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An Extension of the Borel–Weil Construction to the Quantum Group $U_q(n)$*

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Abstract. The Borel–Weil (BW) construction for unitary irreps of a compact Lie group is extended to a construction of all unitary irreps of the quantum group $U_q(n)$. This q-BW construction uses a recursion procedure for $U_q(n)$ in which the fiber of the bundle carries an irrep of $U_q(n-1) \times U_q(1)$ with sections that are holomorphic functions in the homogeneous space $U_q(n)/U_q(n-1) \times U_q(1)$. Explicit results are obtained for the $U_q(n)$ irreps and for the related isomorphism of quantum group algebras.

1. Introduction

There is an elegant geometric procedure for constructing all unitary irreducible representations (irreps) of a compact classical Lie group, the Borel–Weil (BW) construction [8] which realizes irreps as holomorphic sections of homogeneous holomorphic line bundles over Kähler manifolds. Our objective in the present paper is to develop the quantum group extension of this construction for the quantum group $U_q(n)$, a q-deformation of the enveloping algebra of the classical Lie algebra $A_{n-1} \times A_0$.

The Borel–Weil method [14, 24], applied to representations of a compact simple Lie group G, constructs a line bundle over the homogeneous space G/T, where T is the maximal torus of G. This coset space G/T can be made into a complex manifold, as can be seen from the fact that $G/T \cong G_c/B^+$, where G_c is the complexified group G and B^+ is the Borel sub-group. (We will be considering primarily $U(n)$ for which B^+ is the sub-group of upper triangular matrices.) To every character λ of the torus T one associates a homogeneous holomorphic line bundle L_λ over G/T; this uses the fact that every homomorphism λ: $T \to \mathbb{C}^x$, extends uniquely to a holomorphic homomorphism $\lambda: B^+ \to \mathbb{C}^x$, so that one may define the associated line bundle

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$L_\lambda \equiv G_c \times_{B^+} \mathbb{C}$, (which denotes the quotient of $G_c \times \mathbb{C}$ by the equivalence relation $(gb, \xi) = (g, \lambda(b)\xi) \forall b \in B^+$). The group G_c acts on the line bundle L_λ and hence on its cross-sections; this action descends to an action by the group G. The Borel–Weil theorem [14] asserts that if λ is a dominant weight then the space of holomorphic sections of L_λ is an irrep of G with dominant weight λ. (A nice discussion of the Borel–Weil construction, accessible to physicists and interpreted physically in terms of quantal models, is in [1], see also [28].)

In our extension of this construction to the quantum group $U_q(n)$, we will employ a recursive approach which differs from the BW procedure in that the induction will be from an irrep of the subgroup $U_q(n-1) \times U_q(1)$ rather than from a character of the maximal torus T_q. The base manifold is then $U_q(n)/U_q(n-1) \times U_q(1)$, which however, still has a complex structure. The recursive procedure assumes a prior construction of irreps of $U_q(n-1)$ itself by the same method, and similarly. Appealing now to the theorem on "induction-in-stages" [20, 14] one sees that this recursive approach is, despite the seeming difference, equivalent in fact to the BW construction itself. (In the literature, holomorphic induction from a *general* subgroup is ascribed, among other things, to Bott [9] in his extension of the BW theorem to homology and cohomology spaces.)

Our extension of the Borel–Weil construction to quantum groups will differ in still another way from the procedure actually used by BW, in that we will employ analytic methods rather than the purely geometric methods of BW. In part this is due to our preference for the methods of theoretical physics which emphasize explicit, concrete constructions including explicit bases carrying irreps. (In the physics literature induced representations of the BW type are known as "vector coherent states" [23, 27, 25, 11, 17].) The more incisive reason is, however, that although quantum planes with quantum (non-commutative) coordinates are known, we do not know of any quantum extension to a theory of *curved quantum manifolds*, and certainly no theory of "quantum Kähler manifolds" as yet. Such considerations are essential to the geometric concepts employed in the BW construction, but will play no rôle in our q-BW construction. It is our belief that a suitable re-interpretation of the explicit results (obtained below) in the context of the geometry of manifolds may be of use in developing such a curved non-commuting quantum manifold theory.

Let us sketch the plan of this paper. In Sect. 2, we recall the defining algebraic relations for the quantum group $U_q(n)$. To illustrate our explicit constructive methods, we obtain in Sect. 3 all irreps of the q-group $U_q(2)$, using q-boson operator techniques adapted to flat q-manifolds.

In Sect. 4, we develop our q-extension of the Borel–Weil method for the simplest of examples, $U_q(2)$, obtaining again all the irreps of Sect. 3 by this method. In Sect. 5 we extend our q-BW procedure to $U_q(n)$, obtaining thereby all irreps. Theorem (6.23) – establishing the isomorphism of the q-algebras, defined by the q-BW construction, to the standard $U_q(n)$ algebras of Sect. 2 – is one of the basic results of this paper, and is developed, and proved, in Sect. 6, along with Theorem (6.24) the q-analog of the Borel–Weil Theorem (for $U_q(n)$). In Sect. 7, an alternative form for the q-BW irrep bases is established. This result, given in Lemmata (7.1) and (7.3), is another of the basic results of this paper, and defines (by construction) a basic class of $U_q(n)$ q-Wigner–Clebsch–Gordan coefficients (Lemma (7.1)) as well as the matrix elements of the metric (Lemma (7.3)).

2. Resumé of the Defining Relations for $U_q(n)$

The defining algebraic relations for the quantum group $SU_q(n)$ corresponding to a deformation of the classical simple Lie algebra A_{n-1} have been given by Jimbo [13], Drinfel'd [10], among others [31]. Denote the q-generators in the Chevalley–Weyl basis as $\{E_i^{\pm}, H_i\}, i = 1, \ldots, n-1$. Then the $SU_q(n)$ algebra is defined by the following relations:

Commutation Relations.

$$[H_i, H_j] = 0, \tag{2.1}$$

$$[H_i, E_j^{\pm}] = \pm k_{(i,j)} E_j^{\pm}, \tag{2.2a}$$

where

$$k_{(i,j)} = \begin{cases} 1 & i = j \\ -\frac{1}{2} & i = j \pm 1 \\ 0 & \text{otherwise} \end{cases}, \tag{2.2b}$$

$$[E_i^+, E_j^-] = \delta_{ij} \frac{q^{H_i} - q^{-H_i}}{q^{1/2} - q^{-1/2}} \equiv \delta_{ij}[2H_i], \quad \text{(see (2.8) below).} \tag{2.3}$$

Quadratic q-Serre relations.

$$[E_i^{\pm}, E_j^{\pm}] = 0, j \neq i \pm 1, \quad 1 \leqq i, \quad j \leqq n-1. \tag{2.4}$$

Cubic q-Serre relations.

$$\sum_{v=0}^{2} (-1)^v \begin{bmatrix} 2 \\ v \end{bmatrix}_q (E_i^{\pm})^{2-v} E_j^{\pm} (E_i^{\pm})^v = 0, \quad (i,j) = (i, i \pm 1), 1 \leqq i, j \leqq n-1; \tag{2.5}$$

where the q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ m \end{bmatrix}_q \equiv \frac{[n]!}{[m]![n-m]!}, \quad [n]! \equiv [n][n-1]\cdots[1]. \tag{2.6a, b}$$

We use the notation $[n]$ for the q-number

$$[n] \equiv \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} = q^{(n-1)/2} + q^{(n-3)/2} + \cdots + q^{-((n-1)/2)}, \quad n \in \mathbb{Z}, \quad q \in \mathbb{R}^+. \tag{2.7}$$

This notation for q-numbers is extended to the diagonal operators H_i by defining:

$$[H_i] \equiv \frac{q^{H_i/2} - q^{-H_i/2}}{q^{1/2} - q^{-1/2}}. \tag{2.8}$$

Remark. Note that $[n]$ is symmetric under $q \to q^{-1}$, and has precisely n terms whose powers decrease by steps of *unity*. Just as for angular momentum theory [4], these requirements force the use of half-integers, here integral powers of $q^{1/2}$. The literature is not uniform, however, and q often appears for $q^{1/2}$, with corresponding steps of *two* units. (In a similar way $k_{(i,j)}$ in (2.2b) accords with the use of half-integers in quantum physics.)

The Hopf algebra operations take the form:

$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i, \tag{2.9}$$

$$\Delta(E_i^{\pm}) = E_i^{\pm} \otimes q^{H_i/4} + q^{-H_i/4} \otimes E_i^{\pm}, \tag{2.10}$$

$$\varepsilon(1) = 1, \quad \varepsilon(E_i^{\pm}) = \varepsilon(H_i) = 0, \tag{2.11a, b}$$

$$\gamma(E_i^{\pm}) = -q^{\mp 1/2} E_i^{\pm}, \quad \gamma(H_i) = -H_i. \tag{2.12a, b}$$

The quantum group $U_q(n)$ is defined by the generators above plus the additional generator H_n which commutes with all other generators. The Hopf algebra operations for H_n are the same as those for the other H_i.

