AN EXTENSION OF POINCARÉ'S LAST GEOMETRIC THEOREM.

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1. Introduction.

The Crowned Memoir by POINCARÉ, »Le problème de trois corps et les équations de la dynamique», in volume 13 of the *Acta mathematica* contained the first great attack upon the non-integrable problems of dynamics. Under the direction of Professor MITTAG-LEFFLER, the *Acta mathematica* has had many remarkable articles, but perhaps none of larger scientific importance than this one. Its many ideas, in which the periodic motions took a central part, led naturally to POINCARÉ's later dynamical researches.

In a highly interesting paper, »Sur un théorème de géométrie», published shortly before his death in volume 33 of the *Rendiconti del Circolo Matematico di Palermo,* POINCARÉ showed that a certain geometric theorem (proved by him in particular cases) would carry with it the answer to some outstanding questions concerning the periodic motions. The peculiarity of the method by which I obtained a general demonstration of its truth soon afterwards,[1] and the dynamical origin of the theorem itself, have suggested the extension given here.

In thus responding to the kind invitation of Professor NÖRLUND, I desire to render homage to Professor MITTAG-LEFFLER, especially because of the inspiring tradition which he has established for the *Acta mathematica.*

[1] Proof of Poincaré's Geometric Theorem, *Transactions of the American Mathematical Society,* volume 14; or see a translation in volume 42 of the *Bulletin de la Société Mathématique de France.*

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2. Statement of the Theorem.

Let r, θ stand for polar coordinates in the plane, so that $r=a>0$ is the equation of a circle C of radius a. A doubly connected ring R, bounded by the circle C and a closed curve[1] Γ encircling C, as well as a second like ring R_1 bounded by the same circle C and a like encircling curve Γ_1, will engage our attention. The two rings, R and R_1, are taken to be related in that a one-to-one, direct, continuous point-transformation T carries R into R_1. Thus we may write

$$C_1 = T(C),\ \Gamma_1 = T(\Gamma),\ R_1 = T(R),$$
$$C = T_{-1}(C_1),\ \Gamma = T_{-1}(\Gamma_1),\ R = T_{-1}(R_1),$$

where the meaning of the notation is manifest.

The extension of Poincaré's last geometric theorem to be established here is as follows:

Theorem. If Γ and Γ_1 are met only once by any radial line $\theta = \text{constant}$, and if T carries points on C and Γ in opposite angular directions (with respect to θ) to their new positions on C and Γ_1 respectively, then either (a) there are two distinct invariant points P of R and R_1 under T, or (b) there is a ring in R (or R_1) abutting upon C which is carried into part of itself by T (or T_{-1}).[2]

In the form enunciated by Poincaré the boundaries Γ and Γ_1 coincide, while the alternative (b) is excluded by means of the hypothesis that an area integral

$$\int\int Pr\, dr\, d\theta \qquad (P>0)$$

is invariant under T.

The importance of the removal of the condition that Γ and Γ_1 coincide lies in the fact that the extended theorem may be applied to establish the

[1] A *closed curve* will be defined as the common boundary of a finite, simply connected, open continuum and the complementary open outer continuum. A *ring* is the region bounded by two closed curves, one within the other. If these curves do not touch, the ring is a doubly connected open continuum. No other type of ring enters here until the last section 8.

[2] The restriction made on the curves Γ and Γ_1 might be lightened in that these curves need only to be »right-handedly accessible» and »left-handedly accessible», as these terms are defined in my paper, »Surface Transformations and their Dynamical Applications» in volume 43 of the *Acta mathematica*. But the less general and somewhat simpler theorem stated suffices to illustrate the same type of extension, and appears to be adequate for the dynamical applications.

existence of infinitely many periodic motions near a stable periodic motion in a dynamical system with two degrees of freedom. Furthermore the existence of motions which are not periodic but are the uniform limits of periodic motions then follows at once. The actual existence of such quasi-periodic motions has not been proved hitherto as far as I am aware.[1] In the present paper I do not enter into these dynamical applications.

It is also worthy of note that the extended theorem does not involve the hypothesis of an invariant area integral, and so falls essentially in the domain of *analysis situs*. Furthermore the existence of two distinct invariant points is established, whereas the possibility of only a single invariant point has not hitherto been excluded.

