An Extended Topological Yang-Mills Theory

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Introducing infinite number of fields, we construct an extended version of the topological Yang-Mills theory. The properties of the extended topological Yang-Mills theory (ETYMT) are discussed from standpoint of the covariant canonical quantization. It is shown that the ETYMT becomes a cohomological topological field theory or a theory equivalent to a quantum Yang-Mills theory with anti-self-dual constraint according to subsidiary conditions imposed on state-vector space. On the basis of the ETYMT, we may understand a transition from an unbroken phase to a physical phase (broken phase).

§ 1. Introduction

In recent years, "cohomological" topological field theories (CTFT's)[1)~5)] have been studied by many people as an interesting application of quantum field theory to mathematics. Main interest in the CTFT's has been directed to describing topological invariants associated with low dimensional manifolds in terms of quantum field theory and examining mathematical properties of the invariants. For example, Donaldson's topological invariants have been represented as correlation functions in the topological Yang-Mills theory (TYMT).[1),3)]

The TYMT, which is the prototype for the CTFT's, is characterized by the Lagrangian[6)]

$$\mathcal{L}_{\mathrm{TYMT}}=-i\sqrt{g}\,\boldsymbol{\delta}\mathrm{Tr}\left[\frac{1}{2}\chi^{\mu\nu}(F_{\mu\nu}+\alpha B_{\mu\nu})+\bar{\phi}^{-2}\mathcal{D}_{\mu}\psi^{1\mu}-\bar{c}\nabla_{\mu}A^{\mu}\right] \tag{1}$$

with $F=dA+iqAA$ and $\mathcal{D}=\nabla+iq[A,\cdot]$, where A is a Yang-Mills connection one-form, ψ^1 an anticommuting one-form, B a commuting self-dual two-form, χ an anticommuting self-dual two-form, $\bar{\phi}^{-2}$ a commuting zero-form, and $\bar{c}$ an anticommuting zero-form. These fields are defined on a Riemannian four manifold M^4 with a metric $g_{\mu\nu}$ and take values in the Lie algebra $\mathcal{G}$ of a gauge group G. The covariant exterior derivative ∇ is defined from the Riemannian connection on M^4, and α and q are a gauge parameter and a coupling constant, respectively. The BRST transformation $\boldsymbol{\delta}$ is defined by

$$\boldsymbol{\delta}A=\psi^1+D\phi^1\,,\qquad \boldsymbol{\delta}\psi^1=-iD\phi^2-iq\{\psi^1,\phi^1\}\,,$$

$$\boldsymbol{\delta}\chi=iB-iq\{\chi,\phi^1\}\,,\qquad \boldsymbol{\delta}B=iq[B,\phi^1]-iq[\chi,\phi^2]\,,$$

$$\delta\phi^1 = i\phi^2 - \frac{1}{2}iq\{\phi^1, \phi^1\}, \qquad \delta\phi^2 = iq[\phi^2, \phi^1],$$

$$\delta\bar{c} = ib, \qquad \delta b = 0,$$

$$\delta\bar{\phi}^{-2} = \bar{\phi}^{-1} + iq[\bar{\phi}^{-2}, \phi^1], \qquad \delta\bar{\phi}^{-1} = -iq\{\bar{\phi}^{-1}, \phi^1\} + q[\bar{\phi}^{-2}, \phi^2], \tag{2}$$

where $D = d + iq[A, \cdot]$. Here ϕ^1 and $\bar{\phi}^{-1}$ are $\mathcal{G}$-valued anticommuting zero-forms, and b and ϕ^2 $\mathcal{G}$-valued commuting zero-forms. We can verify the nilpotency $\delta^2 = 0$. After being carried out the BRST transformation in (1), $\mathcal{L}_{\text{TYMT}}$ is written as

$$\mathcal{L}_{\text{TYMT}} = \sqrt{g}\,\mathrm{Tr}\Big[\frac{1}{2}B^{\mu\nu}F_{\mu\nu} + i\chi^{\mu\nu}D_\mu\psi^1_\nu - i\bar{\phi}^{-1}\mathcal{D}_\mu\psi^{1\mu} - \bar{\phi}^{-2}(\mathcal{D}_\mu D^\mu\phi^2 - q\{\psi^1_\mu, \psi^{1\mu}\})$$
$$- b\nabla_\mu A^\mu - i\bar{c}\nabla_\mu(D^\mu\phi^1 + \psi^{1\mu}) + \frac{1}{2}aB^{\mu\nu}B_{\mu\nu} - \frac{1}{2}aq\phi^2\{\chi^{\mu\nu}, \chi_{\mu\nu}\}\Big]. \tag{3}$$

Now we assume that the asymptotic fields of the relevant fields are governed by the quadratic part of $\mathcal{L}_{\text{TYMT}}$.[7] Then, from (2) and (3) in the $q=0$ case, we can show that, in the Lorentz frame $p_\mu = (|p_3|, 0, 0, p_3)$, each of $(A_{\text{as}\ i}, \psi'^1_{\text{as}\ i}; \chi_{\text{as}\ i3}, B_{\text{as}\ i3})$, $(A_{\text{as}\ 3}, \psi'^1_{\text{as}\ 3}; \bar{c}_{\text{as}}, b_{\text{as}})$ and $(\phi^1_{\text{as}}, \phi^2_{\text{as}}; \bar{\phi}^{-2}_{\text{as}}, \bar{\phi}^{-1}_{\text{as}})$ forms BRST quartet. Here $\psi'^1_{\text{as}\ \mu} \equiv \psi^1_{\text{as}\ \mu} + \partial_\mu\phi^1_{\text{as}}$ and $\mu = (i, 3)$ $(i = 0, 1, 2)$. The subscript "as" has been used to denote the asymptotic fields of the corresponding fields. Noting the absence of BRST singlets in the TYMT, we find that, as a result of the Kugo-Ojima quartet mechanism,[7] the subspace $\mathcal{V} = \{|\Phi\rangle;\ Q|\Phi\rangle = 0\}$ restricted by the BRST charge Q corresponding to δ does not contain nontrivial states corresponding to observable particles. The same results are also shown in the other CTFT's. Thus, from a physical standpoint, it seems that the CTFT's are not suitable for description of usual physical phenomena.

However, as is mentioned in Witten's articles,[1),5)] the CTFT's are fit to describe an "unbroken phase" in which general covariance and higher gauge symmetries are unbroken and hence all of degree of freedom are gauged away.*) Meaningful objects in the unbroken phase are topological ones alone. Since the unbroken phase is believed to be realized at extremely short distances, it is natural to consider how observable particles and local physical degree of freedom arise in a transition from the unbroken phase to a physical phase (broken phase) which is described by usual field theories. To understand this mechanism, it will be necessary, as a first step, to construct a theory that is applicable to both the unbroken and broken phases.

In the present paper we propose an extended version of the TYMT which, in a certain subspace of state-vector space, is a CTFT and which, in another certain subspace, is equivalent to a quantum anti-self-dual Yang-Mills theory, in which the Yang-Mills field would satisfy the anti-self-dual equation only on shell.[9)] (In abelian case, we can make an extended version that becomes equivalent to the usual quantum Maxwell theory.) The extended topological Yang-Mills theory (ETYMT) may be important for construction of a dynamical model that describes, from viewpoint of the CTFT, a transition from the unbroken phase to a physical phase (broken phase) containing a Yang-Mills field.

*) Independent of Witten's works, Shintani has discussed the large distance phenomena of the strong interactions on the basis of a certain type of CTFT.[8)]

Introducing infinite number of fields, we construct the ETYMT. The fields form *half-infinite chain representation*[10] of the BRST algebra that consists of BRST and anti-BRST transformations. In field theories studied so far which possess both BRST and anti-BRST symmetries,[7),10)~12)] the anti-BRST symmetry is only useful to discuss technical parts in the theories. However, in the ETYMT, anti-BRST symmetry has essential role, since BRST and anti-BRST transformations in the ETYMT are represented in a nonequivalent manner. We can obtain the quantum anti-self-dual Yang-Mills theory by virtue of confinement of surplus ghost fields based on the anti-BRST symmetry. Of course, the BRST symmetry is important for the ETYMT to obtain a topological situation.