The Chevalley–Weyl basis for $U_q(n)$ is very economical in introducing only the minimal number of generators ($3n-2$ for $U_q(n)$). To carry out the q-BW construction for $U_q(n)$ in the required detail, as given in the sections following, we need the full set of generators (n^2 for $U_q(n)$) and not the minimal set alone. To denote these n^2 operators we will use Weyl's notation:

$$E_{ij}, \quad i \neq j, \quad 1 \leqq i, \quad j \leqq n. \tag{2.13}$$

The correspondence between these operators and the minimal set used above is:

$$E_i^{+} = E_{i,i+1} \quad \text{for} \quad i = 1, 2, \ldots, n-1, \tag{2.14}$$

$$E_i^{-} = E_{i+1,1} \quad \text{for} \quad i = 1, 2, \ldots, n-1, \tag{2.15}$$

and

$$H_i = \tfrac{1}{2}(E_{ii} - E_{i+1,i+1}), \quad 1 \leqq i \leqq n-1, \tag{2.16}$$

with

$$H_n = \sum_{i=1}^{n} E_{ii}. \tag{2.17}$$

3. The q-Boson Realization Applied to $U_q(2)$ Irreps

Let us now consider a realization of these algebras, extending an approach – the Jordan–Schwinger mapping [4] – which stems from the physics literature on field quantization. This approach (for classical Lie groups) maps a matrix realization of the Lie algebra generators into bi-linear boson operators [4]. In one dimension these boson operators comprise the *boson creation operator*, a, and its Hermitian conjugate, (the *boson destruction operator*), $\bar{a}$, obeying the Heisenberg algebra relation

$$[\bar{a}, a] = \mathbf{1}, \tag{3.1}$$

with the cyclic ("vacuum") ket-vector $|0\rangle$ defined by the equation

$$\bar{a}|0\rangle = 0. \tag{3.2}$$

(Equivalently, one can construct basis vectors as functions of the complex variable z and regard the operators $a, \bar{a}$ as multiplication by z and differentiation $\partial/\partial z$ respectively, with the vacuum ket omitted.) Orthonormal basis vectors, $|n\rangle$, corresponding to n quanta, that is:

$$N|n\rangle \equiv (a\bar{a})|n\rangle = n|n\rangle, \tag{3.3}$$

can be constructed by repeated application of the creation operator, a, acting on the cyclic vector $|0\rangle$,

$$|n\rangle = (n!)^{-1/2}(a)^n|0\rangle. \tag{3.4}$$

Let us now construct q-analogs to the boson operators [5, 19, 29, 16]. Introduce the *q-creation operator* a^q, its Hermitian conjugate the *q-destruction operator* $\bar{a}^q$, and the *q-boson vacuum ket vector* $|0\rangle_q$ defined by the equation

$$\bar{a}^q|0\rangle_q = 0. \tag{3.5}$$

Instead of the Heisenberg relation (3.1), postulated the algebraic relation:

$$\bar{a}^q a^q - q^{1/2} a^q \bar{a}^q = q^{-N^q/2}, \tag{3.6}$$

where N^q is the (Hermitian) *number operator* satisfying

$$[N^q, a^q] = a^q, \quad [N^q, \bar{a}^q] = -\bar{a}^q, \quad \text{with } N^q|0\rangle_q \equiv 0. \tag{3.7a, b, c}$$

This algebra is a deformation of the Heisenberg algebra, which is obtained in the limit $q \to 1$. (Note that the q-number operator N^q is now no longer the operator $a\bar{a}$ as in the Heisenberg case.) Orthonormal ket vectors corresponding to states of n q-quanta are given by:

$$|n\rangle_q \equiv ([n]!)^{-1/2}(a^q)^n|0\rangle_q, \tag{3.8}$$

$$\text{with:} \quad N^q|n\rangle_q = n|n\rangle_q. \tag{3.9}$$

In (3.8), the expression $[n]!$ denotes the q-number factorial defined in (2.6b).

In the constructions to follow, we will find it useful to introduce the q-exponential function defined by: ($\alpha \in \mathbb{C}$)

$$\exp_q(\alpha a^q)|0\rangle_q \equiv \sum_{n=0}^{\infty} \frac{(\alpha a^q)^n}{[n]!}|0\rangle_q, \tag{3.10}$$

which has the property (which follows from (3.6)) that:

$$\bar{a}^q \exp_q(\alpha a^q)|0\rangle_q = \alpha \exp_q(\alpha a^q)|0\rangle_q. \tag{3.11}$$

This q-exponential is a q-analog of the classical exponential function, although as such it is not unique; it is, however, invariant under $q \to q^{-1}$.

Remark. As in the case of the boson operators, one can equivalently regard the operator a^q as effecting multiplication by the complex variable z, but now $\bar{a}^q$ is represented not by differentiation, but by the finite difference operator D_z defined by

$$D_z f(z) = \frac{f(zq^{1/2}) - f(zq^{-1/2})}{z(q^{1/2} - q^{-1/2})},$$

for suitable functions $f(z)$. The operator D_z and q-extensions to classical functions are not new to quantum groups, having been studied, early in this century, by Jackson [12]. (In fact the theory of q-functions dates back to Heine a century earlier.) The subject has been developed extensively more recently by Andrews [2], Askey [3], Koornwinder [15] and Milne [22].

It is now easy to define a q-analog for the algebra of the generators of the quantum group $SU_q(2)$. In the language of q-boson operators, one defines a pair of *mutually commuting* q-bosons a_i^q and $\bar{a}_i^q$ for $i = 1, 2$. That is, for each i, a_i^q and $\bar{a}_i^q$ obey Eqs. (3.5)–(3.7) and, in addition, the relations:

$$\text{for} \quad i \neq j: \quad [a_i^q, a_j^q] = [\bar{a}_i^q, \bar{a}_j^q] = [a_i^q, \bar{a}_j^q] = 0. \tag{3.12}$$

The generators $\{E_{12}, E_{21}, H_1\}$ of SU_q are then realized by:

$$E_{12} = a_1^q \bar{a}_2^q, \quad E_{21} = a_2^q \bar{a}_1^q, \quad H_1 = \tfrac{1}{2}(N_1 - N_2). \tag{3.13a, b, c}$$

It can be verified, using (3.5)–(3.7), that these generators satisfy under commutation the quantum algebra of $SU_q(2)$, Eqs. (2.1)–(2.3), (there being no Serre relations for this case).

Let us consider now, for $SU_q(2)$, the set of basis vectors defined by:

$$|(m)\rangle_q \equiv \left| \begin{matrix} m_{12} & & 0 \\ & m_{11} & \end{matrix} \right\rangle_q \equiv ([m_{12} - m_{11}]![m_{11}]!)^{-1/2} (a_1^q)^{m_{11}} (a_2^q)^{m_{12} - m_{11}} |0\rangle_q, \tag{3.14}$$

for $0 \leqq m_{11} \leqq m_{12}$, $m_{12} = 0, 1, \ldots$.

Remarks. (1) The vector $|(m)\rangle_q$ in (3.14) is labelled using the *Gel'fand–Weyl pattern* (m) specialized to $SU(2)$. More generally, each basis vector – denoted $|(m)\rangle$ – in the vector space belonging to the unitary irrep $[m_{1n} \cdots m_{nn}]$ of $U(n)$ is labelled *uniquely* by a triangular pattern of integers:

$$(m) \equiv \begin{pmatrix} m_{1n} & & m_{2n} & & \cdots & & m_{nn} \\ & m_{1,n-1} & & m_{2,n-1} & \cdots & m_{n-1,n-1} & \\ & & & \cdots & & & \\ & & & m_{11} & & & \end{pmatrix} \tag{3.15}$$

with the integers m_{ij} in the pattern obeying the *betweenness constraints*:

$$m_{ij} \geqq m_{i,j-1} \geqq m_{i+1,j}. \tag{3.16}$$

This labelling incorporates, by means of the (geometric) betweenness constraints, the content of the *Weyl branching theorem for* $U(n)$; thus for example, the pattern (m) in (3.15) shows that the vector (m) belongs to the irrep $[m_{1n} \cdots m_{nn}]$ in $U(n)$, belongs to the $U(n-1)$ irrep $[m_{1,n-1} \cdots m_{n-1,n-1}]$, to the $U(n-2)$ irrep $[m_{1,n-2} \cdots m_{n-2,n-2}], \ldots$, and finally to the $U(1)$ irrep $[m_{11}]$. The Gel'fand–Weyl labelling is the analog in representation label space of a flag manifold labelling.

(2) It is important to remark for applications to the quantum group $U_q(n)$ that *it follows from the Lusztig–Rosso theorem* [18, 26] *that the Gel'fand–Weyl labels for* $U_q(n)$ *are invariant under deformation and hence properly label, uniquely, the basis vectors for the unitary irreps of the quantum group* $U_q(n)$, *and incorporate* (just as in Remark (1) above) *the Weyl branching theorem for the* $U_q(n)$ *quantum groups.*

(3) For the simple group $SU_q(n)$ the unitary irreps have $m_{nn} = 0$ in the Gel'fand–Weyl pattern.

It is easily verified [5] that the vectors in (3.14) under the action by the generators (3.13) form a basis for the unitary irrep $[m_{12}, 0]$ of $SU_q(2)$, each vector being distinguished by the eigenvalue $(m_{11} - m_{12})/2$ of H_1. The vector space $\mathbf{V}_{[m_{12},0]}$

spanned by the $(m_{12}+1)$ basis vectors $\left\{\left|\left(\begin{matrix} m_{12} & 0 \\ & m_{11} \end{matrix}\right)\right\rangle_q \middle| m_{11}=0,1,\ldots,m_{12}\right\}$ carries a unitary irrep of $SU_q(2)$ and *every* unitary irrep of $SU_q(2)$ is realized in this way. Let us denote the direct sum of these spaces by $\mathbf{V}$:

$$\mathbf{V} \equiv \sum_{m_{12}\in\mathbb{Z}^+} \oplus \mathbf{V}_{[m_{12},0]}. \tag{3.17}$$

Consider now the quantum group $U_q(2)$ and its unitary irreps. The q-boson realization of the algebra of $U_q(2)$ is given by (3.13a–c) with the adjunction of the $U_q(1)$ generator

$$H_2 = (N_1 + N_2), \tag{3.18}$$

or, equivalently, by adjoining N_1 and N_2 independently.