The outstanding question as to the possibility of an n dimensional extension of POINCARÉ's last geometric theorem must now be briefly referred to.

An examination of the analytic properties of the motions near a given stable periodic motion in a dynamical system with n degrees of freedom, and of the corresponding transformation T to which it gives rise, is likely to show that there exist infinitely many nearby periodic motions. The theorem of POINCARÉ appears merely as the qualitative expression of the essential elements of the analytic situation for $n=2$; and in fact the most special case treated by POINCARÉ then suffices to cover the dynamical applications.[2] To achieve the appropriate n dimensional generalization of the theorem, it is necessary to determine the qualitatively essential elements of the n dimensional analytic treatment. Probably this can be accomplished in a simple way.

3. δ-Chains. Lemma 1.

Choose arbitrarily a number $\delta > 0$.

By means of the transformation T any point P_0 on the circle C is carried into a point $T(P_0)$ on C. An outward radial motion through a distance α_0, arbitrary except that $0 \leqq \alpha_0 < \delta$, carries $T(P_0)$ to a point P_1 on the same radial

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[1] The notable investigations of H. BOHR have taken up the analytic representation of such motions. See, for instance, his recent papers: Zur Theorie der fast periodischer Funktionen, volume 45, *Acta mathematica;* Einige Sätze über Fourierreihe fastperiodischer Funktionen, volume 23, *Mathematische Zeitschrift.*

[2] In my Chicago Colloquium Lectures on *Dynamical Systems*, soon to appear in book form, I establish these assertions.

line. Similarly an outward radial motion of $T(P_1)$ through a distance α_1, arbitrary except that $0 \leqq \alpha_1 < \delta$, carries $T(P_1)$ to a point P_2 on the same radial line. By continuing in this manner a *δ-chain* of points

$$P_0, P_1, P_2, \ldots,$$

is obtained, in which each point is derived from its predecessor by the application of T and a subsequent outward radial motion through a distance less than δ. The δ-chain can only terminate at some nth stage when P_n falls outside of R, so that the transformation T is not there defined. Such a terminating δ-chain will be called *finite.*

A precise condition for the non-existence of any finite δ-chain is contained in the following

Lemma 1. A necessary and sufficient condition that there exists no finite δ-chain is that there exists in R an open ring Σ abutting on C, which is carried by T into a ring $T(\Sigma)$ lying in Σ and radially distant from the boundary of Σ by at least δ in the outward direction.

The sufficiency of the condition in obvious. For if a point P lies in such a continuum Σ, its image $T(P)$ does and so do also the points obtained from $T(P)$ by an outward radial motion through a distance less than δ, just because $T(P)$ lies in $T(\Sigma)$. Thus the successive elements $P_1, P_2, \ldots$ of a chain must continue to lie in Σ and so in R, inasmuch as P_0 lies in Σ.

The necessity of the condition is also easily established. We begin by considering the nature of the sets of points $M_0, M_1, \ldots$ constituted by the points $P_0, P_1, \ldots$ respectively.

The set M_0 is the circle C of course.

The set M_1 is evidently the open circular ring

$$a \leqq r < a + \delta$$

It contains the set M_0 and is made up of inner points except for those of C.

The set M_2 contains all the points of M_1 and so of M_0. In fact it is possible to find a single point P_{-1} of C which is taken to P_0 by T. Thus P_{-1}, P_0, P_1 will form a δ-chain of three points so that P_1 is a point of M_2 also.

Furthermore, except for the points of C, all of the points of M_2 are interior points. In showing this to be the case it is clearly unnecessary to consider points P_2 which belong to M_1. For such as do not, the corresponding P_1 is an interior point of M_1. The transformation T, being one-to one and continuous, will take P_1 and its neighborhood into $T(P_1)$ and its neighbourhood. A further outward radial motion through a distance less than δ will take $T(P_1)$ and this neighborhood into P_2 and its neighborhood. Hence P_2 is an interior point of M_2 in this case also.

Finally, the set M_2 is connected, for it is obtained from the connected set $T(P_1)$ by an outward radial motion through a distance less than δ.

Thus it is seen successively that $M_1, M_2, \ldots$ form a series of open connected continua abutting on C, each of which contains its predecessors. If there exists no finite δ-chain, an infinite series of such regions is obtained, all of which will lie in R. These will define a limiting open connected continuum abutting on C. This continuum S is of course nothing but the set of points which belong to some δ-chain.

