-(\alpha+1)}\{1 - \beta^\alpha(\beta + x)^{-\alpha}\}^{a-1}[1 - \{1 - \beta^\alpha(\beta + x)^{-\alpha}\}^c]^\eta}{B(ac^{-1}, \eta + 1)\{1 - I_{\{1-\beta^\alpha(\beta+x)^{-\alpha}\}^c}(ac^{-1}, \eta + 1)\}}, \tag{8}$$

respectively. Application of $X = G^{-1}(V^{1/c})$ to a beta random variable V with positive parameters a/c and $\eta + 1$ yields X with cumulative function (7).

Figures 1 and 2 illustrate some of the possible shapes of the density function (6) and hazard rate function (8), respectively, for selected parameter values. The density function and hazard rate function can take various forms depending on the parameter values.

The McLomax distribution is easily simulated by inverting Equation (7) as follows: if V is a beta random variable with parameters a/c and $\eta + 1$, then

$$X = \frac{\beta\{1 - (1 - V)^{1/\alpha}\}}{(1 - V)^{1/\alpha}}$$

follows the McLomax$(\alpha, \beta, a, \eta, c)$ distribution. This scheme is useful because of the existence of fast generators for beta random variables.

3. Expansion for the density function

We give a very useful representation for the McLomax density function. If $|z| < 1$ and $\rho > 0$ is a real non-integer, we have the power series expansion

$$(1 - z)^\rho = \sum_{j=0}^{\infty} (-1)^j [\rho]_j z^j, \tag{9}$$

where $[\rho]_j = \rho(\rho - 1)\cdots(\rho - j + 1)$, for $j = 0, 1, \ldots$, is the falling factorial. The falling factorial is related to the gamma function by $[\rho]_j = \Gamma(\rho + 1)/\{\Gamma(\rho + 1 - j)j!\} = (-1)^j\Gamma(j - \rho)/\Gamma(-\rho)$. For $\rho \neq 0, -1, -2, \ldots$, we have $\Gamma(-\rho) = \pi\{\sin[\pi(\rho + 1)]\Gamma(\rho + 1)\}^{-1}$. Clearly, if ρ is a positive integer, the power series stops at $j = \rho$ and $[\rho]_j = \binom{\rho}{j}$. By applying the power series (9) in Equation (6), we obtain

$$f(x) = \frac{c\alpha\beta^\alpha(\beta + x)^{-(\alpha+1)}}{B(ac^{-1}, \eta + 1)} \sum_{j,k=0}^{\infty} (-1)^{j+k}[\eta]_j\,[jc + a - 1]_k \left(\frac{\beta}{\beta + x}\right)^{k\alpha},$$

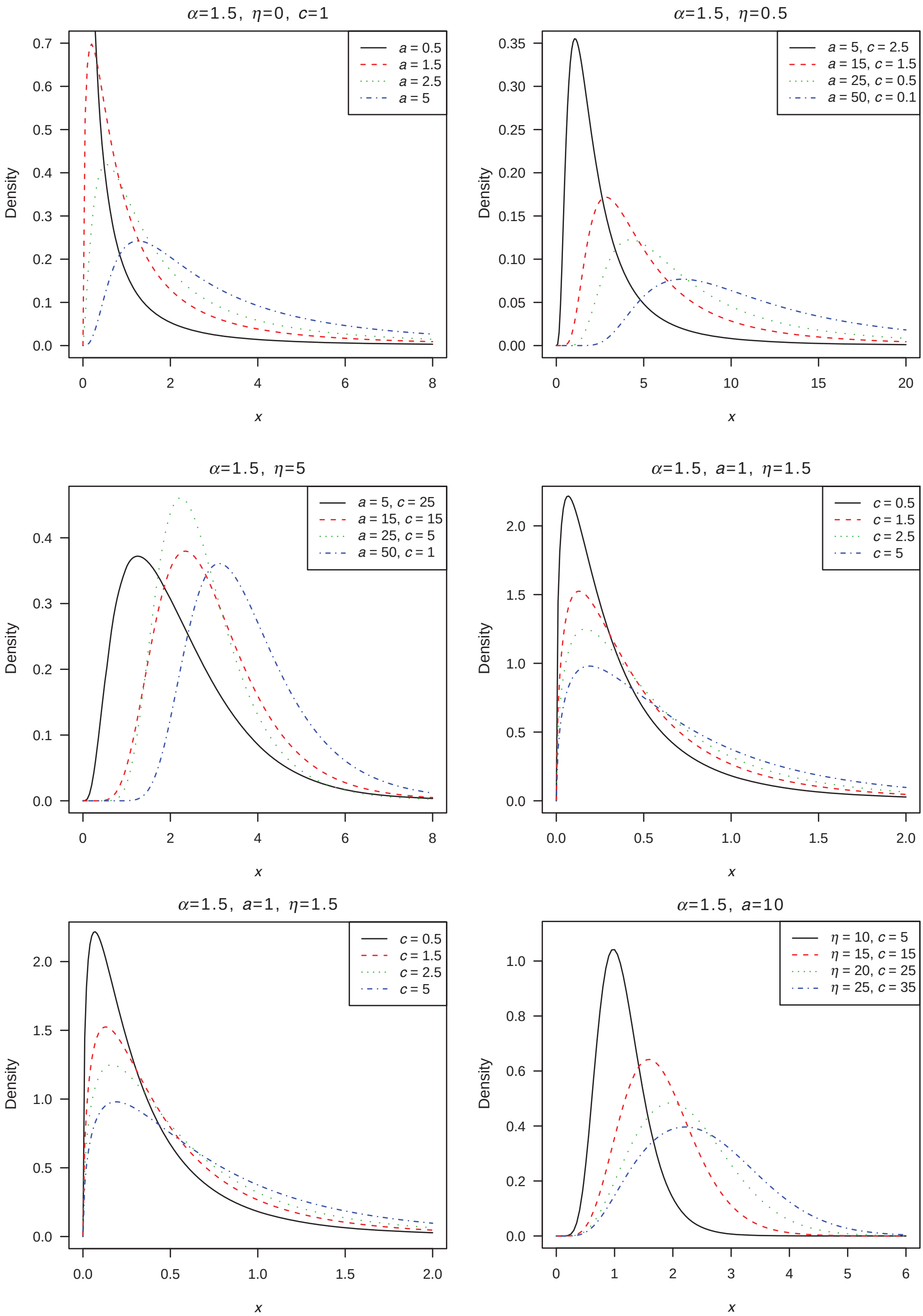

Figure 1. Plots of the density function (6) for some parameter values, $\beta = 1$.

