

An extension of a theorem of Myers

Dedicated to Professor S. Kashiwabara on his 60th birthday

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§ 0. Introduction.

In this article we shall prove a simple extension of a theorem due to Myers [6], which states; if the sectional curvature of a connected and complete Riemannian manifold M has a positive lower bound then M is compact. He proved also that a connected and complete Riemannian manifold is compact if its Ricci curvature is bounded from below by a positive constant. The latter theorem has been generalized in several ways by Ambrose [1], Calabi [3] and Avez [2].

It is clear that a compact Riemannian manifold has a bounded volume. However as is stated below the converse does not hold in general. In this respect, M. Maeda [5] has shown that a connected and complete Riemannian manifold whose sectional curvature lies in an interval $(0, \alpha]$ is compact if and only if its volume is bounded. Recently Wu [8] proved that a complete, non-compact and orientable n -dimensional hypersurface in a Euclidean $(n+1)$ -space has infinite volume if its sectional curvature is non-negative and all positive at one point. Our aim is to prove the following

THEOREM. *Let M be a complete and connected Riemannian n -manifold of non-negative sectional curvature. Then M is compact if and only if its volume is bounded.*

REMARK. In [9] Yau announced the same result by a different method. His proof is based on the existence of a non-trivial convex continuous function on complete open manifold of non-negative curvature, for which he refers to [4].

REMARK. For any positive ε , we can construct an n -dimensional complete, connected and non-compact hypersurface of revolution in a Euclidean $(n+1)$ -space which has a bounded volume and its sectional curvature in $[-\varepsilon, \infty)$.

In order to prove our theorem it suffices to show that a connected, complete and non-compact Riemannian n -manifold M of non-negative sectional curvature has an infinite volume. Thus we may restrict our attention to such an M .

§1. Notations.

From now on let M be as above. A ray is by definition a geodesic $\gamma: [0, \infty) \rightarrow M$ such that any of its subarc is the shortest connection between the end points. In this article we always parametrize geodesics by arclengths. For a subset X of M , let $B_a(X) := \{y \in M; d(y, X) < a\}$, where $d: M \times M \rightarrow R$ is the distance function on M . For each point $x \in M$ we denote by $C(x)$ the cut locus of x . We denote by $\exp_x: M_x \rightarrow M$ the exponential map and by $i: M \rightarrow R$ the injectivity radius function, where M_x is the tangent space to M at x .

Let $\gamma: [0, \infty) \rightarrow M$ be a fixed ray with $\gamma(0) = p$, and $\gamma_t: [0, \infty) \rightarrow M$ be $\gamma_t(s) = \gamma(t+s)$. For each $t \geq 0$, the set $C_{r_t} = M - \bigcup_{t' > 0} B_{t'}(\gamma_t(t'))$ is called a supporting half space and is a totally convex set (see [4]). We denote by H_{r_t} its boundary set. Then we have a filtration $\{C_{r_t}\}_{t \geq 0}$ of totally convex sets (see Proposition 1.3 of [4]) satisfying; (1) $t_1 \leq t_2$ implies $C_{r_{t_1}} \subset C_{r_{t_2}}$ and $C_{r_{t_1}} = \{y \in C_{r_{t_2}}; d(y, H_{r_{t_2}}) \geq t_2 - t_1\}$. In particular $H_{r_{t_1}} = \{y \in C_{r_{t_2}}; d(y, H_{r_{t_2}}) = t_2 - t_1\}$. (2) $\bigcup_{t \geq 0} C_{r_t} = M$. (3) $p \in H_{r_0}$.

For each point $x \in M$ ($x \neq p$) a ray $\sigma: [0, \infty) \rightarrow M$, $\sigma(0) = x$, is by definition asymptotic to γ if there are a divergent sequence $\{t_j\}$ and a family of minimizing geodesics $\{\sigma_{x,t_j}\}$ each emanating from x and ending at $\gamma(t_j)$ such that $\lim_{j \rightarrow \infty} \sigma'_{x,t_j}(0) = \sigma'(0)$.

§2. Properties of asymptotic rays.

LEMMA 2.1. Let $\lambda: [0, a] \rightarrow M$ be a minimizing geodesic joining $p = \lambda(0)$ to $x = \lambda(a)$ and $\sigma: [0, \infty) \rightarrow M$ a ray asymptotic to γ emanating from x . Then $\sphericalangle(\sigma'(0), \lambda'(a)) \leq \sphericalangle(\gamma'(0), \lambda'(0))$.