Consider now the region $T(S)$. Since if a point Q belongs to M_p, the point $T(Q)$ belongs to M_{p+1}, it follows that $T(S)$ is an open connected contiuum abutting on C which lies in S. Moreover if $T(Q)$ be moved in an outward radial direction through a distance less than δ, the point obtained will still belong to M_{p+1}. Thus every point of $T(S)$ is radially distant from the boundary of S by at least δ in the outward direction.

Consequently, if S were a ring it would be a region of the type declared by the Lemma to exist. But it is evidently conceivable that the part of the boundary of S accessible from infinity may not constitute the whole of that boundary. This will be the case when S occludes certain regions or parts of its boundary from infinity, and so is not a ring.

Suppose now that S is not a ring and let S stand for the occluded point set. Clearly the set $S+\overline{S}$ formed by S and $\overline{S}$ does form a proper ring. We proceed to prove that this augmented region $S+\overline{S}$ has the other properties demanded of Σ in Lemma I.

Clearly $S+\overline{S}$ lies upon R, since S does; and so $S+\overline{S}$ may be subjected to the transformation T. Also $S+\overline{S}$ is carried into all or part of itself by T. For if a point belongs to S it has been seen to be carried into a point of S by T; whereas if a point belongs to $\overline{S}$ and so is occluded by S, it is carried into a point occluded by $T(S)$, and all the more occluded by S, so that it belongs to $S+\overline{S}$ also. Moreover a similar reasoning shows that every point of $T(S+\overline{S})$ is radially distant from the boundary of $S+\overline{S}$ by at least δ in the outward direction. For if such a point belongs to $T(S)$ it has this property with reference to the boundary of S, and so of course with respect to the boundary of $S+\bar{S}$; whereas if a point belongs to $T(\bar{S})$ it is derived from a point occluded by S, and must be occluded by $T(S)$, so that a further outward radial motion through a distance less than δ gives rise to a point occluded by S and so in $S+\bar{S}$. This last step involves the previously deduced relation between S and $T(S)$.

Hence in every case $S+\bar{S}$ constitutes a ring Σ having the properties stated in Lemma 1. Thus the proof is completed.

4. Minimal δ-chains.

Suppose now that there exists at least one finite δ-chain. There will then be a least positive integer n, for which a δ-chain $P_0, P_1, \ldots, P_n$ exists such that P_n falls outside of R.

Such *minimal* δ-chains have some interesting properties. For example it is obvious that a point P_i of such a chain belongs to M_i but not M_j, $j<i$; in the contrary case a finite δ-chain of fewer elements could be at once constructed. Thus P_0 is the only point of the δ-chain on C, P_1 is the only point of the δ-chain in the open ring $a<r<a+\delta$, and so on.

The only other property which we shall require is not much less obvious: if P_i and P_j $(i\geqq 1, j\geqq 1)$ lie on one and the same radial line, so that $T(P_{i-1})$ and $T(P_{j-1})$ do also, then $T(P_{i-1})$ and $T(P_{j-1})$ will occur in the same radial order as P_i and P_j.

To establish this fact, we note first that $T(P_{i-1})$ and $T(P_{j-1})$ will not coincide, for then P_{i-1} and P_{j-1} coincide, so that all the points of the chain between P_{i-1} and P_{j-1} as well as one of these two points might be omitted from the minimal chain. This is absurd. For a like reason P_i and P_j will not coincide.

Now suppose that $T(P_{i-1})$ has an r coordinate which is less than that of $T(P_{j-1})$. This condition will be satisfied if i and j are named in the proper order. The only possible radial ordering of the four points in question not in accordance with the statement to be proved is

$$T(P_{i-1}),\ T(P_{j-1}),\ P_j,\ P_i$$

where the radial coordinate increases from left to right; in fact, P_i must lie further out than P_j which in turn is at least as far out as $T(P_{j-1})$. (In this ordering it would be conceivable that $T(P_{j-1})$ and P_j coincide.) But it is apparent that P_j is then obtainable from $T(P_{i-1})$ by an outward radial motion through a distance less than δ, and that P_i is likewise obtainable from $T(P_{j-1})$. This is true because the radial distance from $T(P_{i-1})$ to P_i is less than δ. Consequently it follows that P_j is a point of M_i and also that P_i is a point of M_j. But the property first specified eliminates one of these two possibilities. Therefore the stated ordering must hold.