which leads to an infinite linear combination

$$f(x) = \sum_{k=0}^{\infty} p_k g(x; (k+1)\alpha, \beta), \quad x > 0, \tag{10}$$

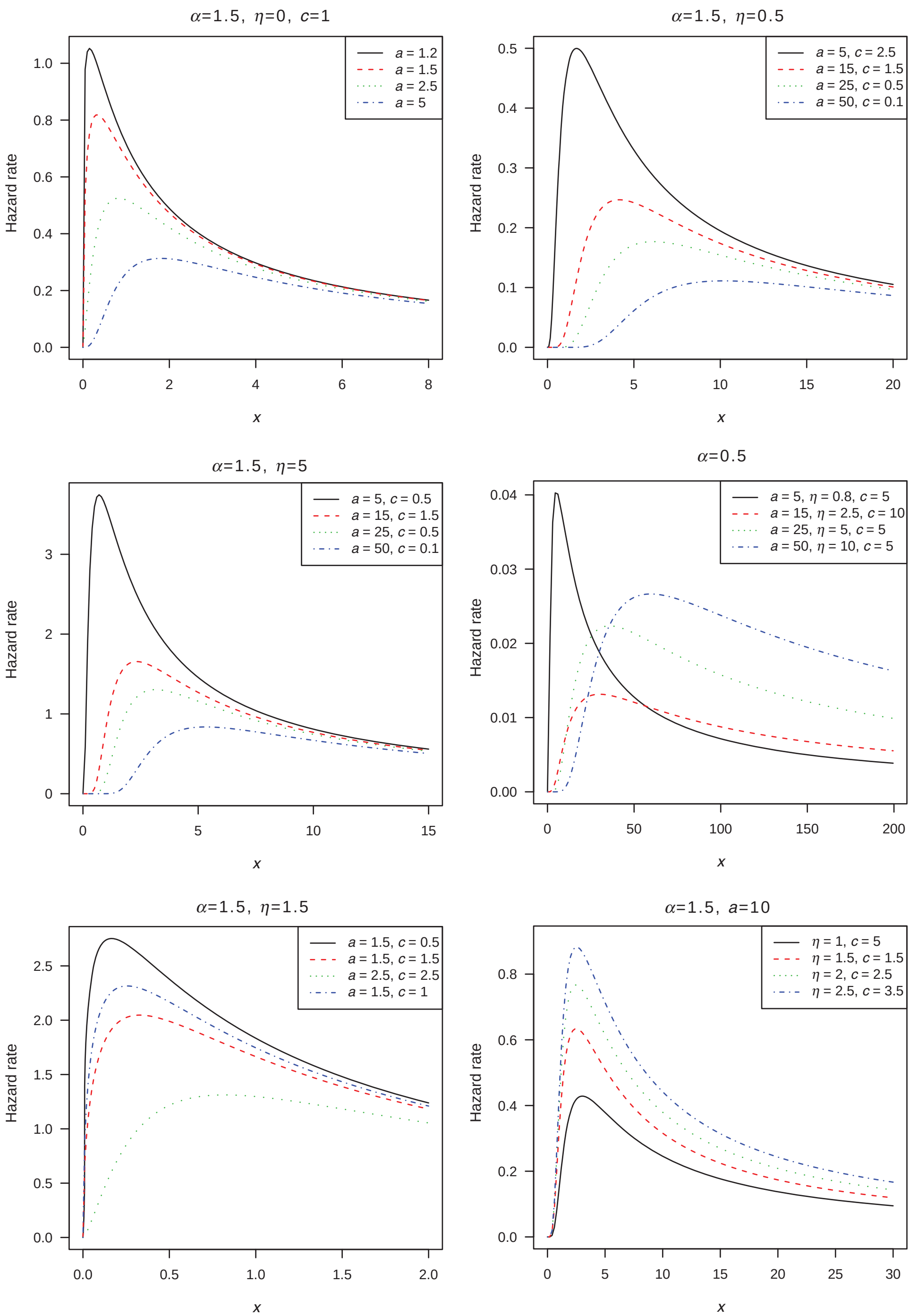

Figure 2. Plots of the hazard rate function (8) for some parameter values, $\beta = 1$.

where $g(x; (k+1)\alpha, \beta)$ denotes the Lomax($(k+1)\alpha, \beta$) density function and the weighted coefficients are

$$p_k = p_k(a, \eta, c) = \frac{c(-1)^k}{(k+1)B(ac^{-1}, \eta+1)} \sum_{j=0}^{\infty} (-1)^j [\eta]_j [jc + a - 1]_k.$$

By integrating both sides of Equation (10) in x from 0 to ∞, we can verify that $\sum_{k=0}^{\infty} p_k = 1$. From Equation (10), we obtain

$$F(x) = \sum_{k=0}^{\infty} p_k G(x; (k+1)\alpha, \beta), \quad x > 0,$$

where $G(x; (k+1)\alpha, \beta)$ denotes the cdf of the Lomax($(k+1)\alpha, \beta$) distribution.

From the infinite linear combination (10), several mathematical quantities of the McLomax distribution can be obtained directly from the quantities of the Lomax distribution.

The density function (10) can be rewritten as

$$f(x) = g(x; \alpha, \beta) \left(\sum_{k=0}^{\infty} p_k \frac{g(x; (k+1)\alpha, \beta)}{g(x; \alpha, \beta)} \right), \quad x > 0,$$

where the multiplier quantity inside the brackets is a kind of 'correction factor'. So, the correction applied to the Lomax density function yields the McLomax density function.

