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AN EXTENSION OF PITMAN'S THEOREM FOR SPECTRALLY POSITIVE LEVY PROCESSES

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If X is a spectrally positive Lévy process, $\overline{X}^c$ the continuous part of its maximum process, and J the sum of the jumps of X across its previous maximum, then $X - 2\overline{X}^c - J$ has the same law as X conditioned to stay negative. This extends a result due to Pitman, who links the real Brownian motion and the three-dimensional Bessel process. Several other relations between the Brownian motion and the Bessel process are extended in this setting.

0. Introduction. Pitman (1975) constructs the three-dimensional Bessel process R as the difference $2\overline{X} - X$, where X is a real Brownian motion (BM) and $\overline{X} = \sup\{X_s: 0 \le s \le t\}$ its maximum process. This result is extended by Rogers and Pitman (1981), who show that R may be replaced by the Euclidean norm of a three-dimensional Brownian motion with constant drift a when X is replaced by a real Brownian motion with drift $\pm|a|$. But, in terms of diffusions, you cannot go much further: Rogers (1981) proves that the class of real diffusions X such that $2\overline{X} - X$ is a strong Markov process is essentially restricted to the preceding processes.

Pitman's theorem is not only a remarkable complement to Lévy's identity for the reflected Brownian motion, but also a cornerstone of other deep connections between BM and Bessel(3). Let us briefly recall the main relations. First, Doob (1957) and McKean (1963) observed that, roughly speaking, Bessel(3) is a BM conditioned to stay positive. Ciesielski and Taylor (1962) noted that the total time spent by a three-dimensional BM in the unit ball has the same law as the first hitting time of 1 by a reflected BM. Further important links are discovered by Williams (1974, 1979). In particular, BES(3) appears when time-reversing BM at its first hitting time of 1 (this explains very nicely the result of Ciesielski and Taylor) and when splitting the Brownian excursion at its maximum. Various other relations can be found in Williams (1979), Rogers and Williams (1987) and Revuz and Yor (1991). In this framework, the links between Pitman's theorem, Lévy's identity, the invariance of the Brownian excursion law by time reversal and Williams' time reversal identity between BM and Bessel(3) are illuminated by Exercise 4.15 in Revuz and Yor [(1991), Chapter VII)].

The main result of this paper is an extension of Pitman's theorem for spectrally positive Lévy processes (i.e., processes with independent homoge-

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neous increments and no negative jumps). Specifically, if $\mathbb{P}$ is the law of a spectrally positive Lévy processes X, if $\overline{X}^c$ is the continuous part of the increasing process $\overline{X}$ and if J_t is the sum of the jumps accomplished by X across its previous maximum before time t, then $X - 2\overline{X}^c - J$ has the same law, say, P, as X "conditioned to stay negative." This conditioning is a priori formal, since in general the event $\{X_t < 0 \text{ for all } t > 0\}$ has zero probability. Nevertheless, just as for the Brownian motion, it can be made rigorous in terms of h-processes. However, the exit properties of spectrally positive Lévy processes enable us to introduce the law P using bare-hands methods without any difficulty, and we will not need any notion of h-processes. The construction presented in this paper is not valid for all Lévy processes. If X is the opposite of a compensated Poisson process, the paths of $X - J - 2\overline{X}^c$ are broken lines with slope ± 1, and $X - 2\overline{X}^c - J$ is clearly not a Markov process. The author has recently obtained further results in this direction [Bertoin (1991)]. Note also that, in the Brownian case, X can be recovered from $2\overline{X} - X$, while in the Lévy setting, this feature has been lost in general.

The original proof of Pitman in the Brownian case was based on a random walk analogue. Some developments in this setting were recently made by Miyazaki and Tanaka (1989) and Tanaka (1989). Here, our approach relies essentially on the following three points: first, the invariance property under time reversal for the excursion law of X [due to Getoor and Sharpe (1982)]; second, an analogue of Lévy's identity from which we deduce a description of the excursion law of the reflected process $X - \overline{X}$ in terms of time reversal of X; and third, a time reversal identity linking $\mathbb{P}$ and P, analogous to Williams' result. Of course, the continuity of the Brownian path is crucial in exercise 4.15 of Revuz and Yor (1991), so we cannot simply repeat their arguments. Nevertheless, the result follows straightforwardly from the preceding three points when X has no Gaussian component. The general case is then deduced by approximation, the difficulty consisting of checking that suitable convergences hold.

The basic results on spectrally positive Lévy processes, their excursions and their exit properties are recalled in Section 1. Section 2 is devoted to the study of the reflected process. Several important results on the conditional law P are presented in Section 3. The main theorem is proved in Section 4.

1. Preliminaries. We begin with some notation concerning the canonical space. Let Ω be the space of right-continuous functions with left limits ω: $[0, \zeta) \to \mathbb{R}$, where $\zeta = \zeta(\omega)$ is the lifetime. Ω is endowed with Skorohod's topology, and $\mathscr{F}$ stands for its Borel σ-algebra. For every Borel set A, we denote by $\tau(A)$ and $\sigma(A)$, respectively, the first hitting time of A and the last exit time from A:

$$\tau(A) = \inf\{t > 0: \omega(t) \in A\}, \qquad \sigma(A) = \sup\{t: \omega(t) \in A\},$$

with the usual convention: $\inf \varnothing = \infty$, $\sup \varnothing = -\infty$. For convenience, we simply write $\tau(x)$ and $\sigma(x)$ instead of $\tau(\{x\})$ and $\sigma(\{x\})$.

We denote by X the canonical process on Ω, $X_t(\omega) = \omega(t)$, and we define

$$\overline{X}_t = \sup\{X_s: 0 \le s \le t\}, \qquad \underline{X}_t = \inf\{X_s: 0 \le s \le t\},$$

$$\overline{X}^c_t = \overline{X}_t - \sum_{s \le t} \left(\overline{X}_s - \overline{X}_{s-}\right), \qquad J_t = \sum_{s \le t} (X_s - X_{s-}) \mathbf{1}_{\{X_s > \overline{X}_{s-}\}}.$$

Finally, $(\mathscr{F}_t)_{t \ge 0}$ stands for the canonical filtration.

We consider a probability measure $\mathbb{P}$ on $(\Omega, \mathscr{F})$ such that, under $\mathbb{P}$, X has independent homogeneous increments, no negative jumps and $\mathbb{P}(X_0 = 0) = 1$. By the Lévy–Khintchine theorem,

$$\mathbb{E}(\exp\{-zX_t\}) = \exp\{t\Psi(z)\}, \qquad t \ge 0, \operatorname{Re} z = 0,$$

$$\Psi(z) = az + \tfrac{1}{2}\sigma_0^2 z^2 + \int_{(0,\infty)} \left(e^{-zx} - 1 + zx\mathbf{1}_{\{x<1\}}\right) d\Pi(x), \qquad \operatorname{Re} z \ge 0,$$

where a is a real number, $\sigma_0 \ge 0$ and Π is a measure on $(0, \infty)$ such that $\int(1 \wedge x^2)\, d\Pi(x) < \infty$. We call Ψ the characteristic exponent and Π the Lévy measure of $(X, \mathbb{P})$. Furthermore, we assume that X and $-X$ are not subordinators under $\mathbb{P}$. The mapping $x \mapsto \Psi(x)$ is convex on $[0, \infty)$, and $\lim_{x \uparrow \infty} \Psi(x) = \infty$.

We denote the largest real solution of $\Psi(x) = 0$ by $\Phi(0)$ and by Φ: $[0, \infty) \to [\Phi(0), \infty]$, the inverse function of $x \mapsto \Psi(x)$ on $[\Phi(0), \infty)$. Since X has no negative jumps, $\underline{X}$ is a continuous process, and the strong Markov property implies that $t \mapsto \tau(-t)$ is a subordinator. Its Laplace transform is obtained by use of the optional sampling theorem on the local martingale $\exp\{-\Phi(\alpha)X_t - \alpha t\}$: for every $x \ge 0$,

$$\mathbb{E}(\exp\{-\alpha\tau(-x)\}) = \exp\{-x\Phi(\alpha)\} \tag{1.1}$$

[see, for instance, Bingham (1975), Section 4].

We say that $\mathbb{P}$ drifts to $+\infty$ (respectively, drifts to $-\infty$, oscillates) if $\lim_{t \uparrow \infty} X_t = +\infty$ (respectively, if $\lim_{t \uparrow \infty} X_t = -\infty$, if $\lim_{t \uparrow \infty} \overline{X}_t = +\infty$ and $\lim_{t \uparrow \infty} \underline{X}_t = -\infty$) $\mathbb{P}$-a.s. In terms of the characteristic exponent, this property is equivalent to $\Phi(0) > 0$ [respectively, $\Phi(0) = 0$ and $\Phi'(0+) < \infty$, $\Phi(0) = 0$ and $\Phi'(0+) = \infty$].

For every $x \in \mathbb{R}$, we denote by $\mathbb{P}_x$ the law of $x + X$ under $\mathbb{P}$. When I is an open interval containing x, $\mathbb{P}_x^I$ stands for the law of the canonical process killed at its first exit from I under $\mathbb{P}_x$. We denote by μ the Itô measure of the excursions away from 0 of $(X, \mathbb{P})$ [see Itô (1972)]. Of course, μ is specified only up to a constant factor (which will be chosen in Section 2 for convenience). When 0 is irregular for itself with respect to $(X, \mathbb{P})$, μ is regarded as being proportional to the law of the canonical process killed at its first return to 0. The constant factor does not play any role is this section.