In order to construct all unitary irreps of $U_q(2)$ *via a q-boson realization* it is necessary, however, to use two independent (commuting) pairs of q-bosons, (a_1^q, a_2^q) and (b_1^q, b_2^q). Using the co-multiplication operation, one finds for the generators (*dropping the q-superscript label henceforth*):

$$E_{12} = a_1\bar{a}_2 \otimes q^{(N_1^b - N_2^b)/4} + q^{-(N_1^a - N_2^a)/4} \otimes b_1\bar{b}_2, \tag{3.19a}$$

$$E_{21} = a_2\bar{a}_1 \otimes q^{(N_1^b - N_2^b)/4} + q^{-(N_1^a - N_2^a)/4} \otimes b_2\bar{b}_1, \tag{3.19b}$$

$$N_1 = N_1^a \otimes \mathbf{1} + \mathbf{1} \otimes N_1^b, \tag{3.19c}$$

$$N_2 = N_2^a \otimes \mathbf{1} + \mathbf{1} \otimes N_2^b. \tag{3.19d}$$

where the superscripts a, b refer to the q-boson sets $\{a\}$, $\{b\}$ respectively. The generators in (3.19a–d) act on the space $\mathbf{V}\otimes\mathbf{V}$ and we seek irreducible subspaces of $\mathbf{V}\otimes\mathbf{V}$ in which the states carry Gel'fand–Weyl labels (m) of $U_q(2)$.

Remarkably, the desired basis vectors can be compactly written in the operator form [6]:

$$\left|\begin{matrix} m_{12} & m_{22} \\ & m_{11} \end{matrix}\right\rangle = M^{-1/2}(a_{12})^{m_{22}}(a_1)^{m_{11}-m_{22}}(a_2)^{m_{12}-m_{11}}|0\rangle, \tag{3.20a}$$

where

$$M = \frac{[m_{12}+1]!\,[m_{12}-m_{11}]!\,[m_{11}-m_{22}]!\,[m_{22}]!}{[m_{12}-m_{22}+1]!} \tag{3.20b}$$

and a_{12} is the operator defined by:

$$a_{12} = q^{(N_2^a + N_1^b + 1)/4} a_1 b_2 - q^{-(N_1^a + N_2^b + 1)/4} a_2 b_1. \tag{3.20c}$$

We remark that the operator a_{12} in (3.20c) is invariant under the action of $SU_q(2)$ generated by an appropriate subset of (3.19).

The proof (by direct verification), that the vectors (3.20) are indeed a realization of the carrier space of the irrep $[m_{12}, m_{22}]$ of $U_q(2)$ in an orthonormal Gel'fand–Weyl basis, proceeds by expanding the operator a_{12} so that (3.20) becomes a sum of homogeneous polynomials in the operators $\{a_i, b_i\}$. In carrying out this expansion we use essentially the q-binomial theorem, which can be expressed elegantly in terms of non-commuting coordinates. In the problem at hand, these coordinates are defined by

$$x = q^{(N_2^a + N_1^b + 1)/4} a_1 b_2, \quad y = q^{-(N_1^a + N_2^b + 1)/4} a_2 b_1 \tag{3.21a}$$

which satisfy

$$xy = qyx, \tag{3.21b}$$

and so can be regarded as *quantum coordinates* [21], [30].

Remark. *Taking the inner product of* (3.20a) *with product states* (*given by* (3.14) for $\{a\}$ and $\{b\}$) (separately) *defines the explicit Wigner–Clebsch–Gordan coefficient reducing* $\mathbf{V} \otimes \mathbf{V}$.

4. The Extension of the Borel–Weil Construction to $U_q(2)$

We are now in a position to develop the q-analog to the Borel–Weil (BW) construction for the simplest case, $U_q(2)$. Recall that the BW construction for the compact classical group $U(2)$, constructs irreps as holomorphic sections of a holomorphic homogeneous line bundle. The homogeneous space $U(2)/T$, with T defined to be the maximal torus is the manifold S^2 which can be made into a one-dimensional complex space, *in fact, a Kähler manifold. To every character of* T one associates a holomorphic homogeneous line bundle over G/T, which carries a G-action. The sections of this bundle are irreps of $U(2)$ with highest weight given by the character of T.

Before embarking on the q-extension of the BW construction, let us first carry out the BW construction using the explicit boson operators and coherent state techniques of the physics literature [23, 27, 25, 11, 17]. For the group $U(2)$, we consider the coset space $U(2)/U(1) \times U(1)$, which will be spanned by polynomials in the single boson, z, (so labelled to emphasize the analogy to the complex variable used in the BW construction). The eigenvectors of the torus group $T = U(1) \times U(1)$ (the fiber vector space) will be one-dimensional irrep vectors $\left|\begin{matrix} m_{12} & & m_{22} \\ & m_{22} & \end{matrix}\right\rangle$, which have *fixed* Gel'fand–Weyl labels (m), corresponding to the character χ_λ having the weights $\lambda = (m_{22}, m_{12})$. These weights result from the action of the two $U(1) \times U(1)$ generators $E_{11} \to m_{22}$ and $E_{22} \to m_{12}$ on the vector $\left|\begin{matrix} m_{12} & & m_{22} \\ & m_{22} & \end{matrix}\right\rangle$.

Note that this character corresponds to *minimal* weight, that is the Gel'fand–Weyl *pattern has the form* (* /), *where the line* / *denotes that the labels* m_{11} *and* m_{22} are identical ("minimally tied patterns"). The reason for using minimal weight ("anti-dominant") instead of maximal weight ("dominant") – as in BW – is technical, and results from the requirement that we *add* "angular momenta," and not *subtract* angular momenta as in the original BW procedure (with the nuisance of many confusing minus signs).

The BW construction augments the vector $\left|\begin{matrix} m_{12} & & m_{22} \\ & m_{22} & \end{matrix}\right\rangle$ by a function which satisfies:

$$f(gt^{-1}) = \chi_\lambda(t) f(g), \tag{4.1}$$

where $t \in T$ and χ_λ is the aforementioned character of T, with λ corresponding to the weight (m_{22}, m_{12}). This condition on f is the constraint that associates the principal bundle $U(2) \to U(2)/T$ to the line bundle determined by the character χ_λ.

Such functions are easily constructed and we will use the function given by:

$$f(g) \equiv \left\langle \begin{matrix} m_{12} & m_{22} \\ & m_{22} \end{matrix} \right| \exp(zE_{21}) \left| \begin{matrix} m_{12} & m_{22} \\ & m_{11} \end{matrix} \right\rangle, \tag{4.2}$$

which are matrix elements in $U(2)$ of the operator $g = \exp(zE_{21})$ in the universal enveloping algebra. Note that the state on the right in the bracket, Eq. (4.2) denotes an *arbitrary* eigenvector in the $U(2)$ irrep $[m_{12} \quad m_{22}]$. Since E_{21} is a *lowering* operator, the exponential series in (4.2) *terminates* so that $f(g)$ is actually a polynomial in the boson operator z, and hence $f(g)$ is clearly holomorphic in z.

The explicit normalized vectors $|m\rangle_{\mathrm{BW}}$ of the irrep $[m_{12} \quad m_{22}]$ are given by:

$$\left| \begin{matrix} m_{12} & m_{22} \\ & m_{11} \end{matrix} \right\rangle_{\mathrm{BW}} = \left\langle \begin{matrix} m_{12} & m_{22} \\ & m_{22} \end{matrix} \right| \exp(zE_{21}) \left| \begin{matrix} m_{12} & m_{22} \\ & m_{11} \end{matrix} \right\rangle \left| \begin{matrix} m_{12} & m_{22} \\ & m_{22} \end{matrix} \right\rangle. \tag{4.3}$$

There are many features to discuss for these irreps, (for example, the explicit G-action), but we will omit these for brevity since such results will be clear from specializing the q-analog results below.

Remark. There is one feature [17] of the vectors given in (4.3) which, however, must be discussed for clarity. The vector $\left| \begin{matrix} m_{12} & m_{22} \\ & m_{22} \end{matrix} \right\rangle$ on the right-hand side is a unique eigenvector of the group $U(1) \times U(1)$ with $E_{11} \to m_{22}$, $E_{22} \to m_{12}$, but it is also (by construction) defined to be a vector in the $U(2)$ irrep $[m_{12} \quad m_{22}]$ in order that the matrix element (4.2) be well-defined. The coefficient on the right-hand side in (4.3) (that is, $f(g)$ in (4.2)) is a function of z, but strictly speaking, since z (and $\bar{z}$) are *operators*, we must make explicit that this operator-valued function acts on the vacuum vector $|0\rangle$ to yield a new vector, called a *coherent* state in the physics literature [23]. To be precise and use the standard notation properly therefore one should re-write (4.3) in the form:

$$\left| \begin{matrix} m_{12} & m_{22} \\ & m_{11} \end{matrix} \right\rangle_{\mathrm{BW}} = (\mathbf{1} \otimes f(g)) \left(\left| \begin{matrix} m_{12} & m_{22} \\ & m_{22} \end{matrix} \right\rangle \otimes |0\rangle \right). \tag{4.4}$$

To proceed to the quantum group $U_q(2)$ is now straightforward. The q-exponential, $\exp_q$ in (3.10), replaces the ordinary exponential in (4.2), and the carrier space for all unitary irreps of $U_q(2)$ is then given by the vectors:

$$\left| \begin{matrix} m_{12} & m_{22} \\ & m_{11} \end{matrix} \right\rangle_{\text{q-BW}} = \left\langle \begin{matrix} m_{12} & m_{22} \\ & m_{22} \end{matrix} \right| \exp_q(zE_{21}^q) \left| \begin{matrix} m_{12} & m_{22} \\ & m_{11} \end{matrix} \right\rangle \left| \begin{matrix} m_{12} & m_{22} \\ & m_{22} \end{matrix} \right\rangle, \tag{4.5}$$

analogous to (4.3), or in the proper form of (4.4),

$$\left| \begin{matrix} m_{12} & m_{22} \\ & m_{11} \end{matrix} \right\rangle_{\text{q-BW}} = \left(\frac{[m_{12} - m_{22}]!}{[m_{12} - m_{11}]!} \right)^{1/2} \left(\mathbf{1} \otimes \frac{z^{m_{11} - m_{22}}}{\sqrt{[m_{11} - m_{22}]!}} \right) \left(\left| \begin{matrix} m_{12} & m_{22} \\ & m_{22} \end{matrix} \right\rangle \otimes |0\rangle \right), \tag{4.6}$$

where we have evaluated explicitly the matrix element $f(g)$.