PROOF. Let $\{t_j\}$ and $\{\sigma_{x,t_j}\}$ be a divergent sequence and a shortest connections each σ_{x,t_j} joining x to $\gamma(t_j)$ such that $\sigma'(0) = \lim_{j \rightarrow \infty} \sigma'_{x,t_j}(0)$. To the geodesic triangle $(\lambda, \sigma_{x,t_j}, \gamma|_{[0, t_j]})$ we apply the angle comparison theorem of Toponogov and let $j \rightarrow \infty$.

LEMMA 2.2. For any $t_0 > 0$ and any two points $x_1, x_2 \in H_{r_{t_0}}$ let $\sigma_1, \sigma_2: [0, \infty) \rightarrow M$ be rays emanating from $x_i = \sigma_i(0)$ and asymptotic to γ , both obtained by the same divergent sequence $\{t_j\}$. Then the function $\rho(t) := d(\sigma_1(t), \sigma_2(t))$ is monotone increasing. Moreover $\rho|_{[t^*, \infty)}$ is constant if and only if there exists a totally geodesic flat surface with boundary $\sigma_1([t^*, \infty))$, $\sigma_2([t^*, \infty))$ and a shortest geodesic segment joining $\sigma_1(t^*)$ to $\sigma_2(t^*)$.

PROOF. From $x_i \in H_{r_{t_0}}$, follows $\lim_{t \rightarrow \infty} [d(x_i, \gamma(t)) - (t - t_0)] = 0$. Let $\{\sigma_{i,t_j}\}$ be the family of shortest connections joining x_i to $\gamma(t_j)$ which satisfy $\sigma'_i(0) =$

$\lim_{j \rightarrow \infty} \sigma'_{i,t_j}(0)$. For each $t > t_0$, $\sigma_i(t) \in H_{r_{t_0+t}}$ follows from $\lim_{j \rightarrow \infty} \sigma_{i,t_j}(t) = \sigma_i(t)$ and $\lim_{j \rightarrow \infty} [d(\sigma_{i,t_j}(t), \gamma(t_j)) - (t_j - t)] = 0$. To a geodesic triangle with vertices $\sigma_{1,t_j}(t)$, $\sigma_{2,t_j}(t)$ and $\gamma(t_j)$ we apply the angle comparison theorem to get (letting $j \rightarrow \infty$) both $\sphericalangle(\beta'(0), \sigma'_1(t)) \geq \pi/2$ and $\sphericalangle(-\beta'(\rho(t)), \sigma'_2(t)) \geq \pi/2$, where $\beta: [0, \rho(t)] \rightarrow M$ is a minimizing geodesic joining $\sigma_1(t)$ to $\sigma_2(t)$. Consider a 1-parameter variation $V: (-\varepsilon, \varepsilon) \times [0, \rho(t)] \rightarrow M$ of β such that $V(u, 0) = \sigma_1(t+u)$, $V(u, \rho(t)) = \sigma_2(t+u)$ for $u \in (-\varepsilon, \varepsilon)$. Then the first variation formula implies $L'(0) \geq 0$, where $L(u)$ is the length of the curve $v \rightarrow V(u, v)$. Clearly $L(u) \geq \rho(t+u)$ for any $u \in (-\varepsilon, 0]$ and $L(0) = \rho(t)$. Therefore for $u \in (-\varepsilon, 0]$ we see $\rho(t) - \rho(t+u) \geq 0$ because of $\liminf_{u \rightarrow 0} \frac{\rho(t) - \rho(t+u)}{-u} \geq \lim_{u \rightarrow 0} \frac{L(0) - L(u)}{-u} \geq 0$. The last statement follows from the convexity theorem due to Alexandrov (for detail see [7]).

LEMMA 2.3. For any $t_0 > 0$ and $x \in H_{r_{t_0}}$, let $\sigma: [0, \infty) \rightarrow M$ be a ray asymptotic to γ such that $\sigma(0) = x$. Then we have

(2.1) For any $t > 0$, σ_t is the unique ray emanating from $\sigma_t(0)$ and asymptotic to γ .

(2.2) $C_{r_{t_0+t}} \subset C_{\sigma_t}$ for any $t > 0$.