The dual law of $(X, \mathbb{P})$ is $(-X, \mathbb{P})$ and, by Getoor and Sharpe [(1982), Section 7], the excursion measure has the following property of invariance under time reversal:

(1.2) The measure $\mathbf{1}_{\{\tau(0)<\infty\}}\mu$ is invariant under the transformation $\omega \mapsto \check{\omega}$, where $\check{\omega}(t) = -\omega((\tau(0)-t)-)$, for $t < \tau(0)$, and $\check{\omega}(t) = 0$ otherwise.

One can also prove (1.2) using the fact that Lebesgue measure is stationary for Lévy processes [see Pitman (1987)].

In the special case where $\mathbb{P}$ has no Gaussian component (i.e., $\sigma_0 = 0$), X does not creep across positive levels, that is, $\mathbb{P}(\tau(y) = \tau([y,\infty)) < \infty) = 0$ for every $y > 0$ [see Millar (1973)]. We deduce that the generic excursion of $(X, \mathbb{P})$ necessarily has one of the following two forms. Either it stays negative or first it is negative, then it jumps across 0 and, finally, it stays positive until it (possibly) returns to 0. The law of the jump across 0 may be computed as a special case of Rogers [(1984), Theorem 1]; see the remark following the proof of Corollary 1 in Section 2 (the arbitrary factor in the definition of μ is then chosen to give the simplest formulas). Let us denote by j the unique time when the excursion jumps across 0. By the strong Markov property, under μ, conditionally on $X_j = x$, $x > 0$, the post-j process $(X_{j+t}\colon 0 \le t \le \tau(0) - j)$ is independent of the pre-j process $(X_t\colon 0 \le t < j)$ and has law $\mathbb{P}_x^{(0,\infty)}$. Moreover, again by the strong Markov property, the pre-j process has the same law under $\mu(\cdot \mid X_j = x)$ as under $\mu(\cdot \mid X_j = x, \tau(0) < \infty)$. Hence, the time reversal identity (1.2) yields the following lemma.

LEMMA 1. *Assume that $\sigma_0 = 0$, and set $j = \tau((0,\infty))$. Under μ, conditionally on $-X_{j-} = y$ and $X_j - X_{j-} = z$, the processes $(-X_{(j-t)-}\colon 0 \le t < j)$ and $(X_{j+t}\colon 0 \le t < \tau(0) - j)$ are independent. The first process has the law $\mathbb{P}_y^{(0,\infty)}(\cdot \mid X_{\zeta-} = 0)$, that is, the law of the initial Lévy process starting from y, conditioned to hit 0 and then killed at 0. The second has the law $\mathbb{P}_{z-y}^{(0,\infty)}$.*

Finally, we recall an important exit property for spectrally positive Lévy processes. The two-sided exit problem consists of determining the probability that X makes its first exit from $[b, c)$, $b < 0 < c$, through b. It is solved in Takács (1967) and Emery (1973); see also Rogers (1990). There is a continuous increasing function $W\colon [0,\infty) \to [0,\infty)$, with Laplace–Stieltjes transform

$$\int_0^\infty e^{-\alpha y} W(y)\, dy = 1/\Psi(\alpha), \qquad \alpha > \Phi(0),$$

such that

$$\mathbb{P}(\tau([c,\infty)) > \tau(b)) = W(c)/W(c-b). \tag{1.3}$$

2. The reflected process. The purpose of this section is to investigate the (strong Markov) reflected process $X - \overline{X}$ [see Bingham (1975), Greenwood and Pitman (1980), Silverstein (1980) and Rogers (1984) for related studies]. In general, X cannot be reconstructed from $X - \overline{X}$, because $X - \overline{X}$ does not contain all the information about the jumps of X across its previous maximum. Since these quantities will play an important role in the sequel, they will be reintroduced in this study. Our main tool will be an identity of Lévy type (Lemma 2); but first, we will need the following notation:

1. OX_t is the amount by which X will first overshoot 0 after time t, that is, $OX_t = X_s$, where $s = \inf\{r > t\colon X_r > 0\}$ ($OX_t = 0$ when $s = \infty$).
2. $O\overline{X}_t$ is the amount by which X will first overshoot $\overline{X}_t$ after time t, that is, $O\overline{X}_t = X_s - \overline{X}_t$, where $s = \inf\{r > t\colon X_r > \overline{X}_t\}$ ($O\overline{X}_t = 0$ when $s = \infty$).
3. $T(t) = \inf\{s\colon \int_0^s \mathbf{1}_{\{X_r < 0\}}\, dr > t\}$.

When $\mathbb{P}$ does not drift to $+\infty$, $T(t)$ is finite for all $t \geq 0$, and we denote by (Y, OY), the process (X, OX) after the time-substitution by T. When $\mathbb{P}$ drifts to $+\infty$, the lifetime of T is $\xi = \int_0^\infty \mathbf{1}_{\{X_r < 0\}}\, dr$. We denote by (Y, OY), the resurrected process associated to $(X, OX) \circ T$, which is constructed by pasting together independent copies of $((X, OX)_{T(t)}\colon t < \xi)$. Now we claim the following.

LEMMA 2. *Under* $\mathbb{P}$, $(X - \overline{X}, O\overline{X})$ *and* (Y, OY) *have the same laws.*

PROOF. We will only prove the lemma when $\mathbb{P}$ does not drift to $+\infty$. The proof in the opposite case is almost the same, but with heavier notation. For every $\varepsilon > 0$, we introduce two sequences of stopping times:

$$\begin{aligned} U(\varepsilon, 0) &= V(\varepsilon, 0) = 0,\\ U(\varepsilon, 2n+1) &= \inf\{s > U(\varepsilon, 2n)\colon X_s \leq -\varepsilon\},\\ U(\varepsilon, 2n+2) &= \inf\{s > U(\varepsilon, 2n+1)\colon X_s \geq 0\},\\ V(\varepsilon, 2n+1) &= \inf\{s > V(\varepsilon, 2n)\colon (X - \overline{X})_s \leq -\varepsilon\},\\ V(\varepsilon, 2n+2) &= \inf\{s > V(\varepsilon, 2n+1)\colon X_s = \overline{X}_s\}. \end{aligned}$$

[$U(\varepsilon, n+1)$ is the first time after $U(\varepsilon, n)$ when X hits $[0, \infty)$ again if n is odd, and if $-\varepsilon$ is n is even; the construction for $V(\varepsilon, \cdot\,)$ is the same, but with respect to $X - \overline{X}$.] We consider the two increasing processes ${}^{\varepsilon}A$ and ${}^{\varepsilon}B$ defined by ${}^{\varepsilon}A(0) = {}^{\varepsilon}B(0) = 0$ and

$$d\,{}^{\varepsilon}A(t) = \sum_n \mathbf{1}_{[U(\varepsilon, 2n+1), U(\varepsilon, 2n+2)]}\, dt,$$

$$d\,{}^{\varepsilon}B(t) = \sum_n \mathbf{1}_{[V(\varepsilon, 2n+1), V(\varepsilon, 2n+2)]}\, dt.$$

We denote by ${}^{\varepsilon}A^{-1}$ and ${}^{\varepsilon}B^{-1}$ their respective right-continuous inverses. Recall that X has no negative jumps and hits every negative real number (because $\mathbb{P}$ does not drift to $+\infty$, this is the reason why we have to consider resurrected processes otherwise). Thus $X_{U(\varepsilon, 2n+1)} = -\varepsilon$ whenever $U(\varepsilon, 2n) < \infty$ and $(X - \overline{X})_{V(\varepsilon, 2n+1)} = -\varepsilon$ whenever $V(\varepsilon, 2n) < \infty$. In $(-\infty, 0)$, $X - \overline{X}$ evolves just like X, so the two sequences of processes,

$$\{((X, OX)_{U(\varepsilon, 2n+1)+t}\colon 0 \leq t < U(\varepsilon, 2n+2) - U(\varepsilon, 2n+1))\colon n \in \mathbb{N}\}$$

and

$$\left\{\left((X-\overline{X}, O\overline{X})_{V(\varepsilon,2n+1)+t}: 0 \le t < V(\varepsilon,2n+2) - V(\varepsilon,2n+1)\right): n \in \mathbb{N}\right\},$$

have the same laws. The process (X, OX) time-changed by ${}^{\varepsilon}A^{-1}$ is obtained after pasting together the first sequence, and $(X-\overline{X}, OX)\circ {}^{\varepsilon}B^{-1}$ is similarly obtained with the second sequence. Thus they are equally distributed.

