AN EXTENSION OF LINDELÖF'S THEOREM TO MEROMORPHIC FUNCTIONS

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1. Introduction. If $f(z)$ be an entire function of finite order ρ, then Lindelöf has obtained a set of conditions in order that $f(z)$ may be of maximum, mean or minimum type [2; 1, 25-30]. A theorem of the similar nature for meromorphic functions is stated by Valiron [8] and Hari Shanker has recently [3] extended the results of Lindelöf by taking comparison function $r^\rho L(r)$. In this note we prove three theorems which will include the theorems of Lindelöf, Valiron and Hari Shanker as special cases.

Let $f(z)$ be a meromorphic function of finite order ρ. We have

$$(1.1) \qquad f(z) = z^k \exp(Cz^{p_3} + \cdots) \prod_1^\infty E(z/a_n, p_1) / \prod_1^\infty E(z/b_n, p_2),$$

where $Q(z) = Cz^{p_3} + \cdots\cdots$ is a polynomial of degree $p_3 \leqq [\rho]$. Write

$$n(r) = n(r, 1/f) + n(r,f), \ N(r) = N(r, 1/f) + N(r,f).$$

When $\rho > 0$, and $N(r)$ is of order ρ, we define a proximate order $\rho(r)$ for $N(r)$ as follows.

(i) $\rho(r)$ is differentiable for $r > r_0$, except at isolated points at which $\rho'(r-0)$ and $\rho'(r+0)$ exist.

(ii) $$\lim_{r\to\infty} \rho(r) = \rho.$$

(iii) $$\lim_{r\to\infty} r\rho'(r) \log r = 0.$$

(iv) $$\limsup_{r\to\infty} N(r)/r^{\rho(r)} = 1.$$

For the existence of a proximate order see [6] where $\rho(r)$ is constructed with $\log M(r)$; the argument given there can be utilised to construct $\rho(r)$ with the above properties (i)—(iv).

When ρ is integer, we can write $f(z)$ in the form

$$(1.2) \qquad f(z) = z^k \exp(cz^\rho + \cdots) \prod_1^\infty E(z/a_n, \rho) / \prod_1^\infty E(z/b_n, \rho)$$

$$= z^k \exp(cz^\rho + \cdots) P_1/P_2 \text{ (say)}$$

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where if f has no poles, P_2 is to be replaced by 1, and if f has a finite number of poles then $$P_2 = \prod_1^M E(z/b_n, \rho).$$

Similarly when $$\lim_{r\to\infty} n(r, 1/f) < \infty.$$

THEOREM 1. *If ρ is integer and $N(r)$ is of order ρ, then*

$$T(r,f) = \frac{r^\rho}{\rho\pi}\left|c\rho + \sum_{|a_n|<r} a_n^{-\rho} - \sum_{|b_n|<r} b_n^{-\rho}\right| + O(r^{\rho(r)}) \tag{1.3}$$

where $f(z)$ is given by (1.2). *If $N(r)$ is of order less than ρ, then we have from* (1.1)

$$p_3 = \rho,\ C \neq 0,\ T(r,f) = \frac{r^\rho}{\pi}|C| + O(r^\mu),\ \mu < \rho. \tag{1.4}$$

THEOREM 2. *Let $f(z)$ be a meromorphic function of integer order ρ and let $L(r)$ be a slowly changing positive function* [4; pp. 52-54] *and write*

$$\limsup_{r\to\infty} T(r)/r^\rho L(r) = T_L;\ \limsup_{r\to\infty} n(r)/r^\rho L(r) = n_L;$$
$$\limsup_{r\to\infty} S(r)/r^\rho L(r) = S_L; \tag{1.5}$$

where

$$S(r) = \frac{r^\rho}{\rho\pi}\left|c\rho + \sum_{|a_n|<r} a_n^{-\rho} - \sum_{|b_n|<r} b_n^{-\rho}\right|. \tag{1.6}$$

(i) *If $N(r)$ is of order ρ, then*

$$0 < T_L < \infty \Longleftrightarrow 0 < \max(n_L, S_L) < \infty,$$
$$T_L = \infty \Longleftrightarrow \max(n_L, S_L) = \infty,$$
$$T_L = 0 \Longleftrightarrow \max(n_L, S_L) = 0. \tag{1.7}$$

(ii) *If $N(r)$ is of order less than ρ, then*

$$T(r) \sim S(r) \sim r^\rho|C|/\pi;\ n(r) = O(r^\alpha),\ \alpha < \rho. \tag{1.8}$$

THEOREM 3. *Let $f(z)$ be a meromorphic function of order ρ.*

(i) *If $\rho > 0$ be non-integer, and T_L etc. be defined as in* (1.5), *then*

$$0 < T_L < \infty \Longleftrightarrow 0 < n_L < \infty,$$
$$T_L = \infty \Longleftrightarrow n_L = \infty,$$
$$T_L = 0 \Longleftrightarrow n_L = 0. \tag{1.9}$$