In § 2, we give the classical Lagrangian of the ETYMT and, taking gauge symmetries into account, we define BRST and anti-BRST transformations. Introducing additional BRST transformations, we carry out in § 3, gauge fixing for the gauge symmetries. In § 4, we classify asymptotic fields following the representation theory of the BRST algebra and, using this result, we discuss properties of the ETYMT from standpoint of the covariant canonical quantization. The additional remarks are given in § 5.

§ 2. Lagrangian and symmetries

We begin our discussion with the following Lagrangian:

$$\mathcal{L}_1 = -i\sqrt{g}\,\boldsymbol{\delta}'_{\mathrm{T}}\mathrm{Tr}\Big[\frac{1}{2}\chi^{-1\mu\nu}F_{\mu\nu} + \sum_{n=1}^{\infty}\chi^{-2n-1\mu\nu}\Big(D_\mu\phi_\nu^{2n} + \frac{1}{2}iq\sum_{i=1}^{n-1}[\phi_\mu^{2i},\ \phi_\nu^{2(n-i)}]\Big)\Big], \qquad (4)$$

where the ϕ^{2n}'s are $\mathcal{G}$-valued commuting one-forms on M^4, and the χ^{-2n-1}'s $\mathcal{G}$-valued anticommuting self-dual two-forms on M^4. For the sake of convenience, we may hereafter use ϕ^0 to denote A. The *topological BRST transformation* $\boldsymbol{\delta}'_{\mathrm{T}}$ is defined by

$$\boldsymbol{\delta}'_{\mathrm{T}}\phi^{2n} = \phi^{2n+1}, \quad \boldsymbol{\delta}'_{\mathrm{T}}\phi^{2n+1} = 0, \quad \boldsymbol{\delta}'_{\mathrm{T}}\chi^{-2n-1} = i\chi^{-2n}, \quad \boldsymbol{\delta}'_{\mathrm{T}}\chi^{-2n} = 0, \qquad (n = 0, 1, 2, 3, \cdots) \qquad (5)$$

where the ϕ^{2n+1}'s are $\mathcal{G}$-valued anticommuting one-forms and the χ^{-2n}'s $\mathcal{G}$-valued commuting self-dual two-forms. The superscripts of the fields refer to *T ghost numbers* of the corresponding fields. Carrying out the topological BRST transformation in (4), we obtain

$$\mathcal{L}_1 = \sqrt{g}\,\mathrm{Tr}\Big[\frac{1}{2}\chi^{0\mu\nu}F_{\mu\nu} + \sum_{n=1}^{\infty}\chi^{-2n\mu\nu}\Big(D_\mu\phi_\nu^{2n} + \frac{1}{2}iq\sum_{i=1}^{n-1}[\phi_\mu^{2i},\ \phi_\nu^{2(n-i)}]\Big) + i\sum_{n=1}^{\infty}\chi^{-2n+1\mu\nu}\Big(D_\mu\phi_\nu^{2n-1} + iq\sum_{i=1}^{n-1}[\phi_\mu^{2i-1},\ \phi_\nu^{2(n-i)}]\Big)\Big]. \qquad (6)$$

In addition to $\boldsymbol{\delta}'_{\mathrm{T}}$, we also define *topological anti-BRST transformation* $\bar{\boldsymbol{\delta}}'_{\mathrm{T}}$ by

$$\bar{\boldsymbol{\delta}}'_{\mathrm{T}}\phi^0 = \bar{\boldsymbol{\delta}}'_{\mathrm{T}}\phi^1 = 0, \qquad \bar{\boldsymbol{\delta}}'_{\mathrm{T}}\phi^{n+2} = \boldsymbol{\delta}'_{\mathrm{T}}\phi^n, \qquad \bar{\boldsymbol{\delta}}'_{\mathrm{T}}\chi^{-n} = \boldsymbol{\delta}'_{\mathrm{T}}\chi^{-n-2}. \qquad (7)$$

Then each of the two sets ϕ^n's and χ^{-n}'s forms the half-infinite chain representation[10] of the BRST algebra $\boldsymbol{\delta}'^2_{\mathrm{T}} = \bar{\boldsymbol{\delta}}'^2_{\mathrm{T}} = \{\boldsymbol{\delta}'_{\mathrm{T}}, \bar{\boldsymbol{\delta}}'_{\mathrm{T}}\} = 0$. We note that although $\mathcal{L}_1$ cannot be written in the form of anti-BRST coboundary term, such as $\bar{\boldsymbol{\delta}}'_{\mathrm{T}}$ (*), it is invariant under $\bar{\boldsymbol{\delta}}'_{\mathrm{T}}$.

The Lagrangian $\mathcal{L}_1$ is also invariant under the following gauge transformation:

$$\delta_{\mathrm{P}}^{A}\psi^{2n}=D\Lambda^{2n}+iq\sum_{i=1}^{n}[\psi^{2i},\Lambda^{2(n-i)}]\,,\quad \delta_{\mathrm{P}}^{A}\psi^{2n+1}=iq\sum_{i=0}^{n}[\psi^{2i+1},\Lambda^{2(n-i)}]\,,$$

$$\delta_{\mathrm{P}}^{A}\chi^{-2n}=iq\sum_{i=0}^{\infty}[\chi^{-2(n+i)},\Lambda^{2i}]\,,\qquad \delta_{\mathrm{P}}^{A}\chi^{-2n-1}=iq\sum_{i=0}^{\infty}[\chi^{-2(n+i)-1},\Lambda^{2i}]\,, \tag{8}$$

where the Λ^{2n}'s are $\mathcal{G}$-valued commuting parameters. We hereafter call the gauge transformation δ_{P}^{A} *primary gauge transformation*. Furthermore, $\mathcal{L}_1$ is invariant under the following gauge transformation generated by $\mathcal{G}$-valued anticommuting parameters Λ^{2n+1}'s:

$$\delta_{\mathrm{S}}^{A}\psi^{2n}=0\,,\quad \delta_{\mathrm{S}}^{A}\psi^{2n+1}=D\Lambda^{2n+1}+iq\sum_{i=1}^{n}[\psi^{2i},\Lambda^{2(n-i)+1}]\,,$$

$$\delta_{\mathrm{S}}^{A}\chi^{-2n}=-q\sum_{i=0}^{\infty}\{\chi^{-2(n+i)-1},\Lambda^{2i+1}\}\,,\quad \delta_{\mathrm{S}}^{A}\chi^{-2n-1}=0\,. \tag{9}$$

The transformation δ_{S}^{A} arises in association with the primary gauge transformation, and we call δ_{S}^{A} *secondary gauge transformation*. As is done in the TYMT, we modify the topological BRST transformation so as to include the primary and secondary gauge transformations. This is achieved by combining $\boldsymbol{\delta}'_{\mathrm{T}}$ and BRST transformations which are obtained by replacing the each parameter Λ^{n} of the gauge transformations δ_{P}^{A} and δ_{S}^{A} with a $\mathcal{G}$-valued zero-form ϕ^{n+1}. Here ϕ^{2n+1}'s are anticommuting and $\phi^{2(n+1)}$'s are commuting zero-forms. As a result, we have the "modified" topological BRST transformation

$$\boldsymbol{\delta}_{\mathrm{T}}\psi^{2n}=\psi^{2n+1}+D\phi^{2n+1}+iq\sum_{i=1}^{n}[\psi^{2i},\phi^{2(n-i)+1}]\,,$$

$$\boldsymbol{\delta}_{\mathrm{T}}\psi^{2n+1}=-iD\phi^{2(n+1)}+q\sum_{i=1}^{n}[\psi^{2i},\phi^{2(n-i+1)}]-iq\sum_{i=0}^{n}\{\psi^{2i+1},\phi^{2(n-i)+1}\}\,,$$

$$\boldsymbol{\delta}_{\mathrm{T}}\chi^{-2n}=iq\sum_{i=0}^{\infty}([\chi^{-2(n+i)},\phi^{2i+1}]-[\chi^{-2(n+i)-1},\phi^{2(i+1)}])\,,$$

$$\boldsymbol{\delta}_{\mathrm{T}}\chi^{-2n-1}=i\chi^{-2n}-iq\sum_{i=0}^{\infty}\{\chi^{-2(n+i)-1},\phi^{2i+1}\}\,. \tag{10}$$