On the other hand, by Bingham [(1975), Proposition 10], the inverse local time at 0 of $X-\overline{X}$ has no drift, that is,

$$\int_0^{\infty} \mathbf{1}_{\{X_r = \overline{X}_r\}}\, dr = 0 \quad \text{a.s.}$$

Hence, a.s. for every s, $\int_0^s \mathbf{1}_{\{(X-\overline{X})_r < -\varepsilon\}}\, dr$ converges to s as $\varepsilon \downarrow 0$. Since

$$\int_0^s \mathbf{1}_{\{(X-\overline{X})_r < -\varepsilon\}}\, dr \le {}^{\varepsilon}B(s),$$

${}^{\varepsilon}B(s)$ increases to s as $\varepsilon \downarrow 0$. Thus ${}^{\varepsilon}B^{-1}(t)$ decreases to t as $\varepsilon \downarrow 0$; similarly, ${}^{\varepsilon}A^{-1}(t)$ decreases to $T(t)$ as $\varepsilon \downarrow 0$. This shows that $(X-\overline{X}, O\overline{X})$ is distributed as (Y, OY) under $\mathbb{P}$. □

Remember that we do not want to lose information about the jumps of X across its maximum. Thus, we denote by $\overline{\mu}$ the excursion law of $X-\overline{X}$ *but where we replaced the value* 0 *at the lifetime of the excursion by the amount by which* X *overshoots its previous maximum* (i.e., $\overline{\mu}$ is the excursion law of $t \mapsto X_t - \overline{X}_{t-}$). The constant factor in the definition of $\overline{\mu}$ is chosen so that Proposition 9 of Bingham (1975) holds.

When $\mathbb{P}$ has no Gaussian component, we deduce by time-change from Lemma 1 the following description of $\overline{\mu}$ [parts (i) and (ii) are special cases of Rogers (1984), Theorem 1].

Corollary 1. *Assume that* $\sigma_0 = 0$, *and remember that* j *is the time of the unique jump across* 0 *for the generic excursion.*

(i) $\overline{\mu}(X_{j-} = 0) = 0$, *and* $\overline{\mu}(j = \infty) = (1/\Phi'(0+))\mathbf{1}_{\{\Phi(0)=0\}}$.

(ii) *For* $0 < y < z$, $\overline{\mu}(-X_{j-} \in dy,\, X_j - X_{j-} \in dz) = e^{-\Phi(0)y}\, dy\, d\Pi(z)$.

(iii) *Under* $\overline{\mu}(\cdot \mid -X_{j-} = y)$ *the process* $(-X_{(j-t)-}: 0 \le t < j)$ *has the law* $\mathbb{P}_y^{(0,\infty)}(\cdot \mid X_{\zeta-} = 0)$.

Proof. (i) and (ii) The equality $\overline{\mu}(X_{j-} = 0) = 0$ follows from the fact that X does not creep across positive levels when $\mathbb{P}$ has no Gaussian component.

Introduce $\mathfrak{e}$, an independent exponential time of parameter $\varepsilon > 0$. According to Rogers [(1984), Theorem 1] and Bingham [(1975), Proposition 9], $\overline{\mu}(j > \mathfrak{e}) = \varepsilon/\Phi(\varepsilon)$. This quantity goes to 0 as $\varepsilon \downarrow 0$ when $\Phi(0) > 0$, and to $1/\Phi'(0+)$ otherwise. Moreover, Rogers obtains

$$\overline{\mu}(X_{j-} \in dy,\, X_j - X_{j-} \in dz,\, j < \mathfrak{e}) = (1/\Phi(\varepsilon))\, d\Pi(z)\mathbb{P}(\underline{X}_{\mathfrak{e}} \in dy).$$

Since $-\underline{X}_\varepsilon$ has an exponential distribution of parameter $\Phi(\varepsilon)$ under $\mathbb{P}$, we get $\overline{\mu}(-X_{j-} \in dy,\ X_j - X_{j-} \in dz) = \exp(-\Phi(0)y)\,dy\,d\Pi(z)$.

(iii) When 0 is irregular for itself under $\mathbb{P}$, the statement is a direct consequence of Lemma 1. Suppose that 0 is regular and that $\mathbb{P}$ does not drift to $+\infty$, and recall that ℓ is the local time at 0 for $(X, \mathbb{P})$. We check that $\ell \circ T$ is a local time at 0 for Y. Indeed, if $\ell \circ T(t-) < \ell \circ T(t)$, then $T(t-) < T(t)$. By the definition of $T(\cdot)$, $X_s \geq 0$ for Lebesgue a.e. s in $[T(t-), T(t)]$. So, by the description of the excursions of $(X, \mathbb{P})$, $X_s > 0$ for every s in the open interval $(T(t-), T(t))$. But ℓ only increases when X is null, so $\ell \circ T(t-) = \ell \circ T(t)$, and $\ell \circ T$ is continuous a.s. Furthermore, $\ell \circ T$ is a positive additive functional of $Y = X \circ T$ which increases only when $Y = 0$, and $\ell \circ T$ is a local time at 0 for Y. Consequently, the Itô measure of the excursions of Y (where we replaced the value 0 at the lifetime of the excursion by the value that takes OY over the excursion interval) is simply the image of μ by the mapping $\omega \mapsto \omega'$, $\omega'(r) = \omega(r)\mathbf{1}_{\{\omega(r-)<0\}}$ [see, e.g., Bertoin (1990), Lemma I.1]. We need only apply Lemma 1.

When $\mathbb{P}$ drifts to $+\infty$, we denote by Z the process obtained by killing X immediately after its last jump across 0, and then resurrected (at 0). If ν stands for the Itô measure of the excursions of Z, then, by Lemma 1 (recall that j is the time of the jump across 0 for the generic excursion), the following holds:

> Under $\nu(\cdot \mid -X_{j-} = y)$, $(-X_{(j-t)-}\colon 0 \leq t < j)$ has the law $\mathbb{P}_y^{(0,\infty)}(\cdot \mid X_{\zeta-} = 0)$ and $(X_{j+t}\colon 0 \leq t < \tau(0) - j)$ is nonnegative.

Corollary 1 then follows from this description and the same arguments as before. □

REMARK. Viewed from Lemma 2 and Corollary 1, there is now a natural choice for the local time ℓ of X at 0. Namely, $\ell \circ T$ should be the local time at 0 for Y, which has already been specified by Proposition 9 of Bingham (1975). This forces the choice for the constant factor in μ. The reader can check easily that the description of μ given in Lemma 1 when $\sigma_0 = 0$ can now be completed by the following:

(i) $\mu(X_{j-} = 0) = 0$ and $\mu(j = \infty) = (1/\Phi'(0+))\mathbf{1}_{\{\Phi(0)=0\}}$.
(ii) For every $0 < y < z$, $\mu(-X_{j-} \in dy,\ X_j - X_{j-} \in dz) = e^{-\Phi(0)y}\,dy\,d\Pi(z)$.

3. Spectrally positive Lévy processes conditioned to stay negative. We want now to introduce, as simply as possible, a probability measure P on $(\Omega, \mathscr{F})$ which is defined intuitively as $\mathbb{P}$ conditioned by the event "X stays negative." The function W introduced in Section 1 plays the role of a scale function for $(X, \mathbb{P})$, and we simply follow the approach of Williams (1974).

First, we consider $y < x < 0$, and we introduce

$$P_x^{(y,0)} = \mathbb{P}_x^{(y,0)}\big(\cdot \mid X_{\zeta-} = y\big),$$

that is, the law of the initial Lévy process starting from x, conditioned to make its first exit from $(y, 0)$ through y, and killed when it hits y. For every $t \geq 0$ and $\Lambda \in \mathscr{F}_t$, the Markov property at time t and (1.3) imply that

$$P_x^{(y,0)}(\Lambda, t < \zeta) = \frac{1}{W(-x)} \mathbb{E}_x\big(\mathbf{1}_\Lambda \mathbf{1}_{\{t < \tau((y,0)^c)\}} W(-X_t)\big).$$

By monotone convergence this quantity has a finite limit,

$$(3.1) \qquad P_x(\Lambda, t < \zeta) = \frac{1}{W(-x)} \mathbb{E}_x\big(\mathbf{1}_\Lambda \cdot \mathbf{1}_{\{t < \tau([0,\infty))\}} W(-X_t)\big)$$

as y goes to $-\infty$, and, according to the Vitali–Hahn–Saks theorem, P_x is a probability measure on $(\Omega, \mathscr{F})$. Roughly speaking, P_x is the law of the initial Lévy process starting from $x < 0$ and conditioned to stay negative. Note that, when $\mathbb{P}$ drifts to $-\infty$, this conditioning has a rigorous meaning, and P_x fits it.

In terms of h-processes, we conditioned $\mathbb{P}_x^{(-\infty,0)}$ by the positive invariant function $W(-x)$. In particular, the lifetime ζ is infinite P_x-a.s. It should be clear that $(\Omega, \mathscr{F}, \mathscr{F}_t, X, P_x)$ has the strong Markov property. Finally, it follows easily from Section 1 that P_x drifts to $-\infty$.

The famous Spitzer–Rogozin identity enables us to compute the Green kernel for P_x.