Equation (10) (and other expansions in this article) can be computed numerically in the software such as MAPLE [28], MATLAB [29] and MATHEMATICA [30]. These symbolic software have currently the ability to deal with analytical expressions of formidable size and complexity.

4. Moments

Some of the most important features and characteristics of a distribution can be studied through moments. If X has the McLomax($\alpha, \beta, a, \eta, c$) distribution, the sth moment of X for $s < \alpha$ comes immediately from Equations (5) and (10) as

$$\mu'_s = E(X^s) = \beta\Gamma(s+1) \sum_{k=0}^{\infty} p_k \frac{\Gamma((k+1)\alpha - s)}{\Gamma((k+1)\alpha)}. \tag{11}$$

For example, the moments of the BLomax and KwLomax distributions are obtained from Equation (11) for $c = 1$ and $a = c$, respectively. Furthermore, the central moments (μ_p) and cumulants (κ_p) of X are obtained from Equation (11) by

$$\mu_p = \sum_{k=0}^{p} \binom{p}{k} (-1)^k \mu_1'^p \mu'_{p-k} \quad \text{and} \quad \kappa_p = \mu'_p - \sum_{k=1}^{p-1} \binom{p-1}{k-1} \kappa_k \mu'_{p-k},$$

respectively, where $\kappa_1 = \mu'_1$. Thus, $\kappa_2 = \mu'_2 - \mu_1'^2$, $\kappa_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu_1'^3$, etc. The pth descending factorial moment of X is

$$\mu'_{(p)} = E[X^{(p)}] = E[X(X-1) \times \cdots \times (X-p+1)] = \sum_{m=0}^{p} s(p,m)\mu'_m,$$

where $s(r,m) = (m!)^{-1}[\mathrm{d}^m m^{(r)}/\mathrm{d}x^m]_{x=0}$ is the Stirling number of the first kind. The factorial moments of X for $p < \alpha$ are given by

$$\mu'_{(p)} = \beta \sum_{k=0}^{\infty} \frac{p_k}{\Gamma((k+1)\alpha)} \sum_{m=0}^{p} s(p,m)\Gamma((k+1)\alpha - m)\Gamma(m+1).$$

5. Generating function

We now derive two explicit expressions for the mgf $M(t;\alpha,\beta)$ of the Lomax distribution. First, we have from Equation (4),

$$M(t;\alpha,\beta)=\alpha\beta^{\alpha}\int_0^{\infty}\exp(tx)(\beta+x)^{-(\alpha+1)}\,\mathrm{d}x=\alpha\int_0^{\infty}\exp(\beta ty)(1+y)^{-(\alpha+1)}\,\mathrm{d}y.$$

We use Equations (9.210.1), (9.210.2) and (9.211.4) given by Gradshteyn and Ryzhik [7] to write the Kummer function $\Psi(\alpha,\gamma;z)$ as

$$\begin{aligned}\Psi(\alpha,\gamma;z)&=\frac{1}{\Gamma(\alpha)}\int_0^{\infty}\exp(-zt)t^{\alpha-1}(1+t)^{\gamma-(\alpha+1)}\,\mathrm{d}t\\&=\frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}{}_1F_1(\alpha;\gamma;z)+\frac{\Gamma(1-\gamma)}{\Gamma(\alpha)}{}_1F_1(1+\alpha-\gamma;2-\gamma;z),\end{aligned}$$

where ${}_1F_1(\alpha;\gamma;z)$ is the confluent hypergeometric function defined by

$${}_1F_1(\alpha;\gamma;z)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)}\sum_{j=0}^{\infty}\frac{\Gamma(\alpha+j)}{\Gamma(\gamma+j)}\frac{z^j}{j!}.$$

Hence, from the above facts, we obtain for $t<0$,

$$M(t;\alpha,\beta)=\Psi(1,1-\alpha;-\beta t)=\alpha^{-1}{}_1F_1(1;1-\alpha;-\beta t)+\mathrm{e}^{-\beta t}.$$

From Equation (10) and the last equation, the McLomax generating function reduces to

$$M(t)=\mathrm{e}^{-\beta t}+\alpha^{-1}\sum_{k=0}^{\infty}\frac{p_k\,{}_1F_1(1;1-(k+1)\alpha;-\beta t)}{(k+1)}. \qquad (12)$$

Evidently, Equation (12) involves computation of the confluent hypergeometric function, and routines for this are widely available, such as the `hypergeom(·)` in MAPLE.

A second representation follows from Equation (4) by setting $u=\beta/(\beta+x)$. We have

$$M(t;\alpha,\beta)=\alpha\int_0^{1}\exp\{t\beta(1-u)/u\}u^{\alpha-1}\,\mathrm{d}u.$$

We can use MAPLE to calculate the above integral for $t<0$ as

$$M(t;\alpha,\beta)=-\alpha\,\mathrm{e}^{-\beta t}(-\beta t)^{\alpha}\left[\frac{\pi(\pi\alpha)}{\Gamma(\alpha+1)}+\Gamma(-\alpha)-\Gamma(-\alpha,-\beta t)\right],$$

where $\Gamma(\alpha,x)=\int_x^{\infty}w^{\alpha-1}\,\mathrm{e}^{-w}\,\mathrm{d}w$ is the complementary incomplete gamma function. From Equation (10) and the last equation, we can express the McLomax generating function as

$$\begin{aligned}M(t)=&-\alpha\,\mathrm{e}^{-\beta t}\sum_{k=0}^{\infty}p_k(k+1)(-\beta t)^{(k+1)\alpha}\\&\times\left\{\frac{\pi[\pi(k+1)\alpha]}{\Gamma[(k+1)\alpha+1]}+\Gamma(-(k+1)\alpha)-\Gamma(-(k+1)\alpha,-\beta t)\right\}.\end{aligned} \qquad (13)$$

Equations (12) and (13) are the main results of this section.