The simplicity of the explicit q-BW result (4.6) for the vectors carrying all irreps of $U_q(2)$ – as contrasted with the standard q-boson realization for the same irreps as given by (3.20) – is striking.

It remains to give in detail the action of the group $U_q(2)$ on the basis vectors of (4.6). Let us express this action as a mapping Γ, from the four abstract $U_q(2)$ generators E_{12}, E_{21}, N_1 and N_2 to the induced G-action on (4.6) denoted by $\Gamma(g)$. That is, we define the mapping $\Gamma: g \to g' \equiv \Gamma(g)$, where g is one of the generators E_{12}, E_{21}, N_1 or N_2. It is easily shown that the explicit action on (4.6) requires that Γ have the form:

$$\Gamma(E_{21}) = \bar{z}, \qquad \Gamma(N_1) = N_1 + N, \tag{4.7a,b}$$

$$\Gamma(E_{12}) = z[N_2 - N_1 - N] \quad \Gamma(N_2) = N_2 - N, \tag{4.7c,d}$$

where the q-boson creation (destruction) operator is z (respectively $\bar{z}$) acting on $|0\rangle$, N is the number operator as given by (3.9), and N_1, N_2 denote the $U_q(1) \times U_q(1)$ generators which act on the fiber vector $\left|\begin{matrix} m_{12} & & m_{22} \\ & m_{22} & \end{matrix}\right\rangle$ in (4.6). (To interpret $\Gamma(E_{12})$, recall that (2.28) extends the q-number notation to diagonal operators.)

Remark. The notation used in (4.7) for the q-BW generators is an abuse of the proper notation, but is, however, clearer and certainly more convenient. The reason is that the generator $\Gamma(E_{12})$ must be "taken apart" in order to write it in the proper co-multiplicative tensor product form. Written properly the generators in (4.7) have the form:

$$\Gamma(E_{12}) = (1 \otimes z)([N_2 - N_1] \otimes q^{N/2} - q^{(N_2 - N_1)/2} \otimes [N]), \tag{4.8a}$$

$$\Gamma(E_{21}) = 1 \otimes \bar{z}, \tag{4.8b}$$

$$\Gamma(N_1) = N_1 \otimes 1 + 1 \otimes N, \tag{4.8c}$$

$$\Gamma(N_2) = N_2 \otimes 1 - 1 \otimes N. \tag{4.8d}$$

In this form the q-BW generators are not only more complicated in appearance, and less understandable, but are seemingly not invariant under $q \to q^{-1}$, unlike (4.7).

Let us sum up the results, obtained above, in the form of a lemma.

Lemma (4.9). *The map $\Gamma: g \to g' \equiv \Gamma(g)$, given in (4.7), where g is one of the four generators E_{12}, E_{21}, N_1, N_2 of $U_q(2)$, is an isomorphism of quantum group algebras.*

Proof. Direct verification. ■

Remarks. (1) The factor defined by $K_q^2(m) = [m_{12} - m_{22}]!/[m_{12} - m_{11}]!$ in (4.6) has the significance of defining the metric in the Kähler manifold (S^2 in the classical case). This construction (for the q-analog) case thus allows one to define the metric for the analog to a "q-Kähler" manifold for the quantum group $U_q(2)$.

(2) It is noteworthy that in this q-BW construction, (4.6), all irrep vectors of $U_q(2)$ appear as monomials, involving only a *single* q-boson, in sharp contrast to the *two* q-bosons involved in (3.14) for $SU_q(2)$, or the *four* bosons in (3.20) for $U_q(2)$. This suggests that the recursive construction of all unitary irreps of $U_q(n)$ from $U_q(n-1) \times U(1)$ can be achieved with only $n-1$ q-bosons; this is indeed correct as developed in the next sections. As noted in the introduction, such induction by stages is equivalent to induction from a representation of the maximal torus. It

follows that to induce $U_q(n)$ from a one-dimensional torus representation requires $\frac{n(n-1)}{2}$ q-bosons in the q-BW construction versus n^2 q-bosons in the standard q-boson approach.

5. The q-BW Generalization for $U_q(n)$

The q-BW generalization, to be developed in this section, corresponds – for the classical group $U(n)$ – to constructing holomorphic sections of an associated bundle whose fibers carry irreps of the sub-group $U(n-1) \times U(1)$, in contrast (as mentioned in Sect. 1) to the standard BW construction where the fibres carry one-dimensional irreps of the maximal torus. The base manifold is accordingly $U(n)/(U(n-1) \times U(1))$ having $(n-1)$ complex dimensions. Such a procedure is well adapted to a recursive approach to the unitary group $U(n)$ since at each stage in the construction one needs only explicit results for a group known, by construction, from the previous step, that is $U(n-1)$ for $U(n)$, $U(n-2)$ for $U(n-1), \ldots$. For $U(2)$ itself one has, of course, the BW construction. This approach is well known in the physics literature [11, 17], in contrast to the quantum group analog where the construction has not been hitherto obtained. It is clear that from this recursive approach the BW result itself is easily obtained by substituting explicit prior results for all recursive steps.

Our q-BW construction begins by assuming (as the recursion hypothesis) the existence and knowledge of a realization of the quantum group generators of $U_q(n-1)$ and explicit basis vectors carrying any given unitary irrep $[m_{1,n-1}, m_{2,n-1}, \ldots, m_{n-1,n-1}]$ of $U_q(n-1)$. We augment this $U_q(n-1)$ space by a tensor product with the one-dimensional vector space $|(\nu)\rangle$ carrying an irrep of the quantum group $U_q(1)$, generated by E_{nn}. The fibers then carry an irrep of $U_q(n-1) \times U_q(1)$ and belong to the vector space spanned by the vectors $|(\mu)\rangle \otimes |(\nu)\rangle$, (where $|(\mu)\rangle \in [m_{1,n-1} \cdots m_{n-1,n-1}]$ and $|(\nu)\rangle \in [m_{nn}]$) with an action on this vector space by the generators, $E_{ij} \otimes \mathbf{1}$ and $\mathbf{1} \otimes E_{nn}$ belonging to the quantum group $U_q(n-1) \times U_q(1)$.

Just as in our defining example ($U_q(2)$ in Sect. 4) where it proved convenient in constructing the function $f(g)$ (Eq. 4.2) to embed the one-dimensional fiber vector carrying an irrep of $U_q(1) \times U_q(1)$, in the larger space of an irrep of $U_q(2)$, so too is it useful to embed the vectors carrying irreps of the quantum group $U_q(n-1) \times U_q(1)$ in the larger space of an irrep of $U_q(n)$. Accordingly we assume that the vector space of the fibers carries an irrep $[m_{1n} m_{2n} \cdots m_{nn}]$ of $U_q(n)$ transforming irreducibly under the q-group $U_q(n)$ whose n^2 generators are denoted $\{E_{ij}\}, 1 \leqq i, j \leqq n$. The fiber vectors carrying irreps of $U_q(n-1) \times U_q(1)$ are a subset of this $U_q(n)$ irrep, transforming under the $U_q(n-1) \times U_q(1)$ subset of the generators $\{E_{ij}\}$.

It is essential to point out that the associated bundle achieved in this q-BW construction will actually involve only that subset of $U_q(n)$ irrep vectors which belong to a *single* $U_q(n-1) \times U_q(1)$ irrep, exactly as the vectors in Eq. (4.6), for the q-BW construction for $U_q(2)$, explicitly are restricted to a single one-dimensional irrep of $U_q(1) \times U_q(1)$ for the vector space of the fiber.

The base manifold of the bundle has complex dimension $n-1$ and correspondingly $n-1$ *commuting* q-boson coordinates denoted by $\{z_1, z_2, \ldots, z_{n-1}\}$. We define a $U_q(n-1)$ quantum group action on this base manifold by the realization:

$$E_{ij} \to z_i \bar{z}_j, \tag{5.1a}$$

$$E_{ii} \to N_i \equiv \text{the } i^{th} \text{ number operator, with } i,j = 1,2,\ldots,n-1. \tag{5.1b}$$

Here $\{z_i, \bar{z}_i\}$ denotes, for $i = 1,2,\ldots,n-1$, the $n-1$ commuting q-bosons obeying, for each i, Eqs. (3.5–7).

The vector space $\mathbf{V}$ of the base manifold carrying representations of $U_q(n-1)$ is the space of all polynomials $\mathscr{P}$ in the $n-1$ q-bosons $\{z_i\}$ terminated by the vacuum ket $|0\rangle$, where, for all i, $\bar{z}_i|0\rangle = 0$; that is:

$$\mathbf{V} \equiv \{\mathscr{P}(\{z_i\})|0\rangle\}. \tag{5.2}$$

These two realizations of $U_q(n-1)$ – one generated by $\{E_{ij}\}$, the other by $\{z_i\bar{z}_j, N_i\}$ acting on the base manifold – admit a co-product carrying tensor product representations initially of $U_q(n-1)$, which can then be extended to $U_q(n-1) \times U_q(1)$, by adjoining the generator E_{nn} of $U_q(1)$.