Proposition 1. *Let f: $[0,\infty) \to \mathbb{R}$ be a continuous function with compact support. We have*

$$E_x\left(\int_0^\infty f(-X_t)\, dt\right) = \frac{1}{W(-x)} \int_0^\infty f(u) W(u) \big(\exp\{-u\Phi(0)\} W(-x) - \mathbf{1}_{\{u < -x\}} W(-x-u)\big)\, du.$$

Proof. Consider $\mathfrak{e}$, an independent exponential time with parameter $\alpha > 0$. By (3.1),

$$E_x(f(-X_{\mathfrak{e}})) = \frac{1}{W(-x)} \mathbb{E}\big(f((\overline{X} - X)_{\mathfrak{e}} - \overline{X}_{\mathfrak{e}} - x) \times W((\overline{X} - X)_{\mathfrak{e}} - \overline{X}_{\mathfrak{e}} - x) \mathbf{1}_{\{\overline{X}_{\mathfrak{e}} < -x\}}\big).$$

According to the Spitzer–Rogozin identity for spectrally positive Lévy processes [see Bingham (1975), Section 5], $(\overline{X} - X)_{\mathfrak{e}}$ and $\overline{X}_{\mathfrak{e}}$ are independent under $\mathbb{P}$, $(\overline{X} - X)_{\mathfrak{e}}$ has an exponential distribution with parameter $\Phi(\alpha)$ and $\overline{X}_{\mathfrak{e}}$ has law $q_\alpha(dy)$ and Laplace transform

$$\int_{[0,\infty)} e^{-\beta y} q_\alpha(dy) = \left(\frac{\alpha}{\alpha - \Psi(\beta)}\right)\left(1 - \frac{\beta}{\Phi(\alpha)}\right), \qquad \beta > 0.$$

It follows that

$$E_x\left(\int_0^\infty e^{-\alpha t} f(-X_t)\,dt\right) = \frac{\Phi(\alpha)}{\alpha W(-x)} \int_0^\infty du\, f(u) W(u) \times\left(\int_{[-x-u,\,-x]} \exp\{-(u+x+y)\Phi(\alpha)\} q_\alpha(dy)\right).$$

Since the measure $(\Phi(\alpha)/\alpha) q_\alpha(dy)$ has Laplace transform $(\Phi(\alpha) - \beta)/(\alpha - \Psi(\beta))$, a quantity which converges to $(\beta - \Phi(0))/\Psi(-\beta)$ as $\alpha \downarrow 0$, $\Phi(\alpha) q_\alpha(dy)/\alpha$ converges to $dW(y) - \Phi(0) W(y)\,dy$, and the proposition is obtained by letting $\alpha \downarrow 0$ in the preceding equality. □

It is time now to introduce the probability P which motivates this section.

PROPOSITION 2. *The family $(P_x\colon x \in (-\infty, 0))$ converges in Skorohod's sense as $x \uparrow 0$ to a probability measure P such that*

$$P(X_0 = 0,\ \zeta = \infty,\ X_t < 0 \text{ for all } t > 0) = 1.$$

Moreover, the semigroup associated to $(P_x\colon x \in (-\infty, 0])$ fulfills the Feller property.

PROOF. Denote by $P^{(x,0)}$ the law of the first excursion of $(X, \mathbb{P})$ that hits x and is killed at x, that is, the law of $(X_{\sigma(0)+t}\colon 0 \le t < \zeta - \sigma(0))$ under $\mathbb{P}^{(x,\infty)}(\cdot\,|X_{\zeta-} = x)$, or equivalently the law of $(X_t\colon 0 \le t < \tau(x))$ under $\mu(\cdot\,|\tau(x) < \infty)$.

By the strong Markov property, for every $y \in (x, 0)$, the processes $(X_t\colon 0 \le t < \tau(y))$ and $(X_{\tau(y)+t}\colon 0 \le t < \zeta - \tau(y))$ are independent under $P^{(x,0)}$, and their respective laws are $P^{(y,0)}$ and $P_y^{(x,0)}$. Thus, the law of the process obtained by pasting together two independent processes, the first with law $P^{(y,0)}$ and the second with law P_y, does not depend on y; we denote it by P.

By construction, $\tau(x) < \infty$ for every $x < 0$ and $\lim_{x \uparrow 0} \tau(x) = 0$ P-a.s. Hence, the family of the distributions of processes $(X_{\tau(x)+t}\colon t \ge 0)$ under P converges in Skorohod's sense to P as $x \uparrow 0$, that is,

$$P_x \Rightarrow_{\mathscr{L}} P.$$

Finally, the Feller property of the associated semigroup follows from (3.1), the continuity of W, and the fact that for every $t > 0$, the $\mathbb{P}$-law of $\overline{X}_t$ is atomless on $(0, \infty)$ [see, e.g., Bingham (1975), Proposition 10-c]. □

REMARKS. (i) It is well-known that, if $\mathbb{P}$ is the Wiener measure ($a = 0$, $\sigma_0 = 1$, $\Pi \equiv 0$), then P is the law of the opposite of a three-dimensional Bessel process [Doob (1957) and Williams (1974)]. If $\mathbb{P}$ is the law of a Brownian motion with drift ($a \ne 0$, $\sigma_0 = 1$, $\Pi \equiv 0$), then P is the diffusion process in $(-\infty, 0)$ starting at 0 and with generator $\frac{1}{2}(d^2 f/dx^2) + a \coth(ax)\,df/dx$. In other words, P is the law of the opposite of the radial part of a three-dimensional Brownian motion with drift a in the direction of the first coordinate axis [see Williams (1974) and Rogers and Pitman (1981)].

(ii) When $\mathbb{P}$ drifts to $-\infty$, the process $(X_{\sigma(0)+t}\colon t \geq 0)$ is independent of the pre-$\sigma(0)$ process $(X_t\colon 0 \leq t \leq \sigma(0))$ and has law P (last exit decomposition). Similarly, the last excursion of the reflected process, $((X - \overline{X})_{\bar{\sigma}(0)+t}\colon t \geq 0)$, where $\bar{\sigma}(0) = \sup\{s\colon X_s = \overline{X}_s\}$, is independent of the pre-$\bar{\sigma}(0)$ process and has law P (decomposition at the maximum). When $\sigma_0 = 0$, this completes the description of the excursion measure $\bar{\mu}$ initiated in Corollary 1.

An important connection between the Brownian motion and the three-dimensional Bessel process is the identity between the law of a Brownian motion starting from 1 and time-reversed when it first hit 0, and the law of a Bessel(3) starting from 0 and killed at its last exit time from 1 [Williams (1974)]. We deduce from (1.2) an analogous identity for the spectrally positive Lévy processes:

THEOREM 1. *For every $x < 0$, the law of $(-X_{(\tau(0)-t)-}\colon 0 \leq t < \tau(0))$ under $\mathbb{P}_{-x}(\cdot \mid \tau(0) < \infty)$ is the same as the law of $(X_t\colon 0 \leq t < \sigma(x))$ under P.*

PROOF. Assume that 0 is regular for itself under $\mathbb{P}$. For every $z > 0$, the process of the excursions of $(X_t\colon 0 \leq t < \sigma(0))$ under $\mathbb{P}^{(x,z)}$ is a Poisson point process (p.p.p.) with characteristic measure $\mathbf{1}_{\{x<X<z\}}\mu$ and killed at an independent exponential time $\mathfrak{e}$. By (1.2), the process of the excursions of the opposite of the time reversed process $(-X_{(\sigma(0)-t)-}\colon 0 \leq t < \sigma(0))$ under $\mathbb{P}^{(x,z)}$ is a p.p.p. with characteristic measure $\mathbf{1}_{\{-z<X<-x\}}\mu$ and killed at time $\mathfrak{e}$. Thus, the law of $(-X_{(\sigma(0)-t)-}\colon 0 \leq t < \sigma(0))$ under $\mathbb{P}^{(x,z)}$, is the same as the law of $(X_t\colon 0 \leq t < \sigma(0))$ under $\mathbb{P}^{(-z,-x)}$. But, under $\mathbb{P}^{(x,z)}$ as well as under $\mathbb{P}^{(-z,-x)}$, the pre-$\sigma(0)$ process and the post-$\sigma(0)$ process are independent, and in the former assertion we may replace $\mathbb{P}^{(x,z)}$ by $\mathbb{P}^{(x,z)}(\cdot \mid X_{\zeta-} = x)$ and $\mathbb{P}^{(-z,-x)}$ by $\mathbb{P}^{(-z,-x)}(\cdot \mid X_{\zeta-} = -z)$. Taking the limit as $z \uparrow \infty$, we get

$$\Big(\big(-X_{(\sigma(-x)-t)-}\colon 0 \leq t < \sigma(-x)\big), \mathbb{P}_{-x}^{(0,\infty)}\big(\cdot \mid X_{\zeta-} = 0\big)\Big)$$
$$=_{\mathscr{L}} \big(\big(X_t\colon 0 \leq t < \sigma(x)\big), P_x\big).$$

That is, the law of the right-continuous version of the opposite of the initial Lévy process, starting from $-x$, conditioned to hit 0, and time-reversed at its last exit from $-x$ before it hits 0, is the same as the law of the initial Lévy process, starting from x, "conditioned to stay negative" and killed at its last exit from x. A simple modification of these arguments shows that the same holds when 0 is irregular for itself under $\mathbb{P}$.

Remember (see the proof of Proposition 2) that $(X_t\colon 0 \leq t < \tau(x))$ has the same law under P as under $\mu(\cdot \mid \tau(x) < \infty, \tau(0) < \infty)$, and note that, by (1.2), it is also the law of $(-X_{(\tau(0)-t)-}\colon 0 \le t < \tau(0) - \sigma(-x))$ under $\mu(\cdot|\tau(-x) < \infty, \tau(0) < \infty)$. By excursion theory, this distribution is also the law of $(-X_{(\zeta-t)-}\colon 0 \le t < \zeta - \sigma(-x))$ under $\mathbb{P}^{(0,\infty)}_{-x}(\cdot|X_{\zeta-} = 0)$. Thus,

$$\begin{aligned}&\left(\left(-X_{(\zeta-t)-}\colon 0 \le t < \zeta - \sigma(-x)\right), \mathbb{P}^{(0,\infty)}_{-x}(\cdot|X_{\zeta-} = 0)\right)\\ &\quad =_{\mathscr{L}} \left(\left(X_t\colon 0 \le t < \tau(x)\right), P\right).\end{aligned}$$

Since $\sigma(-x)$ and $\tau(x)$ are splitting times for $\mathbb{P}^{(0,\infty)}_{-x}(\cdot|X_{\zeta-} = 0)$ and P, respectively, the lemma is proved by putting the pieces together. □

This time reversal identity yields the following last exit decomposition for P.