6. Quantile function

The McLomax quantile function, say $Q(u) = F^{-1}(u)$, can be written in terms of the quantile function of the beta random variable. By inverting Equation (7), we can write

$$x = Q(u) = \frac{\beta[1 - (1 - t^{1/c})^{1/\alpha}]}{(1 - t^{1/c})^{1/\alpha}}, \tag{14}$$

where $t = Q_{ac^{-1},\eta+1}(u) = I_u^{-1}(ac^{-1}, \eta + 1)$ denotes the quantile function of the beta distribution with parameters ac^{-1} and $\eta + 1$. The following expansion for the beta quantile function $Q_{ac^{-1},\eta+1}(u)$ can be found on the Wolfram website (http://functions.wolfram.com/06.23.06.0004.01)

$$Q_{ac^{-1},\eta+1}(u) = \sum_{i=1}^{4} A_i \left[B\left(\frac{a}{c}, \eta + 1\right) \frac{au}{c}\right]^{(ic)/a} + O(u^{(5c)/a}),$$

whose coefficients are given by

$$A_1 = 1, \quad A_2 = \frac{\eta}{[(a/c) + 1]}, \quad A_3 = \frac{\eta[(a/c)^2 + 3(a/c)(\eta + 1) - (a/c) + 5\eta + 1]}{2[(a/c) + 1]^2[(a/c) + 2]}$$

and

$$A_4 = \frac{\eta[(a/c)^4 + (6\eta + 5)(a/c)^3 + (\eta + 3)(8\eta + 3)(a/c)^2]}{3[(a/c) + 1]^3[(a/c) + 2][(a/c) + 3]}.$$

7. Rényi entropy

Entropy has been used in various situations in science and engineering. Numerous entropy measures have been studied and compared in the literature. The entropy of a random variable X with density function $f(x)$ is a measure of variation of the uncertainty. The Rényi entropy is defined by

$$I_{\mathrm{R}}(\delta) = (1 - \delta)^{-1} \log\left\{\int_{-\infty}^{\infty} f(x)^\delta \,\mathrm{d}x\right\},$$

where $\delta > 0$ and $\delta \neq 1$. For further details, the reader is referred to [31].

Let X be a random variable following the McLomax$(\alpha, \beta, a, \eta, c)$ distribution. After some algebra, we have

$$f(x)^\delta = \sum_{k=0}^{\infty} v_k g(x; \alpha(k + \delta) + \delta - 1, \beta), \quad x > 0,$$

where

$$v_k = v_k(\alpha, \beta, a, \eta, c) = \frac{(-1)^k c^\delta \alpha^{\delta-1} \beta^{1-\delta} B(ac^{-1}, \eta + 1)^{-\delta}}{\alpha(k + \delta) + \delta - 1} \sum_{j=0}^{\infty} (-1)^j [\delta\eta]_j [jc + \delta(a - 1)]_k.$$

Note that $f(x)^\delta$ is an infinite linear combination of Lomax density functions. Hence, the Rényi entropy of X reduces to

$$I_{\mathrm{R}}(\delta) = (1 - \delta)^{-1} \log\left(\sum_{k=0}^{\infty} v_k\right).$$

8. Mean deviations

The deviations from the mean and from the median can be used as a measure of spread in a population. Let X be a random variable having the McLomax$(\alpha, \beta, a, \eta, c)$ distribution. We can derive the mean deviations about the mean and about the median from the relations

$$\delta_1 = E(|X - \mu_1'|) = \int_0^\infty |x - \mu_1'| f(x)\,\mathrm{d}x \quad \text{and} \quad \delta_2 = E(|X - M|) = \int_0^\infty |x - M| f(x)\,\mathrm{d}x,$$

respectively, where $\mu_1' = E(X)$ is obtained from Equation (11) and M is the median of X. From Equation (7), the median M is the solution of the nonlinear equation

$$I_{\{1-\beta^\alpha(\beta+M)^{-\alpha}\}^c}(ac^{-1}, \eta + 1) = \frac{1}{2}.$$

The measures δ_1 and δ_2 can be expressed as

$$\delta_1 = 2\mu_1' F(\mu_1') - 2J(\mu_1') \quad \text{and} \quad \delta_2 = \mu_1' - 2J(M),$$

where $F(\mu_1')$ is calculated from Equation (7) and $J(q) = \int_0^q x\, f(x)\,\mathrm{d}x$. We can write from Equation (10),

$$J(q) = \sum_{k=0}^\infty p_k \int_0^q x g(x; (k+1)\alpha, \beta)\,\mathrm{d}x. \tag{15}$$

After some algebra, we obtain

$$\int_0^q x g(x; (k+1)\alpha, \beta)\,\mathrm{d}x = (k+1)\alpha\beta^{-1} \sum_{j=0}^\infty \frac{(-1)^j q^{j+2}[(k+1)\alpha + j]_j}{(j+2)\beta^j},$$

and hence, $J(q)$ can be rewritten as

$$J(q) = \frac{\alpha}{\beta} \sum_{k,j=0}^\infty (k+1) p_k u_{k,j},$$

where $u_{k,j} = u_{k,j}(q, \alpha, \beta) = (-1)^j q^{j+2}[(k+1)\alpha + j]_j \{(j+2)\,\beta^j\}^{-1}$.

Equation (15) can be used to obtain the Bonferroni and Lorenz curves that have applications in fields such as economics, reliability, demography, insurance and medicine. They are defined by

$$B(p) = \frac{J(q)}{p\mu_1'} \quad \text{and} \quad L(p) = \frac{J(q)}{\mu_1'},$$

respectively, where $q = Q(p)$ is determined from the quantile function (14) for a given probability p.