5. The auxiliary transformation E.

Let now $P_0, P_1, \ldots, P_n$ be the points of any minimal δ-chain. From the property just established it follows at once that if $P_i, P_j, P_k, \ldots (i \geqq 1, j \geqq 1, k \geqq 1, \ldots)$ are the points of this chain which lie on a given radial line, then $T(P_{i-1})$, $T(P_{j-1})$, $T(P_{k-1}), \ldots$ occur in precisely the same radial order.

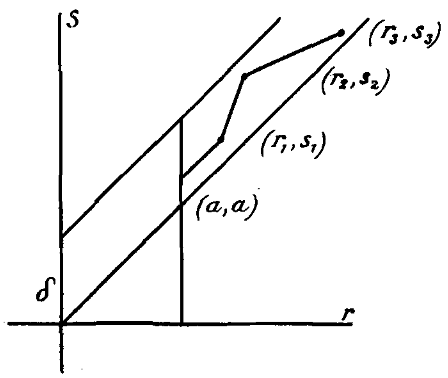

Fig. 1.

Imagine a point Q to move outward from $r=a$ along this radial line. It is nearly self-evident that a second point $\bar{Q}$ may be made to move simultaneously on the same line, so as to be always at as least as great a radial distance as Q but never exceeding it by as much as δ, and furthermore so that when Q coincides with $T(P_{i-1})$, $T(P_{j-1})$, $T(P_{k-1}), \ldots$, $\bar{Q}$ will coincide with $P_i, P_j, P_k, \ldots$ respectively.

This fact may be made graphically more evident as follows. Let $r_1, r_2, \ldots$ be the radial distances of $T(P_{i-1})$, $T(P_{j-1}), \ldots$ arranged in order of increasing radial magnitude, and let $s_1, s_2, \ldots$ be the corresponding distances of $P_i, P_j, \ldots$ so that the inequalities obtain:

$$r_1 < r_2 < r_3 \ldots,$$

$$s_1 < s_2 < s_3 \ldots,$$

$$0 \leqq s_1 - r_1 < \delta, \quad 0 \leqq s_2 - r_2 < \delta \quad \ldots\ldots$$

If we take the number pairs (r_1, s_1), (r_2, s_2), ... as the cartesian coordinates of points in the plane, join these points in succession by straight line segments (see Figure 1), and extend the broken line so obtained to right and left from the two end points by lines making an angle of 45° with the positive r axis, the graph of a function $s=f(r)$ is given by the broken line. If r be regarded as the radial coordinate of Q and s as that of $\bar{Q}$, the correspondence between Q and $\bar{Q}$ so defined has the desired properties.

It is conceivable that Q coincides with $\bar{Q}$ at $r=a$, in which case, however, Q is of course not a point $T(P_{i-1})$; for, if it were, Q must be $T(P_0)$ and $\bar{Q}$ must be P_1 distinct from Q. In any case, by replacing the rectilinear part of the graph for $r \leqq r_1$ by another of slightly less slope, a modified correspondence is obtained which makes $\bar{Q}$ fall beyond Q at the outset when $r=a$. It is convenient in what follows to suppose this to have been done.

In this way there is defined along every radial line on which points P_i, P_j, ... of the *minimal* δ-*chain* falls, a *one-to-one, continuous, outward* radial motion through a distance less than δ which takes every point $T(P_{i-1})$, $T(P_{j-1})$, ... into its corresponding P^i, P_j, ...

All of these linear radial motions may be effected by a single one-to-one, continuous, outward radial motion of the plane through a distance less than δ and defined for $r \geqq a$. For imagine in the above figure (Figure 1) a third θ axis perpendicular to the plane of the r, s axes, and imagine all of the graphs drawn in their appropriate planes $\theta =$ constant. These broken lines all rise in the r direction and lie at a vertical distance less than δ above the plane $s=r$. Join the pairs of points of adjacent broken lines with the same r coordinate by straight line segments. These evidently define a function $s=f(r, \theta)$ giving rise to an outward radial motion E for $r \geqq a$ having the character required.

