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AN EXTENSION OF ROSEN'S THEOREM TO NON-IDENTICALLY DISTRIBUTED RANDOM VARIABLES[1]

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1. Introduction and summary. In [5], B. Rosen showed that if

$$\{X_k : k = 1, 2, \cdots\}$$

is an independent sequence of identically distributed random variables with $EX_k = 0$ and $\operatorname{Var} X_k = \sigma^2, 0 < \sigma^2 < \infty$ and if $S_n = X_1 + \cdots + X_n$, then the series $\sum_{n=1}^{\infty} n^{-1}(P(S_n < 0) - \frac{1}{2})$ is absolutely convergent. This theorem was motivated by a result of Spitzer [6] who, under the same conditions, established the convergence of this series as a corollary to a result in the theory of random walks. Rosen's theorem was generalized by Baum and Katz [1] who showed that if $EX_k = 0$ and $E|X_k|^{2+\alpha} < \infty$ for $0 \leqq \alpha < 1$ then

$$\sum_{n=1}^{\infty} n^{-(1-\alpha/2)} |P(S_n < 0) - \tfrac{1}{2}| < \infty.$$

These results led to the study of series convergence rate criteria for the central limit theorem and a partial solution of this problem was obtained for the case of identically distributed random variables in [2]. A more complete solution has been recently obtained by Heyde [4].

The first study of series convergence rates for $P(S_n < 0)$ in the case of independent but non-identically distributed random variables was made by Heyde [3]. Based on an extension of Rosen's theorem utilizing certain uniform bounds on the characteristic function of the X_k's he concluded the absolute convergence of the series $\sum_{n=1}^{\infty} n^{-(1-\alpha/2)}(P(S_n < n^p x) - \frac{1}{2})$ for $-\infty < x < \infty$ and $0 \leqq p < \frac{1}{2}(1 - \alpha)$, $0 \leqq \alpha < 1$, thus obtaining what he termed small deviation convergence rates. In the present paper two more extensions of Rosen's theorem to independent but non-identically distributed random variables are given under different hypotheses than Heyde's. The first (Theorem 1) reduces to Rosen's theorem in the case of identically distributed random variables. The second (Theorem 2) results in a theorem similar to that of Baum and Katz [1] as required in Heyde's small deviation result. This will make it possible to obtain his conclusion by simply carrying out the last step in his proof. These results are obtained in Section 3. In Section 2 some preliminary results are stated and examples are given in Section 4 to show that the first two hypotheses of Theorem 1 cannot, in general, be relaxed.

2. Preliminary results. The proof of Rosen's theorem depends on the following inversion theorem [3], [4]:

(1) *If X is a random variable with characteristic function $\psi(t)$ and distribution*

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function $F(x)$ *such that* $\int_{-\infty}^{\infty} (1 + |x|)F(dx) < \infty$, *then for every* $\delta > 0$ *and* x,

$$\tfrac{1}{2}[F(x+0) - F(x-0)] = \tfrac{1}{2} + 1/2\pi i \int_0^{\delta} t^{-1}\{e^{ixt}\psi(-t) - e^{-itx}\psi(t)\}dt + R(1, x, \delta),$$

where

$$R(1, x, \delta) = \pi^{-1}\int_{-\infty}^{\infty} F(dy)\int_{\delta}^{\infty} [\sin (x - y)t/t]\, dt.$$

Now, let $\{X_k : k = 1, 2, \cdots\}$ be independent random variables with characteristic functions $\varphi_k(t)$ and let $S_n = X_1 + \cdots + X_n$. Let $F_n(x)$ denote the distribution function of S_n and $\psi_n(t) = \prod_1^n \varphi_k(t)$ its characteristic function. Then, by (1),

$$\begin{aligned} P(S_n < 0) - \tfrac{1}{2} &= \tfrac{1}{2}\{F_n(0+) - F_n(0-)\} - \tfrac{1}{2}P\{S_n = 0\} - \tfrac{1}{2} \\ &= (2\pi i)^{-1}\int_0^{\delta} t^{-1}[\psi_n(-t) - \psi_n(t)]\, dt + R(n, 0, \delta) \\ &\quad - \tfrac{1}{2}P\{S_n = 0\} \\ &= \pi^{-1}\int_0^{\delta} t^{-1}\prod_1^n |\varphi_k(t)| \sin [\textstyle\sum_1^n \arg \varphi_k(t)]\, dt \\ &\quad + R(n, 0, \delta) - \tfrac{1}{2}P(S_n = 0), \end{aligned} \tag{2}$$