PROOF. Let $\{l_j\}$ and $\{\sigma_{t,j}\}$ be a divergent sequence and a family of shortest connections, each $\sigma_{t,j}$ joining $\sigma(t)$ to $\gamma(l_j)$, such that $\{\sigma'_{t,j}(0)\}$ has a limit, say v . Suppose $v \neq \sigma'(t)$. To a geodesic triangle $(\sigma_{x,l_j}, \sigma_{t,j}, \sigma|_{[0,t]})$ we apply the angle comparison theorem. Looking at the angle at $\sigma(t)$ we can derive a contradiction from $\lim_{j \rightarrow \infty} [(l_j - t_0) - d(x, \gamma(l_j))] = \lim_{j \rightarrow \infty} [l_j - t_0 - t - d(\sigma(t), \gamma(l_j))] = 0$. Next we take a point $z \in M - C_{\sigma_t}$. Then there exist $\alpha > 0$ and s_0 such that $(s_0 - t) - d(z, \sigma(s_0)) > \alpha$. Since $C_{r_{s_0+t_0}}$ is convex and $\sigma(t)$ is in it, $\sigma_{t,j}$ has at least one intersection with $H_{r_{s_0+t_0}}$. Let z_j be the intersection furthest from $\sigma(t)$. From (2.1) follows $\lim_{j \rightarrow \infty} z_j = \sigma(s_0)$ and hence there is j_0 such that $d(z_j, \sigma(s_0)) < \alpha/3$, $|d(z_j, \sigma(t)) - (s_0 - t)| < \alpha/3$ and $|d(z_j, \gamma(l_j)) - l_j + s_0 + t_0| < \alpha/3$ for all $j > j_0$. Therefore we have $d(z, \gamma(l_j)) < l_j - t_0 - t - \alpha/3$ for $j > j_0$.

For each $t > 0$ let S_{r_t} be the set such that $S_{r_t} = \{\exp_{r(t)} uX; 0 \leq u < r, X \in M_{r(t)}, \|X\| = 1 \text{ and } \langle X, \gamma'(t) \rangle = 0\}$ and $S_{r_t} := S_{r_t}^\infty$. Then we have the

LEMMA 2.4. S_{r_t} lies in the opposite side of $\partial B_t(p)$ with respect to H_{r_t} . If x is a point on $H_{r_t} \cap \partial B_t(p)$ then there exists a unique ray starting from x and asymptotic to γ .

PROOF. From the argument in Lemma 2.2 we see that any asymptotic ray emanating from $x \in H_{r_t}$ has an intersection with S_{r_t} , and hence $S_{r_t} \subset M - \text{Int } C_{r_t}$. It is evident that $\overline{B_t(p)} \subset C_{r_t}$. Suppose $x \in H_{r_t} \cap \partial B_t(p)$. Then $\lim_{j \rightarrow \infty} [(t_j - t) - d(x, \gamma(t_j))] = 0$. Let $\lambda: [0, t] \rightarrow M$ be a minimizing geodesic joining $p = \lambda(0)$ to $x = \lambda(t)$. By applying the angle comparison theorem to the triangle with vertices

p , x and $\gamma(t_j)$, we see $\lambda'(t) = \sigma'(0)$. Thus the uniqueness is proved.

REMARK 1. Let y be any point on H_{r_t} ($t > 0$). Then for any $s > 0$ and any asymptotic ray σ starting from y , S_{σ_s} lies in the opposite side of $\partial B_s(y)$ with respect to $H_{r_{t+s}}$.

REMARK 2. If γ is a ray on the complete simply connected hyperbolic space form of constant curvature -1 , S_{r_t} lies between $\partial B_t(p)$ and the horosphere H_{r_t} , where $p = \gamma(0)$.

REMARK 3. Let $x \in H_{r_t} \cap S_{r_t}$ and $\beta: [0, b] \rightarrow M$ be a geodesic segment joining $\gamma(t)$ to x such that $\beta([0, b]) \subset S_{r_t}$. Then in the same manner as in the proof of Lemma 2.2, there exists a totally geodesic flat surface with boundary $\gamma([t, \infty))$, $\beta([0, b])$ and $\sigma([0, \infty))$, where σ is the asymptotic ray whose initial vector is obtained by the parallel translation of $\gamma'(t)$ along β .

§ 3. Fermi coordinates along a ray.