The results of the last two sections may now be incorporated in the following

Lemma 2. If there exists a finite δ-chain and so a minimal δ-chain $P_0, P_1, \ldots, P_n$ (n a minimum), then there exists a one-to-one continuous outward radial motion E through a distance less than δ, defined for $r \geqq a$, which carries C outward and takes

$$T(P_0),\ T(P_1),\ \ldots\ T(P_{n-1})$$

into

$$P_1,\ P_2,\ \ldots\ P_n$$

respectively.

6. The auxiliary curve. Lemma III.

We consider next the compound transformation TE obtained by following T with such a transformation E. Clearly TE is a one-to-one direct transformation of R into a ring $E(R_1)$, and carries the circle C into a distinct continuous closed curve C_1 which surrounds C. Furthermore TE takes each point P_0, $P_1, \ldots P_{n-1}$ of the minimal δ-chain corresponding to E into $P_1, P_2, \ldots P_n$ respectively. In fact, any point P_{i-1} is carried by T into $T(P_{i-1})$ and then by E into P_i. Since P_0 lies on $C_0=C$, P_1 will lie on C_1.

By the application of TE the doubly connected ring bounded by C_0 and C_1, is taken into a like ring bounded by C_1 and C_2. This second ring abuts on the outer side C_1 of the first ring, and the point P_2 lies on C_2. Thus, by performing TE successively, a succession of expanding rings $C_0 C_1$, $C_1 C_2, \ldots, C_{n-1} C_n$ is obtained, each abutting on its predecessor, while $P_0, P_1, \ldots, P_n$ lie on C_0, $C_1, \ldots, C_n$ respectively.

Of course this process would terminate earlier if any ring $C_{r-1} C_r$ $(r<n)$ extended beyond R. But all points in $C_0 C_1$ evidently belong to M_1, all points in $C_1 C_2$ to M_2, and so on, so that those in $C_{r-1} C_r$ belong to M_r, and cannot lie outside of R by the very definition of a minimal δ-chain. On the other hand P_n on C_n does lie outside of R, so that part of the ring $C_{n-1} C_n$ does extend beyond R.

At this stage it is convenient to take r and θ as the rectangular coordinates of a point in the r, θ plane. From any one selected determination of the transformation T in this plane, all the others can be obtained by a translation in the θ direction through any distance $2k\pi (k=1, 2, \ldots)$. The circle C appears as a straight line $r=a$, parallel to the θ axis; Γ and Γ_1 appear as open curves lying above this line and extending indefinitely far to right and left, while C_1, $C_2, \ldots$ are similar curves, C_1 lying above C, C_2 above C_1, and so on. All of these curves are congruent in each interval

$$2k\pi \leqq \theta \leqq 2(k+1)\pi.$$

The rings $C C_1$, $C_1 C_2, \ldots$ appear as adjoining strips. The compound transformation TE carries each strip into the one immediately above it.

In this new plane join P_0, P_1 by a continuous arc $P_0 P_1$ without multiple points, crossing the strip $C_0 C_1$ and having only P_0 and P_1 on C_0 and C_1 respectively. The arc $P_0 P_1$ is evidently carried by TE into $P_1 P_2$ crossing the second strip $C_1 C_2$. Again this arc $P_1 P_2$ is carried by TE into an arc $P_2 P_3$ on the third strip, and so on (see Figure 2). Obviously in this way a continuous curve $P_0 P_1 P_2 \ldots P_n$ without multiple points is obtained.

Let Q_0 be the first point of $P_0 P_1 \ldots P_n$ to cross the boundary Γ of R. The point Q_0 evidently falls on $P_{n-1} P_n$ but is not the end point P_n. Let us consider the image of $P_0 P_1 \ldots P_{n-1} Q_0$ under TE. The transformed curve $P_1 \ldots P_n Q_1$ is made up of the arcs

$$P_1 P_2,\ P_2 P_3,\ \ldots,\ P_{n-1} P_n,\ P_n Q_1,$$

and is obviously without multiple points. It is clear also that the transformed curve has no point in common with $P_0 P_1$. Hence the auxiliary curve $P_0 Q_1$ has