COROLLARY 2. *Under P, the process $(X_{\sigma(x)+t} - x\colon t \ge 0)$ is independent of the pre-$\sigma(x)$ process $(X_t\colon 0 \le t < \sigma(x))$ and has law P. In particular, the process $t \mapsto \sigma(-t)$ is a subordinator, with Laplace transform*

$$E(\exp\{-\alpha\sigma(-t)\}) = \exp\{-t(\Phi(\alpha) - \Phi(0))\}, \qquad \alpha > 0.$$

We are now able to make explicit the entrance law $P(X_t \in dx)$.

COROLLARY 3. *The following identity between measures holds ($x \in \mathbb{R}_-$, $t \in \mathbb{R}_+$):*

$$P(X_t \in dx)\, dt = W(-x)\mathbb{P}(\tau(x) \in dt)\, dx.$$

PROOF. By (1.1), $\mathbb{P}(\underline{X}_\infty < y) = \exp\{y\Phi(0)\}$, $y < 0$, and, by (1.3), $\lim_{n\uparrow\infty} W(n)/W(n+y) = \exp\{-y\Phi(0)\}$. But, for every $n > -x$, we deduce from (3.1) that

$$P_x(\tau([y,0)) > \tau(-n)) = \frac{W(y-x)}{W(n+y)}\frac{W(n)}{W(-x)}, \qquad x < y < 0.$$

Taking the limit as $n \uparrow \infty$, we get

$$P_x\left(\overline{X}_\infty < y\right) = \exp\{-y\Phi(0)\}\frac{W(y-x)}{W(-x)}. \tag{3.2}$$

Denote by $\overline{\overline{X}}_t = \sup\{X_s\colon s \ge t\}$, the future maximum process. We deduce from (3.2) and the Markov property that, if $\mathfrak{e}$ is an independent exponential time with parameter θ, then

$$P\left(\overline{\overline{X}}_{\mathfrak{e}} < y\right) = \int_{(-\infty,y)} \frac{W(y-x)}{W(-x)} \exp\{-y\Phi(0)\}P(X_{\mathfrak{e}} \in dx).$$

Since the law of $\sigma(y)$ under P is the same as the law of $\tau(y)$ under $\mathbb{P}(\cdot|\tau(y) < \infty)$, we have

$$\begin{aligned}P\left(\overline{\overline{X}}_{\mathfrak{e}} < y\right) &= \mathbb{P}(\tau(y) < \mathfrak{e}|\tau(y) < \infty)\\ &= \exp\{y(\Phi(\theta) - \Phi(0))\}, \quad \text{by (1.1)}.\end{aligned}$$

Hence,

$$\int_{(-\infty, y]} W(y-x) \frac{P(X_e \in dx)}{W(-x)} = \exp\{y\Phi(\theta)\}. \tag{3.3}$$

Assume for the moment that there is a signed measure $dM_\lambda(t)$, $\lambda > 0$, on $[0, \infty)$ such that $W * M_\lambda(dt) = (1 - e^{-\lambda t})\,dt$, where $*$ stands for the convolution operator. Remember that the Laplace–Stieltjes transform of W is $1/\Psi(\alpha)$, so

$$\int_{[0,\infty)} e^{-\alpha t}\, dM_\lambda(t) = \frac{\lambda \Psi(\alpha)}{\alpha(\alpha + \lambda)}.$$

It follows now from (3.3) that

$$\int_{(-\infty, 0)} (1 - e^{\lambda x}) \frac{P(X_e \in dx)}{W(-x)} = \int_{(-\infty, 0)} \exp\{y\Phi(\theta)\}\, dM_\lambda(-y)$$
$$= \frac{\lambda\theta}{\Phi(\theta)(\lambda + \Phi(\theta))}.$$

Hence, $P(X_e \in dx)/W(-x) = \theta e^{\Phi(\theta)x}\,dx$, $x < 0$, that is, by (1.1),

$$\int_0^\infty e^{-\theta t} P(X_t \in dx)\, dt = W(-x)\left(\int_{[0,\infty)} e^{-\theta t}\mathbb{P}(\tau(x) \in dt)\right) dx.$$

We still have to prove that M_λ exists. The functions

$$\varphi_1(t) = \int_{(0,1)} (z - t)\mathbf{1}_{\{t \le z\}}\, d\Pi(z) \quad \text{and} \quad \varphi_2(t) = \int_{[1,\infty)} (t - z)\mathbf{1}_{\{t \ge z\}}\, d\Pi(z)$$

have respective Laplace transforms

$$\frac{1}{\alpha^2}\int_{(0,1)} (e^{-\alpha z} - 1 + \alpha z)\, d\Pi(z) \quad \text{and} \quad \frac{1}{\alpha^2}\int_{[1,\infty)} (e^{-\alpha z} - 1)\, d\Pi(z).$$

Thus, if we set $\varphi = \varphi_1 + \varphi_2$, then the signed measure

$$\lambda\big(\varphi(t) - \lambda e^{-\lambda \cdot} * \varphi(\cdot)(t)\big)\, dt$$

has Laplace transform

$$\frac{\lambda\big(\Psi(\lambda) + a\alpha - \frac{1}{2}\sigma_0^2\alpha^2\big)}{\alpha(\alpha + \lambda)},$$

and, if δ_0 stands for the Dirac mass at 0,

$$dM_\lambda(t) = \lambda\big(\varphi(t) - \lambda e^{-\lambda \cdot} * \varphi(\cdot)(t) + \big(\tfrac{1}{2}\sigma_0^2 + a\big)\lambda e^{-\lambda t}\big)\, dt - \tfrac{1}{2}\sigma_0^2\delta_0$$

is the desired signed measure. □

REMARKS. (i) When $\mathbb{P}$ does not drift to $+\infty$, we deduce from Lemma 2 an analogue of the Ciesielski–Taylor [Ciesielski and Taylor (1962)] identity: The law of the first hitting time of $x < 0$ by the reflected process $X - \overline{X}$ under $\mathbb{P}$ is the same as the law of the total time spent by X in $[x, 0]$ under P.

(ii) It is interesting to note the following analogue of the Spitzer–Rogozin identity which is an immediate consequence of Corollary 3 and (3.2). Remember that $\overline{\overline{X}}$ is the future maximum process, and consider $\mathfrak{e}$, an independent exponential time with parameter θ. One checks that, under P, $(\overline{\overline{X}} - X)_{\mathfrak{e}}$ and $\overline{\overline{X}}_{\mathfrak{e}}$ are independent, that $(\overline{\overline{X}} - X)_{\mathfrak{e}}$ has Laplace transform

$$\beta \mapsto \frac{\beta + \Phi(\theta) - \Phi(0)}{\Psi(\beta + \Phi(\theta))} \frac{\theta}{\Phi(\theta) - \Phi(0)},$$

and that $-\overline{\overline{X}}_{\mathfrak{e}}$ has an exponential distribution with parameter $\Phi(\theta) - \Phi(0)$.

4. An extension of Pitman's theorem. Remember that $\overline{X}^c$ is the continuous part of the supremum process and that J is the sum of the jumps of X across its previous maximum. By Corollary 1,

$$\int (1 \wedge y) \overline{\mu}(X_j - X_{j-} \in dy) = \begin{cases} \int y(1 \wedge y)\, d\Pi(y), & \text{if } \Phi(0) = 0, \\ (1/\Phi(0)) \int (1 - e^{-\Phi(0)y})(1 \wedge y)\, d\Pi(y), & \\ & \text{if } \Phi(0) > 0, \end{cases}$$

and this quantity is always finite. This implies (by excursion theory) that J_t is finite for every t, $\mathbb{P}$-a.s. We claim the following:

THEOREM 2. *Under $\mathbb{P}$, the processs $X - 2\overline{X}^c - J$ has law P.*

The rest of this section is devoted to the proof of Theorem 2, which is broken into three lemmas.

LEMMA 3. *Assume that $\sigma_0 = 0$. Under $\mathbb{P}$, the process $X - J$ has law P.*

PROOF. We first assume that $\mathbb{P}$ does not drift to $-\infty$. Let $(x_n: n \in \mathbb{N})$ be a sequence of negative real numbers, with $\sum_n x_n = -\infty$. Consider $(Z^n: n \in \mathbb{N})$, a sequence of independent processes, where $Z^n = (Z_t^n: 0 \le t < \xi^n)$ has the same law as $(X_t: 0 \le t < \sigma(x_n))$ under P. Introduce Z:

$$\text{for } 0 \le t < \xi^n, \quad Z(\xi^0 + \cdots + \xi^{n-1} + t) = Z_t^n + \sum_{i=0}^{n-1} Z^i(\xi^i -),$$

the process obtained after pasting together the processes $(Z^n + \sum_{i=0}^{n-1} Z^i(\xi^i -)$: $n \in \mathbb{N})$. By Corollary 2, $(Z_t: 0 \le t < \xi^0 + \cdots + \xi^n)$ has the same law as $(X_t$: $0 \le t < \sigma(x_0 + \cdots + x_n))$ under P. Since $\sum_n x_n = -\infty$, Z has law P.