Let E_1, \dots, E_n be unit parallel fields along γ such that $E_n(t) = \gamma'(t)$ and $(E_1(t), \dots, E_n(t))$ is an orthonormal basis for $M_{r(t)}$. Set $W_0 := B_s(\gamma([0, 1]))$ and let κ_0 be the maximum of sectional curvature on \bar{W}_0 and i_0 the minimum of the injectivity radius on the set. Let $T_r(\gamma([0, 1])) := \bigcup_{0 \leq t \leq 1} S_{r_t}$. Since $\gamma([0, \infty)) \subset M - C(p)$ there is $r > 0$ satisfying

$$(3.1) \quad B_r(\gamma([0, 1])) \subset M - C(p)$$

and

$$(3.2) \quad r \leq \text{Min} \left\{ \pi/2 \sqrt{\kappa_0}, \frac{1}{2} i_0, 1 \right\}.$$

If r satisfies the above conditions then so does any $r^* \in (0, r]$. Let $\hat{R}_0 := \sup \{ r > 0; r \text{ satisfies (3.1) and (3.2)} \}$. Then clearly \hat{R}_0 satisfies them. For any $r \in (0, \hat{R}_0]$, $T_r(\gamma([0, 1]))$ is the disjoint union of $\bigcup_{0 \leq t \leq 1} S_{r_t}$ and hence we can introduce Fermi coordinates in $T_r(\gamma([0, 1]))$. Indeed, suppose $x = \exp_{r(t_1)} u X = \exp_{r(t_2)} v Y$ holds for some $0 \leq t_1 \leq t_2 \leq 1$, $0 < u, v < \hat{R}_0$ and unit vectors X, Y each perpendicular to $\gamma'(t_1), \gamma'(t_2)$ respectively. The circumference of the triangle $(x, \gamma(t_1), \gamma(t_2))$ is less than $4\hat{R}_0 \leq 2\pi/\sqrt{\kappa_0}$, which enables us to draw the corresponding triangle on the standard n -sphere $S^n(\kappa_0)$ of constant curvature κ_0 . From (3.2), each of the edges does not intersect the cut locus of the corresponding vertex. Thus by means of Rauch's theorem, we derive a contradiction from $\hat{R}_0 \leq \pi/2 \sqrt{\kappa_0}$. We denote by $\varphi: T_{\hat{R}_0}(\gamma([0, 1])) \rightarrow R^n$ the coordinate map, i. e., $\varphi^{-1}(x_1, \dots, x_n) = \exp_{r(x_n)} \sum_{i=1}^{n-1} x_i E_i(x_n)$ for $0 \leq x_n \leq 1$ and $\sum_{i=1}^{n-1} x_i^2 < \hat{R}_0^2$. For $z \in T_{\hat{R}_0}(\gamma([0, 1]))$, we denote by $dM(z)$ and $dR^n(\varphi(z))$ the volume element of M and R^n at the point.

LEMMA 3.1. For any $x_1, x_2 \in \partial B_1(p) \cap T_{\hat{R}_0}(\gamma([0, 1]))$ such that $d(x_i, H_{r_1}) < \frac{1}{2} \hat{R}_0$ ($i=1, 2$), let $\sigma_i: [-a_i, \infty) \rightarrow M$ ($i=1, 2$) be rays each asymptotic to γ and emanating from $x_i = \sigma_i(-a_i)$, where $a_i \geq 0$ and $\sigma_i(0) \in H_{r_1}$. Then we have for any numbers $r_i \in (0, \frac{1}{2} \hat{R}_0]$ such that $d(\sigma_1(0), \sigma_2(0)) \geq r_1 + r_2$,

$$(3.3) \quad T_{r_1}(\sigma_1([0, 1])) \cap T_{r_2}(\sigma_2([0, 1])) = \emptyset.$$