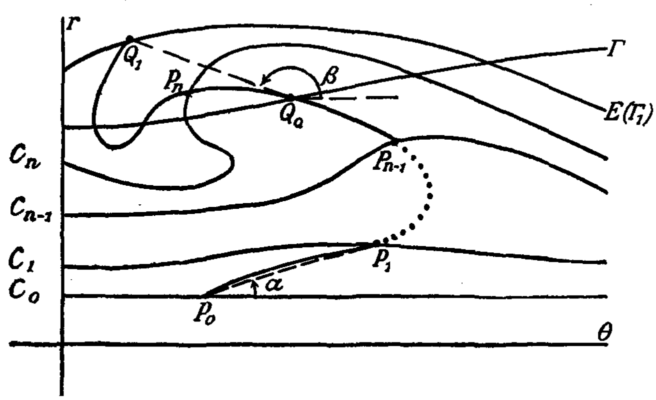

Fig. 2.

no multiple points. It has the further property that TE takes the part of it, $P_0 Q_0$, which crosses R, into $P_1 Q_1$ which lies partly outside of R and crosses $E(R_1)$.

Strictly speaking, since P_0 has an infinite series of representative points when r, θ are taken as rectangular coordinates, namely those obtained from any one by a motion to right or left through a distance $2k\pi$, an infinite series of congruent curves $P_0 Q_1$ are obtained. However, if we revert to r, θ as polar coordinates and choose the arc $P_0 P_1$ so as not to have multiple points in this plane, then it is evident that the curves $P_0 Q_1$ represented in the new plane are distinct from one another and without multiple points.

The results thus obtained may be summarized in the

Lemma 3. *Under the hypotheses and notation of Lemma 2, there exists a continuous curve without multiple points,*

$$P_0 P_1 \cdots P_{n-1} Q_0 P_n Q_1,$$

such that the compound transformation TE carries the arc $P_0 Q_0$ crossing R into $P_1 Q_1$ crossing $E(R_1)$, while $P_0 P_1$ crosses the ring bounded by C and $E(C)$.

7. The δ-Theorem.

On the basis of the above three preparatory lemmas, we can now prove a theorem, out of which the extension of Poincaré's last geometric theorem stated in section 2 follows:

δ-Theorem. If Γ and Γ_1 are met only once by any radial line $\theta=$constant, and if T carries points on C and Γ in opposite angular directions (with respect to θ) to their new positions on C and Γ_1 respectively, then for any $\delta > 0$ either (a) there is a point P of R such that $T(P)$ of R_1 is on the same radial line and is distant by less than δ from P, or (b) there is an open ring Σ in R (or R_1) abutting on C, which is carried by T (or T_{-1}) into a ring lying in Σ and radially distant from the boundary of Σ by at least δ in the outward direction.

To establish this theorem it is evidently sufficient to prove that if there exists no region Σ, there must exist a point P.

If there exists no region Σ there will exist finite δ-chains by Lemma I, and then in virtue of the properties developed in Lemmas 2, 3 there will exist an auxiliary transformation E and a curve $P_0 P_1 \cdots P_{n-1} Q_0 P_n Q_1$.

Imagine now a point A to move along this curve from P_0 to Q_0, so that its image A_1 under TE moves from P_1 to Q_1. The vector AA_1 represented in the plane in which r, θ are rectangular coordinates (Figure 2) will rotate through a definite angle during this process, which we will designate by $rot\ AA_1$.

For definiteness let us assume that points of C have their θ coordinate increased under T, so that then, by hypothesis, points of Γ have their θ coordinate decreased under T. If α denotes the positive acute angle that the vector $P_0 P_1$ makes with the positive θ axis, while β denotes the angle between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ which $Q_0 Q_1$ makes with the same line, then the rotation in question is clearly $\beta-\alpha$, or else differs from $\beta-\alpha$ by a multiple of 2π. It is of central importance for what follows to establish that this rotation is precisely $\beta-\alpha$.

Suppose that the strip bounded by C and $E(\Gamma_1)$ which the auxiliary curve $P_0 Q_1$ crosses, is deformed by a purely radial distortion so that $E(C)$ and $E(\Gamma_1)$

(which is continuous and meets each radial line precisely once, because of the hypothesis made about Γ_1) become straight lines $r=b$ and $r=c$ while the line C is not moved. Meanwhile $rot\ AA_1$ taken along the deformed curves will alter continuously. Hence $\beta-\alpha$ measured in a similar manner will continue to be the precise value of $rot\ AA_1$, or will continue to differ from it by one and the same multiple of 2π. Moreover, α and β will continue subject to the same inequalities as before:

$$0<\alpha<\frac{\pi}{2}, \quad \frac{\pi}{2}<\beta<\frac{3\pi}{2}.$$

Suppose now that the auxiliary curve as thus modified into a curve crossing the strip $a\leqq r\leqq c$, is deformed further on this strip while P_0, P_1, Q_0, Q_1 are held fixed. It is again clear that, because of the continuity in the variation of $rot\ AA_1$ so long as the curve does not acquire multiple points, the stated formula will continue true or false in this second process of variation.