The topological BRST transformation of ϕ^{n+1}'s is determined to be

$$\boldsymbol{\delta}_{\mathrm{T}}\phi^{2n+1}=i\phi^{2(n+1)}-\frac{1}{2}iq\sum_{i=0}^{n}\{\phi^{2i+1},\phi^{2(n-i)+1}\}\,,$$

$$\boldsymbol{\delta}_{\mathrm{T}}\phi^{2(n+1)}=iq\sum_{i=0}^{n}[\phi^{2(i+1)},\phi^{2(n-i)+1}]\,. \tag{11}$$

We define "modified" topological anti-BRST transformation $\bar{\boldsymbol{\delta}}_{\mathrm{T}}$ by

$$\bar{\boldsymbol{\delta}}_{\mathrm{T}}\psi^{0}=\bar{\boldsymbol{\delta}}_{\mathrm{T}}\psi^{1}=0\,,\quad \bar{\boldsymbol{\delta}}_{\mathrm{T}}\psi^{n+2}=\boldsymbol{\delta}_{\mathrm{T}}\psi^{n}\,,\quad \bar{\boldsymbol{\delta}}_{\mathrm{T}}\chi^{-n}=\boldsymbol{\delta}_{\mathrm{T}}\chi^{-n-2}\,,$$

$$\bar{\boldsymbol{\delta}}_{\mathrm{T}}\phi^{1}=\bar{\boldsymbol{\delta}}_{\mathrm{T}}\phi^{2}=0\,,\quad \bar{\boldsymbol{\delta}}_{\mathrm{T}}\phi^{n+3}=\boldsymbol{\delta}_{\mathrm{T}}\phi^{n+1}\,. \tag{12}$$

The transformation rules (10)~(12) satisfy the BRST algebra

$$\boldsymbol{\delta}_T{}^2 = \bar{\boldsymbol{\delta}}_T{}^2 = \{\boldsymbol{\delta}_T, \bar{\boldsymbol{\delta}}_T\} = 0 . \tag{13}$$

Even though $\boldsymbol{\delta}_T'$ in (4) is replaced with $\boldsymbol{\delta}_T$, the Lagrangian $\mathcal{L}_1$ itself does not change; $\mathcal{L}_1$ can be written as

$$\mathcal{L}_1 = -i\sqrt{g}\,\boldsymbol{\delta}_T \mathrm{Tr}\Big[\frac{1}{2}\chi^{-1\mu\nu}F_{\mu\nu} + \sum_{n=1}^{\infty}\chi^{-2n-1\mu\nu}\Big(D_\mu\phi_\nu^{2n} + \frac{1}{2}iq\sum_{i=1}^{n-1}[\phi_\mu^{2i}, \phi_\nu^{2(n-i)}]\Big)\Big] . \tag{14}$$

Clearly, $\mathcal{L}_1$ is invariant under $\boldsymbol{\delta}_T$. The invariance of $\mathcal{L}_1$ under $\bar{\boldsymbol{\delta}}_T$ can readily be seen by rewritting $\mathcal{L}_1$ as

$$\mathcal{L}_1 = \sqrt{g}\,\mathrm{Tr}\Big[\frac{1}{2}\chi'^{0\mu\nu}F_{\mu\nu} - i\bar{\boldsymbol{\delta}}_T\Big\{\sum_{n=1}^{\infty}\chi^{-2n+1\mu\nu}\Big(D_\mu\phi_\nu^{2n} + \frac{1}{2}iq\sum_{i=1}^{n-1}[\phi_\mu^{2i}, \phi_\nu^{2(n-i)}]\Big)\Big\}\Big] , \tag{15}$$

where $\chi'^0 \equiv -i\boldsymbol{\delta}_T\chi^{-1}$. The Lagrangian $\mathcal{L}_1$ includes the first term of $\mathcal{L}_{\mathrm{TYMT}}$, $-i\sqrt{g}\,\boldsymbol{\delta}\mathrm{Tr}[(1/2)\chi^{\mu\nu}F_{\mu\nu}]$. The BRST transformation rules of A, ψ^1, ϕ^1 and ϕ^2 in (10) and (11) are the same as in (2). From these facts, we can regard our theory as an extension of the TYMT.

§ 3. Gauge fixing

In order to quantize our theory, we have to carry out gauge fixing for the primary and secondary gauge transformations. After determination of gauge conditions, we choose between the following two methods to construct gauge fixing terms: (i) Introducing antighosts and multiplier fields, we construct a gauge fixing term in the form of topological-BRST coboundary term. (ii) Introducing a new BRST transformation, new ghosts, anti-ghosts and multiplier fields, we construct a gauge fixing term in the form of coboundary term with respect to both the topological and new BRST transformations. The method (i) has been used in Ref. 6). The method (ii) is the procedure that Horne has proposed to fix the Yang-Mills gauge symmetry in the TYMT.[13] (In § 1, we have used the method (i) for simplicity.) In this section, we apply the method (ii) to both of the primary and secondary gauge transformations, since the use of (ii) is essential to our discussion, as we will see in the next section.

(I) *Gauge fixing for the secondary gauge transformation*

Let us introduce new BRST transformation $\boldsymbol{\delta}_S$ and $\mathcal{G}$-valued commuting zero-forms γ^{2n+1}'s, $\bar{\gamma}^{-2n-1}$'s and $\beta^{-2(n+1)}$'s, and $\mathcal{G}$-valued anticommuting zero-forms $\gamma^{2(n+1)}$'s, $\bar{\gamma}^{-2(n+1)}$'s and β^{-2n-1}'s. We call $\boldsymbol{\delta}_S$ *secondary BRST transformation.* We assign *S ghost numbers* 1, -1 and 0 to γ^{n+1}'s, $\bar{\gamma}^{-n-1}$'s and β^{-n-1}'s, respectively, while to the first stage fields, namely ψ^n's, χ^{-n}'s and ϕ^{n+1}'s, we assign S ghost number zero. The secondary BRST transformation of the first stage fields is given by

$$\boldsymbol{\delta}_S\psi^{2n} = 0 , \quad \boldsymbol{\delta}_S\psi^{2n+1} = -iD\gamma^{2n+1} + q\sum_{i=1}^{n}[\phi^{2i}, \gamma^{2(n-i)+1}] ,$$

$$\boldsymbol{\delta}_S\chi^{-2n} = -iq\sum_{i=0}^{\infty}[\chi^{-2(n+i)-1}, \gamma^{2i+1}] , \quad \boldsymbol{\delta}_S\chi^{-2n-1} = 0 ,$$

$$\boldsymbol{\delta}_S \phi^{2n+1} = i\gamma^{2n+1}\,, \quad \boldsymbol{\delta}_S \phi^{2(n+1)} = -\gamma^{2(n+1)}\,. \tag{16}$$

Here we have defined $\boldsymbol{\delta}_S \phi^n$ and $\boldsymbol{\delta}_S \chi^{-n}$ by replacing the each parameter Λ^{2n+1} of the secondary gauge transformation with the new ghost γ^{2n+1}, and hence the Lagrangian $\mathcal{L}_1$ is invariant under $\boldsymbol{\delta}_S$. The second stage fields, namely γ^{n+1}'s, $\bar{\gamma}^{-n-1}$'s and β^{-n-1}'s, obey the following transformation rules:

$$\boldsymbol{\delta}_S \gamma^{2n+1} = \boldsymbol{\delta}_S \gamma^{2(n+1)} = 0\,, \quad \boldsymbol{\delta}_S \bar{\gamma}^{-2n-1} = \beta^{-2n-1} - iq \sum_{i=0}^{\infty} [\bar{\gamma}^{-2(n+i+1)}, \gamma^{2i+1}]\,,$$

$$\boldsymbol{\delta}_S \bar{\gamma}^{-2(n+1)} = -i\beta^{-2(n+1)}\,, \quad \boldsymbol{\delta}_S \beta^{-2n-1} = q \sum_{i=0}^{\infty} [\beta^{-2(n+i+1)}, \gamma^{2i+1}]\,, \quad \boldsymbol{\delta}_S \beta^{-2(n+1)} = 0\,. \tag{17}$$

