AN EXTENSION OF A RESULT OF LIAPOUNOFF ON THE RANGE OF A VECTOR MEASURE[1]

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Liapounoff[2] established in 1940 that the range of a countably additive finite measure with values in a finite-dimensional real vector space is bounded and closed and in the nonatomic case convex. A simplified proof of this result was given by Halmos[3] in 1948. The aim of the present paper is to extend this result to the following case. Let μ_{it}, $1 \leqq i \leqq k$, $1 \leqq t \leqq n_i$, be a set of countably additive, finite measures. If $\{(E_1, E_2, \cdots, E_k)\}$ is the totality of decompositions of a space X into k pairwise disjoint measurable sets, the range R of the vector ψ with components $\mu_{it}(E_i)$, $i = 1, 2, \cdots, k$, $t = 1, 2, \cdots, n_i$, is bounded, closed, and in the nonatomic case convex.

Let X be any set and let S be a σ-field of subsets of X (called the measurable sets of X). A measure μ (one-dimensional) is non-negative if $\mu(E) \geqq 0$ for every $E \in \mathsf{S}$; $\mu^*(E)$ will denote the total variation of μ on E.[4] The measure μ is absolutely continuous with respect to the measure ν if μ and ν are defined on S and $\mu^*(E) = 0$ for every $E \in \mathsf{S}$ for which $\nu^*(E) = 0$. A necessary and sufficient condition that μ be absolutely continuous with respect to ν is that for every $\epsilon > 0$ there exist a $\delta > 0$ such that $\mu^*(E) < \epsilon$ for all $E \in \mathsf{S}$ so that $\nu^*(E) < \delta$. $\{E_i\}$, $i = 1, 2, \cdots, k$, is said to be a decomposition of F if the E_i are pairwise disjoint measurable subsets of X and $\bigcup_i E_i = F$. Let μ_{it},

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[1] The extension of Liapounoff's result was obtained by a different method and previous to the writing of this paper by A. Dvoretzky, A. Wald, and J. Wolfowitz as a by-product of the proof of another theorem. A generalization of this other theorem in the case of finite measures was also obtained by the author before discovering the work of Dvoretzky, Wald, and Wolfowitz. See their papers, *Elimination of randomization in certain problems of statistics and of the theory of games*, Proc. Nat. Acad. Sci. U.S.A. vol. 36 (1950) pp. 256–259; also, *Relations among certain ranges of vector measures*, Pacific Journal of Mathematics (1951). Overlapping results have also been obtained by D. Blackwell. See *On a theorem of Lyapunov*, Ann. Math. Statist. vol. 22 (1951) pp. 112–115 and *The range of certain vector integrals*, Proceedings of the American Mathematical Society vol. 2 (1951) pp. 390–395.

[2] A. Liapounoff, *Sur les fonctions vecteurs complètement additives*, Bull. Acad. Sci. URSS. Sér. Math. vol. 4 (1940) pp. 465–478.

[3] P. Halmos, *The range of a vector measure*, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 416–421. (Much of our notation is taken from this paper.)

[4] For classical definitions and results of measure theory we refer to S. Saks, *Theory of the integral*, Warsaw, 1937, and P. Halmos, *Measure theory*, Nostrand, 1950.

$i=1, 2, \cdots, k$, $t=1, 2, \cdots n_i$, be a set of countably additive finite measures defined on $\mathcal{S}$. We define ψ on the decompositions of X as the vector whose components are $\mu_{it}(E_i)$, $i=1, 2, \cdots, k$, $t=1, 2, \cdots, n_i$. The range of ψ is R.

Note 1. Since all the μ_{it} are obviously absolutely continuous with respect to the non-negative measure $\nu(E) = \sum_{i,t} \mu_{it}^*(E)$, the Radon-Nikodým theorem permits us to represent the μ_{it} as integrals, that is, $\mu_{it}(E) = \int_E f_{it}(x) d\nu(x)$.

A measurable set E is an atom of a measure μ if $\mu(E) \neq 0$ and if for every measurable set $F \subset E$ either $\mu(F)=0$ or $\mu(F)=\mu(E)$. A measurable set E is said to be an atom of ψ if the vector $\phi(E)$ whose components are $\mu_{it}(E)$, $i=1, 2, \cdots, k$, $t=1, 2, \cdots, n_i$, is not zero and if for every measurable $F \subset E$ either $\phi(F)=\phi(E)$ or $\phi(F)=0$. ψ is said to be nonatomic on $F \in \mathcal{S}$ if none of the measures μ_{it} has an atom on a subset of F. ψ is said to be purely atomic on F if there is a denumerable sequence $\{F_i\}$, where the F_i are pairwise disjoint atoms of ψ and $F = \bigcup_i F_i$.

Note 2. It is easy to see that corresponding to any atom E of any of the measures μ_{it} there is an atom $F \subset E$ of ψ, and that X may be expressed as the union of two disjoint sets X_1, X_2 where ψ is nonatomic on X_1 and ψ is purely atomic on X_2.