But in the first place the arc P_0P_1 crosses the strip $a\leqq r\leqq b$ while P_1Q_1 lies outside of it. Hence P_0P_1 can be deformed on the strip into a rectilinear segment P_0P_1. Moreover the arc $P_1Q_0Q_1$ crosses the strip $b\leqq r\leqq c$, and can obviously be continuously deformed into the broken line $P_1Q_0Q_1$ without changing the position of P_1, Q_0 or Q_1. Hence we obtain by legitimate modification a broken line $P_0P_1Q_0Q_1$ where these points are arranged in order of increasing r coordinates, while P_1 has a greater θ coordinate than P_0, and Q_1 has a lesser θ coordinate than Q_0. In this normal position the validity of the expression $\beta-\alpha$ for $rot\ AA_1$ is self-evident. Hence it must have held along the auxiliary curve with which we started, no matter how complicated that curve may have been.

On account of the inequalities to which β and α were subjected, we conclude therefore that $rot\ AA_1$ is positive as the point A moves from P_0 to Q_0 across R along the auxiliary curve.

Now consider the modified transformation TE_λ where E_λ denotes that radial displacement which moves a point by λ times the distance that E does. Thus E_1 is the same as E, while E_0 is the identical transformation in which no point is displaced. If A_1 designates $TE_\lambda(A)$, it is plain that as λ diminishes from 1 to 0, $rot\ AA_1$ changes continuously unless A and A_1 coincide for some λ. But this would give rise a point P as specified in alternative (a) of the theorem. Hence this possibility may be excluded. Consequently, since as λ diminishes, P_1 and Q_1 merely move along lines $\theta=$constant to the right and left of P_0

and Q_0 respectively, the inequality $rot\, A A_1 > 0$ must continue to hold until λ reaches 0.

Therefore the angular rotation of a vector drawn from a point A to its image A_1 under $T = TE_0$, as A moves along the auxiliary curve $P_0 Q_0$, is positive. If the auxiliary curve be continuously varied into any other curve across the ring R, this rotation must vary continuously, or we are led to an invariant point and thus to the alternative (*a*). Hence it never reduces to 0, since the hypothesis of the theorem ensures that the vector $A A_1$ has a positive θ component for A on C and a negative θ component for A on Γ. Thus the total rotation of the vector $A A_1$ is positive along any curve crossing R.

It is now necessary to note the complete symmetry between T and T_{-1} in the hypothesis and conclusion of the δ-theorem. On this account we may take the inverse transformation T_{-1} as fundamental, in which case the rôles of R and R_1, of Γ and Γ_1 are merely interchanged. Furthermore, the transformation T_{-1} carries points on C and Γ_1 in just the opposite θ direction. For definiteness it has been assumed that T moves points on C and Γ to right and left respectively in the plane in which r and θ appear as rectangular coordinates. Consequently T_{-1} moves points on C and Γ_1 to the left and right respectively in that plane.

With this slight modification in mind we arrive at the conclusion that the total rotation of the vector drawn from a point B to its image B_{-1} under T_{-1} along any curve crossing the ring R_1 is negative.

But as B crosses R_1, B_{-1} crosses R of course, and may be taken as a point A. Hence we infer that $rot\, A_1 A$ is negative along any curve across R.

This is absurd, since the total rotation of the vector $A_1 A$ is precisely the same as that of $A A_1$ which has already been proved to be positive under the stated circumstances.

Consequently the δ-theorem is established.

8. Completion of the proof.

The hypotheses of the theorem stated in section 2 include those of the δ-theorem, and in addition we may exclude the alternative (*b*) of the δ-theorem for any positive δ. Hence for every positive δ there exists a point P of R which is carried by T into a point $T(P)$ of R_1 on the same radial line and distant from P by not more than δ. A sequence of such points P with δ approaching o evidently has at least one limiting point in R and R_1, which is invariant under T.