The topological BRST transformation of the second stage fields is defined by

$$\boldsymbol{\delta}_T \gamma^{2n+1} = \gamma^{2(n+1)} + iq \sum_{i=0}^{n} [\gamma^{2i+1}, \phi^{2(n-i)+1}]\,,$$

$$\boldsymbol{\delta}_T \gamma^{2(n+1)} = q \sum_{i=0}^{n} ([\gamma^{2i+1}, \phi^{2(n-i+1)}] - i\{\gamma^{2(i+1)}, \phi^{2(n-i)+1}\})\,,$$

$$\boldsymbol{\delta}_T \bar{\gamma}^{-2n-1} = iq \sum_{i=0}^{\infty} ([\bar{\gamma}^{-2(n+i)-1}, \phi^{2i+1}] - [\bar{\gamma}^{-2(n+i+1)}, \phi^{2(i+1)}])\,,$$

$$\boldsymbol{\delta}_T \bar{\gamma}^{-2(n+1)} = i\bar{\gamma}^{-2n-1} - iq \sum_{i=0}^{\infty} \{\bar{\gamma}^{-2(n+i+1)}, \phi^{2i+1}\}\,,$$

$$\boldsymbol{\delta}_T \beta^{-2n-1} = q \sum_{i=0}^{\infty} (-i\{\beta^{-2(n+i)-1}, \phi^{2i+1}\} + [\beta^{-2(n+i+1)}, \phi^{2(i+1)}])\,,$$

$$\boldsymbol{\delta}_T \beta^{-2(n+1)} = \beta^{-2n-1} + iq \sum_{i=0}^{\infty} [\beta^{-2(n+i+1)}, \phi^{2i+1}]\,, \tag{18}$$

with which we define the topological anti-BRST transformation of the second stage fields by

$$\bar{\boldsymbol{\delta}}_T \gamma^1 = \bar{\boldsymbol{\delta}}_T \gamma^2 = 0\,, \qquad \bar{\boldsymbol{\delta}}_T \gamma^{n+3} = \boldsymbol{\delta}_T \gamma^{n+1}\,,$$

$$\bar{\boldsymbol{\delta}}_T \bar{\gamma}^{-n-1} = \boldsymbol{\delta}_T \bar{\gamma}^{-n-3}\,, \quad \bar{\boldsymbol{\delta}}_T \beta^{-n-1} = \boldsymbol{\delta}_T \beta^{-n-3}\,. \tag{19}$$

The transformation rules (18) and (19) satisfy the BRST algebra (13). We can verify the relations

$$\boldsymbol{\delta}_S{}^2 = \{\boldsymbol{\delta}_T, \boldsymbol{\delta}_S\} = \{\bar{\boldsymbol{\delta}}_T, \boldsymbol{\delta}_S\} = 0 \tag{20}$$

for all the first and second stage fields.

Now we choose a gauge fixing term for the secondary gauge transformation δ_S^A so that it, as one of gauge conditions, gives the gauge condition $\mathcal{D}_\mu \phi^{1\mu} = 0$ derived from (3), and so that it remains invariant under the (anti)BRST transformations $\boldsymbol{\delta}_T$, $\bar{\boldsymbol{\delta}}_T$, and $\boldsymbol{\delta}_S$. Such a gauge fixing term is indeed given by

$$\mathcal{L}_2 = \sqrt{g}\, \boldsymbol{\delta}_T \boldsymbol{\delta}_S \mathrm{Tr}[\sum_{n=0}^{\infty} \bar{\gamma}^{-2(n+1)} (\mathcal{D}_\mu \phi^{2n+1\mu} + iq \sum_{i=1}^{n} [\phi_\mu^{2i}, \phi^{2(n-i)+1\mu}])]\,. \tag{21}$$

The invariance of $\mathcal{L}_1$ under δ_S^A is broken by adding $\mathcal{L}_2$ to $\mathcal{L}_1$. Carrying out the BRST transformations in (21), we obtain

$$\mathcal{L}_2=\sqrt{g}\sum_{n=0}^{\infty}\mathrm{Tr}[-i\beta^{-2n-1}(\mathcal{D}_\mu\psi^{2n+1\mu}+iq\sum_{i=1}^{n}[\psi_\mu^{2i},\psi^{2(n-i)+1\mu}])$$
$$-\beta^{-2(n+1)}(\mathcal{D}_\mu D^\mu\phi^{2(n+1)}-q\sum_{i=0}^{n}\{\psi_\mu^{2i+1},\psi^{2(n-i)+1\mu}\}+iq\sum_{i=1}^{n}[\mathcal{D}_\mu\psi^{2i\mu},\phi^{2(n-i+1)}]$$
$$+2iq\sum_{i=1}^{n}[\psi_\mu^{2i},D^\mu\phi^{2(n-i+1)}]-q^2\sum_{i=1}^{n}\sum_{j=1}^{i-1}[\psi_\mu^{2j},[\psi^{2(i-j)\mu},\phi^{2(n-i+1)}]])$$
$$-\bar{\gamma}^{-2n-1}(\mathcal{D}_\mu D^\mu\gamma^{2n+1}+iq\sum_{i=1}^{n}[\mathcal{D}_\mu\psi^{2i\mu},\gamma^{2(n-i)+1}]$$
$$+2iq\sum_{i=1}^{n}[\psi_\mu^{2i},D^\mu\gamma^{2(n-i)+1}]-q^2\sum_{i=1}^{n}\sum_{j=1}^{i-1}[\psi_\mu^{2j},[\psi^{2(i-j)\mu},\gamma^{2(n-i)+1}]])$$
$$-i\bar{\gamma}^{-2(n+1)}(\mathcal{D}_\mu D^\mu\gamma^{2(n+1)}+iq\sum_{i=0}^{n}[\mathcal{D}_\mu\psi^{2i+1\mu},\gamma^{2(n-i)+1}]$$
$$+iq\sum_{i=1}^{n}[\mathcal{D}_\mu\psi^{2i\mu},\gamma^{2(n-i+1)}]+2iq\sum_{i=0}^{n}[\psi_\mu^{2i+1},D^\mu\gamma^{2(n-i)+1}]$$
$$+2iq\sum_{i=1}^{n}[\psi_\mu^{2i},D^\mu\gamma^{2(n-i+1)}]-q^2\sum_{i=1}^{n}\sum_{j=0}^{i-1}[\psi_\mu^{2j+1},[\psi^{2(i-j)\mu},\gamma^{2(n-i)+1}]]$$
$$-q^2\sum_{i=1}^{n}\sum_{j=1}^{i}[\psi_\mu^{2j},[\psi^{2(i-j)+1\mu},\gamma^{2(n-i)+1}]]-q^2\sum_{i=1}^{n}\sum_{j=1}^{i-1}[\psi_\mu^{2j},[\psi^{2(i-j)\mu},\gamma^{2(n-i+1)}]])]\,. \quad (22)$$