9. Maximum-likelihood estimation

We consider estimation of the parameters of the McLomax distribution by the method of maximum likelihood. Let $\boldsymbol{x} = (x_1, \ldots, x_n)^\top$ be a random sample of size n of the McLomax distribution with unknown parameter vector $\boldsymbol{\theta} = (\alpha, \beta, a, \eta, c)^\top$. The total log-likelihood function for $\boldsymbol{\theta}$ is

$$\ell(\boldsymbol{\theta}) = n\log(\alpha) + n\alpha\log(\beta) + n\log(c) - n\log\{B(ac^{-1}, \eta+1)\} - (\alpha+1)\sum_{i=1}^{n}\log(\beta+x_i) + (a-1)\sum_{i=1}^{n}\log(\dot{z}_i) + \eta\sum_{i=1}^{n}\log(1-\dot{z}_i^c),$$

where $\dot{v}_i = \dot{v}_i(\alpha,\beta) = \beta^\alpha/(\beta+x_i)^\alpha$, $\dot{z}_i = \dot{z}_i(\alpha,\beta) = 1-\dot{v}_i$, for $i = 1,\ldots,n$. By taking the partial derivatives of the above log-likelihood function with respect to α, β, a, η and c, we obtain the components of the score vector $\boldsymbol{U_\theta} = (U_\alpha, U_\beta, U_a, U_\eta, U_c)^\top$:

$$U_\alpha = \frac{n}{\alpha} + n\log(\beta) - \sum_{i=1}^{n}\log(\beta+x_i) - \frac{(a-1)}{\alpha}\sum_{i=1}^{n}\frac{\dot{v}_i\log(\dot{v}_i)}{\dot{z}_i} + \frac{\eta c}{\alpha}\sum_{i=1}^{n}\frac{\dot{v}_i\dot{z}_i^{c-1}\log(\dot{v}_i)}{1-\dot{z}_i^c},$$

$$U_\beta = \frac{n\alpha}{\beta} - (\alpha+1)\sum_{i=1}^{n}\frac{1}{\beta+x_i} - \frac{\alpha(a-1)}{\beta}\sum_{i=1}^{n}\frac{\dot{w}_i\dot{v}_i}{\dot{z}_i} + \frac{\alpha\eta c}{\beta}\sum_{i=1}^{n}\frac{\dot{w}_i\dot{v}_i\dot{z}_i^{c-1}}{1-\dot{z}_i^c},$$

$$U_a = \frac{n\psi(a/c+\eta+1)}{c} - \frac{n\psi(a/c)}{c} + \sum_{i=1}^{n}\log(\dot{z}_i),$$

$$U_\eta = n\psi(a/c+\eta+1) - n\psi(\eta+1) + \sum_{i=1}^{n}\log(1-\dot{z}_i^c),$$

$$U_c = \frac{n}{c} + \frac{na}{c^2}\{\psi(a/c) - \psi(a/c+\eta+1)\} - \eta\sum_{i=1}^{n}\frac{\dot{z}_i^c\log(\dot{z}_i)}{1-\dot{z}_i^c},$$

where $\psi(\cdot)$ is the digamma function and $\dot{w}_i = \dot{w}_i(\beta) = x_i/(\beta+x_i)$, for $i = 1,\ldots,n$. The maximum-likelihood estimate (MLE) $\hat{\boldsymbol{\theta}} = (\hat{\alpha}, \hat{\beta}, \hat{a}, \hat{\eta}, \hat{c})^\top$ of $\boldsymbol{\theta} = (\alpha, \beta, a, \eta, c)^\top$ is obtained by setting $U_\alpha = 0$, $U_\beta = 0$, $U_a = 0$, $U_\eta = 0$ and $U_c = 0$ and by solving them simultaneously. These equations cannot be solved analytically, and statistical software can be used to solve them numerically via iterative methods. We can use iterative techniques such as a Newton–Raphson-type algorithm to obtain the estimate $\hat{\boldsymbol{\theta}}$. The Broyden–Fletcher–Goldfarb–Shanno method (see e.g. [32,33]) with analytical derivatives has been used for maximizing the log-likelihood function $\ell(\boldsymbol{\theta})$.

The normal approximation of the MLE of $\boldsymbol{\theta}$ can be used for constructing approximate confidence intervals and for testing hypotheses on the parameters α, β, a, η and c. Under conditions that are fulfilled for the parameters in the interior of the parameter space, we have $\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \overset{A}{\sim} \mathcal{N}_5(\mathbf{0}, \boldsymbol{K}_{\boldsymbol{\theta}}^{-1})$, where $\overset{A}{\sim}$ means approximately distributed and $\boldsymbol{K_\theta}$ is the unit expected information matrix. We have the asymptotic result $\boldsymbol{K_\theta} = \lim_{n\to\infty} n^{-1}\boldsymbol{J}_n(\boldsymbol{\theta})$, where $\boldsymbol{J}_n(\boldsymbol{\theta})$ is the observed information matrix. The average matrix evaluated at $\hat{\boldsymbol{\theta}}$, say $n^{-1}\boldsymbol{J}_n(\hat{\boldsymbol{\theta}})$, can estimate $\boldsymbol{K_\theta}$. The observed information matrix $\boldsymbol{J}_n(\boldsymbol{\theta}) = -\partial^2\ell(\boldsymbol{\theta})/\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^\top$ is given in the appendix.