LEMMA 1. *If* $\psi = (\mu_{11}(E_1), \mu_{12}(E_1), \cdots, \mu_{kn_k}(E_k))$ *is nonatomic on* X, *the range* R *of* ψ *is convex.*

PROOF. Suppose that $\psi = a$ for the decomposition $E_1, E_2, \cdots, E_k$ and $\psi = b$ for the decomposition $F_1, F_2, \cdots, F_k$. Suppose $0 \leqq \lambda \leqq 1$. Consider the vector measure whose components are $\mu_{rt}(E)$, $r=i, j$, $t=1, 2, \cdots, n_r$, for the measurable subsets E of $E_i \cap F_j$. By the Liapounoff Theorem the range of this vector measure is convex and hence $E_i \cap F_j$ may be decomposed into two disjoint measurable sets V_{ij}, W_{ij} so that $\mu_{rt}(V_{ij}) = \lambda \mu_{rt}(E_i \cap F_j)$, $r=i, j$, $t=1, 2, \cdots, n_r$, and hence $\mu_{jt}(W_{ij}) = (1-\lambda)\mu_{jt}(E_i \cap F_j)$, $t=1, 2, \cdots, n_j$. Consider the decomposition $G_1, G_2, \cdots, G_k$ where $G_i = \bigcup_j [V_{ij} \cup W_{ji}]$. It is easily seen that for this decomposition $\psi = \lambda a + (1-\lambda) b$.

Our proof that R is closed will consist of showing that any terminal point of the closure of R is in R.[5]

LEMMA 2. *For a given set of constants* α_{it}, *the function of* ψ, $\sum_{i,t} \alpha_{it}\mu_{it}(E_i)$ *attains its maximum.*

PROOF. Let

[5] We use terminal point of a convex set to mean boundary point with respect to the lowest-dimensional hyperplane containing the set.

$$\nu_i(E) = \sum_{t=1}^{n_i} \alpha_{it}\mu_{it}(E), \qquad g_i(x) = \sum_{t=1}^{n_i} \alpha_{it} f_{it}(x).$$

Then

$$\nu_i(E) = \int_E g_i(x) d\nu(x).$$

Let

$$T_{i_1 i_2 \cdots i_r} = \{x : g_{i_1}(x) = g_{i_2}(x) = \cdots = g_{i_r}(x) > g_j(x) \text{ for all } j \notin \{i_1, i_2, \cdots, i_r\}\}.$$

It is easily seen now that not only is this lemma true, but that a necessary and sufficient condition that a decomposition maximize $\sum_{i,t} \alpha_{it}\mu_{it}(E_i)$ is that except for a set of ν measure 0, $T_{i_1 i_2 \cdots i_r} \subset (E_{i_1} \cup E_{i_2} \cup \cdots \cup E_{i_r})$.

In the nonatomic case the closure $\bar{R}$ of the convex set R is convex and a supporting plane Π of $\bar{R}$ may be written $\sum_{i,t} \alpha_{it} x_{it} = c$ where not all α_{it} are zero and $c = \sup \{ \sum_{i,t} \alpha_{it}\mu_{it}(E_i) \}$. Hence we have the following corollary.

COROLLARY 1. *If ψ is nonatomic with range R and Π is a supporting plane of $\bar{R}$, $\Pi \cap R \neq 0$.*

LEMMA 3. *If ψ is nonatomic, every point of $\bar{R} \cap \Pi$ is a limit point of $R \cap \Pi$.*

PROOF. If $\{a_n\}$ is a sequence of points of R converging to a point a of $\bar{R} \cap \Pi$ and the decomposition corresponding to a_n is given by $E_{1n}, E_{2n}, \cdots, E_{kn}$, we have $\sum_{i=1}^{k} \nu_i(E_{in}) = \sum_{i=1}^{k} \int_{E_{in}} g_i(x) d\nu(x) \to c = \max \{ \sum_{i,t} \alpha_{it}\mu_{it}(E_i) \}$. It follows that the ν measure of the set $\{x : x \in T_{i_1 i_2 \cdots i_r} \text{ and } x \notin (E_{i_1 n} \cup \cdots \cup E_{i_r n})\}$ must approach zero. If the decomposition corresponding to a_n is modified to $F_{1n}, F_{2n}, \cdots, F_{kn}$ by adjusting the elements of the E_{in} so that $x \in T_{i_1 i_2 \cdots i_r}$ implies $x \in F_{i_1 n} \cup F_{i_2 n} \cup \cdots \cup F_{i_r n}$, then $\sum_{i,t} \alpha_{it}\mu_{it}(F_{in}) = c$ and $\mu_{it}(F_{in}) - \mu_{it}(E_{in}) \to 0$ which gives us our result.