where $R(n, x, \delta)$ is obtained from the equation for $R(1, x, \delta)$ by replacing F by F_n. It follows that

$$\begin{aligned} \sum_{n=1}^{\infty} n^{-(1-\alpha/2)} |P(S_n < 0) - \tfrac{1}{2}| &\leqq \pi^{-1}\sum_{n=1}^{\infty} n^{-(1-\alpha/2)} \\ &\quad \cdot \int_0^{\delta} t^{-1}(\prod_1^n |\varphi_k(t)|)\, |\sin [\textstyle\sum_1^n \arg \varphi_k(t)]|\}\, dt \\ &\quad + \sum_{n=1}^{\infty} n^{-(1-\alpha/2)}|R(n, 0, \delta)| + \sum_{n=1}^{\infty} n^{-(1-\alpha/2)}P(S_n = 0). \end{aligned} \tag{3}$$

Rosen's theorem and its extensions are all based on establishing the convergence of the three series on the right hand side of (3). The convergence of the last two series depends only on the uniform non-degeneracy of the X_k's in the following sense:

LEMMA 1. *If there exists* $K > 0$ *and* $\delta > 0$ *such that for* $|t| < \delta$, $|\varphi_k(t)| \leqq 1 - Kt^2$ *uniformly in* k,
then

(i) *if* I_n *is any interval on the real line of length not exceeding* n^p, $0 < p < \frac{1}{2}$, *then* $P(S_n \varepsilon I_n) \leqq Cn^{p-\frac{1}{2}}$, *where* C *is a constant independent of* I_n *and* n,

(ii) $\sup_a P(S_n = a) \leqq Cn^{-\frac{1}{2}}$ *where* C *is independent of* n,

(iii) *for every* ϵ, $0 < \epsilon < \frac{1}{2}$, *there exists a constant* C, *independent of* n *and* x, *such that* $|R(n, x, \delta)| \leqq Cn^{\epsilon-\frac{1}{2}}$.

Thus, for $0 \leqq \alpha < 1$

(iv) $\sum_{n=1}^{\infty} n^{-(1-\alpha/2)} |R(n, 0, \delta)| < \infty$, *and*

(v) $\sum_{n=1}^{\infty} n^{-(1-\alpha/2)} P(S_n = 0) < \infty$.

PROOF. Parts (i) and (ii) are proved exactly as in [4], noting that the proof depends only on the bound $|\psi_n(t)| \leqq (1 - Kt^2)^n$ for $|t| < \delta$. Part (iii) follows from (i) exactly as shown in [1]. Parts (iv) and (v) follow immediately from (ii) and (iii). (Take $\epsilon < \frac{1}{2}(1 - \alpha)$ in (iii).)

In the identically distributed case, the condition of Lemma 1 is satisfied for any non-degenerate random variable ([4]). Thus, the finiteness of σ^2 in Rosen's theorem is needed only to guarantee the convergence of the first series on the right hand side of (3) when $\alpha = 0$. The corresponding conditions required in the non-identically distributed case are given in Theorem 1.

3. The extensions of Rosen's Theorem. The basic result of the paper is the following theorem.

THEOREM 1. *Let $\{X_k : k = 1, 2, \cdots\}$ be an independent sequence of random variables with distribution functions $F_k(x)$, variances σ_k^2, and characteristic functions $\varphi_k(t)$ and assume $EX_k = 0$, $k = 1, 2, \cdots$. Let $S_n = X_1 + \cdots + X_n$. If*

(A) *x^2 is uniformly integrable with respect to F_k, $k = 1, 2, \cdots$,*

(B) *$\sigma_k^2 \geqq \sigma^2 > 0$, $k = 1, 2, \cdots$, and*

(C) *there exists a Borel measurable function $B(t)$ and a $\delta > 0$ such that $|\mathrm{Im}\, \varphi_k(t)| \leqq B(t)$ for $|t| < \delta$, $k = 1, 2, \cdots$, and $\int_0^\delta (B(t)/t^3)\, dt < \infty$, then*

$$\sum_{n=1}^{\infty} n^{-1} |P(S_n < 0) - \tfrac{1}{2}| < \infty.$$

Note that in the case of identically distributed random variables this reduces to Rosen's theorem. We will need the following lemmas to establish Theorem 1 and succeeding results.