We can compute the maximum values of the unrestricted and restricted log-likelihood functions to obtain the likelihood ratio (LR) statistics for testing some sub-models of the McLomax distribution. For example, we can use the LR statistic to check if the fit using the McLomax distribution is statistically 'superior' to a fit using the BLomax distribution for a given data set. We consider the partition $\boldsymbol{\theta} = (\boldsymbol{\theta}_1^\top, \boldsymbol{\theta}_2^\top)^\top$ of the parameter of the McLomax distribution, where $\boldsymbol{\theta}_1$ is a subset of parameters of interest and $\boldsymbol{\theta}_2$ is a subset of the remaining parameters. The LR statistic for testing the null hypothesis $\mathcal{H}_0 : \boldsymbol{\theta}_1 = \boldsymbol{\theta}_1^{(0)}$ against the alternative hypothesis $\mathcal{H}_1 : \boldsymbol{\theta}_1 \neq \boldsymbol{\theta}_1^{(0)}$ is given by $w = 2\{\ell(\hat{\boldsymbol{\theta}}) - \ell(\tilde{\boldsymbol{\theta}})\}$, where $\hat{\boldsymbol{\theta}}$ and $\tilde{\boldsymbol{\theta}}$ are the MLEs under the alternative and null hypotheses, respectively, and $\boldsymbol{\theta}_1^{(0)}$ is a specified parameter vector. The statistic w is asymptotically $(n \to \infty)$ distributed as χ_k^2, where k is the dimension of the subset $\boldsymbol{\theta}_1$ of interest. Then, we can compare the McLomax model against the BLomax model by testing $\mathcal{H}_0 : c = 1$ versus $\mathcal{H}_1 : c \neq 1$. The LR statistic becomes $w = 2\{\ell(\hat{\alpha}, \hat{\beta}, \hat{a}, \hat{\eta}, \hat{c}) - \ell(\tilde{\alpha}, \tilde{\beta}, \tilde{a}, \tilde{\eta}, 1)\}$, where $\hat{\alpha}$, $\hat{\beta}$, $\hat{a}$, $\hat{\eta}$ and $\hat{c}$ are the MLEs under $\mathcal{H}_1$ and $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{a}$ and $\tilde{\eta}$ are the MLEs under $\mathcal{H}_0$.

10. Application

We provide an application of the McLomax distribution and their sub-models: BLomax, KwLomax, ELomax, ESLomax and Lomax distributions. We compare the results of the fits of these models. We consider an uncensored data set corresponding to remission times (in months) of a random sample of 128 bladder cancer patients reported in [34]. The data are as follows: 0.08, 2.09, 3.48, 4.87, 6.94 , 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46 , 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69. All the computations were done using the Ox matrix programming language [35]. Ox is freely distributed for academic purposes and is available at http://www.doornik.com.

Table 1 lists the MLEs (and the corresponding standard errors in parentheses) of the model parameters and the following statistics: Akaike information criterion (AIC), Bayesian information criterion (BIC) and Hannan–Quinn information criterion (HQIC). These results show that the ELomax distribution has the lowest AIC, BIC and HQIC values among all the fitted models, and so it could be chosen as the best model. Additionally, it is evident that the ESLomax distribution proposed in [27] presents the worst fit to the current data and that the proposed models outperform this distribution. In order to assess if the model is appropriate, the histogram of the data and plots of the fitted McLomax, BLomax, KwLomax, ELomax, ESLomax and Lomax distributions are shown in Figure 3. From these plots, we can conclude that the McLomax, BLomax, KwLomax and ELomax models yield the best fits and hence can be adequate for these data.

In addition to comparing the models, we use two other criteria. First, we consider the LR statistic and next we consider formal goodness-of-fit tests. The McLomax model includes some sub-models (described in Section 2), thus allowing their evaluation relative to each other and to a more general model. As mentioned before, we can compute the maximum values of the unrestricted and restricted log-likelihoods to obtain the LR statistics for testing some sub-models of the McLomax distribution. The values of the LR statistics are given in Table 2. From the figures in this table, we can conclude that there is no difference among the fits to the current data using the McLomax, BLomax, KwLomax and ELomax models. In addition, these models provide a better representation of the data than the ESLomax and Lomax models based on the LR test at the 5% significance level.

Now, we apply the formal goodness-of-fit tests in order to verify which distribution fits better to these data. We consider the Cramér–von Mises (W^*) and Anderson–Darling (A^*) statistics. The statistics W^* and A^* are described in detail in [36]. In general, the smaller the values of the statistics W^* and A^*, the better the fit to the data. Let $H(x; \boldsymbol{\theta})$ be the cdf, where the form of H is known, but $\boldsymbol{\theta}$ (a k-dimensional parameter vector, say) is unknown. To obtain the statistics W^* and A^*, we can proceed as follows: (i) compute $v_i = H(x_i; \hat{\boldsymbol{\theta}})$, where the x_i's are in ascending order;

Table 1. MLEs (standard errors in parentheses) and the measures AIC, BIC and HQIC.

	Estimates					Statistic		
Distribution	α	β	a	η	c	AIC	BIC	HQIC
McLomax	0.8085 (3.364)	11.2929 (15.818)	1.5060 (0.243)	4.1886 (25.029)	2.1046 (3.079)	829.82	844.09	835.62
BLomax	3.9191 (18.192)	23.9281 (27.338)	1.5853 (0.280)	0.1572 (5.024)		828.14	839.55	832.78
KwLomax	0.3911 (2.386)	12.2973 (17.316)	1.5162 (0.228)	11.0323 (87.144)		827.88	839.29	832.52
ELomax	4.5857 (2.227)	24.7414 (16.686)	1.5862 (0.280)			826.14	834.70	829.62
ESLomax	1.0877 (0.086)		4.6575 (0.686)			856.61	862.31	858.93
Lomax	13.9384 (15.386)	121.0225 (142.714)				831.67	837.37	833.98

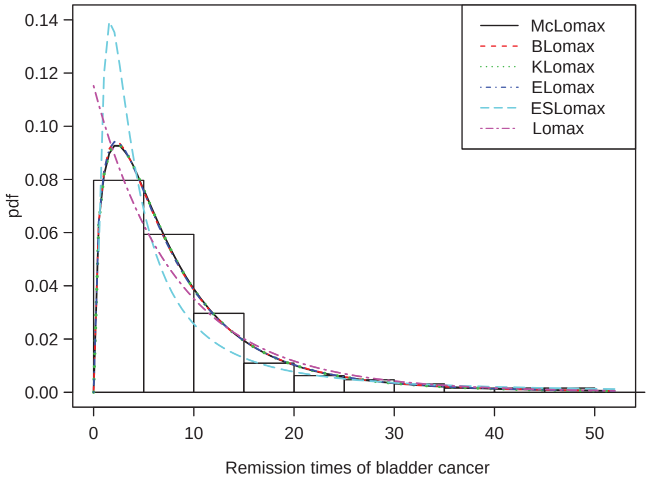

Figure 3. Estimated densities of the McLomax, BLomax, KwLomax, ELomax, ESLomax and Lomax distributions.