Now, we randomize the sequence $(x_n: n \in \mathbb{N})$ as follows: We fix a positive ε, and we pick the x_n's according to a sequence of i.i.d. r.v.'s with the same law as X_{j-} under $\overline{\mu}(\cdot | X_j - X_{j-} > \varepsilon)$. The resulting $Z = Z^\varepsilon$ still has law P.

On the other hand, under $\mathbb{P}$, the sequence of the excursions of $X - \overline{X}$, killed just before time j and such that $X_j - X_{j-} > \varepsilon$, has the same law as the sequence $(Z^n = Z^{n,\varepsilon}: n \in \mathbb{N})$ (by Corollary 1 and Theorem 1). Remember that $\bar{\mu}(X_j - X_{j-} = 0) = 0$ and that the inverse local time at 0 for $X - \bar{X}$ has no drift [Bingham (1975), Proposition 10]. Thus, the process obtained by pasting together the sequence of the excursions of $X - \bar{X}$ such that the jump at the lifetime is larger than ε converges a.s. when $\varepsilon \downarrow 0$, in the (localized) Skorohod topology, to $X - \bar{X}$. Consequently, under $\mathbb{P}$, the process

$$t \mapsto (X - \bar{X})_t - \sum_{s \le t} (\bar{X}_{s-} - X_{s-}) \mathbf{1}_{\{X_s > \bar{X}_{s-}\}}$$

has law P.

Since we assumed that $(X, \mathbb{P})$ has no Gaussian component, $\bar{X}$ is $\mathbb{P}$-a.s. pure jumps, that is, $\bar{X}_t = \sum_{s \le t}(X_s - \bar{X}_{s-})\mathbf{1}_{\{X_s > \bar{X}_{s-}\}}$. Thus the process $X - J$ has law P, and the lemma is proved when $\mathbb{P}$ does not drift to $-\infty$.

The proof when $\mathbb{P}$ drifts to $-\infty$ is an immediate modification of the preceding arguments [simply consider a finite sequence $(x_0, \dots, x_n, x_{n+1})$, with $x_0, \dots, x_n \in (-\infty, 0)$ and $x_{n+1} = -\infty$ instead of the infinite sequence of finite negative real numbers and remember Remark (ii) after Proposition 2]. □

Recall that $\bar{X}^c \equiv 0$ $\mathbb{P}$-a.s. when $\sigma_0 = 0$, so in this case, Lemma 3 rephrases Theorem 2. The case when $\sigma_0 = 1$ will follow from Lemma 3 by approximation. We work under $\mathbb{P}$. Denote by B a Brownian motion, and by $({}^nB)_{n \ge 1}$ a sequence of processes such that the following hold:

1. $(1/\varepsilon_n)^n B$ is a compensated Poisson process with intensity $1/\varepsilon_n^2$, where (ε_n) is a sequence of positive real numbers, $\varepsilon_n \downarrow 0$ and $\Pi(\{\varepsilon_n\}) = 0$.
2. ${}^nB$ converges uniformly over every compact interval to B as $n \uparrow \infty$, $\mathbb{P}$-a.s.

The existence of such processes is guaranteed by Skorohod's theorem, once we remember that convergence in Skorohod's topology to a continuous limit is equivalent to uniform convergence. Also consider a process A which is independent of the ${}^nB$'s and which is a Lévy process with characteristic exponent

$${}^0\Psi(z) = az + \int_{(0,\infty)} \left(e^{-zx} - 1 + zx\mathbf{1}_{\{x<1\}}\right) d\Pi(x).$$

We set ${}^0X = A$, ${}^nX = A + {}^nB$, and $X = A + B$. Left superscripts, as in ${}^n\mathbb{P}, {}^n\bar{X}, {}^nJ, \dots,$ will now refer to laws or processes related to ${}^nX$.

Since A and ${}^nB$ are independent, their respective jump times are a.s. distinct. We decompose ${}^nJ$ as ${}^nJ = {}^nK + {}^nL + {}^nM + {}^nN$, where

$${}^nK_t = \sum_{s \le t} \left({}^nX_s - {}^n\bar{X}_{s-}\right)\mathbf{1}_{\{{}^nX_s > {}^n\bar{X}_{s-},\, {}^nX_s - {}^nX_{s-} \ne \varepsilon_n\}}$$

$${}^nL_t = \sum_{s \le t} \left({}^n\bar{X}_{s-} - {}^nX_{s-}\right)\mathbf{1}_{\{{}^nX_s > {}^n\bar{X}_{s-},\, {}^nX_s - {}^nX_{s-} \ne \varepsilon_n\}}$$

$${}^nM_t = \sum_{s \le t} \left({}^nX_s - {}^n\bar{X}_{s-}\right)\mathbf{1}_{\{{}^nX_s > {}^n\bar{X}_{s-},\, {}^nX_s - {}^nX_{s-} = \varepsilon_n\}}$$

$${}^nN_t = \sum_{s \le t} \left({}^n\bar{X}_{s-} - {}^nX_{s-}\right)\mathbf{1}_{\{{}^nX_s > {}^n\bar{X}_{s-},\, {}^nX_s - {}^nX_{s-} = \varepsilon_n\}},$$

that is, ${}^nK$ (respectively, ${}^nM$) is the sum of the overshoots of ${}^nX$ over its previous maximum, where we only consider the jumps of ${}^nX$ which are jumps of A (respectively, of ${}^nB$) and ${}^nL$ (respectively, ${}^nN$) is the sum of the differences $\overline{X}_{s-} - X_{s-}$ at times when ${}^nX$ jumps across its previous maximum, and where we only consider the jumps of ${}^nX$ which are jumps of A (respectively, of ${}^nB$). First, we study the convergence of ${}^nK$ and ${}^nL$ as $n \uparrow \infty$.

LEMMA 4. *There is a subsequence of* (ε_n), *also denoted by* (ε_n) *for convenience, such that*, $\mathbb{P}$-*a.s.*,

> ${}^nK_t$ *(respectively*, ${}^nL_t$*) converges uniformly over every compact interval to* $\sum_{s \le t}(X_s - \overline{X}_{s-})\mathbf{1}_{\{X_s > \overline{X}_{s-}\}}$ *(respectively*, $\sum_{s \le t}(\overline{X}_{s-} - X_{s-})\mathbf{1}_{\{X_s > \overline{X}_{s-}\}}$*).*

PROOF. Fix a positive ε, consider $\mathfrak{e}$, an independent exponential time with parameter 1, and introduce

$$ {}^nJ_t^\varepsilon = \sum_{s \le t} ({}^nX_s - {}^nX_{s-})\mathbf{1}_{\{{}^nX_s > {}^n\overline{X}_{s-},\, {}^nX_s - {}^nX_{s-} \in (0,\varepsilon) \setminus \{\varepsilon_n\}\}}. $$

(${}^nJ_t^\varepsilon$ is the sum of the jumps of ${}^nX$ across its previous maximum, where we only consider jumps of size less than ε and which are jumps of A.) First, we show that $\mathbb{E}({}^nJ_\mathfrak{e}^\varepsilon)$ goes to 0 uniformly in n as $\varepsilon \downarrow 0$. By excursion theory, for every $\alpha > 0$,

$$ \mathbb{E}\big(\exp\{-\alpha\, {}^nJ_\mathfrak{e}^\varepsilon\}\big) = \mathbb{E}\left(\int_0^\infty dt \exp\{-\bar{\rho}(t) - \alpha\, {}^nJ^\varepsilon(\bar{\rho}(t))\}\right)\int(1 - e^{-j})\, d\,{}^n\overline{\mu} $$

$$ \big(\text{where } \bar{\rho} \text{ is the inverse local time at 0 for } X - \overline{X}\big) $$

$$ = \frac{\int(1 - e^{-j})\, d\,{}^n\overline{\mu}}{\int\Big(1 - \exp\Big\{-j - \alpha(X_j - X_{j-})\mathbf{1}_{\{X_j - X_{j-} \in (0,\varepsilon)\setminus\{\varepsilon_n\}\}}\Big\}\Big)\, d\,{}^n\overline{\mu}}. $$

In particular,

$$ (3.4)\quad \mathbb{E}({}^nJ_\mathfrak{e}^\varepsilon) \le \int (X_j - X_{j-})\mathbf{1}_{\{X_j - X_{j-} \in (0,\varepsilon)\setminus\{\varepsilon_n\}\}}\, d\,{}^n\overline{\mu} \times \left(\int(1 - e^{-j})\, d\,{}^n\overline{\mu}\right)^{-1}. $$