Explicitly we have as generators of this direct product quantum group, $U_q(n-1) \times U_q(1)$, the realization:

$$\Gamma(E_{ij}) \equiv E_{ij} \otimes q^{((N_i - N_j)/4)} + q^{-((E_{ii} - E_{jj})/4)} \otimes z_i \bar{z}_j, \quad i<j, \tag{5.3a}$$

$$\Gamma(E_{ij}) = E_{ij} \otimes q^{((N_j - N_i)/4)} + q^{-((E_{jj} - E_{ii})/4)} \otimes z_i \bar{z}_j, \quad i>j, \tag{5.3b}$$

$$\Gamma(E_{ii}) \equiv E_{ii} \otimes \mathbf{1} + \mathbf{1} \otimes N_i, \tag{5.3c}$$

where i,j range over $1,2,\ldots,n-1$, and for the n^{th} generator:

$$\Gamma(E_{nn}) \equiv E_{nn} \otimes \mathbf{1} - \sum_{i=1}^{N-1} \mathbf{1} \otimes N_i. \tag{5.3d}$$

We have labelled these generators of $U_q(n-1) \times U_q(1)$ as $\{\Gamma(E_{ij})\}$ to distinguish the action of these generators as *the left action* on the bundle.

The problem posed now for completing the q-BW construction is to extend this left action by generators of the sub-group $U_q(n-1) \times U_q(1)$ of $U_q(n)$ to an action realizing the group $U_q(n)$ itself. Phrased differently, the problem now to be resolved is the imposition of constraints analogous to (4.1) and (4.2) such that one obtains an associated bundle from the principal bundle. This step requires the introduction of an invariant operator $\mathcal{O}$, the analog of zE_{21} in (4.3).

Definition. *Define the operator $\mathcal{O}$ by:*

$$\mathcal{O} \equiv \sum_{i=1}^{n-1} \left(q^{-\frac{1}{4}\sum_{j=1}^{i-1} E_{jj} + \frac{1}{4}\sum_{j=i+1}^{n-1} E_{jj}} E_{ni} \otimes q^{\frac{1}{4}\sum_{j=1}^{i-1} N_j - \frac{1}{4}\sum_{j=i+1}^{n-1} N_j} z_i \right). \tag{5.4}$$

Lemma (5.5). *The operator $\mathcal{O}$ in (5.4) is invariant under the commutator action of the $U_q(n-1) \times U_q(1)$ generators $\{\Gamma(E_{ij})\}$ defined in (5.3).*

Proof. The invariance under the generators of the Chevalley–Weyl basis $\Gamma(E_{ii})$, $(i = 1,\ldots,n)$ and for $\Gamma(E_{i,i-1})$, $\Gamma(E_{i-1,i})$ with $i = 1,\ldots,n-1$, is verified directly. Defining the elements of $\Gamma(\cdot)$ not in this set by the q-commutator: $[A,B]_q \equiv AB - q^{-1/2}BA$ – which can be shown to satisfy the Serre relations – extends the invariance to all of $\Gamma(\cdot)$ defined in (5.3). ■

The construction which we have carried out so far in this section is a tensor product of fiber vectors belonging to irreps of $U_q(n)$ tensored with vectors in the base manifold $\mathbf{V}$. Consider now the sub-set $\mathbf{W}$ of these tensor product vectors defined by:

$$\mathbf{W} \equiv \sum_{(m)} \oplus (|(m)\rangle \otimes |0\rangle), \tag{5.6}$$

where (m) is a Gel'fand–Weyl pattern belonging to $U_q(n)$. Under the action of the generators $\{E_{ij}\} \otimes \mathbf{1}$, with $i, j = 1, 2, \ldots, n$, the space $\mathbf{W}$ splits into a direct sum of $U_q(n)$ irreps labelled by Young frames $[m_{1n} m_{2n} \cdots m_{nn}]$, with individual vectors in these irreps labelled by the Gel'fand–Weyl patterns (m).

Consider next the vector space $\mathbf{W}(w)$ defined by:

$$\mathbf{W}(w) \equiv (\mathcal{O})^w \mathbf{W}, \tag{5.7}$$

where $(\mathcal{O})^w$ denotes the invariant operator $\mathcal{O}$ multiplied by itself w times, with w a non-negative integer.

It is easily shown that:

Lemma (5.8). *If a vector* $\mathbf{v}$ *in* $\mathbf{W}$ *is labelled under the action of* $E_{ij} \otimes \mathbf{1}$ *by the* $U_q(n)$ *labels* $(m) = (m_{ij})$ *with* $1 \leqq i, j \leqq n$ *– see (3.15–3.16) – then the vector* $\mathcal{O}^w \mathbf{v}$ *carries the same* $U_q(n-1) \times U_q(1)$ *labels* (m_{ij}) *with* $1 \leqq i, j \leqq n-1$ *under the action of the generators* $\Gamma(E_{ij})$ *defined in (5.3a–d).*

Proof. This is clear from Lemma (5.5) since the action of $\Gamma(E_{ij})$ commutes with the invariant $\mathcal{O}$ and $\Gamma(E_{ij}) \to E_{ij}$ on $\mathbf{W}$. ■

In fact one can state much more about the vectors in $\mathbf{W}(w)$.

Lemma (5.9). *Let a sub-space of* $\mathbf{W}(w)$ *be defined as the vectors* $(\mathcal{O})^w \mathbf{v}$, *where* $\mathbf{v}$ *belongs to the* $U_q(n)$ *irrep* $[m_{1n} \cdots m_{nn}]$ *of* $\mathbf{W}$. *Under the action of* $E_{ij} \otimes \mathbf{1}$, *this subspace splits into a direct sum of* $U_q(n-1) \times U_q(1)$ *irreps with labels* $[m'_{1,n-1} \cdots m'_{n-1,n-1}] \times [m'_{1n}]$, *where* $\sum_{i=1}^{n-1} (m_{i,n-1} - m'_{i,n-1}) = w, m'_{1n} = \sum_{i=1}^{n} m_{in} - \sum_{i=1}^{n-1} m_{i,n-1} + w$, *and the* $U_q(n-1)$ *irrep labels* $[m'_{1,n-1} \cdots m'_{n-1,n-1}]$ *are compatible, as a sub-Gel'fand–Weyl pattern, with the* $U_q(n)$ *labels* $[m_{1n}, m_{2n}, \ldots, m_{nn}]$.

Proof. This follows from the fact that the expansion of $(\mathcal{O})^w$ as a sum of monomial products in the q-boson operators $\{z_1, \ldots, z_{n-1}\}$ is homogeneous of degree w. This implies that the action of $\Gamma(E_{nn})$, on a given term in the expansion of $\mathcal{O}^w \mathbf{v}$, yields:

$$\Gamma(E_{nn}) = E_{nn} \otimes \mathbf{1} - \sum_{i=1}^{n-1} N_i \to \left(\sum_{i=1}^{n} m_{in} - \sum_{i=1}^{n-1} m'_{i,n-1} \right) - w. \tag{5.10}$$

However, from Lemma (5.8) we know that:

$$\Gamma(E_{nn}) \to \sum_{i=1}^{n} m_{in} - \sum_{i=1}^{n-1} m_{i,n-1}. \tag{5.11}$$

Hence, taking the difference, we find:

$$\sum_{i=1}^{n-1} (m_{i,n-1} - m'_{i,n-1}) = w, \tag{5.12}$$

as asserted. This also implies that the label for the $U_q(1)$ vector is:

$$m'_{1n} = \sum_{i=1}^{n} m_{in} - \sum_{i=1}^{n-1} m_{i,n-1} + w, \tag{5.13}$$

since this is the eigenvalue of $E_{nn} \otimes \mathbf{1}$. Since the vectors in $U_q(n-1) \times U_q(1)$ are generated from the irrep $[m_{1n} \cdots m_{nn}]$ of $U_q(n)$ by the *lowering operators* $E_{ni} \otimes \mathbf{1}$ in $\mathcal{O}$, it follows that the irrep labels $[m'_{1,n-1} \cdots m'_{n-1,n-1}]$ of $U_q(n-1)$ are compatible as sub-Gel'fand–Weyl pattern labels. ■

The restriction to a specified value of w in $\mathbf{W}(w)$ can be removed. Using the q-exponential defined in (3.10) we can now construct the vector space $\mathbf{W}(e)$:

$$\mathbf{W}(e) \equiv \exp_q(\mathcal{O})\mathbf{W}. \tag{5.14}$$

Just as in our defining example ($U_q(2)$ in Sect. 4) the space $\mathbf{W}(e)$ is too large and we must project this space onto the $U_q(n-1) \times U_q(1)$ sub-space that is anti-dominant in $U_q(n)$. For this purpose we use the projection operator $\mathbf{P}$ defined by:

$$\mathbf{P} \equiv \left(\sum_{\substack{m_{1n} \\ (\mu)}} \left| \begin{matrix} m_{1n} m_{2n} \cdots m_{nn} \\ m_{2n} \cdots m_{nn} \\ (\mu) \end{matrix} \right\rangle \left\langle \begin{matrix} m_{1n} m_{2n} \cdots m_{nn} \\ m_{2n} \cdots m_{nn} \\ (\mu) \end{matrix} \right| \right) \otimes \mathbf{1}. \tag{5.15}$$