It should be noted that $\mathcal{L}_2$, as well as $\mathcal{L}_1$, does not contain ϕ^{2n+1}'s and is invariant under the primary gauge transformation δ_P^A defined by (8) and

$$\delta_P^A\phi^{2n+1}=iq\sum_{i=0}^{n}[\phi^{2i+1},\Lambda^{2(n-i)}]\,,\qquad \delta_P^A\phi^{2(n+1)}=iq\sum_{i=0}^{n}[\phi^{2(i+1)},\Lambda^{2(n-i)}]\,,$$
$$\delta_P^A\gamma^{2n+1}=iq\sum_{i=0}^{n}[\gamma^{2i+1},\Lambda^{2(n-i)}]\,,\qquad \delta_P^A\gamma^{2(n+1)}=iq\sum_{i=0}^{n}[\gamma^{2(i+1)},\Lambda^{2(n-i)}]\,,$$
$$\delta_P^A\bar{\gamma}^{-2n-1}=iq\sum_{i=0}^{\infty}[\bar{\gamma}^{-2(n+i)-1},\Lambda^{2i}]\,,\qquad \delta_P^A\bar{\gamma}^{-2(n+1)}=iq\sum_{i=0}^{\infty}[\bar{\gamma}^{-2(n+i+1)},\Lambda^{2i}]\,,$$
$$\delta_P^A\beta^{-2n-1}=iq\sum_{i=0}^{\infty}[\beta^{-2(n+i)-1},\Lambda^{2i}]\,,\qquad \delta_P^A\beta^{-2(n+1)}=iq\sum_{i=0}^{\infty}[\beta^{-2(n+i+1)},\Lambda^{2i}]\,. \quad (23)$$

From this fact, $\mathcal{L}_1+\mathcal{L}_2$ is a "basic" quantity in the sense of a certain basic cohomology,[14),15)] which is important for examining non-triviality of observables (and topological invariants) in the ETYMT.

The gauge fixing term $\mathcal{L}_2$ can be written as

$$\mathcal{L}_2=-\sqrt{g}\,\boldsymbol{\delta}_S\mathrm{Tr}[i\bar{\gamma}'^{-1}\mathcal{D}_\mu\psi^{1\mu}+\bar{\boldsymbol{\delta}}_T\{\sum_{n=1}^{\infty}\bar{\gamma}^{-2n}(\mathcal{D}_\mu\psi^{2n+1\mu}+iq\sum_{i=1}^{n}[\psi_\mu^{2i},\psi^{2(n-i)+1\mu}])\}]\,, \quad (24)$$

where $\bar{\gamma}'^{-1}\equiv -i\boldsymbol{\delta}_T\bar{\gamma}^{-2}$. From (24), we see that $\mathcal{L}_2$ is invariant under $\bar{\boldsymbol{\delta}}_T$.

(II) *Gauge fixing for the primary gauge transformation*

As is done in (I), we introduce new BRST transformation $\boldsymbol{\delta}_P$ and $\mathcal{G}$-valued commuting zero-forms c^{2n+1}'s, $\bar{c}^{-2n-1}$'s and b^{-2n}'s, and $\mathcal{G}$-valued anticommuting zero-forms c^{2n}'s, $\bar{c}^{-2n}$'s and b^{-2n-1}'s. We hereafter call δ_P *primary BRST transformation.* We assign *P ghost numbers* 1, -1 and 0 to c^n's, $\bar{c}^{-n}$'s and b^{-n}'s, respectively and assign S ghost number zero to them. The first and second stage fields are assigned zero P ghost number. The primary BRST transformation of the first and second stage fields is defined by replacing the each parameter Λ^{2n} of the primary gauge transformation with the new ghost c^{2n}. We thus have

$$\boldsymbol{\delta}_P\psi^{2n}=Dc^{2n}+iq\sum_{i=1}^{n}[\psi^{2i},c^{2(n-i)}]\,,\qquad \boldsymbol{\delta}_P\psi^{2n+1}=-iq\sum_{i=0}^{n}\{\psi^{2i+1},c^{2(n-i)}\}\,,$$

$$\boldsymbol{\delta}_P\chi^{-2n}=iq\sum_{i=0}^{\infty}[\chi^{-2(n+i)},c^{2i}]\,,\qquad \boldsymbol{\delta}_P\chi^{-2n-1}=-iq\sum_{i=0}^{\infty}\{\chi^{-2(n+i)-1},c^{2i}\}\,,$$

$$\boldsymbol{\delta}_P\phi^{2n+1}=-ic^{2n+1}-iq\sum_{i=0}^{n}\{\phi^{2i+1},c^{2(n-i)}\}\,,\qquad \boldsymbol{\delta}_P\phi^{2(n+1)}=iq\sum_{i=0}^{n}[\phi^{2(i+1)},c^{2(n-i)}]\tag{25}$$

and

$$\boldsymbol{\delta}_P\gamma^{2n+1}=iq\sum_{i=0}^{n}[\gamma^{2i+1},c^{2(n-i)}]\,,\qquad \boldsymbol{\delta}_P\gamma^{2(n+1)}=-iq\sum_{i=0}^{n}\{\gamma^{2(i+1)},c^{2(n-i)}\}\,,$$

$$\boldsymbol{\delta}_P\bar{\gamma}^{-2n-1}=iq\sum_{i=0}^{\infty}[\bar{\gamma}^{-2(n+i)-1},c^{2i}]\,,\qquad \boldsymbol{\delta}_P\bar{\gamma}^{-2(n+1)}=-iq\sum_{i=0}^{\infty}\{\bar{\gamma}^{-2(n+i+1)},c^{2i}\}\,,$$

$$\boldsymbol{\delta}_P\beta^{-2n-1}=-iq\sum_{i=0}^{\infty}\{\beta^{-2(n+i)-1},c^{2i}\}\,,\qquad \boldsymbol{\delta}_P\beta^{-2(n+1)}=iq\sum_{i=0}^{\infty}[\beta^{-2(n+i+1)},c^{2i}]\,.\tag{26}$$

Here $\boldsymbol{\delta}_P\phi^{2n+1}$ has been modified by adding $-ic^{2n+1}$ to the original one. It is evident that $\mathcal{L}_1$ and $\mathcal{L}_2$ are invariant under $\boldsymbol{\delta}_P$. The third stage fields, namely c^n's, $\bar{c}^{-n}$'s and b^{-n}'s, obey the following transformation rules:

$$\boldsymbol{\delta}_P c^{2n}=-\frac{1}{2}iq\sum_{i=0}^{n}\{c^{2i},c^{2(n-i)}\}\,,\quad \boldsymbol{\delta}_P c^{2n+1}=iq\sum_{i=0}^{n}[c^{2i+1},c^{2(n-i)}]\,,$$

$$\boldsymbol{\delta}_P\bar{c}^{-2n}=ib^{-2n}\,,\quad \boldsymbol{\delta}_P\bar{c}^{-2n-1}=-b^{-2n-1}\,,\quad \boldsymbol{\delta}_P b^{-2n}=\boldsymbol{\delta}_P b^{-2n-1}=0\,.\tag{27}$$

The topological BRST transformation of the third stage fields is defined by

$$\boldsymbol{\delta}_T c^{2n}=ic^{2n+1}\,,\qquad \boldsymbol{\delta}_T c^{2n+1}=0\,,\qquad \boldsymbol{\delta}_T\bar{c}^{-2n}=0\,,$$

$$\boldsymbol{\delta}_T\bar{c}^{-2n-1}=\bar{c}^{-2n}\,,\quad \boldsymbol{\delta}_T b^{-2n}=0\,,\qquad \boldsymbol{\delta}_T b^{-2n-1}=ib^{-2n}\,,\tag{28}$$

with which we define the topological anti-BRST transformation of the third stage fields by

$$\bar{\boldsymbol{\delta}}_T c^0=\bar{\boldsymbol{\delta}}_T c^1=0\,,\qquad \bar{\boldsymbol{\delta}}_T c^{n+2}=\boldsymbol{\delta}_T c^n\,,$$

$$\bar{\boldsymbol{\delta}}_T\bar{c}^{-n}=\boldsymbol{\delta}_T\bar{c}^{-n-2}\,,\quad \bar{\boldsymbol{\delta}}_T b^{-n}=\boldsymbol{\delta}_T b^{-n-2}\,.\tag{29}$$

In addition, we set

$$\boldsymbol{\delta}_S c^n=\boldsymbol{\delta}_S\bar{c}^{-n}=\boldsymbol{\delta}_S b^{-n}=0\,.\tag{30}$$

The transformation rules (28)~(30) then satisfy the relations (13) and (20). We can verify the relations

$$\boldsymbol{\delta}_{\mathrm{P}}{}^2=\{\boldsymbol{\delta}_{\mathrm{T}}, \boldsymbol{\delta}_{\mathrm{P}}\}=\{\bar{\boldsymbol{\delta}}_{\mathrm{T}}, \boldsymbol{\delta}_{\mathrm{P}}\}=\{\boldsymbol{\delta}_{\mathrm{S}}, \boldsymbol{\delta}_{\mathrm{P}}\}=0 \tag{31}$$

for all the fields. As a result, the (anti)BRST transformations $\boldsymbol{\delta}_{\mathrm{T}}$, $\bar{\boldsymbol{\delta}}_{\mathrm{T}}$, $\boldsymbol{\delta}_{\mathrm{S}}$ and $\boldsymbol{\delta}_{\mathrm{P}}$ are off-shell nilpotent and anticommute one another.