PROOF. By means of Lemma 2.4 each $\sigma_i|_{[0, \infty)}$ is the unique ray emanating from $\sigma_i(0)$ and asymptotic to γ . Hence the function $\rho(t) := d(\sigma_1(t), \sigma_2(t))$ is monotone increasing. Suppose there is a point q on $T_{r_1}(\sigma_1([0, 1])) \cap T_{r_2}(\sigma_2([0, 1]))$. Then $q = \exp_{\sigma_1(t_1)} r_1^* u_1 = \exp_{\sigma_2(t_2)} r_2^* u_2$ holds for some $0 < r_i^* < r_i$ and unit vectors $u_i \in M_{\sigma_i(t_i)}$, $\langle u_i, \sigma_i'(t_i) \rangle = 0$. Without loss of generality we may assume $t_2 \geq t_1$. Clearly $q \in W_0$ follows from $d(q, p) < r_1^* + t_1 + \frac{1}{2} \hat{R}_0 + 1 < 3$. From $t_2 - t_1 = d(\sigma_1(t_1), H_{r_{t_2+1}}) \leq r_1^* + r_2^*$, the circumference of the triangle $(\sigma_1(t_1), \sigma_2(t_2), \sigma_2(t_1))$ is less than $4(r_1^* + r_2^*) \leq 4\hat{R}_0 \leq 2\pi/\sqrt{\kappa_0}$. Since $\sigma_2([t_1, t_2]) \subset M - C(\sigma_1(t_1))$ the function $f: [0, \infty) \rightarrow R$ defined by $f(t) := d(\sigma_1(t_1), \sigma_2(t))$ is smooth on an open interval containing $[t_1, t_2]$. Suppose $f(t_2) < f(t_1)$. Then there is $t_3 \in (t_1, t_2)$ such that $f'(t_3) < 0$. Hence the edge angles of the triangle $(\sigma_1(t_1), \sigma_2(t_1), \sigma_2(t_3))$ at $\sigma_2(t_3)$ and $\sigma_2(t_1)$ are not smaller than $\pi/2$. Thus we derive a contradiction from Rauch's theorem. Therefore we have $f(t_2) \geq f(t_1) = \rho(t_1) \geq \rho(0) \geq r_1 + r_2$. On the other hand $f(t_2) \leq r_1^* + r_2^* < r_1 + r_2$ is a contradiction.

Now we take $R_0 \in (0, \hat{R}_0]$ in such a way that $T_{R_0}(\gamma([0, 1]))$ satisfies:

$$(3.4) \quad \text{For each } x \in \partial B_1(p) \cap T_{R_0}(\gamma([0, 1])) \text{ the minimizing geodesic } \lambda: [0, 1] \rightarrow M \text{ joining } p \text{ to } x \text{ has the extension } \lambda|_{[0, b]} \text{ such that } \lambda(b) \in S_{\hat{R}_1}^{\hat{R}_0} \text{ and moreover } \sphericalangle(\lambda'(0), \gamma'(0)) \leq \pi/6.$$

Then we have for any $x \in \partial B_1(p) \cap T_{R_0}(\gamma([0, 1]))$ and any ray $\sigma: [-a, \infty) \rightarrow M$ asymptotic to γ such that $x = \sigma(-a)$ ($a \geq 0$) and $\sigma(0) \in H_{r_1}$,

$$(3.5) \quad a < \frac{1}{2} \hat{R}_0.$$

This follows immediately from $a = d(x, H_{r_1}) \leq b - 1 < (1 + \hat{R}_0^2)^{\frac{1}{2}} - 1 < \frac{1}{2} \hat{R}_0^2 < \frac{1}{2} \hat{R}_0$.

PROOF OF THEOREM. For each point $x \in \partial B_1(p) \cap T_{R_0}(\gamma([0, 1]))$ we choose an asymptotic ray $\sigma^x: [-a_x, \infty) \rightarrow M$ such that $\sigma^x(-a_x) = x$, $\sigma^x(0) \in H_{r_1}$, and we denote by $\lambda_x: [0, 1] \rightarrow M$ the shortest connection joining p to x . We denote by A_x^r the area of $\partial B_1(p) \cap T_r(\sigma^x([-a'_x, 1]))$, where a'_x is chosen so that $a'_x = \text{Min} \{a''_x \in [a_x, 1]; \partial B_1(p) \cap T_r(\sigma^x([-a''_x, 1])) \text{ is a connected neighborhood of } x \text{ on } \partial B_1(p) \cap T_{R_0}(\gamma([0, 1]))\}$. We also denote by $A_r(\sigma^x(t))$ the area of $S_{\sigma^x(t)}^r$. Then

$$(3.6) \quad 1 \leq \lim_{r \rightarrow 0} \frac{A_r^x}{A_r(\sigma^x(-a_x))} \leq \langle \lambda'_x(1), \sigma^{x'}(-a_x) \rangle^{-1} \\ \leq \langle \lambda'_x(0), \gamma'(0) \rangle^{-1}.$$

Thus we can find (making use of Lemma 2.1) R_1^* such that

$$(3.7) \quad 1 \leq \frac{A_r^x}{A_r(\sigma^x(-a_x))} \leq 2 \langle \lambda'_x(0), \gamma'(0) \rangle^{-1} \leq 4/\sqrt{3}$$

holds for any $x \in \partial B_1(p) \cap T_{R_0}(\gamma([0, 1]))$ and any $r \in (0, R_1^*]$.