We assert, and will prove below, that the vector space (call it $\mathbf{W}_{\text{q-BW}}$) carrying all unitary irreps of $U_q(n)$ in this q-BW construction is the vector space given by:

$$\mathbf{W}_{\text{q-BW}} \equiv \mathbf{P}\mathbf{W}(e). \tag{5.16}$$

Specializing to a generic vector in the irrep $[m_{1n} m_{2n} \cdots m_{nn}]$ having the Gel'fand–Weyl labels (m_{ij}), $1 \leqq i, j \leqq n$, we find:

$$|(m)\rangle_{\text{q-BW}} = \sum_{\substack{(\mu) \in U_q(n-1) \times U_q(1) \\ (\mu) \subsetneq [m]}} \langle (\mu) | \exp_q(\mathcal{O}) | (m) \rangle \, |(\mu)\rangle \otimes |0\rangle, \tag{5.17}$$

where the meaning of $(\mu) \subsetneq [m]$ is that the pattern (μ) "fits into" (injects into) the Gel'fand–Weyl patterns of the $U_q(n)$ irrep $[m_{1n} \cdots m_{nn}]$ as:

$$\begin{pmatrix} m_{1n} & & m_{2n} & & m_{3n} & & \cdots & & m_{n-1n} & & m_{nn} \\ & m_{2n} & & m_{3n} & & & \cdots & m_{n-1n} & & m_{nn} & \\ & & \mu_{1,n-2} & & \mu_{2,n-2} & & \cdots & \mu_{n-2,n-2} & & & \\ & & & & & & \cdots & & & & \\ & & & & & & \mu_{11} & & & & \end{pmatrix},$$

so that the vectors $|(\mu)\rangle$ carry the $U_q(n-1) \times U_q(1)$ irrep labels:

$$|(\mu)\rangle = \left| \begin{pmatrix} m_{2n} & m_{3n} & \cdots & m_{nn} \\ & \mu_{1,n-2} & \cdots & \mu_{n-2,n-2} \\ & & \cdots & \\ & & \mu_{11} & \end{pmatrix} \right\rangle \otimes \left| (m_{1n}) \right\rangle.$$

The constraint expressed by the projection operator $\mathbf{P}$ is exactly the geometric constraint imposed by the construction of an associated bundle from the principal bundle. In fact, one sees that the matrix element in (5.17) is the precise analog of the constraint in the $U_q(2)$ construction, Eq. (4.1), imposed by the $U_q(2)$ matrix element (4.2).

To be fully explicit as to the meaning of Eq. (5.17), let us note that the sum in Eq. (5.17) is a *finite* sum (since $\mathcal{O}$, cf. [5.4], consists of lowering operators E_{ni} only), and that the matrix element $\langle(\mu)|\exp_q(\mathcal{O})|(m)\rangle$ is therefore a *polynomial*, with numerical coefficients $\in \mathbb{C}[q, q^{-1}]$, over powers of the q-boson operators $\{z_1, \ldots, z_{n-1}\}$. This *operator-valued polynomial* acts on the vacuum ket $|0\rangle$ to produce an orthonormalized vector in the carrier space of $U_q(n-1)$ with an action by the generators $E_{ij} \to z_i \bar{z}_j$, $E_{ii} \to N_i$ $(1 \leqq i, j \leqq n-1)$. The vectors $|(\mu)\rangle$, as discussed above, are orthonormal vectors in the irrep $[\mu_{1n-1}, \mu_{2n-1}, \ldots, \mu_{n-1,n-1}] \times [\mu_{1n}]$ of $U_q(n-1) \times U_q(1)$, with an action by the generators E_{ij}. (As also remarked we inject this (μ) pattern into $U_q(n)$ by identifying (for $U_q(n-1)$) $\mu_{i,n-1} = m_{i+1n}$, (with $i = 1, 2, \ldots, n-1$) and for $U_q(1)\mu = m_{1n}$.)

To clarify the bundle nature of (5.17) let us note that since the fiber is the vector space $\{|\mu\rangle\}$ and the base manifold is the space $\{\mathscr{P}(z)|0\rangle\}$ we can identify the left-action of the $U_q(n-1) \times U_q(1)$ generators as action by the generators $\Gamma(E_{ij})$ given in Eqs. (5.3a–d). (We will shortly extend this action to all of $U_q(n)$.) The right action on the bundle is defined by the generators $\{E_{ij}\}$ of $U_q(n)$ now acting directly on the vectors $|(m)\rangle$ *inside the matrix element* – yielding $\sum_{(m')} \langle(m')|g|(m)\rangle|(m')\rangle$, where the $\langle\cdots\rangle$ are matrix elements of the given generator $g \in \{E_{ij}\}$ – and hence the right action on Eq. (5.17) realizes by definition the unitary irrep $[m_{1n} \cdots m_{nn}]$ of $U_q(n)$. (We remark again that these two actions do not commute.)

What remains to be proved is that the left-action $\Gamma(E_{ij})$ extends from $U_q(n-1) \times U_q(1)$ to $U_q(n)$, that the map Γ is an isomorphism of q-Lie algebras, and that (5.17) defines orthonormal vectors carrying the irrep $[m]$ with Gel'fand–Weyl labels (m) under the action of the $U_q(n)$ generators $\Gamma(E_{ij})$.

6. The Isomorphism of q-Lie Algebras

The invariant operator $\mathcal{O}$, Eq. (5.4), has several properties which will be of importance for our construction. As written, (5.4), the invariant operator $\mathcal{O}$ is the sum of $n-1$ monomials, x_i, and may be written in the form:

$$\mathcal{O} \equiv \sum_{i=1}^{n-1} x_i, \tag{6.1}$$

with

$$x_i \equiv q^{-\frac{1}{4}\sum_{j=1}^{i-1} E_{jj} + \frac{1}{4}\sum_{j=i+1}^{n-1} E_{jj}} E_{ni} \otimes q^{\frac{1}{4}\sum_{j=1}^{i-1} N_j - \frac{1}{4}\sum_{j=i+1}^{n-1} N_j} z_i. \tag{6.2}$$

We state the most significant fact about the operators $\{x_i\}$ in the form of a lemma.

Lemma (6.3). *The operators $x_1, x_2, \ldots, x_{n-1}$ obey the q-commutation rule:*

$$x_i x_j = q^{-1} x_j x_i, \quad \textit{for} \quad i < j. \tag{6.4}$$

Accordingly the operators $\{x_i\}$ are an operator realization of the quantum coordinates of an $n-1$ hyperplane.

Proof. Direct verification. ■

It is useful to introduce a second set of operator coordinates, the set $\{X_i\}$, which have the definition:

$$X_i \equiv q^{-\frac{1}{4}\sum_{j=1}^{i-1} H_j + \frac{1}{4}\sum_{j=i+1}^{n-1} H_j} E_{ni} \otimes q^{-\frac{1}{4}\sum_{j=1}^{i-1} N_j + \frac{1}{4}\sum_{j=i+1}^{n-1} N_j} z_i. \tag{6.5}$$

Lemma (6.6). *The operators $\{X_i\}$ have the property that $X_iX_j = X_jX_i$ $\forall i,j = 1,\ldots,n-1$.*

Proof. Direct verification. ■

We are now in a position to state a very useful identity for the q-exponential function of Eq. (3.10).

Lemma (6.7). *Using the operators $\{x_i\}$ and $\{X_i\}$ of Eqs. (6.1) and (6.4), respectively, the q-exponential identity:*

$$\exp_q(\mathcal{O})|0\rangle \equiv \exp_q\left(\sum_{i=1}^{n-1} x_i\right)|0\rangle = \sum_{i=1}^{n-1} (\exp_q(X_i))|0\rangle, \tag{6.8}$$

is valid. Since the $\{X_i\}$ commute, the product on the right-hand side of (6.7) may be taken in any order.

Proof. Direct verification. Note, however, that this identity is *not* valid if the terminating vacuum ket $|0\rangle$ is absent. ■

These ancillary results have an immediate application to the determination of the left-action $U_q(n)$ generators denoted by $\Gamma(E_{ni})$.

Lemma (6.9). *The left-action generators corresponding to the abstract $U_q(n)$ generators E_{ni} are given by:*

$$E_{ni} \to \Gamma(E_{ni}) = q^{\frac{1}{4}\left(\sum_{j=1}^{i-1} E_{jj} - \sum_{j=i+1}^{n-1} E_{jj}\right)} \otimes \bar{z}_i, \tag{6.10}$$

for $i = 1,2,\ldots,n-1$.

Proof. Consider the action of $\Gamma(E_{ni})$ on the vector $|(m)\rangle_{\text{q-BW}}$ given in (5.17). The operators E_{jj} in the q-factor of $\Gamma(E_{ni})$ have an action on the ket vectors $|(\mu)\rangle$ in the tensor product, $|(\mu)\rangle \otimes |0\rangle$, giving the eigenvalues $E_{jj} \to \sum_{i=1}^{j} \mu_{ij} - \sum_{i=1}^{j-1} \mu_{i,j-1}$. These same eigenvalues would be produced by the action of the E_{jj} acting to the left immediately *inside* the matrix element $\langle(\mu)|\exp_q(\mathcal{O})|(m)\rangle$. Since the $\bar{z}_i$ act on the $\exp_q(\mathcal{O})$ inside the matrix element as well, we can therefore take $\Gamma(E_{ni})$ to act *inside* the matrix element. To determine the action of $\Gamma(E_{ni})$ on $\exp_q(\mathcal{O})$ we note that this operator is terminated on the right by the vacuum ket, so that we may use Lemma (6.7).

One next observes that $\Gamma(E_{ni})$ commutes with all X_j for $j \neq i$. Hence we need only calculate that:

$$\Gamma(E_{ni})\exp_q(X_i)|0\rangle = \exp_q(X_i)E_{ni}|0\rangle \tag{6.11}$$

as can be shown directly.