(ii) *When $\rho = 0$, let the comparison function be $L(r) = (\log r) L_1(r)$ where $L_1(r)$ is slowly changing and $\uparrow \infty$ with r. Write*

$$T_L = \limsup_{r\to\infty} T(r)/L(r);\ N_L = \limsup_{r\to\infty} \frac{\{N(r, 1/f) + N(r,f)\}}{L(r)}.$$

Then

$$(1.10)\qquad T_L \leqq N_L \leqq 2T_L,$$

and the relations (1.9) *hold with* N_L *instead of* n_L. *If* $f(z)$ *be entire function of zero order, then* $T_L = N_L$.

2. Proof of Theorem 1. Write (1.2) in the form

$$f(z) = \phi_0\,\phi_1\,\phi_2\,\phi_3\,\phi_4^{-1}\,\phi_5,$$

where

$$\phi_0(z) = z^{\kappa} \exp(c_1 z^{\rho-1} + \cdots),$$

$$\phi_1(z) = \exp\left\{cz^\rho + \frac{z^\rho}{\rho}\left(\sum_{|a_n|<2r} a_n^{-\rho} - \sum_{|b_n|<2r} b_n^{-\rho}\right)\right\},$$

$$\phi_2(z) = \prod_1^{n_0} E(z/a_n, \rho-1)/\prod_1^{n_0} E(z/b_n, \rho-1),$$

$$\phi_3(z) = \prod_{n>n_0}^{|a_n|<2r} E(z/a_n, \rho-1),$$

$$\phi_4(z) = \prod_{n>n_0}^{|b_n|<2r} E(z/b_n, \rho-1),$$

$$\phi_5(z) = \prod_{|a_n|\geqq 2r} E(z/a_n, \rho)\Big/\prod_{|b_n|\geqq 2r} E(z/b_n, \rho).$$

Now for ϕ_0, ϕ_2

$$T(r) = O(r^{\rho-1}) + O(\log r).$$

Further

$$\log^+|\phi_3(z)| = O\left(N(r, 1/f) + r^{\rho-1}\int_{r_0}^{2r} n(x, 1/f)x^{-\rho}\,dx + n(2r, 1/f)\right).$$

Hence

$$T(r, \phi_3) = O(r^{\rho(r)}).$$

Similarly

$$T(r, \phi_4) = O(r^{\rho(r)}),$$

$$T(r, \phi_5) = O\left(r^{\rho+1}\int_{2r}^{\infty} x^{\rho(x)-\rho-2}\,dx\right) = O(r^{\rho(r)}).$$

Hence

$$T(r,f)=\frac{r^\rho}{\rho\pi}\left|c\rho+\sum_{|a_n|\leqq r}a_n^{-\rho}-\sum_{|b_n|\leqq r}b_n^{-\rho}\right|+O(r^{\rho(r)})$$

$$=S(r)+O(r^{\rho(r)}).$$

If $N(r)=O(r^\alpha)$, where $\alpha<\rho$ then

$$T(r,P_i)=O(r^\beta),\ i=1,2,\ \alpha<\beta<\rho.$$

Hence

$$P_3=\rho,\ C\neq 0,\ T(r,f)=\frac{|C|r^\rho}{\pi}+O(r^\mu),\ \mu<\rho.$$

3. Proof of Theorem 2. (i) Suppose $N(r)$ is of order ρ. If $n_L<\infty$, then we obtain as in the first part of Theorem 1,

(3. 1) $$T(r,f)=S(r)+O(r^\rho L(r));$$

and if $n_L=0$, then

(3. 2) $$T(r,f)=S(r)+o(r^\rho L(r)).$$

Hence if $S_L<\infty$, then $T_L<\infty$. If $T_L<\infty$, then $T(r)<Ar^\rho L(r)$, $N(r)<A_1r^\rho L(r)$, $n(r)<A_2r^\rho L(r)$. Hence $n_L<\infty$ and from (3. 1) $S_L<\infty$. If $n_L>0$ then $T_L>0$. If $T_L>0$, then $\max(n_L,S_L)>0$, for if $n_L=0$, then from (3. 2), $T_L=S_L>0$. If $T_L=\infty$, then $\max(n_L,S_L)=\infty$, for if this expression is less than ∞, then from (3. 1) we get $T_L<\infty$. If $n_L=\infty$ then $T_L=\infty$ and if $S_L=\infty$, $n_L<\infty$ then from (3. 1) $T_L=\infty$. If $T_L=0$, then $n_L=0$ and from (3. 2) $S_L=0$. If $n_L=S_L=0$ then $T_L=0$.