Setting $W_1 := \bigcup_x B_3(\sigma^x([0, 1])) \cup W_0$, where the union is taken over all points on $\partial B_1(p) \cap T_{R_0}(\gamma([0, 1]))$, we see from (3.5) that $d(p, w) \leq 1 + \frac{1}{2} \hat{R}_0 + 4$ for any $w \in W_1$. Thus \bar{W}_1 is compact and hence the sectional curvature and the injectivity radius take the maximum κ_1 and the minimum i_1 on \bar{W}_1 . Therefore we can find \hat{R}_1 such that (a) $0 < \hat{R}_1 < R_1^*$, (b) $\hat{R}_1 \leq \text{Min} \{ \pi/2\sqrt{\kappa_1}, \frac{1}{2}i_1 \}$, (c) $T_{\hat{R}_1}(\sigma^x[0, 1]) \cap C(\sigma^x(0)) = \emptyset$ for any $x \in \partial B_1(p) \cap T_{R_0}(\gamma([0, 1]))$.

Next for any $\xi \in (0, 1)$, there exists $\hat{R}_1(\xi) \in (0, \hat{R}_1]$ such that for any $x \in \partial B_1(p) \cap T_{R_0}(\gamma([0, 1]))$ and any $y \in \partial B_1(\sigma^x(0)) \cap T_{\hat{R}_1(\xi)}(\sigma^x([0, 1]))$, if $\lambda_y : [0, 1] \rightarrow M$ is the shortest connection from $\sigma^x(0)$ to y , then

$$(3.8) \quad \sphericalangle(\lambda'_y(0), \sigma^{x'}(0)) < \frac{1}{2} \cos^{-1} \xi,$$

and

$$(3.9) \quad \lambda_y \text{ has the extension } \lambda_y|_{[0, b]}, (b > 1) \text{ such that } \lambda_y(b) \in S_{\sigma^x}^{\hat{R}_1}.$$

Let $D^{n-1}(r)$ be the volume of the $(n-1)$ -dimensional disk of radius r in R^{n-1} . Then we can choose $R_1(\xi) \in (0, \hat{R}_1(\xi)]$ such that for any $x \in \partial B_1(p) \cap T_{R_0}(\gamma[0, 1])$, we have (making use of Fermi coordinates φ along $\sigma^x|_{[-a_x, 1]}$)

$$(3.10) \quad dM(z)/dR^n(\varphi(z)) \geq \xi \quad \text{for any } z \in T_{R_1(\xi)}(\sigma^x([0, 1])),$$

$$(3.11) \quad \hat{\xi} \leq A_r(\sigma^x(t))/D^{n-1}(r) \leq 1 \quad \text{for any } t \in [-a_x, 1]$$

and

$$(3.12) \quad \hat{\xi} \leq \text{area of } \partial B_1(\sigma^x(0)) \cap T_r(\sigma^x([0, 1])) / A_r(\sigma^x(1)) \\ \text{for any } r \in (0, R_1(\xi)].$$

We are now ready to prove the theorem. It follows from (3.7) and (3.11) that for any $r \in (0, R_1(\xi)]$

$$\text{the area of } [\partial B_1(p) \cap T_r(\sigma^x([-a'_x, 1]))] \leq \frac{4}{\sqrt{3}} A_r(\sigma^x(-a_x)) \leq \frac{4}{\sqrt{3}} D^{n-1}(r).$$