Thus we find that:

$$\Gamma(E_{ni})|(m)\rangle_{\text{q-BW}} = \sum_{\substack{(\mu)\in U_q(n-1)\times U_q(1) \\ (\mu) \lhd [m]}} \langle(\mu)|\Gamma(E_{ni})\exp_q(\mathcal{O})|(m)\rangle\,|(\mu)\rangle \otimes |0\rangle$$

$$= \sum_{\substack{(\mu)\in U_q(n-1)\times U_q(1)\\ (\mu)\lhd[m]}} \langle(\mu)|\Gamma(E_{ni})\prod_{i=1}^{n-1}\exp_q(X_i)|(m)\rangle|(\mu)\rangle\otimes|0\rangle,$$

using (6.8)

$$= \sum_{\substack{(\mu)\in U_q(n-1)\times U_q(1)\\ (\mu)\lhd[m]}} \langle(\mu)|\prod_{i=1}^{n-1}\exp_q(X_i)E_{ni}|(m)\rangle|\mu\rangle\otimes|0\rangle, \tag{6.12}$$

using (6.11).

By definition, however, the matrix elements $\langle\cdot|\cdot|\cdot\rangle$ of the q-Lie algebra generators E_{ni} acting on the standard Gel'fand–Weyl (abstract) basis vectors $\{|(m)\rangle\}$ are given as:

$$E_{ni}|(m)\rangle = \sum_{(m')}\langle(m')|E_{ni}|(m)\rangle|(m')\rangle, \tag{6.13}$$

so that Eq. (6.12) becomes

$$\begin{aligned}&\Gamma(E_{ni})|(m)\rangle_{\text{q-BW}}\\ &= \sum_{(m')}\langle(m')|E_{ni}|(m)\rangle \sum_{\substack{(\mu)\in U_q(n-1)\times U_q(1)\\ (\mu)\lhd[m]}} \langle(\mu)|\prod_{i=1}^{n-1}\exp_q(X_i)|(m')\rangle|(\mu)\rangle\otimes|0\rangle,\\ &= \sum_{(m')}\langle(m')|E_{ni}|(m)\rangle|(m')\rangle_{\text{q-BW}},\end{aligned} \tag{6.14}$$

where in the last step we used Eq. (6.8) once again. Thus the action of $\Gamma(E_{ni})$ on the q-Borel–Weil basis yields precisely the same action as the (abstract) generators E_{ni} on the (abstract) standard Gel'fand–Weyl basis. ■

The determination of the generators $\Gamma(E_{in})$ is – in contrast to Lemma (6.9) – a *much* more difficult task. The reason for this difficulty lies in the problems associated with the existence of the Serre relations. To calculate one needs to find an explicit form for the generators E_{ij} of $U_q(n)$ including those which are not in the set $\{E_{i,i+1}, E_{i,i-1}, E_{ii}\}$. The choice of any explicit form must be required to be *stable* under co-multiplication, and since $\Gamma(g)$ and g ($g\in$generators of $U_q(n)$) are to have the *same* abstract matrix elements, this necessary requirement can easily lead to contradictions or inconsistencies for a particular (incorrect) explicit form. (Of course one could always work directly with the generic matrix-elements, but this is a task of daunting complexity.)

The procedure we will use is based on the following observation:

Lemma (6.15). *The generators $\Gamma(E_{ni})$, $i = 1, 2, \ldots, n-1$, defined in Eq. (6.10) all commute.*

Proof. Direct calculation. ■

Moreover, we find that the realization by $\Gamma(E_{ni})$ in Eq. (6.10) and $\Gamma(E_{i,i\pm 1})$ in Eq. (5.3a, b) have the property:

Lemma (6.16). *The generators $\Gamma(E_{ni})$, Eq. (6.10), and $\Gamma(E_{i,i\pm 1})$, Eqs. (5.3a, b) satisfy the relations*

$$\Gamma(E_{ni})\Gamma(E_{i,i\pm 1}) - q^{\mp 1/2}\Gamma(E_{i,i\pm 1})\Gamma(E_{ni}) = q^{\pm 1/2\Gamma(E_{ii})}\Gamma(E_{n,i\pm 1}). \tag{6.17}$$

Proof. Direct calculation. ■

Remark. These relations are quite special to the present construction and are not necessarily stable under co-multiplication. However, since $\Gamma(g)$ and g, where $g \in U_q(n)$ generators, have the *same* generic matrix elements, these particular relations are, in fact, stable. One must be very cautious with such relations; for example, it is easily shown that using $E_{i,j}$, with $j \neq i+1$ in Eq. (6.17) leads to a contradiction.

The construction $\Gamma: g \to \Gamma(g)$, $g \in \mathrm{Lie}(U_q(n))$, has the implication that the matrix elements of both g and $\Gamma(g)$, evaluated on their associated irrep bases, are not only isomorphic, but *identical*.

Since we explicitly assume that the generators $\{E_{ij}\}$ of $U_q(n)$ have the standard matrix elements of the q-Gel'fand–Weil basis, it follows that these generators possess an involution: $g \to g^\dagger$ such that $E_{ij}^\dagger = E_{ji}$. It follows that $\Gamma(g)$ also possesses the same involution, defined via explicit matrix elements. We conclude that:

Lemma (6.18). *The generators $\Gamma(E_{in})$, $i = 1, 2, \ldots, n-1$, all commute.*

Lemma (6.19). *The generators $\Gamma(E_{in})$, and $\Gamma(E_{i\pm 1,i})$, satisfy the relations*

$$\Gamma(E_{i\pm 1,i})\Gamma(E_{in}) - q^{\pm 1/2}\Gamma(E_{in})\Gamma(E_{i\pm 1,i}) = q^{\mp 1/2\Gamma(E_{ii})}\Gamma(E_{i\pm 1,n}). \tag{6.20}$$

We are now in a position where the explicit form of $\Gamma(E_{in})$ can be given. The procedure is to calculate, using only those quadratic commutation relations of the $U_q(n)$ Lie algebra which are independent of the Serre relations, to determine that component of $\Gamma(E_{in})$ involving the operator $z_i\left[E_{nn} - E_{ii} - \sum_{k=1}^{n-1} N_k\right]$. This is a straightforward calculation (the result of which is given below.) One then uses Lemma (6.19) and the known operators $\Gamma(E_{i\pm 1,i})$ to determine all remaining components of $\Gamma(E_{in})$.

The result of this calculation is:

Lemma (6.21). *The left-action generators $\Gamma(E_{in})$, $i = 1.2, \ldots, n-1$, corresponding to the abstract $U_q(n)$ generators E_{in} by the mapping Γ are given by:*

$$\Gamma(E_{in}) = q^{\alpha_{ii}} z_i\left[E_{nn} - E_{ii} - \sum_{k=1}^{n-1} N_k\right] - \sum_{\substack{j=1\\ j\neq i}}^{n-1} q^{\alpha_{ij}} z_j E_{ij}, \tag{6.22}$$

where $i = 1, 2, \ldots, n-1$, and:

$$q^{\alpha_{ij}} \equiv q^{-\frac{1}{4}\left(\sum_{k=1}^{j-1} E_{kk} - \sum_{k=j+1}^{n-1} E_{kk}\right)} \cdot q^{\left(\substack{\pm\\ 0}\right)\left(\frac{1}{2}\left(E_{nn} - \sum_{k=1}^{n-1} N_k\right) + \frac{3}{4}\right)}$$
$$\cdot q^{\left(\substack{\pm\\ 0}\right)\left(\frac{1}{4}(N_i + N_j)\right)} \cdot q^{\left(\substack{\pm\\ 0}\right)\frac{1}{2}\left(\sum_\tau N_k\right)},$$

with

$$\left(\substack{+\\ -\\ 0}\right) = \begin{cases} + & \textit{for} \quad i<j \\ - & \textit{for} \quad i>j, \\ 0 & \textit{for} \quad i=j \end{cases}$$

*and the sum $\sum_\tau N_k$ in the last term is over k-*values lying strictly inside the interval *i to j (if $i<j$) or j to i (if $i>j$).*

Proof. By construction, the generators $\Gamma(E_{in})$ validate the map $\Gamma: g \to \Gamma(g)$, for which the matrix elements of both g and $\Gamma(g)$ (on their associated irrep bases) are identical. Since the generators $g \in U_q(n)$ certainly obey the $U_q(n)$ algebra, including all Serre relations, this must be true of the generators $\Gamma(E_{in})$, which establishes the Lemma. ■

Remark. It is all too easy in a proof such as the one above to make subtle errors, so that it is reassuring to know that we have actually demonstrated, by direct calculation, that *all* of the defining relations Eqs. (2.1)–(2.5) of the $U_q(n)$ algebra are valid for $\Gamma(E_{in})$, $\Gamma(E_{ni})$ and, of course, for the remaining elements of the $\Gamma(E_{ij})$ mapping (this is automatically true via co-multiplication). A direct verification is such an exceedingly tedious and uninstructive task that it is not useful even to sketch the procedure here. Surprisingly the cubic Serre relations, and the relations asserted in Lemma (6.19) are not the most difficult. The hardest relations to prove by direct calculation are the quadratic Serre relations, *each of which involve* $(n-1)^2$ *commutators.* Almost all of these commutators evaluate to zero or cancel with the result of other commutators in pairs. There are, however, a special class of terms – beginning at $U_q(5)$ – *which do not vanish* (seemingly in contradiction to our assertion). For these terms one can show that the constraints imposed by the form of Eq. (5.17) – the constraint being expressed by Lemma (5.9) and the projection (5.15) – cause these terms *when operating on the basis* (Eq. (5.17)) to vanish, thus validating the claimed commutation relations.