Thus the existence of at least one invariant point of R and R_1 is established.

If now we recur to the auxiliary plane in which r and θ appear as rectangular coordinates, and allow a point A to make a circuit in a positive sense of that part of R contained between two parallels to the r axis at a distance 2π apart, it is clear that *rot* AA_1 over the circuit vanishes since the rotation is zero along the arc of C and the arc of Γ, and cancels along the other two boundaries.

Evidently this circuit contains within it each invariant point only once, and thus the total rotation is the algebraic sum of the rotations over small circuits about the separate invariant points.[1] But at a *simple* invariant point this rotation is $\pm 2\pi$, by definition. Hence there are at least two distinct invariant points, unless there is a single *multiple* invariant point K with a rotation o about it.

From the existence of a single invariant point the existence of a second invariant point follows in the »general case» by the above argument due to Poincaré. However, the proof that there does exist a second *distinct* invariant point is a much more delicate matter.

We will suppose that there exists one and only one invariant point K, and show that we are then led to a contradiction by means of a slight extension of our earlier argument.

Instead of considering a fixed positive δ, we shall employ a $\delta(\theta)$ which varies from one radial line to another. An outward radial motion of a point P through a distance less than δ refers then to the value of δ along the radial line on which P lies. If $\delta = 0$, the point P is to be held fixed. Evidently δ-chains and minimal δ-chains may be defined with respect to such a variable $\delta(\theta)$.

In the case before us we propose to select δ as small and positive except along the single radial line through the single invariant point K. Furthermore it is obviously possible to select δ so that it varies continuously with θ and is always less than the distance of P to $T(P)$ or of $T(P)$ to K for any point P on the radial line in question, these distances being reckoned in the plane in which r and θ appear as rectangular coordinates.

If δ is so selected, no point of any δ-chain can be the invariant point K,

[1] The case where there are infinitely many invariant points may be excluded from consideration.

for such a point is obtained from its predecessor $P+K$ by imposing upon $T(P)$ an outward radial motion through a distance less than that of $T(P)$ from K.

Lemma 1 will continue to hold for this slightly modified type of δ-chain provided that the outer boundary of the ring Σ is allowed to touch C at the point where the radial line through K meets C. But no such region Σ can exist in consequence of the exclusion of alternative (*b*) of section 2. Hence there will exist a finite δ-chain, and thus a minimal chain $P_0, P_1 \cdots P_n$ corresponding to this particular $\delta(\theta)$.

With reference to this particular minimal chain we can set up an auxiliary transformation E having the properties incorporated in Lemma 2.

There then arises by consideration of a compound transformation TE a series of rings $CC_1, C_1C_2, \ldots$ as before with the single difference that the two boundaries of a ring may touch at one point. The points P_0P_1 may be joined by a curve in CC_1 as before and so an auxiliary curve

$$P_0P_1 \cdots P_{n-1}Q_0P_nQ_1$$

is obtained as in Lemma 3, except that this curve may possess double points without crossing, on account of the possibility that successive curves $C, C_1, C_2, \ldots$ may touch at a single point. Of course this auxiliary curve cannot pass through the invariant point K, which lies outside of the series of rings.

Proceeding now as in section 7 we consider *rot* AA_1 along the curve P_0Q_0 where A_1 is the image of A under TE. The mode of determination of $\delta(\theta)$ ensures that A_1 is always distinct from A. It follows then as before that this rotation is positive along the auxiliary curve and remains so under the parametric transformation TE_λ as λ decreases from 1 to 0. Hence *rot* AA_1 along this curve when A_1 is the image of A under T must be positive. It will therefore remain positive along any curve which crosses R and can be obtained from P_0Q_0 by a continuous deformation without passing over K. But even if the curve does pass over K, *rot* AA_1 is not thereby affected since the rotation is 0 about K. Hence along any curve whatsoever that crosses R, *rot* AA_1 is positive.

But operating with T_{-1}, we infer that *rot* $A_1A =$ *rot* AA_1 is negative, and a contradiction is obtained.

Thus the theorem is established.

An easy extension of the above argument shows that either there exist two invariant points about each of which *rot* AA_1 is not 0, or there exist infinitely many invariant points.