LEMMA 2. *If $|x|^{2+\alpha}$ is uniformly integrable with respect to F_k, $k = 1, 2, \cdots$, for $0 \leqq \alpha < 1$, then*

(i) *there exists $M < \infty$ such that for all β, $0 \leqq \beta < \alpha$,*

$$\int |x|^{2+\beta} F_k(dx) \leqq M, \qquad \text{uniformly in } k,$$

(ii) *$|x|^{2+\beta}$ is uniformly integrable with respect to F_k, $k = 1, 2, \cdots$, for every β, $0 \leqq \beta < \alpha$,*

(iii) *there exists $\delta > 0$ such that for $|t| < \delta$,*

$$\tfrac{1}{4}\sigma_k^2 t^2 + a_k(t)t^2 \leqq 1 - \mathrm{Re}\, \varphi_k(t) \leqq \tfrac{1}{2}\sigma_k^2 t^2 + A_k(t)t^2, \tag{4}$$

where $a_k(t)$ and $A_k(t)$ are functions for which

$$\lim_{t \to 0} a_k(t) = \lim_{t \to 0} A_k(t) = 0, \qquad \text{uniformly in } k.$$

PROOF. (i) By hypothesis, there exists $M > 2$ such that

$$\int_{|x| > (M-1)^{1/(2+\alpha)}} |x|^{2+\alpha} F_k(dx) \leqq 1, \qquad \text{uniformly in } k.$$

Then,

$$\int |x|^{2+\beta} F_k(dx) \leqq \int_{|x| \leqq (M-1)^{1/(2+\alpha)}} |x|^{2+\beta} F_k(dx) + \int_{|x| > (M-1)^{1/(2+\alpha)}} |x|^{2+\alpha} F_k(dx) \leqq M.$$

(ii) follows from the inequality

$$\int_{|x|>M} |x|^{2+\beta} F_k(dx) \leqq M^{-(\alpha-\beta)} \int_{|x|>M} |x|^{2+\alpha} F_k(dx).$$

(iii) It is easily seen that there exists δ, $0 < \delta < \pi/2$ such that $\frac{1}{4}x^2 \leqq 1 - \cos x \leqq \frac{1}{2}x^2$ for $|x| \leqq \delta$. Thus,

$$\tfrac{1}{4} \int_{|tx| \leqq \delta} (tx)^2 F_k(dx) \leqq \int_{|tx| \leqq \delta} (1 - \cos tx) F_k(dx) \leqq \tfrac{1}{2} \int_{|tx| \leqq \delta} (tx)^2 F_k(dx),$$

which implies

$$\tfrac{1}{4}\sigma_k^2 t^2 - \tfrac{1}{4}t^2 \int_{|x|>\delta/|t|} x^2 F_k(dx) \leqq 1 - \operatorname{Re} \varphi_k(t) - \int_{|tx|>\delta} (1 - \cos tx) F_k(dx) \leqq \tfrac{1}{2}\sigma_k^2 t^2 - \tfrac{1}{2}t^2 \int_{|x|>\delta/|t|} x^2 F_k(dx).$$

Thus, (4) holds with

$$a_k(t) = t^{-2} \int_{|tx|>\delta} (1 - \cos tx) F_k(dx) - \tfrac{1}{4} \int_{|x|>\delta/|t|} x^2 F_k(dx)$$

and

$$A_k(t) = t^{-2} \int_{|tx|>\delta} (1 - \cos tx) F_k(dx) - \tfrac{1}{2} \int_{|x|>\delta/|t|} x^2 F_k(dx).$$

Now, $t^{-2} \int_{|tx|>\delta} |1 - \cos tx| F_k(dx) \leqq \frac{1}{2} \int_{|x|>\delta/|t|} x^2 F_k(dx)$, because $|1 - \cos x| \leqq x^2/2$ for all x. Thus, (iii) follows from (ii) with $\beta = 0$.