Table 2. LR tests.

Model	w	p-Value
McLomax versus BLomax	0.3173	0.5732
McLomax versus KwLomax	0.0556	0.8135
McLomax versus ELomax	0.3187	0.8527
McLomax versus ESLomax	32.7853	0.0000
McLomax versus Lomax	7.8409	0.0494
BLomax versus ELomax	0.0014	0.9704
BLomax versus ESLomax	32.4679	0.0000
BLomax versus Lomax	7.5235	0.0232
KwLomax versus ELomax	0.2631	0.6080
KwLomax versus ESLomax	32.7296	0.0000
KwLomax versus Lomax	7.7852	0.0204
ELomax versus ESLomax	32.4666	0.0000
ELomax versus Lomax	7.5222	0.0061

(ii) compute $y_i = \Phi^{-1}(v_i)$, where $\Phi(\cdot)$ is the standard normal cdf and $\Phi^{-1}(\cdot)$ is its inverse; (iii) compute $u_i = \Phi\{(y_i - \bar{y})/s_y\}$, where $\bar{y} = (1/n)\sum_{i=1}^{n} y_i$ and $s_y^2 = (n-1)^{-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$; (iv) calculate

$$W^2 = \sum_{i=1}^{n}\left\{u_i - \frac{(2i-1)}{2n}\right\}^2 + \frac{1}{12n}$$

and

$$A^2 = -n - \frac{1}{n}\sum_{i=1}^{n}\{(2i-1)\ln(u_i) + (2n+1-2i)\ln(1-u_i)\};$$

and (v) modify W^2 into $W^* = W^2(1+0.5/n)$ and A^2 into $A^* = A^2(1+0.75/n+2.25/n^2)$. For further details, the reader is refereed to [36]. The values of the statistics W^* and A^* for all models are given in Table 3. Based on these statistics, we conclude that the McLomax model fits the current data better than the other models.

In summary, the proposed McLomax distribution (and its sub-models) produces better fits for the data from Lee and Wang [34] than the Lomax and ESLomax distributions considered by Gupta

Table 3. Goodness-of-fit tests.

Distribution	Statistic	
	W^*	A^*
McLomax	0.02535	0.16851
BLomax	0.02831	0.19001
KwLomax	0.02586	0.17265
ELomax	0.02832	0.19021
ESLomax	0.38594	2.45671
Lomax	0.08068	0.48736

et al. [27]. In this case, the ELomax distribution could be chosen since it has less parameters to be estimated and, according to the LR statistic (Table 2), it presents a similar fit to those of the McLomax, BLomax and KwLomax models.

11. Concluding remarks

For the first time, we introduce a five-parameter continuous distribution which generalizes the Lomax [9] and the exponentiated standard Lomax [27] distributions. Furthermore, the new distribution includes as special sub-models other distributions. We refer to the new model as the McLomax distribution and study some of its mathematical and statistical properties. We demonstrate that the McLomax density function can be expressed as a linear combination of the Lomax density functions. We provide the moments and two representations for the mgf. We also present an expression for the quantile function. Explicit expressions are derived for the mean deviations, Bonferroni and Lorenz curves and Rényi entropy.

Our formulas related to the McLomax model are manageable, and with the use of modern computer resources with analytical and numerical capabilities, they may turn into adequate tools comprising the arsenal of applied statisticians. Parameter estimation is approached by maximum likelihood and the observed information matrix is derived. The usefulness of the new distribution is illustrated in an analysis of real data. We hope that the proposed extended model may attract wider applications in survival analysis.

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Appendix

The observed information matrix for the parameter vector $\boldsymbol{\theta} = (\alpha, \beta, a, \eta, c)^\top$ is given by

$$\boldsymbol{J}_n(\boldsymbol{\theta}) = -\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^\top} = -\begin{pmatrix} U_{\alpha\alpha} & U_{\alpha\beta} & U_{\alpha a} & U_{\alpha\eta} & U_{\alpha c} \\ \cdot & U_{\beta\beta} & U_{\beta a} & U_{\beta\eta} & U_{\beta c} \\ \cdot & \cdot & U_{aa} & U_{a\eta} & U_{ac} \\ \cdot & \cdot & \cdot & U_{\eta\eta} & U_{\eta c} \\ \cdot & \cdot & \cdot & \cdot & U_{cc} \end{pmatrix},$$

whose elements are

$$U_{\alpha\alpha} = -\frac{n}{\alpha^2} - \frac{(a-1)}{\alpha^2}\sum_{i=1}^{n}\frac{\dot{v}_i[\log(\dot{v}_i)]^2}{\dot{z}_i}\left(1+\frac{\dot{v}_i}{\dot{z}_i}\right) + \frac{\eta c(1-c)}{\alpha^2}\sum_{i=1}^{n}\frac{\dot{v}_i^2\dot{z}_i^{c-2}[\log(\dot{v}_i)]^2}{1-\dot{z}_i^c}$$
$$+\frac{\eta c}{\alpha^2}\sum_{i=1}^{n}\frac{\dot{v}_i\dot{z}_i^{c-1}[\log(\dot{v}_i)]^2}{1-\dot{z}_i^c}\left(1-\frac{c\dot{v}_i\dot{z}_i^{c-1}}{1-\dot{z}_i^c}\right),$$