As a gauge fixing term for the primary gauge transformation, we take

$$\mathcal{L}_3=-i\sqrt{g}\,\boldsymbol{\delta}_{\mathrm{T}}\boldsymbol{\delta}_{\mathrm{P}}\mathrm{Tr}[\sum_{n=0}^{\infty}\bar{c}^{-2n-1}\nabla_\mu\phi^{2n\mu}] \tag{32}$$

so as to obtain the Landau gauge condition $\nabla_\mu A^\mu=0$. After being carried out the BRST transformations, $\mathcal{L}_3$ is written as

$$\begin{aligned}\mathcal{L}_3=\sqrt{g}\sum_{n=0}^{\infty}\mathrm{Tr}[&-b^{-2n}\nabla_\mu\phi^{2n\mu}-ib^{-2n-1}\nabla_\mu\phi'^{2n+1\mu}\\ &-i\bar{c}^{-2n}\nabla_\mu(D^\mu c^{2n}+iq\sum_{i=1}^{n}[\phi^{2i\mu}, c^{2(n-i)}])\\ &+\bar{c}^{-2n-1}\nabla_\mu(D^\mu c^{2n+1}+iq\sum_{i=1}^{n}[\phi^{2i\mu}, c^{2(n-i)+1}]+q\sum_{i=0}^{n}\{\phi'^{2i+1\mu}, c^{2(n-i)}\})]\end{aligned} \tag{33}$$

with

$$\phi'^{2n+1}\equiv\phi^{2n+1}+D\phi^{2n+1}+iq\sum_{i=1}^{n}[\phi^{2i}, \phi^{2(n-i)+1}]\,.$$

Similar to $\mathcal{L}_1$ and $\mathcal{L}_2$, $\mathcal{L}_3$ can also be written as

$$\mathcal{L}_3=i\sqrt{g}\,\boldsymbol{\delta}_{\mathrm{P}}\mathrm{Tr}[\bar{c}^0\nabla_\mu\phi^{0\mu}+\bar{\boldsymbol{\delta}}_{\mathrm{T}}\{\sum_{n=1}^{\infty}\bar{c}^{-2n+1}\nabla_\mu\phi^{2n\mu}\}]\,. \tag{34}$$

We easily find that $\mathcal{L}_3$ is invariant under all the (anti)BRST transformations. The invariance of $\mathcal{L}_1+\mathcal{L}_2$ under the primary gauge transformation is broken by adding $\mathcal{L}_3$ to $\mathcal{L}_1+\mathcal{L}_2$, and hence we can take the total Lagrangian

$$\mathcal{L}=\mathcal{L}_1+\mathcal{L}_2+\mathcal{L}_3 \tag{35}$$

as a Lagrangian for quantum theory.

The total Lagrangian $\mathcal{L}$ never yields the Yang-Mills equation of A. If

$$\mathcal{L}_4=-i\sqrt{g}\,\boldsymbol{\delta}_{\mathrm{T}}\mathrm{Tr}\left[\frac{1}{8}\chi'^{0\mu\nu}\chi^{-1}_{\mu\nu}\right]=\sqrt{g}\,\mathrm{Tr}\left[\frac{1}{8}\chi'^{0\mu\nu}\chi'^{0}_{\mu\nu}\right] \tag{36}$$

is added to $\mathcal{L}$, we can derive from it the Yang-Mills equation after the elimination of χ'^0. However, although $\mathcal{L}_4$ is invariant under $\boldsymbol{\delta}_{\mathrm{T}}$, $\bar{\boldsymbol{\delta}}_{\mathrm{T}}$ and $\boldsymbol{\delta}_{\mathrm{S}}$, it is not invariant under $\boldsymbol{\delta}_{\mathrm{P}}$ except for the abelian case $G=[U(1)]^m$. For this reason, the use of $\mathcal{L}+\mathcal{L}_4$ is not allowed in the case with non-abelian gauge group.

§ 4. Classification of asymptotic fields and properties of the theory*)

In a field theory having the BRST symmetry, any asymptotic field is classified into BRST-singlet or BRST-doublet according to its BRST transformation property.[7),10)]

*) In this section, we specialize M^4 to an asymptotically flat manifold in order to treat asymptotic fields as free fields.

It is reasonable to assume that the asymptotic fields are governed by the quadratic part of the Lagrangian of the theory.*) If gauge fixing has been completely done in a BRST invariant manner, there necessarily exists a BRST-doublet FP-conjugate to every BRST-doublet. We can find from the quadratic part of Lagrangian which two BRST-doublets constitute FP-conjugate pair called BRST-quartet. In what follows, we denote BRST-quartet as $(p, d; p', d')$, where p and p' (d and d') stand for parent fields (daughter fields), and two doublets (p, d) and (p', d') constitute a FP-conjugate pair.

In our theory, the Lagrangian for the asymptotic fields is given by

$$\begin{aligned}\mathcal{L}_{\mathrm{quad}}=\sum_{n=0}^{\infty}\mathrm{Tr}[&\chi_{\mathrm{as}}^{-2n\mu\nu}\partial_\mu\psi_{\mathrm{as}\ \nu}^{2n}+i\chi_{\mathrm{as}}^{-2n-1\mu\nu}\partial_\mu\psi'^{2n+1}_{\mathrm{as}\ \nu}-i\partial_\mu\beta_{\mathrm{as}}^{-2n-1}\partial^\mu\phi_{\mathrm{as}}^{2n+1}\\&+\partial_\mu\beta_{\mathrm{as}}^{-2(n+1)}\partial^\mu\phi_{\mathrm{as}}^{2(n+1)}+\partial_\mu\bar{\gamma}_{\mathrm{as}}^{-2n-1}\partial^\mu\gamma_{\mathrm{as}}^{2n+1}+i\partial_\mu\bar{\gamma}_{\mathrm{as}}^{-2(n+1)}\partial^\mu\gamma_{\mathrm{as}}^{2(n+1)}\\&+\psi_{\mathrm{as}\ \mu}^{2n}\partial^\mu b_{\mathrm{as}}^{-2n}-i\psi'^{2n+1}_{\mathrm{as}\ \mu}\partial^\mu b'^{-2n-1}_{\mathrm{as}}+i\partial_\mu\bar{c}_{\mathrm{as}}^{-2n}\partial^\mu c_{\mathrm{as}}^{2n}\\&-\partial_\mu\bar{c}_{\mathrm{as}}^{-2n-1}\partial^\mu c_{\mathrm{as}}^{2n+1}]\,,\end{aligned}\tag{37}$$