Lemma 3. *If $|x|^{2+\alpha}$ is uniformly integrable with respect to F_k, $k = 1, 2, \cdots$, for some α, $0 \leqq \alpha < 1$, then*

$$\lim_{t\to 0} \operatorname{Im} \varphi_k(t)/t^{2+\alpha} = 0, \qquad \textit{uniformly in } k.$$

Proof. The well known inequalities

$$|\sin x - x| \leqq \tfrac{1}{6}|x|^3 \quad \text{for} \quad |x| \leqq 1$$
$$\leqq 2|x| \quad \text{for} \quad |x| > 1$$

imply $|\sin x - x| \leqq 2x^{2+\alpha}$ for all x, $0 \leqq \alpha < 1$. Since $EX_k = 0$ for all k, the following inequalities are valid for $|t| \leqq 1$:

$$\begin{aligned}
&|\operatorname{Im} \varphi_k(t)/t^{2+\alpha}| \\
&= \left|\int_{-\infty}^{\infty} ((\sin tx - tx)/t^{2+\alpha}) F_k(dx)\right| \\
&\leqq \int_{|xt| \leqq t^{\frac{1}{2}}} (|\sin tx - tx|/|t|^{2+\alpha}) F_k(dx) + \int_{|xt|>t^{\frac{1}{2}}} (|\sin tx - tx|/|t|^{2+\alpha}) F_k(dx) \\
&\leqq \tfrac{1}{6} |t|^{1-\alpha} \int_{|x| \leqq 1/t^{\frac{1}{2}}} |x|^3 F_k(dx) + 2 \int_{|x|>1/t^{\frac{1}{2}}} |x|^{2+\alpha} F_k(dx) \\
&\leqq \tfrac{1}{6} |t|^{(1-\alpha)/2} \int |x|^{2+\alpha} F_k(dx) + 2 \int_{|x|>1/t^{\frac{1}{2}}} |x|^{2+\alpha} F_k(dx) \\
&\leqq \tfrac{1}{6} M |t|^{(1-\alpha)/2} + 2 \int_{|x|>1/t^{\frac{1}{2}}} |x|^{2+\alpha} F_k(dx).
\end{aligned}$$

The result now follows from the hypothesis of the lemma.

Lemma 4. *If Conditions* (A) *and* (B) *of Theorem* 1 *are satisfied, there exists K, $0 < K < \infty$ and $\delta > 0$ such that for $|t| \leqq \delta$,*

$$|\varphi_k(t)| \leqq 1 - Kt^2, \qquad \textit{uniformly in } k.$$

PROOF. Let $b_k(t) = |\text{Im}\,\varphi_k(t)|/t^2$. By Lemma 2,

$$|\varphi_k(t)| \leqq |\text{Re}\,\varphi_k(t)| + |\text{Im}\,\varphi_k(t)| \leqq 1 - \tfrac{1}{4}\sigma_k^2 t^2 - a_k(t)t^2 + b_k(t)t^2$$

for sufficiently small values of $|t|$. However, it follows from Lemmas 2 and 3 with $\alpha = 0$ and Condition (B) that there exists $\delta > 0$ and $0 < K < \infty$ such that for $|t| \leqq \delta$, $\frac{1}{4}\sigma_k^2 + a_k(t) - b_k(t) \geqq K$ uniformly in k.

We are now in a position to prove Theorem 1.

PROOF OF THEOREM 1. Because of Lemmas 1 and 4 it suffices to establish the convergence of the first series on the right hand side of (3) with $\alpha = 0$. Now, by Lemma 2, $\text{Re}\,\varphi_k(t) \geqq 1 - \frac{1}{2}\sigma_k^2 t^2 - A_k(t)t^2$, $\sigma_k^2 \leqq M$ and $A_k(t) \to 0$ as $t \to 0$ uniformly in k. Thus, there exists $\delta_1 > 0$ and $1 < C < \infty$ such that for $|t| \leqq \delta_1$, $\text{Re}\,\varphi_k(t) \geqq 1/C$. Thus, since $|\sin x| \leqq |x|$ and $|\arctan x| \leqq |x|$ for all x,

$$|\sin\,(\textstyle\sum_1^n \arg\varphi_k(t))| \leqq \sum_1^n |\arctan\,[\text{Im}\,\varphi_k(t)/\text{Re}\,\varphi_k(t)]| \leqq C\sum_1^n |\text{Im}\,\varphi_k(t)|$$

$\leqq nCB(t)$ for $|t| \leqq \delta'$, where δ' is the minimum of δ_1 and the δ of Condition (C).