By means of (3.10) the volume of $[T_r(\sigma^x([0, 1-R_0]))] \geq (1-R_0) \cdot D^{n-1}(r) \cdot \xi$. Then we can find at most countably many points $\{x_i\}$ on $\partial B_1(p) \cap T_{R_0}(\gamma([0, 1]))$, real

numbers $\{r_i\}$, $0 < r_i \leq \frac{1}{2}R_1(\xi)$ and asymptotic rays $\sigma^i : [-a_i, \infty) \rightarrow M$ such that $\sigma^i(-a_i) = x_i$, $\sigma^i(0) \in H_{r_1}$, and $d(\sigma^i(0), \sigma^j(0)) \geq r_i + r_j$ for $i \neq j$, and such that

$$(3.13) \quad \partial B_1(p) \cap T_{R_0}(\gamma([0, 1])) - \bigcup_{i=1}^{\infty} T_{r_i}(\sigma^i([-a'_i, 1])) \cap \partial B_1(p)$$

has measure zero on $\partial B_1(p) \cap T_{R_0}(\gamma([0, 1]))$,

$$(3.14) \quad T_{r_i}(\sigma^i([0, 1])) \cap T_{r_j}(\sigma^j([0, 1])) = \emptyset \quad \text{for } i \neq j.$$

Setting $c :=$ the area of $\partial B_1(p) \cap T_{R_0}(\gamma([0, 1]))$, we have

$$(3.15) \quad \sum_{i=1}^{\infty} \text{vol} [T_{r_i}(\sigma^i([0, 1-R_0]))] \geq \frac{\sqrt{3}}{4}(1-R_0) \cdot c \cdot \xi.$$

Next, setting $W_2 := \bigcup_{i=1}^{\infty} \bigcup_y B_3(\sigma^y([0, 1])) \cup W_1 \cup W_0$ where y ranges over all points on $\partial B_1(\sigma^i(0)) \cap T_{r_i}(\sigma^i([0, 1]))$ and $\sigma^y : [-a_y, \infty) \rightarrow M$ is an asymptotic ray such that $\sigma^y(-a_y) = y$, $\sigma^y(0) \in H_{r_2}$, we see that \bar{W}_2 is compact. Hence by means of (3.8) there exists R_2^* such that

$$(3.7)' \quad 1 \leq \frac{A_r^y}{A_r(\sigma^y(-a_y))} \leq \xi^{-1}$$

for any $y \in \partial B_1(\sigma^i(0)) \cap T_{r_i}(\sigma^i([0, 1]))$ and any $r \in (0, R_2^*]$. Clearly \hat{R}_2 can also be found in the same way as \hat{R}_1 , and $\hat{R}_0 \geq \hat{R}_1 \geq \hat{R}_2$. In the same manner as we find $\hat{R}_1(\xi)$, we can choose $\hat{R}_2(\xi)$ satisfying (3.8) and (3.9) for any $y \in \partial B_1(\sigma^i(0)) \cap T_{r_i}(\sigma^i([0, 1]))$, any $z \in \partial B_1(\sigma^y(0)) \cap T_{\hat{R}_2(\xi)}(\sigma^y([0, 1]))$ and $\lambda_z : [0, 1] \rightarrow M$ joining $y = \lambda_z(0)$ to $z = \lambda_z(1)$. Thus $R_2(\xi)$ can be chosen so as to satisfy (3.10), (3.11) and (3.12) for any σ^y . Hence for each i there exist at most countably many points $\{x_{ij}\}$ on $\partial B_1(\sigma^i(0)) \cap T_{r_i}(\sigma^i([0, 1]))$, real numbers $\{r_{ij}\}$, $0 < r_{ij} \leq \frac{1}{2}R_2(\xi)$ and asymptotic rays $\sigma^{ij} : [-a_{ij}, \infty) \rightarrow M$ such that $\sigma^{ij}(-a_{ij}) = x_{ij}$, $\sigma^{ij}(0) \in H_{r_2}$ and $d(\sigma^{ij}(0), \sigma^{ik}(0)) \geq r_{ij} + r_{ik}$ for any j and $k \neq j$, and such that

$$(3.13)' \quad \partial B_1(\sigma^i(0)) \cap T_{r_i}(\sigma^i([0, 1])) - \bigcup_{j=1}^{\infty} T_{r_{ij}}(\sigma^{ij}([-a'_{ij}, 1])) \cap B_1(\sigma^i(0))$$

has measure zero on $\partial B_1(\sigma^i(0)) \cap T_{r_i}(\sigma^i([0, 1]))$

and

$$(3.14) \quad T_{r_{ij}}(\sigma^{ij}([0, 1])) \cap T_{r_{ik}}(\sigma^{ik}([0, 1])) = \emptyset \quad \text{for } j \neq k,$$