$$U_{\alpha\beta} = \frac{n}{\beta} - \sum_{i=1}^{n}\frac{1}{\beta+x_i} - \frac{(a-1)}{\beta}\sum_{i=1}^{n}\frac{\dot{v}_i\dot{w}_i}{\dot{z}_i}\left(1+\log(\dot{v}_i)+\frac{\dot{v}_i\log(\dot{v}_i)}{\dot{z}_i}\right)$$
$$+\frac{\eta c(1-c)}{\beta}\sum_{i=1}^{n}\frac{\dot{w}_i\dot{v}_i^2\dot{z}_i^{c-2}\log(\dot{v}_i)}{1-\dot{z}_i^c} + \frac{\eta c}{\beta}\sum_{i=1}^{n}\frac{\dot{v}_i\dot{w}_i\dot{z}_i^{c-1}\{1+\log(\dot{v}_i)\}}{1-\dot{z}_i^c}$$
$$-\frac{\eta c^2}{\beta}\sum_{i=1}^{n}\frac{\dot{w}_i\dot{v}_i^2\dot{z}_i^{2(c-1)}\log(\dot{v}_i)}{(1-\dot{z}_i^c)^2},$$

$$U_{\alpha a} = -\frac{1}{\alpha}\sum_{i=1}^{n}\frac{\dot{v}_i\log(\dot{v}_i)}{\dot{z}_i},\quad U_{\alpha\eta} = \frac{c}{\alpha}\sum_{i=1}^{n}\frac{\dot{v}_i\dot{z}_i^{c-1}\log(\dot{v}_i)}{1-\dot{z}_i^c},$$

$$U_{\alpha c} = \frac{\eta}{\alpha}\sum_{i=1}^{n}\frac{\dot{v}_i\dot{z}_i^{c-1}\log(\dot{v}_i)\{1+c\log(\dot{z}_i)\}}{1-\dot{z}_i^c} + \frac{\eta c}{\alpha}\sum_{i=1}^{n}\frac{\dot{v}_i\dot{z}_i^{2c-1}\log(\dot{v}_i\dot{z}_i)}{(1-\dot{z}_i^c)^2},$$

$$U_{\beta\beta} = -\frac{n\alpha}{\beta^2} + (\alpha+1)\sum_{i=1}^{n}\frac{1}{(\beta+x_i)^2} - \frac{\alpha(a-1)}{\beta^2}\sum_{i=1}^{n}\frac{\dot{v}_i\dot{w}_i}{\dot{z}_i}\left(1-\alpha\dot{w}_i-\frac{\alpha\dot{v}_i\dot{w}_i}{\dot{z}_i}\right)$$
$$-\frac{\alpha(a-1)}{2\beta}\sum_{i=1}^{n}\frac{\dot{v}_i\dot{y}_i}{\dot{z}_i} - \frac{\alpha^2\eta c^2}{\beta^2}\sum_{i=1}^{n}\frac{\dot{w}_i^2\dot{v}_i^2\dot{z}_i^{c-2}}{(1-\dot{z}_i^c)^2}$$
$$+\frac{\alpha^2\eta c}{\beta^2}\sum_{i=1}^{n}\frac{\dot{v}_i\dot{w}_i^2\dot{z}_i^{c-1}}{1-\dot{z}_i^c}\left(1+\frac{\dot{v}_i}{\dot{z}_i}\right) + \frac{\alpha\eta c}{\beta}\sum_{i=1}^{n}\frac{\dot{v}_i\dot{z}_i^{c-1}}{1-\dot{z}_i^c}\left(\frac{\dot{y}_i}{2}-\frac{\dot{w}_i}{\beta}\right),$$

$$U_{\beta a} = -\frac{\alpha}{\beta}\sum_{i=1}^{n}\frac{\dot{v}_i\dot{w}_i}{\dot{z}_i},\quad U_{\beta\eta} = \frac{\alpha c}{\beta}\sum_{i=1}^{n}\frac{\dot{v}_i\dot{w}_i\dot{z}_i^{c-1}}{1-\dot{z}_i^c},$$

$$U_{\beta c} = \frac{\alpha\eta}{\beta}\sum_{i=1}^{n}\frac{\dot{w}_i\dot{v}_i\dot{z}_i^{c-1}}{1-\dot{z}_i^c} + \frac{\alpha\eta c}{\beta}\sum_{i=1}^{n}\frac{\dot{w}_i\dot{v}_i\dot{z}_i^{c-1}\log(\dot{z}_i)}{(1-\dot{z}_i^c)^2},$$

$$U_{aa} = -\frac{n}{c^2}\left\{\psi'\left(\frac{a}{c}\right)-\psi'\left(\eta+1+\frac{a}{c}\right)\right\},\quad U_{a\eta} = \frac{n\psi'(\eta+1+a/c)}{c},$$
$$U_{ac} = \frac{n}{c^2}\left\{\psi\left(\frac{a}{c}\right)-\psi\left(\eta+1+\frac{a}{c}\right)\right\} + \frac{n\,a}{c^3}\left\{\psi'\left(\frac{a}{c}\right)-\psi'\left(\eta+1+\frac{a}{c}\right)\right\},$$

$$U_{\eta\eta} = n\left\{\psi'\left(\eta+1+\frac{a}{c}\right)-\psi'(\eta+1)\right\},\quad U_{\eta c} = -\frac{na\psi'(\eta+1+a/c)}{c^2} - \sum_{i=1}^{n}\frac{\dot{z}_i^c\log(\dot{z}_i)}{1-\dot{z}_i^c},$$

$$U_{cc} = -\frac{n}{c^2} - \frac{2na\psi(a/c)}{c^3} - \frac{na^2\psi'(a/c)}{c^4} + \frac{2na\psi(\eta+1+a/c)}{c^3}$$
$$+\frac{na^2\psi'(\eta+1+a/c)}{c^4} - \eta\sum_{i=1}^{n}\frac{\dot{z}_i^c[\log(\dot{z}_i)]^2}{(1-\dot{z}_i^c)^2},$$

where $\psi'(\cdot)$ is the trigamma function, and $\dot{y}_i = \dot{y}_i(\beta) = -2\dot{w}_i/(\beta+x_i)$ and $\dot{w}_i$, $\dot{v}_i$ and $\dot{z}_i$ are defined in Section 9.