Now, take $\delta =$ minimum of δ' and the δ of Lemma 4. Then,

$$\begin{aligned}\textstyle\sum_{n=1}^\infty n^{-1}\pi^{-1}\int_0^\delta t^{-1}(\prod_1^n |\varphi_k(t)|)|\sin\,(\sum_1^n \arg\varphi_k(t))|\,dt \\ \leqq C\pi^{-1}\int_0^\delta t^{-1}\textstyle\sum_{n=1}^\infty (1 - Kt^2)^n B(t)\,dt \\ = C(K\pi)^{-1}\int_0^\delta (B(t)/t^3)\,dt < \infty.\end{aligned}$$

In Section 4 we will show that Conditions (A) and (B) are sharp in the sense that examples violating exactly one of the conditions can be constructed which satisfy Condition (C) and for which the conclusion of Theorem 1 is false. Neither of these two conditions is necessary since the conclusion of the theorem is trivially true for any independent sequence of non-degenerate, symmetric random variables for which $P(X_k = 0) \equiv 0$. It is easy to construct such sequences which violate Conditions (A) and (B) simultaneously. On the other hand, we have not been able to establish that Condition (C) is sharp in the same sense. However, Condition (C) *is* sharp in the sense that it is a consequence of a slightly strengthened version of Condition (A):

LEMMA 5. *If* $|x|^{2+\alpha}$ *is uniformly integrable with respect to* F_k, $k = 1, 2, \cdots$, *for some* $\alpha > 0$ *then for every* $\epsilon > 0$ *there exists* $\delta > 0$ *such that Condition* (C) *is satisfied for* $|t| \leqq \delta$ *with* $B(t) = \epsilon|t|^{2+\alpha}$.

PROOF. This follows immediately from Lemma 3.

This allows us to establish a theorem resembling that of Baum and Katz [1]:

THEOREM 2. *If* $|x|^{2+\gamma}$ *is uniformly integrable with respect to* F_k, $k = 1, 2, \cdots$, *for some* γ, $0 < \gamma \leqq 1$ *and if* $\sigma_k^2 \geqq \sigma^2 > 0$ *uniformly in* k, *then for every* α, $0 \leqq \alpha < \gamma$,

$$\textstyle\sum_{n=1}^\infty n^{-(1-\alpha/2)}|P(S_n < 0) - \frac{1}{2}| < \infty.$$

PROOF. Because of Lemma 2 (ii) and Lemma 5 all of the hypotheses of Theorem 1 are satisfied. Consequently, the proof again reduces to establishing the convergence of the first series on the right side of (3). By the same steps as in the proof of Theorem 1 we have

$$\begin{aligned} S &= \sum_{n=1}^{\infty} n^{-(1-\alpha/2)}\pi^{-1} \int_0^\delta t^{-1} \left(\prod_1^n |\varphi_k(t)|\right)\left|\sin\left(\sum_1^n \arg\varphi_k(t)\right)\right| dt \\ &\leqq (\pi C)^{-1} \int_0^\delta t^{-1} \sum_{n=1}^{\infty} n^{\alpha/2}(1 - Kt^2)^n B(t)\, dt. \end{aligned}$$

But, by an Abelian theorem used in [1],

$$\sum_{n=1}^{\infty} n^{\alpha/2} u^n < C'(1-u)^{-(1+\alpha/2)} \qquad \text{for} \quad 0 \leqq u < 1.$$

Thus,

$$S \leqq C'(\pi C)^{-1} \int_0^\delta t^{-1}(Kt^2)^{-(1+\alpha/2)} B(t)\, dt \leqq C'' \int_0^\delta t^{-(\alpha-\gamma-1)}\, dt \qquad \text{by Lemma 5},$$

where $C'' = C'\epsilon/\pi C K^{1-(\alpha/2)}$.

Since $\alpha < \gamma$, the last integral converges, thus $S < \infty$.

A new version of Heyde's small deviation convergence theorem can now be obtained as a Corollary to Theorem 2. The proof is due to Heyde [3] and, being very brief, is included for completeness.