where $a'_{ij} \in (a_{ij}, 1)$ is taken so that $\partial B_1(\sigma^i(0)) \cap T_{r_{ij}}(\sigma^{ij}([-a'_{ij}, 1]))$ is a connected open neighborhood of x_{ij} on $\partial B_1(\sigma^i(0)) \cap T_{\hat{R}_i}(\sigma^i([0, 1]))$. Here we note that each $\sigma^{ij} | [0, \infty)$ is the *unique* asymptotic ray starting from $\sigma^{ij}(0) \in H_{r_2}$ and $\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} T_{r_{ij}}(\sigma^{ij}([0, 1-R_0]))$ is a disjoint union. Thus we obtain

$$\begin{aligned}
 \sum_{j=1}^{\infty} \text{vol} [T_{r_{ij}}(\sigma^{ij}([0, 1-R_0]))] &\geq \sum_{j=1}^{\infty} (1-R_0) D^{n-1}(r_{ij}) \cdot \xi \quad (\text{by (3.10)}) \\
 &\geq \sum_{j=1}^{\infty} (1-R_0) \cdot \xi \cdot A_{r_{ij}}(\sigma^{ij}(-a_{ij})) \quad (\text{by (3.11)}) \\
 &\geq \sum_{j=1}^{\infty} (1-R_0) \xi^2 A_{r_{ij}}^{x_{ij}} \quad (\text{by (3.7)'}) \\
 &\geq (1-R_0) \cdot \xi^2 \cdot \text{area of } [\bigcup_{j=1}^{\infty} \partial B_1(\sigma^i(0)) \cap T_{r_{ij}}(\sigma^{ij}([-a'_{ij}, 1]))] \\
 &\geq (1-R_0) \cdot \xi^2 \cdot \text{area of } [\partial B_1(\sigma^i(0)) \cap T_{r_i}(\sigma^i([0, 1]))] \quad (\text{by (3.13)'}) \\
 &\geq (1-R_0) \cdot \xi^3 \cdot A_{r_i}(\sigma^i(1)) \quad (\text{by (3.12)}) \\
 &\geq (1-R_0) \cdot \xi^4 \cdot A_{r_i}(\sigma^i(-a_i)).
 \end{aligned}$$

Thus we have

$$(3.15)' \quad \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \text{vol} [T_{r_{ij}}(\sigma^{ij}([0, 1-R_0]))] \geq \frac{\sqrt{3}}{4} (1-R_0) \cdot c \cdot \xi^4.$$

Repeating this process N times, we obtain compact subsets $\bar{W}_1, \dots, \bar{W}_N$ on which $R_1(\xi), \dots, R_N(\xi)$ are well defined. Letting i_1, \dots, i_N natural numbers, we have at most countably many points $\{x_{i_1, \dots, i_N}\}$ on $\partial B_1(\sigma^{i_1, \dots, i_{N-1}}(0)) \cap T_{r_{i_1, \dots, i_{N-1}}}(\sigma^{i_1, \dots, i_{N-1}}([0, 1]))$, real numbers $\{r_{i_1, \dots, i_N}\}$, $0 < r_{i_1, \dots, i_N} < \frac{1}{2} R_N(\xi)$ and asymptotic rays $\sigma^{i_1, \dots, i_N} : [-a_{i_1, \dots, i_N}, \infty) \rightarrow M$ emanating from x_{i_1, \dots, i_N} such that $\sigma^{i_1, \dots, i_N}(0) \in H_{r_N}$, e. t. c. Then the above computations imply

$$\begin{aligned}
 (3.16) \quad &\sum_{i_1=1}^{\infty} \dots \sum_{i_N=1}^{\infty} \text{vol} [T_{r_{i_1, \dots, i_N}}(\sigma^{i_1, \dots, i_N}([0, 1-R_0]))] \\
 &\geq \frac{\sqrt{3}}{4} (1-R_0) \cdot c \cdot \xi^{3N-2}.
 \end{aligned}$$

Hence for any number ν , there exist $\xi \in (0, 1)$ and N such that

$$\begin{aligned}
 \text{vol} [M] &> \frac{\sqrt{3}}{4} (1-R_0) c \cdot \xi \{1 + \xi^3 + \dots + \xi^{3(N-1)}\} \\
 &= \frac{\sqrt{3}}{4} (1-R_0) \cdot c \cdot \xi \frac{1 - \xi^{3N}}{1 - \xi^3} > \nu.
 \end{aligned}$$

Thus the proof is complete.

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