where $\psi'^{2n+1}_{\mathrm{as}\ \mu}\equiv\psi^{2n+1}_{\mathrm{as}\ \mu}+\partial_\mu\phi_{\mathrm{as}}^{2n+1}$ and $b'^{-2n-1}_{\mathrm{as}}\equiv b_{\mathrm{as}}^{-2n-1}+\beta_{\mathrm{as}}^{-2n-1}$. From $\boldsymbol{\delta}_{\mathrm{T}}$ in the $q=0$ case and $\mathcal{L}_{\mathrm{quad}}$, we find that, in the Lorentz frame $p_\mu=(|p_3|, 0, 0, p_3)$, $(\psi^{2n}_{\mathrm{as}\ i}, \psi'^{2n+1}_{\mathrm{as}\ i}; \chi^{-2n-1}_{\mathrm{as}\ i3}, \chi^{-2n}_{\mathrm{as}\ i3})$, $(\psi^{2n}_{\mathrm{as}\ 3}, \psi'^{2n+1}_{\mathrm{as}\ 3}; b'^{-2n-1}_{\mathrm{as}}, b^{-2n}_{\mathrm{as}})$, $(\phi^{2n+1}_{\mathrm{as}}, \phi^{2(n+1)}_{\mathrm{as}}; \beta^{-2(n+1)}_{\mathrm{as}}, \beta^{-2n-1}_{\mathrm{as}})$, $(\gamma^{2n+1}_{\mathrm{as}}, \gamma^{2(n+1)}_{\mathrm{as}}; \bar{\gamma}^{-2(n+1)}_{\mathrm{as}}, \bar{\gamma}^{-2n-1}_{\mathrm{as}})$ and $(c^{2n}_{\mathrm{as}}, c^{2n+1}_{\mathrm{as}}; \bar{c}^{-2n-1}_{\mathrm{as}}, \bar{c}^{-2n}_{\mathrm{as}})$ are BRST-quartets with respect to $\boldsymbol{\delta}_{\mathrm{T}}$. Here $\mu=(i, 3)$ $(i=0, 1, 2)$. We note that there exist no BRST-singlets with respect to $\boldsymbol{\delta}_{\mathrm{T}}$. From $\bar{\boldsymbol{\delta}}_{\mathrm{T}}$ in the $q=0$ case and $\mathcal{L}_{\mathrm{quad}}$, we find that $(\psi^{2(n+1)}_{\mathrm{as}\ i}, \psi'^{2n+1}_{\mathrm{as}\ i}; \chi^{-2n-1}_{\mathrm{as}\ i3}, \chi^{-2(n+1)}_{\mathrm{as}\ i3})$, $(\psi^{2(n+1)}_{\mathrm{as}\ 3}, \psi'^{2n+1}_{\mathrm{as}\ 3}; b'^{-2n-1}_{\mathrm{as}}, b^{-2(n+1)}_{\mathrm{as}})$, $(\phi^{2n+3}_{\mathrm{as}}, \phi^{2(n+1)}_{\mathrm{as}}; \beta^{-2(n+1)}_{\mathrm{as}}, \beta^{-2n-3}_{\mathrm{as}})$, $(\gamma^{2n+3}_{\mathrm{as}}, \gamma^{2(n+1)}_{\mathrm{as}}; \bar{\gamma}^{-2(n+1)}_{\mathrm{as}}, \bar{\gamma}^{-2n-3}_{\mathrm{as}})$ and $(c^{2(n+1)}_{\mathrm{as}}, c^{2n+1}_{\mathrm{as}}; \bar{c}^{-2n-1}_{\mathrm{as}}, \bar{c}^{-2(n+1)}_{\mathrm{as}})$ are BRST-quartets with respect to $\bar{\boldsymbol{\delta}}_{\mathrm{T}}$ and that $\psi^0_{\mathrm{as}\ \mu}$, $\chi^0_{\mathrm{as}\ i3}$, ϕ^1_{as}, γ^1_{as}, $\bar{\gamma}^{-1}_{\mathrm{as}}$, β^{-1}_{as}, c^0_{as}, $\bar{c}^0_{\mathrm{as}}$ and b^0_{as} are BRST-singlets with respect to $\bar{\boldsymbol{\delta}}_{\mathrm{T}}$. Similary, we find that $(\phi^{n+1}_{\mathrm{as}}, \gamma^{n+1}_{\mathrm{as}}; \bar{\gamma}^{-n-1}_{\mathrm{as}}, \beta^{-n-1}_{\mathrm{as}})$ are BRST-quartets with respect to $\boldsymbol{\delta}_{\mathrm{S}}$ and that $\psi^{2n}_{\mathrm{as}\ \mu}$ $\psi'^{2n+1}_{\mathrm{as}\ \mu}$, $\chi^{-n}_{\mathrm{as}\ i3}$, c^n_{as}, $\bar{c}^{-n}_{\mathrm{as}}$, b^{-2n}_{as} and b'^{-2n-1}_{as} are BRST-singlets with respect to $\boldsymbol{\delta}_{\mathrm{S}}$. We also find that $(\psi^{2n}_{\mathrm{as}\ L}, c^{2n}_{\mathrm{as}}; \bar{c}^{-2n}_{\mathrm{as}}, b^{-2n}_{\mathrm{as}})$ and $(\phi^{2n+1}_{\mathrm{as}}, c^{2n+1}_{\mathrm{as}}; \bar{c}^{-2n-1}_{\mathrm{as}}, b^{-2n-1}_{\mathrm{as}})$ are BRST-quartets with respect to $\boldsymbol{\delta}_{\mathrm{P}}$ and that $\psi^{2n}_{\mathrm{as}\ T}$, $\psi^{2n}_{\mathrm{as}\ S}$, $\psi^{2n+1}_{\mathrm{as}\ \mu}$, $\chi^{-n}_{\mathrm{as}\ i3}$, $\phi^{2(n+1)}_{\mathrm{as}}$, $\gamma^{n+1}_{\mathrm{as}}$, $\bar{\gamma}^{-n-1}_{\mathrm{as}}$ and $\beta^{-n-1}_{\mathrm{as}}$ are BRST-singlets with respect to $\boldsymbol{\delta}_{\mathrm{P}}$. Here the subscripts L, T and S stand for the longitudinal, transverse and scalar components, respectively. Taking into account the following constraints in the phase space:

$$\begin{aligned}&(\pi_{\psi^0})_i^{\mathrm{as}}\equiv\frac{\partial\mathcal{L}_{\mathrm{quad}}}{\partial(\partial^0\psi^{0\ i}_{\mathrm{as}})}=\chi^0_{\mathrm{as}\ 0i}\,, \qquad (\pi_{\chi^0})_{0i}^{\mathrm{as}}\equiv\frac{\partial\mathcal{L}_{\mathrm{quad}}}{\partial(\partial^0\chi^{0\ 0i}_{\mathrm{as}})}=0\,,\\&(\pi_{b^0})^{\mathrm{as}}\equiv\frac{\partial\mathcal{L}_{\mathrm{quad}}}{\partial(\partial^0 b^0_{\mathrm{as}})}=\psi^0_{\mathrm{as}\ 0}\,, \qquad (\pi_{\psi^0})_0^{\mathrm{as}}\equiv\frac{\partial\mathcal{L}_{\mathrm{quad}}}{\partial(\partial^0\psi^{0\ 0}_{\mathrm{as}})}=0\,,\end{aligned}\tag{38}$$

we conclude that the genuinely independent BRST-singlets with respect to all of $\bar{\boldsymbol{\delta}}_{\mathrm{T}}$, $\boldsymbol{\delta}_{\mathrm{S}}$ and $\boldsymbol{\delta}_{\mathrm{P}}$ are only the transverse components of Yang-Mills field $A_{\mathrm{as\ T}}$ $(\equiv\psi^0_{\mathrm{as\ T}})$.