Corollary. *If $|x|^{2+\gamma}$ is uniformly integrable with respect to F_k, $k = 1, 2, \cdots$, for some γ, $0 < \gamma \leqq 1$ and if $\sigma_k^2 \geqq \sigma^2 > 0$ uniformly in k, then for every α, $0 \leqq \alpha < \gamma$, every p, $0 \leqq p < (1-\alpha)/2$ and every x, $-\infty < x < \infty$,*

$$\sum_{n=1}^{\infty} n^{-(1-\alpha/2)} |P(S_n < n^p x) - \tfrac{1}{2}| < \infty.$$

Proof. If $x < 0$, $[S_n < n^p x] \subset [S_n < 0]$ and if $x > 0$, $[S_n < n^p x] \subset [S_n < 0] \cup [|S_n| < n^p x]$. For any ρ, $p < \rho < (1-\alpha)/2$, there exists $N = N(\rho, x)$ such that for $n \geqq N$, $[|S_n| < n^p x] \subset [|S_n| < n^\rho/2]$. By Lemma 1 (i), $P(|S_n| < n^\rho/2) \leqq Cn^{\rho-\frac{1}{2}}$. Thus, taking $N = 1$ for $x < 0$,

$$\begin{aligned} \sum_{n=N}^{\infty} n^{-(1-\alpha/2)} |P(S_n < n^p x) - \tfrac{1}{2}| &\leqq \sum_{n=N}^{\infty} n^{-(1-\alpha/2)} |P(S_n < 0) - \tfrac{1}{2}| \\ &\quad + \sum_{n=N}^{\infty} n^{-(1-\alpha/2)} P(|S_n| < n^p/2) < \infty, \end{aligned}$$

since $\sum_{n=N}^{\infty} n^{-(1-\alpha/2)} n^{\rho-\frac{1}{2}} < \infty$ for $\rho < (1-\alpha)/2$.

This theorem is not true even in the case of identically distributed random variables for $p = \frac{1}{2}$, $\alpha = 0$ and $x \neq 0$. It was shown in [2] that if

$$\operatorname{Var}(X_k) = 1, \qquad E(X_k) = 0,$$

then

$$\sum_{n=1}^{\infty} n^{-1} |P(S_n < n^{\frac{1}{2}} x) - \Phi(x/\sigma_n)| < \infty$$

when Φ is the standard normal distribution function and $\{\sigma_n\}$ is a certain sequence of positive numbers such that $\sigma_n \uparrow 1$ as $n \to \infty$. Consequently,

$$\sum_{n=1}^{\infty} n^{-1} |P(S_n < n^{\frac{1}{2}} x) - \tfrac{1}{2}|$$

will diverge for all $x \neq 0$. Thus, one cannot in general extend the Corollary to the case $p = (1-\alpha)/2$.

4. Examples. In this section we establish by two examples that Conditions (A) and (B) of Theorem 1 cannot in general be relaxed.

EXAMPLE 1. In this example Conditions (A) and (C) of Theorem 1 are satisfied but Condition (B) is violated and the conclusion of the theorem is false. Let γ satisfy $\frac{1}{2} < \gamma < 1$ and let $\delta_k = \gamma^{\frac{1}{2}^{k-1}}$. It follows that $\prod_{k=1}^{\infty} \delta_k = \gamma$. Let $\{Y_k : k = 1, 2, \cdots\}$ be an independent sequence of random variables with the following two-point distributions:

$$\begin{aligned} P(Y_k = x) &= \delta_k & &\text{if} \quad x = \delta_k - 1 \\ &= 1 - \delta_k & &\text{if} \quad x = \delta_k \\ &= 0 & &\text{otherwise.} \end{aligned}$$

Then, $E(Y_k) \equiv 0$ and $\operatorname{Var}(Y_k) = \delta_k(1 - \delta_k) \to 0$ as $k \to \infty$. Thus, Condition (B) is violated. If $\xi_k(t)$ is the characteristic function of Y_k, a simple calculation establishes that

$$\begin{aligned} \operatorname{Im} \xi_k(t) &= \delta_k \sin t(\delta_k - 1) + (1 - \delta_k) \sin t\delta_k \\ &= \delta_k(1 - \delta_k)\theta_k(t)t^3/6 \quad \text{where} \quad |\theta_k(t)| \leqq 1. \end{aligned}$$