Furthermore, by Corollary 1, we have

$$ \int (X_j - X_{j-})\mathbf{1}_{\{X_j - X_{j-} \in (0,\varepsilon)\setminus\{\varepsilon_n\}\}}\, d\,{}^n\overline{\mu} = \int_{(0,\varepsilon)} d\Pi(z)\left(\int_0^z y \exp\{-{}^n\Phi(0)y\}\, dy\right) $$

$$ \le \int_{(0,\varepsilon)} z^2\, d\Pi(z). $$

This last quantity is independent of n and converges to 0 as $\varepsilon \downarrow 0$. Moreover, Fristedt's identity for spectrally positive Lévy processes [see Bingham (1975), Proposition 9] yields $\mathbb{E}(\exp\{-\bar{\rho}(t)\}) = \exp\{-t/{}^n\Phi(1)\}$. Hence,

$$\left(\int (1 - e^{-j})\, d^n\bar{\mu}\right)^{-1} = \mathbb{E}\left(\int_0^\infty dt \exp\{-\bar{\rho}(t)\}\right) = {}^n\Phi(1),$$

and this quantity converges to $\Phi(1)$ as $n \uparrow \infty$. In particular, it is bounded. It follows from (3.4) that

$$\lim_{\varepsilon \downarrow 0} \sup\{\mathbb{E}({}^nJ_{\mathrm{e}}^{\varepsilon}) : n \in \mathbb{N}\} = 0. \tag{3.5}$$

Recall now that $\mathbb{P}(X_{s-} = \bar{X}_{s-} < X_s \text{ or } X_{s-} < \bar{X}_{s-} = X_s \text{ for some } s) = 0$ [see Rogers (1984), Corollary 1]. Since ${}^nX$ converges $\mathbb{P}$-a.s. to X uniformly over every compact interval, for every $\varepsilon > 0$, if X jumps at time $s \le \mathrm{e}$ across its previous maximum and if the size of the jump, $X_s - X_{s-}$, is larger than ε, then, for every sufficiently large n [i.e., $n \ge n_0(\omega)$], ${}^nX$ jumps too at time s across its previous maximum, and the size of its jump is ${}^nX_s - {}^nX_{s-} = X_s - X_{s-}$. Conversely, $\mathbb{P}$-a.s., for every sufficiently large n [i.e., $n \ge n_0(\omega)$], if ${}^nX$ jumps at time $s \le \mathrm{e}$ across its previous maximum and if ${}^nX_s - {}^nX_{s-} \ge \varepsilon$, then X also jumps across its previous maximum at time s, with a jump of size $X_s - X_{s-} = {}^mX_s - {}^mX_{s-}$ [for all $m \ge n_0(\omega)$].

Put

$$\begin{aligned} {}^n\mathrm{I} = \sup\Bigg\{\Bigg| &\sum_{s \le t} ({}^nX_s - {}^n\bar{X}_{s-}) \mathbf{1}_{\{{}^nX_s > {}^n\bar{X}_{s-},\, {}^nX_s - {}^nX_{s-} > \varepsilon\}} \\ &- \sum_{s \le t} (X_s - \bar{X}_{s-}) \mathbf{1}_{\{X_s > \bar{X}_{s-},\, X_s - X_{s-} > \varepsilon\}} \Bigg| : 0 \le t \le \mathrm{e}\Bigg\}, \end{aligned}$$

$$\begin{aligned} {}^n\mathrm{II} = \sup\Bigg\{\Bigg| &\sum_{s \le t} ({}^n\bar{X}_{s-} - {}^nX_{s-}) \mathbf{1}_{\{{}^nX_s > {}^n\bar{X}_{s-},\, {}^nX_s - {}^nX_{s-} > \varepsilon\}} \\ &- \sum_{s \le t} (\bar{X}_{s-} - X_{s-}) \mathbf{1}_{\{X_s > \bar{X}_{s-},\, X_s - X_{s-} > \varepsilon\}} \Bigg| : 0 \le t \le \mathrm{e}\Bigg\}, \end{aligned}$$

$${}^n\mathrm{III} = {}^nJ_{\mathrm{e}}^{\varepsilon},$$

$$\mathrm{IV} = \sum_{s \le \mathrm{e}} (X_s - X_{s-}) \mathbf{1}_{\{X_s > \bar{X}_{s-},\, X_s - X_{s-} < \varepsilon\}}.$$

For each integer $k \ge 1$, we first choose $\varepsilon'_k > 0$ so that [by (3.5)]

$$\mathbb{P}(\mathrm{III}_n > 1/k) < 1/k^2, \qquad \text{with } \varepsilon = \varepsilon'_k \text{ and for all } n \ge 1,$$

$$P(\mathrm{IV} > 1/k) < 1/k^2, \qquad \text{with } \varepsilon = \varepsilon'_k \text{ (because } J_{\mathrm{e}} < \infty),$$

and then an integer n_k so that

$$\mathbb{P}(\mathrm{I}_n > 1/k) < 1/k^2, \qquad \text{with } \varepsilon = \varepsilon'_k \text{ and for all } n \ge n_k,$$

$$\mathbb{P}(\mathrm{II}_n > 1/k) < 1/k^2, \qquad \text{with } \varepsilon = \varepsilon'_k \text{ and for all } n \ge n_k,$$

hold. Then

$$\mathbb{P}\left[\sup\left\{\left|{}^nK_t - \sum_{s\le t}(X_s - \overline{X}_{s-})\mathbf{1}_{\{X_s > \overline{X}_{s-}\}}\right| : 0 \le t \le \mathrm{e}\right\} > 3/k\right]$$
$$\le \mathbb{P}(\mathrm{I}_n > 1/k) + \mathbb{P}(\mathrm{III}_n > 1/k) + \mathbb{P}(\mathrm{IV} > 1/k)$$
$$\le 3/k^2, \quad \text{for every } n \ge n_k;$$

hence, an application of the Borel–Cantelli lemma yields

$$\sup\left\{\left|{}^nK_t - \sum_{s\le t}(X_s - \overline{X}_{s-})\mathbf{1}_{\{X_s > \overline{X}_{s-}\}}\right| : 0 \le t \le \mathrm{e}\right\} \to 0, \quad \mathbb{P}\text{-a.s.},$$

as $n \uparrow \infty$ via $n_1 < n_2 < \cdots$, and a similar result for ${}^nL_t$. Note that the subsequence $\{n_k\}$ corresponds to a subsequence of $\{\varepsilon_n\}$. □

Then, we study the convergence of ${}^nM$ and ${}^nN$.

LEMMA 5. *There is a subsequence of (ε_n) such that $\mathbb{P}$-a.s., ${}^nM$ (respectively, ${}^nN$) converges to $\overline{X}^c$ uniformly over every compact interval.*

PROOF. We will only prove this assertion when 0 is regular for itself with respect to A (and hence to all the ${}^nX$'s), because the proof for the opposite case is simply the discrete analogue of the following arguments.

Recall that ${}^nX$ has no Gaussian component and that ${}^n\overline{X}$ is pure jumps a.s., that is, ${}^n\overline{X} = {}^nK + {}^nM$. Since ${}^n\overline{X}$ converges uniformly to $\overline{X}$ and ${}^nK$ to $\overline{X} - \overline{X}^c$ (by Lemma 4), ${}^nM$ converges uniformly to $\overline{X}^c$. It is now sufficient to prove that ${}^nM - {}^nN$ converges uniformly to 0.

First, we assume that ${}^0\mathbb{P}$ oscillates (then so do all the ${}^n\mathbb{P}$'s). We denote by ${}^n\bar{\ell}$ the local time at 0 for ${}^nX - {}^n\overline{X}$, by ${}^n\bar{\rho}$ its right-continuous inverse, and by $(\mathscr{G}_t)_{t\ge 0} = (\mathscr{F}_{{}^n\bar{\rho}(t)})_{t\ge 0}$ the filtration of the excursion process of ${}^nX - {}^n\overline{X}$.

For every $t \ge 0$, ${}^n\bar{\ell}_t$ is a $(\mathscr{G}_t)$-stopping time, ${}^n\mathscr{m} = ({}^nM - {}^nN)_{{}^n\bar{\rho}(\cdot)}$ has independent homogeneous increments and, by Corollary 1, it is symmetric and square-integrable. In particular, it is a square-integrable $(\mathscr{G}_t)$-martingale and, by Doob's inequality,

$$\mathbb{E}\Big(\sup\big\{({}^nM_s - {}^nN_s)^2 : 0 \le s \le t\big\}\Big) \le \mathbb{E}\Big(\sup\big\{{}^n\mathscr{m}_s^2 : 0 \le s \le {}^n\bar{\ell}_t\big\}\Big)$$
$$\le 4\mathbb{E}\Big([{}^n\mathscr{m}]_{{}^n\bar{\ell}_t}\Big),$$

where $[\cdot]$ is the usual bracket for square-integrable $(\mathscr{G}_t)$-martingales. This quantity is less than

$$4\mathbb{E}\left(\sum_{s\le t}\left({}^nX_s + {}^nX_{s-} - 2\,{}^n\overline{X}_{s-}\right)^2 \mathbf{1}_{\{{}^nX_s > {}^n\overline{X}_s,\, {}^nX_s - {}^nX_{s-} = \varepsilon_n\}}\right) + 4\varepsilon_n^2$$

[the term $4\varepsilon_n^2$ is an upper bound for $4({}^nX_s + {}^nX_{s-} - 2\,{}^n\overline{X}_{s-})^2$ evaluated at $s = \inf\{t' > t : {}^nX_{t'} > {}^n\overline{X}_t\}$], that is, the sum of the mean of an additive functional taken at time t and $4\varepsilon_n^2$. We deduce from techniques similar to the ones used in the proof of (3.4) that, if $\mathfrak{e}$ is an independent exponential time with parameter 1,

$$\mathbb{E}\Big(\sup\Big\{\big({}^nM_s - {}^nN_s\big)^2 : s \le \mathfrak{e}\Big\}\Big) - 4\varepsilon_n^2$$

$$\le 4\int \big(X_j + X_{j-}\big)^2 \mathbf{1}_{\{X_j - X_{j-} = \varepsilon_n\}}\, d\,{}^n\overline{\mu} \times \left(\int (1 - e^{-j})\, d\,{}^n\overline{\mu}\right)^{-1}$$

$$= 4\varepsilon_n^2\, {}^n\overline{\mu}(X_j - X_{j-} = \varepsilon_n)\left(\int (1 - e^{-j})\, d\,{}^n\overline{\mu}\right)^{-1}.$$

By Corollary 1, ${}^n\overline{\mu}(X_j - X_{j-} = \varepsilon_n) = 1/\varepsilon_n$, and we saw in the proof of Lemma 4 that $(\int(1 - e^{-j})\, d\,{}^n\overline{\mu})^{-1}$ is bounded. Thus, Lemma 5 is proved when ${}^0\mathbb{P}$ oscillates.