Let us consider the following two subspaces of state-vector space:

$$\begin{aligned}&\mathcal{V}_1=\{|\Phi\rangle;\ Q_{\mathrm{T}}|\Phi\rangle=\bar{Q}_{\mathrm{T}}|\Phi\rangle=Q_{\mathrm{S}}|\Phi\rangle=Q_{\mathrm{P}}|\Phi\rangle=0\}\,,\\&\mathcal{V}_2=\{|\Phi\rangle;\ \bar{Q}_{\mathrm{T}}|\Phi\rangle=Q_{\mathrm{S}}|\Phi\rangle=Q_{\mathrm{P}}|\Phi\rangle=0\}\,,\end{aligned}\tag{39}$$

*) We assume that all relevant fields have their own asymptotic fields.

where $Q_{\rm T}$, $\bar{Q}_{\rm T}$, $Q_{\rm S}$ and $Q_{\rm P}$ are the BRST charges corresponding to $\boldsymbol{\delta}_{\rm T}$, $\bar{\boldsymbol{\delta}}_{\rm T}$, $\boldsymbol{\delta}_{\rm S}$ and $\boldsymbol{\delta}_{\rm P}$, respectively. The each (anti)BRST charge Q_* generates the corresponding (anti) BRST transformation $\boldsymbol{\delta}_*$:

$$[Q_*,\,\cdot\,]_\pm = -i\boldsymbol{\delta}_*(\cdot)\,. \tag{40}$$

We assume here that $|0\rangle \in \mathcal{V}_1$ and $|0\rangle' \in \mathcal{V}_2$. (The vacuum $|0\rangle'$ may be different from the vacuum $|0\rangle$.) As a result of the Kugo-Ojima quartet mechanism[7),10)] based on the subsidiary condition $Q_{\rm T}|\Phi\rangle=0$,*) all elements of $\mathcal{V}_1$ are degenerate with the vacuum $|0\rangle$, since there are no BRST-singlets with respect to $\boldsymbol{\delta}_{\rm T}$; the subspace $\mathcal{V}_1$ does not contain nontrivial states corresponding to observable particles. On the other hand, the subspace $\mathcal{V}_2$ contains nontrivial states that are constructed by applying the creation operators of $A_{{\rm as}\;T}$ to $|0\rangle'$. The other independent states in $\mathcal{V}_2$ are degenerate with the vacuum $|0\rangle'$ owing to the quartet mechanism based on the subsidiary conditions defining $\mathcal{V}_2$. Hence, except zero norm parts, the subspace $\mathcal{V}_2$ has the same structure as the physical subspace of the usual quantum Yang-Mills theory.

From (14), (21), (32), (39) and (40), we find that the total Lagrangian $\mathcal{L}$ is equivalent to zero in the subspace $\mathcal{V}_1$: $\langle\Phi_1|\mathcal{L}|\Psi_1\rangle=0$ for any $|\Phi_1\rangle, |\Psi_1\rangle \in \mathcal{V}_1$, while, from (15), (24), (34), (39) and (40), we find that $\mathcal{L}$ is equivalent to $\sqrt{g}\,{\rm Tr}[(1/2)\chi'^{0\mu\nu}F_{\mu\nu}]$ in the subspace $\mathcal{V}_2$: $\langle\Phi_2|\mathcal{L}|\Psi_2\rangle=\langle\Phi_2|\sqrt{g}\,{\rm Tr}[(1/2)\chi'^{0\mu\nu}F_{\mu\nu}]|\Psi_2\rangle$ for any $|\Phi_2\rangle, |\Psi_2\rangle \in \mathcal{V}_2$. Thus, the ETYMT based on $\mathcal{L}$ becomes a CTFT or a theory equivalent to a quantum anti-self-dual Yang-Mills theory according as the subsidiary condition $Q_{\rm T}|\Phi\rangle=0$ is imposed or not, in addition to the subsidiary conditions $\bar{Q}_{\rm T}|\Phi\rangle=Q_{\rm S}|\Phi\rangle=Q_{\rm P}|\Phi\rangle=0$. As for the abelian case, we can adopt $\mathcal{L}+\mathcal{L}_4$ as an invariant Lagrangian. It is obvious that $\mathcal{L}+\mathcal{L}_4$ is equivalent to zero in $\mathcal{V}_1$ and is locally equivalent to $\sqrt{g}\,{\rm Tr}[(-1/4)F^{\mu\nu}F_{\mu\nu}]$ in $\mathcal{V}_2$: $\langle\Phi_2|(\mathcal{L}+\mathcal{L}_4)|\Psi_2\rangle=\langle\Phi_2|\sqrt{g}\,{\rm Tr}[(1/2)\chi'^{0\mu\nu}(F_{\mu\nu}+(1/4)\chi'^{0}_{\mu\nu})]|\Psi_2\rangle \simeq \langle\Phi_2|\sqrt{g}\,{\rm Tr}[(-1/4)(F^{\mu\nu}F_{\mu\nu}+F^{\mu\nu}\tilde{F}_{\mu\nu})]|\Psi_2\rangle$ for any $|\Phi_2\rangle, |\Psi_2\rangle \in \mathcal{V}_2$, where $\tilde{F}$ is the dual of F. Thus, in the abelian case, we can derive the usual quantum Maxwell theory.

§ 5. Remarks

We found that the ETYMT with the subspace $\mathcal{V}_1$ becomes equivalent to the quantum anti-self-dual Yang-Mills theory (or the quantum Maxwell theory) when the subsidiary condition $Q_{\rm T}|\Phi\rangle=0$ is removed from the conditions defining $\mathcal{V}_1$. Then, on the basis of the ETYMT, we may construct a model that describes, from viewpoint of the CTFT, a transition from a topological phase (unbroken phase) having no local physical degree of freedom to a physical phase (broken phase) characterized by the quantum anti-self-dual Yang-Mills theory (or the quantum Maxwell theory). A mechanism by which breaking of the topological BRST symmetry occurs, while maintaining the other (anti)BRST symmetries, will be important for the transition.**)

The energy-momentum tensor, $T_{\mu\nu}$, derived from $\mathcal{L}$ (or $\mathcal{L}+\mathcal{L}_4$) is written in the following forms:

*) We assume the asymptotic completeness.[7)]

**) The breaking of BRST symmetry has been discussed in relation with the Gribov problem[16)] and non-triviality of observables in the TYMT.[17)]

$$T_{\mu\nu}=\{Q_T, \lambda_{\mu\nu}\}=t_{\mu\nu}+\{\bar{Q}_T, \bar{\lambda}_{\mu\nu}\}, \tag{41}$$

where $\lambda_{\mu\nu}$ and $\bar{\lambda}_{\mu\nu}$ are certain anticommuting tensors. From (39) and (41), we have $\langle\Phi_1|T_{\mu\nu}|\Psi_1\rangle=0$ and $\langle\Phi_2|T_{\mu\nu}|\Psi_2\rangle=\langle\Phi_2|t_{\mu\nu}|\Psi_2\rangle$. Since $t_{\mu\nu}$ depends on the background metric $g_{\mu\nu}$, the energy-momentum tensor $T_{\mu\nu}$ gets the metric dependence as a result of the transition. Similar results are also shown for other conserved quantities.

As mentioned already, we cannot take $\mathcal{L}+\mathcal{L}_4$ as an invariant Lagrangian in the nonabelian case. However, application of the method (i) in § 3 to the primary gauge transformation enables us to construct an invariant Lagrangian such as $\mathcal{L}'=\mathcal{L}_1+\mathcal{L}_4$ +(gauge fixing terms) even in non-abelian case, since the primary BRST transformation is not necessary in the application of the method (i). Unfortunately, there exists no appropriate subsidiary conditions defining a subspace of state-vector space in which the ETYMT based on $\mathcal{L}'$ can be identified with the quantum Yang-Mills theory.

The following procedure may be appropriate to obtain an ETYMT which can be equivalent to the quantum Yang-Mills theory without constraints: We first construct an extended abelian cohomological topological gauge theory in a higher dimension so that it, in a subspace of state-vector space, becomes equivalent to the usual abelian gauge theory. Fixing the spacetime topology on $M^4\times B^N$, we next carry out the Kaluza-Klein like dimensional reduction for the extended theory. Here B^N is a symmetric space. Then we will be able to obtain an ETYMT in M^4 whose non-abelian gauge symmetry is due to the symmetry of B^N.

Acknowledgements

I would like to thank Professor S. Naka for critical discussions. Thanks are also due to Dr. H. Kan for his kind help.

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