LEMMA 4. *If ψ is nonatomic its range R is closed.*

PROOF. Through any terminal point of R there is a supporting hyperplane Π so that $\sum_{i,t} \alpha_{it}\mu_{it}$ is not identically constant for all points of R. It suffices to show that $R \cap \Pi$ is closed. We shall proceed by induction on k and the number n of non-null measures involved in ψ. *Case* 1. $k=1$. This case is trivial. *Case* 2. $k>1$, $n=1$. The closure follows from the Liapounoff theorem for one-dimensional measures. *Case* 3. $k>1$, $n>1$. Let $R_{i_1 i_2 \cdots i_r}$ be the range of ψ on the decompositions of $T_{i_1 i_2 \cdots i_r}$ where $E_i = 0$ if $i \notin \{i_1, i_2, \cdots, i_r\}$. It will suffice to show that the $R_{i_1 i_2 \cdots i_r}$ are closed. The induction establishes this immediately for all sets except $R_{12\cdots k}$. On this set $g_1 = g_2 = \cdots = g_k$. Corresponding to one of the non-null measures μ_{js} there is a nonzero α_{js}. By induction the range $R'_{12\cdots k}$, of the vector ψ' which has all components of ψ except μ_{js}, is closed. μ_{js} is a linear function of the components of ψ' since $\sum_{i,t} \alpha_{it}\mu_{it} = \int_{T_{1,2,\cdots,k}} g_1(x)d\nu(x)$. Hence $R_{12\cdots k}$ is closed.

LEMMA 5. *If ψ is purely atomic, its range R is closed.*

PROOF. This proof is an obvious extension of Liapounoff's. Consider the sequence of atoms $\{F_n\}$ of ψ. If E_i contains F_n (except possibly for a null set), let $a_{in} = 2$ and otherwise 0. Let $a = (a_1, a_2, \cdots, a_n)$ where $a_i = \sum_{n=1}^{\infty} a_{in} 3^{-n}$. This relation gives a one-to-one correspondence with the decompositions of X (excepting deviations by sets of measure zero) and a bounded closed set of vectors. ψ considered as a function of a is continuous and hence R is closed.

THEOREM 1. *The range R of $\psi = (\mu_{11}(E_1), \mu_{12}(E_1), \cdots, \mu_{kn_k}(E_k))$ on the decompositions of X is bounded and closed and in the nonatomic case convex.*

PROOF. Lemma 1 gives the convexity, Lemmas 4 and 5 give the closure when considered in connection with Note 2. The boundedness is trivial because the measures are finite.

COROLLARY 2. *The range of the vector $\Phi = (\mu_{11}(E_1), \mu_{12}(E_1), \cdots, \mu_{kn_k}(E_k))$, where the E_i are pairwise disjoint measurable sets, is bounded and closed and in the nonatomic case convex.*

PROOF. Let $E_{k+1} = X - \bigcup_{i=1}^{k} E_i$. The range of ψ on the decompositions $E_1, E_2, \cdots, E_{k+1}$ has the desired property. The range of Φ is a projection of the range of ψ and also has the desired property.

A more trivial result would arise in the case where the assumption of disjoint sets is removed.

Let I be the unit interval $(0, 1)$, and m the Lebesgue measure on I. The measures μ_{it} on X may be extended to $\eta_{it} = \mu_{it} \times m$ on $X \times I$. Let ϕ be the vector whose components are $\eta_{it}(F_i)$ where $F_1, F_2, \cdots, F_k$ is a decomposition of $X \times I$ into k measurable pairwise disjoint sets.

THEOREM 2. *The range H of ϕ is the convex hull of the range R of ψ.*[6]

[6] This theorem is the generalization of the result of Dvoretzky, Wald, and Wolfowitz referred to in footnote 1.

PROOF. H is convex because the η_{it} are obviously nonatomic. It is evident that $H \supset R$. Hence it suffices to show that

$$\sup_{\phi \in H} \sum_{i,t} \alpha_{it}\eta_{it} \leqq \sup_{\psi \in R} \sum_{i,t} \alpha_{it}\mu_{it}.$$

Since $\mu_{it}(E) = \int_E f_{it}(x) d\nu(x)$, $\eta_{it}(F) = \int_F f_{it}(x, y) d[\nu \times m](x, y)$ where $f_{it}(x, y) = f_{it}(x)$. Then $g_i(x, y) = \sum_t \alpha_{it} f_{it}(x, y) = g_i(x)$. Hence $\sum_{i,t} \alpha_{it}\eta_{it}$ attains its maximum when

$$\{x, y : g_{i_1}(x) = g_{i_2}(x) = \cdots = g_{i_r}(x) > g_i(x) \text{ for all } i \notin \{i_1, i_2, \cdots, i_r\}\} \subset F_{i_1} \text{ if } i_1 < i_2 < \cdots < i_r.$$

But this defines a decomposition of $X \times I$ which corresponds to a decomposition of X for which $\sum_{i,t} \alpha_{it}\mu_{it} = \sum_{i,t} \alpha_{it}\eta_{it}$.

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