(ii) Since $N(r)$ is of order $<\rho$, $\sum a_n^{-\rho}$, $\sum b_n^{-\rho}$ are both convergent, and if f has an infinity of zeros and an infinity of poles,

$$c\rho=C\rho-\sum_1^\infty a_n^{-\rho}+\sum_1^\infty b_n^{-\rho}.$$

Hence

$$S(r)=\frac{r^\rho}{\rho\pi}\left|C\rho-\sum_{|a_n|>r}a_n^{-\rho}+\sum_{|b_n|>r}b_n^{-\rho}\right|$$

and so

(3. 3) $$S(r)\sim r^\rho|C|/\pi.$$

Similarly we can prove (3. 3) when f has a finite number of poles or zeros or both. Hence from (1. 4) and (3. 3)

$$T(r) \sim S(r);\ n(r) = O(N(2r)) = O(r^{\alpha}),\ \alpha < \rho.$$

4. Proof of Theorem 3. (i) We have from (1. 1), [7]

$$T(r, f) \leqq O(r^{p_3}) + O(\log r) + \log M(r, P_1) + \log M(r, P_2)$$

$$\leqq O(r^{\mu}) + O(\log r) + A \int_0^{\infty} \frac{n(x)\, r^{1+\mu}}{x^{1+\mu}(x+r)}\, dx$$

where $\mu = [\rho]$. Hence if $n_L < \infty$,

$$T(r, f) \leqq A_1 \left\{ r^{\mu} + \log r + r^{\mu} \int_1^{r} x^{\rho-\mu-1} L(x)\, dx \right.$$

$$\left. + r^{1+\mu} \int_r^{\infty} x^{\rho-\mu-2} L(x)\, dx \right\}$$

$$\leqq A_1 \left\{ r^{\mu} + \log r + \frac{r^{\rho} L(r)}{\rho - \mu} + \frac{r^{\rho} L(r)}{1 + \mu - \rho} \right\}$$

$$< A_2\, r^{\rho} L(r),$$

and (1. 9) follows provided $n_L < \infty$. If $n_L = \infty$ then $T_L = \infty$ and from above if $T_L = \infty$ then $n_L = \infty$.

(ii) We have [5]

$$N(r) \leqq 2T(r, f) + O(1),$$

$$T(r, f) \leqq \{1 + o(1)\} r \int_r^{\infty} \{N(t)/t^2\}\, dt$$

and (1. 10) follows. If f be entire then $N(r) \leqq T(r, f) + O(1)$, $T_L = N_L$.

5. Remarks. (i) If f is entire and $p_1 = \rho$ then from (1. 3) we have, since $c = C$,

$$T(r, f) = \frac{r^{\rho}}{\rho\pi} \left| C\rho + \sum_{|a_n| < r} a_n^{-\rho} \right| + O(r^{\rho(r)}). \tag{4. 1}$$

Further since

$$T(r, f) \sim T(r, fP)$$

where P is any polynomial, (4. 1) holds also for functions with a finite number of poles and $p_1 = \rho$. We can get this result directly from (1. 3) for we will have

$$T(r, f) = \frac{r^{\rho}}{\rho\pi} \left| \left(C + \frac{1}{\rho} \sum_1^{k} b_n^{-\rho} \right) \rho + \sum_{|a_n| < r} a_n^{-\rho} - \sum_1^{k} b_n^{-\rho} \right| + O(r^{\rho(r)}).$$

A similar remark applies when f has a finite number of zeros.

(ii) The formula (1. 3) is useful when $S(r)$ is large compared to $r^{\rho(r)}$. For instance if

$$f(z) = \Gamma(z),\ S(r) \sim \{r \log r\}/\pi,\ r^{\rho(r)} = O(r).$$

But for functions of the form (1. 2) with

$$c\rho + \sum_{|a_n|<r} a_n^{-\rho} - \sum_{|b_n|<r} b_n^{-\rho} = O(1);\ N(r) > \alpha r^{\rho},\ \alpha > 0,\ r > r_0,$$

(1. 3) does not give much information.

(iii) If f be meromorhic function of integer order ρ and such that $\max(p_1, p_2) = \rho - 1$, then we get from (1. 3)

$$T(r,f) = \frac{r^\rho}{\rho\pi} \left| C\rho - \sum_{|a_n|>r} a_n^{-\rho} + \sum_{|b_n|>r} b_n^{-\rho} \right| + O(r^{\rho(r)}). \tag{4. 2}$$

If f be entire function with $p_1 = \rho - 1$, $p_3 < \rho$ and such that $n_L < \infty$, then from (3. 1) and (4. 2)

$$T(r,f) = \frac{r^\rho}{\rho\pi} \left| \sum_{|a_n|>r} a_n^{-\rho} \right| + O(r^\rho L(r)).$$

It is not necessary to suppose here that $L(r)$ be monotone except when $\rho = 0$ (see Theorem 3(ii)).

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