The proof when ${}^0\mathbb{P}$ drifts to $-\infty$ is an immediate modification (the only difference is that the process of the excursions of ${}^nX - {}^n\overline{X}$ is now a killed p.p.p.).

Assume now that ${}^0\mathbb{P}$ drifts to $+\infty$. The process $({}^nM - {}^nN)_{{}^n\overline{\rho}(\cdot)}$ still has homogeneous independent increments, but it is no longer a $(\mathscr{G}_t)$-martingale. However, we have

$$-\int X_{j-} \mathbf{1}_{\{{}^nX_j - {}^n\overline{X}_{j-} = \varepsilon_n\}}\, d\,{}^n\overline{\mu}$$

$$= \varepsilon_n^{-2} \int_0^{\varepsilon_n} x \exp\{-{}^n\Phi(0)x\}\, dx$$

$$= \big(1 - \exp\{-{}^n\Phi(0)\varepsilon_n\} - {}^n\Phi(0)\varepsilon_n \exp\{-{}^n\Phi(0)\varepsilon_n\}\big)\big({}^n\Phi(0)\varepsilon_n\big)^{-2}$$

$$:= {}^nc.$$

Similarly,

$$\int X_j \mathbf{1}_{\{{}^nX_j - {}^n\overline{X}_{j-} = \varepsilon_n\}}\, d\,{}^n\overline{\mu} = \varepsilon_n^{-2} \int_0^{\varepsilon_n} (\varepsilon_n - x)\exp\{-{}^n\Phi(0)x\}\, dx$$

$$= \big(1 - \exp\{-{}^n\Phi(0)\varepsilon_n\}\big)/\big({}^n\Phi(0)\varepsilon_n\big) - {}^nc$$

$$:= {}^nc'.$$

Note that ${}^n\Phi(0)$ converges to $\Phi(0) > 0$, and that ${}^nc$ and ${}^nc'$ both converge to $\frac{1}{2}$ as $n \uparrow \infty$. Now $({}^nc\,{}^nM - {}^nc'\,{}^nN)_{{}^n\overline{\rho}(\cdot)}$ is a square-integrable $(\mathscr{G}_t)$-martingale. The very same arguments as when ${}^0\mathbb{P}$ was assumed to oscillate prove that there is a subsequence of (ε_n) such that $\mathbb{P}$-a.s.,

> ${}^nc\,{}^nM - {}^nc'\,{}^nN$ converges uniformly over every compact interval to 0.

Since ${}^nM$ converges uniformly to $\overline{X}^c$, so does ${}^nN$. □

Recapitulating Lemmas 4 and 5, we have constructed a subsequence of (ε_n), still denoted by (ε_n), such that $\mathbb{P}$-a.s.,

$$^nJ \text{ converges to } J + 2\overline{X}^c \text{ uniformly over every compact interval.}$$

Let us denote by Q the law of $X - 2\overline{X}^c - J$ (under $\mathbb{P}$) and by ^nP the law of $^nX - {}^nJ$. We know that Q is the limit of the ^nP's in Skorohod's sense. We still need to prove that $^nP \Rightarrow_{\mathscr{L}} P$.

On $\{\tau(-1) < \infty\}$, X enters $(-\infty, -1)$ immediately after time $\tau(-1)$ $\mathbb{P}$-a.s., and $\liminf_{n\uparrow\infty} {}^n\tau(-1) \leq \tau(-1)$. On the other hand, $X_t > -1$ for all $t < \tau(-1)$, so $^n\tau(-1)$ converges to $\tau(-1)$ $\mathbb{P}$-a.s. Consequently, the sequence $(\mathbf{1}_{\{^n\tau(-1)<\infty\}}{}^nX_{\{^n\tau(-1)-t\}-}: 0 \leq t < {}^n\tau(-1))$ converges $\mathbb{P}$-a.s. for Skorohod's topology to $(\mathbf{1}_{\{\tau(-1)<\infty\}}X_{\{\tau(-1)-t\}-}: 0 \leq t < \tau(-1))$. By Theorem 1, the law of the canonical process killed at its last exit from -1 under ^nP converges in Skorohod's sense to the law of the same process under P. It follows now from Corollary 2 that ^nP converges to P in Skorohod's sense. The proof of Theorem 2 is now complete. □

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REFERENCES

BERTOIN, J. (1990). Sur une horloge fluctuante pour les processus de Bessel de petites dimensions. *Séminaire de Probabilités XXIV. Lecture Notes in Math.* **1426** 117–136. Springer, Berlin.

BERTOIN, J. (1991). Sur la décomposition de la trajectoire d'un processus de Lévy spectralement positif en son minimum. *Ann. Inst. H. Poincaré* **27–4**.

BINGHAM, N. (1975). Fluctuation theory in continuous time. *Adv. in Appl. Probab.* **7** 705–766.

CIESIELSKI, Z. and TAYLOR, S. J. (1962). First passage times and sojourn times for Brownian motion in space and the exact Hausdorff measure of the sample path. *Trans. Amer. Math. Soc.* **103** 434–450.

DOOB, J. L. (1957). Conditional Brownian motion and the boundary limits of harmonic functions. *Bull. Soc. Math. France* **85** 431–458.

EMERY, D. J. (1973). Exit problem for a spectrally positive Lévy process. *Adv. in Appl. Probab.* **5** 498–520.

GETOOR, R. K. and SHARPE, M. J. (1982). Excursions of dual processes. *Adv. in Math.* **45** 259–309.

GREENWOOD, P. and PITMAN, J. W. (1980). Fluctation identities for Lévy processes and splitting at the maximum. *Adv. in Appl. Probab.* **12** 893–902.

ITÔ, K. (1972). Poisson point processes attached to Markov processes. *Proc. Sixth Berkeley Symp. Math. Statist. Probab.* **3** 225–239. Univ. California Press, Berkeley.

MCKEAN, H. P. (1963). Excursions of a non-singular diffusion. *Z. Wahrsch. Verw. Gebiete* **1** 230–239.

MILLAR, P. W. (1973). Exit properties of stochastic processes with stationary independent increments. *Trans. Amer. Math. Soc.* **178** 459–479.

MIYAZAKI, H. and TANAKA, H. (1989). A theorem of Pitman type for simple random walks in $\mathbb{Z}^d$. *Tokyo J. Math.* **12-1** 159–174.

PITMAN, J. W. (1975). One-dimensional Brownian motion and the three-dimensional Bessel process. *Adv. in Appl. Probab.* **7** 511–526.

PITMAN, J. W. (1987). Stationary excursions. *Séminaire de Probabilités XXI. Lecture Notes in Math.* **1247** 289–302. Springer, Berlin.

REVUZ, D. and YOR, M. (1991). *Continuous Martingales and Brownian Motion.* Springer, Berlin.

ROGERS, L. C. G. (1981). Characterizing all diffusions with the $2M - X$ property. *Ann. Probab.* **9** 561–572.

ROGERS, L. C. G. (1984). A new identity for real Lévy processes. *Ann. Inst. H. Poincaré Sect. B* **20** 21–34.

ROGERS, L. C. G. (1990). The two-sided problem for spectrally positive Lévy processes. *Adv. in Appl. Probab.* **22-2** 486–487.

ROGERS, L. C. G. and PITMAN, J. (1981). Markov functions. *Ann. Probab.* **9** 573–582.

ROGERS, L. C. G. and WILLIAMS, D. (1987). *Diffusions, Markov Processes, and Martingales* **2**: *Itô Calculus*. Wiley, New York.

SILVERSTEIN, M. L. (1980). Classification of coharmonic and coinvariant functions for a Lévy process. *Ann. Probab.* **8** 539–575.

TAKÁCS, L. (1967). *Combinatorial Methods in the Theory of Stochastic Processes*. Wiley, New York.

TANAKA, H. (1989). Time reversal of random walks in one-dimension. *Tokyo J. Math.* **12-1** 159–174.

WILLIAMS, D. (1974). Path decomposition and continuity of local time for one dimensional diffusions. *Proc. London Math. Soc.* **28** 738–768.

WILLIAMS, D. (1979). *Diffusions, Markov Processes, and Martingales* **1**: *Foundations*. Wiley, New York.

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