Thus,

$$|\operatorname{Im} \xi_k(t)| \leqq |t|^3/24$$

uniformly in k, and Condition (C) is satisfied. Condition (A) is satisfied trivially since the Y_k's are uniformly bounded random variables. To see that the conclusion of Theorem 1 is violated by this sequence, note that if $Y_k < 0$, $k = 1, 2, \cdots, n$, then $V_n = Y_1 + \cdots + Y_n < 0$. Thus,

$$P(V_n < 0) \geqq \prod_1^n P(Y_k < 0) = \prod_1^n \delta_k > \gamma > \tfrac{1}{2} \qquad \text{for all} \quad n,$$

and it follows that

$$\sum_{n=1}^{\infty} n^{-1} |P(V_n < 0) - \tfrac{1}{2}| \geqq (\gamma - \tfrac{1}{2}) \sum_{n=1}^{\infty} n^{-1} = \infty.$$

EXAMPLE 2. In this example Condition (A) of Theorem 1 is violated, yet $\sigma_k^2 < 2$ for all k, while (B) and (C) are satisfied. Again, the conclusion of the theorem is false. This shows that the uniform integrability of x^2 is required—a uniform bound for the variances is not sufficient.

Let $\{h(n)\}$ be a sequence of positive real numbers such that $\sum_{k=1}^{n-1} h(k) < h(n)$ for $n = 2, 3, \cdots$. Let $\{Z_k\}$ be a sequence of random variables such that $\{Y_k, Z_k : k = 1, 2, \cdots\}$ is an independent family of random variables, where $\{Y_k\}$ is the sequence defined in Example 1, and such that the Z_n's are symmetrically distributed with the following probability function:

$$\begin{aligned} P(Z_k = x) &= 1/2h^2(k) & &\text{if} \quad x = -h(k) \quad \text{or} \quad h(k) \\ &= 1 - 1/h^2(k) & &\text{if} \quad x = 0 \\ &= 0 & &\text{otherwise.} \end{aligned}$$

Now, let $X_k = Z_k + Y_k$, $k = 1, 2, \cdots$. Then $S_n = X_1 + \cdots + X_n = U_n + V_n$, where $U_n = Z_1 + \cdots + Z_n$, $V_n = Y_1 + \cdots + Y_n$. Note that $[U_n = 0$ and $V_n < 0] \subset [S_n < 0]$. Thus,

$$P(S_n < 0) \geqq P(U_n = 0)P(V_n < 0).$$

But by the construction of the Z_k's, $U_n = 0$ if and only if $Z_k = 0, k = 1, 2, \cdots, n$. Thus,

$$P(S_n < 0) \geqq [\textstyle\prod_1^n P(Z_n = 0)]P(V_n < 0)$$
$$\geqq [\textstyle\prod_1^n (1 - 1/h^2(k))]P(V_n < 0).$$

By selecting the $h(n)$ sequence sufficiently rapidly increasing and by taking $h(1)$ sufficiently large we can guarantee that

$$\textstyle\prod_1^\infty (1 - 1/h^2(k)) = \epsilon \quad \text{for} \quad \frac{1}{2} < \epsilon < 1.$$

Then,

$$P(S_n < 0) - \tfrac{1}{2} \geqq \epsilon(P(V_n < 0) - \gamma'),$$

where $\gamma' = 1/2\epsilon$. It follows that $\frac{1}{2} < \gamma' < 1$ and by selecting the γ of Example 1 such that $\gamma > \gamma'$ we have

$$P(S_n < 0) - \tfrac{1}{2} \geqq \epsilon(\gamma - \gamma') > 0.$$

Thus, $\sum_{n=1}^\infty n^{-1} |P(S_n < 0) - \frac{1}{2}| = \infty$.

Now, Var $(Z_k) = 1$, thus $\sigma_k^2 =$ Var $(X_k) = 1 + \delta_k(1 - \delta_k)$. Thus, $1 \leqq \sigma_k^2 < 2$ and Condition (B) is satisfied. However, it is easily checked that for every $M > 1$, $\int_{|x|>M} x^2 F_k(dx) \geqq (h(k) - 1)^2/h^2(k)$ for $h(k) > M + 1$. Since $h(k) \to \infty$ as $k \to \infty$, Condition (A) is violated. Finally, to establish Condition (C) it suffices to note that the characteristic function, $\psi_k(t)$, of Z_k is real. Thus, if $\varphi_k(t)$ is the characteristic function of X_k,

$$|\text{Im } \varphi_k(t)| = |\psi_k(t)| \cdot |\text{Im } \xi_k(t)| \leqq |t|^3/24,$$

as in Example 1.

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