We may sum up all of the above results by stating them collectively as a theorem:

Theorem (6.23). *The map $\Gamma: g \to \Gamma(g)$ – where g is a generator of $U_q(n)$ – given in Eqs. (5.3a–d), Eq. (6.10) and Eq. (6.22), is an isomorphism of $U_q(n)$ q-algebras.*

It is a straightforward task – now that we have the isomorphism of Theorem 6.23 – to complete the proof that the vectors $|(m)\rangle_{\text{q-BW}}$ of Eq. (5.17) are the carrier space of the irrep $[m]$ of the quantum group $U_q(n)$ generated by the $\Gamma(E_{ij})$, $1 \leqq i, j \leqq n$. We need only apply the lowering operators $\Gamma(E_{ni})$ to Eq. (5.17), thereby eliminating all of the q-boson operators $\{z_i\}$. Using the lowering operators $\Gamma(E_{ij})$, $i > j$ in Eq. (5.3b), then carries this vector to a lowest weight vector, whose weight is the *anti-dominant weight*, $\Gamma(E_{ii}) \to m_{n+1-i,n}$. This establishes the theorem:

Theorem (6.24). *The* q-BW *vectors constructed in (5.17) are the carrier space of a unitary irrep $[m]$ of the quantum group $U_q(n)$ generated by the operators $\Gamma(E_{ij})$ in Theorem (6.23).*

7. An Alternative Form for the Irrep Vectors $|(m)\rangle_{\text{q-BW}}$

The explicit vectors carrying an arbitrary irrep $[m]$ of the quantum group $U_q(n)$ have been given in Eq. (5.17). Each such vector is a sum of vectors which are tensor products of a homogeneous holomorphic vector over the (quantum space) base manifold and fiber vectors carrying $U_q(n-1) \times U_q(1)$.

It is useful, however, to express these vectors in an alternative form which makes explicit the matrix-elements of the Kähler potential which normalizes the irrep vectors. To obtain this alternative form we recall that the left action generators,

$\Gamma(\dot{E}_{ij})$, contain a sub-set of generators that realize the co-product action for the sub-group $U_q(n-1) \times U_q(1)$. Since the Gel'fand–Weyl pattern, (m), of the irrep vector $|(m)\rangle_{\text{q-BW}}$ uniquely specifies the irrep labels of this sub-group, we conclude that we may write the vector $|(m)\rangle_{\text{q-BW}}$ in the following form:

Lemma (7.1). *The basis vectors* $|(m)\rangle_{\text{q-BW}}$ *defined in (5.17) validate the relation*:

$$|(m)\rangle_{\text{q-BW}} = K\begin{pmatrix} [m_n] \\ [m_{n-1}] \end{pmatrix} \cdot \sum_{\substack{(\mu) \\ (\mu')}} {}_qC^{[\mu][w\dot{0}][m_{n-1}]}_{(\mu)(\mu')(m_{n-1})} \left(\left| \begin{matrix} [w\dot{0}] \\ (\mu') \end{matrix} \right\rangle_{\text{base}} \otimes \left| \begin{matrix} [\mu] \\ (\mu) \end{matrix} \right\rangle_{\text{fiber}} \right), \tag{7.2}$$

where:

(i) *the numerical constant K depends only on the* $U_q(n)$ *irrep labels* $[m_n]$ *and the* $U_q(n-1)$ *irrep labels* $[m_{n-1}]$;

(ii) *the q-Wigner–Clebsch–Gordon coefficient* ${}_qC^{\cdots}_{\cdots}$ *effects the tensor coupling*: $[w\dot{0}] \times [\mu] \to [m_{n-1}]$;

(iii) *the irrep vector* $\left| \begin{matrix} [w\dot{0}] \\ (\mu') \end{matrix} \right\rangle_{\text{base}}$ *is homogeneous and holomorphic in the q-bosons* $\{z_i\}$ *acting on the ket* $|0\rangle$ *with the* $U_q(n-1)$ *irrep labels*: $[w0\cdots0]$, *where* $w = \sum_{i=1}^{n-1} (m_{i,n-1} - \mu_{i,n-1})$;

(iv) *the* $U_q(n-1)$ irrep *labels of the fiber vector* $\left| \begin{matrix} [\mu] \\ (\mu) \end{matrix} \right\rangle_{\text{fiber}}$ *are given by*: $\mu_{i,n-1} = m_{i+1,n}$ *for* $i = 1, 2, \ldots, n-1$. *(These fiber vectors are actually tensored with a fixed* $U_q(1)$ *vector carrying* irrep $[m_{1n}]$ *but this is suppressed to avoid complication.)*

Proof. This may appear to be a rather complicated result, but the underlying structure is simple and actually implicit in the construction for Eq. (5.17). The vectors in the fiber, by construction (this is the projection (5.15)), carry the minimal $(U_q(n-1))$ weight contained in the irrep $[m]$. (These are the labels $[\mu]$.) The homogeneous holomorphic vectors have a unique degree, w, which labels the (symmetric) $U_q(n-1)$ irrep that these vectors carry, $[w\dot{0}]$. The label w is at the same time the total degree of the polynomial in $\{z_i\}$, and this is the number of q-boson quanta in the irrep, namely $w = \sum_i (m_{i,n-1} - \mu_{i,n-1}) \geqq 0$. The known action by the generators $\Gamma(E_{ij})$ then completes the proof. ■

To determine the explicit value of the function $K\begin{pmatrix} [m_n] \\ [m_{n-1}] \end{pmatrix}$, we remark that the $\Gamma(E_{ij})$ realize, by construction, the standard matrix elements of the generators on the basis $\{|(m)\rangle_{\text{q-BW}}\}$. To determine the function K then, one need only evaluate the left-action on the q-WCG coupled vectors and compare. This calculation shows that:

Lemma (7.3). *The normalizing factor* $K\begin{pmatrix} [m_n] \\ [m_{n-1}] \end{pmatrix}$ *in Eq. (7.1) for* $|(m)\rangle_{\text{q-BW}}$ *has the value*:

$$K\begin{pmatrix} [m_n] \\ [m_{n-1}] \end{pmatrix} = \left(\prod_{i=1}^{n-1} \frac{[p_{in} - p_{nn} - 1]!}{[p_{i,n-1} - p_{nn}]!} \right)^{1/2}, \tag{7.4}$$

where: $p_{ij} \equiv m_{ij} + j - i$, *and* $[n]!$ *denotes the q-factorial.*

Remarks. (1) The parameters p_{ij} are called "partial hooks" in the literature since differences of the p_{ij} are precisely Nakayama's hook parameters.

(2) The q-WCG coefficients that appear in Eq. (7.2) are known, in principle, via the recursion process which assumes $U_q(n-1)$ information to obtain $U_q(n)$ results. It will be observed, however, that these coefficients are actually *determined explicitly by evaluating the q-exponentials in* Eq. (5.17) and putting the results in the form of Eq. (7.2). For reasons of brevity we will not give the explicit result here, although it is useful to remark that for $w=1$ these results validate the q-pattern calculus results obtained earlier [7].

(3) For the $q=1$ limit these results go over, as they must, into the well-known Borel–Weil results for Eq. (5.17), in particular, and into the so-called "vector-coherent states" which have been much discussed in the physics literature [11]. These explicit ($q=1$) bases have proved to be very convenient for the optimal construction of tensor operators and their matrix elements [17]. The recursion aspect has been of considerable significance in that the procedure relates (3-j) coefficients of, say, the $U(n)$ group, to (6-j) coefficients in the $U(n-1)$ sub-group [17]. It is reasonable to expect that structural results of this type will carry over to the generic q case, implying the existence of significant new q-analog identities.

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References

1. Alvarez, O., Singer, I. M., Windey, P.: Quantum mechanics and the geometry of the Weyl character formula. Nucl. Phys. **B337**, 467–486 (1990)
2. Andrews, G. E.: q-Series: Their development and application in analysis, number theory, combinatorics, physics, and computer algebra. Conference Board of the Mathematical Sciences Vol. **66**, Providence RI: American Mathematical Society
3. Askey, R., Wilson, J. A.: Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials. Mem. Am. Math. Soc. **319** (1985)
4. Biedenharn, L. C., Louck, J. D.: Angular momentum in quantum physics, theory and application, encyclopedia of mathematics and its applications, Vol. **8**. Cambridge; UK: Cambridge University Press 1984, reprinted 1989
5. Biedenharn, L. C.: The quantum group $SU(2)$ and a q-analogue of the boson operators. J. Phys. A. Math. Gen. **22**, L873–L878 (1989)
6. Biedenharn, L. C., Lohe, M. A.: Some remarks on quantum groups – a new variation on the theme of symmetry. From symmetries to strings: forty years of Rochester conferences. Das, A. (ed.), pp. 189–206. Singapore: World Scientific 1990
7. Biedenharn, L. C.: A q-Boson realization of the quantum group $SU(2)$ and the theory of q-tensor operators. Doebner, H.-D., Hennig, J.-D. (Eds.), Quantum Groups, Proceedings of the 8th International Workshop of Mathematical Physics, July 1989. Lecture Notes in Physics Vol. **370**, p. 67. Berlin, Heidelberg, New York: 1990
8. Borel, A., Weil, A.: Reprêsentations linèaires et espaces homogènes Kählerians des groupes de Lie compacts, Séminaire Bourbaki, May 1954 (Exposé by J.-P. Serre)
9. Bott, R.: Homogeneous vector bundles. Ann. Math. **66**, (2) (1957)
10. Drinfeld, V. G.: Quantum groups: pp. 798–820 ICM Proceedings, Berkeley 1986

11. Hecht, K. T.: The vector coherent state method and its application to problems of higher symmetries. Lecture Notes in Physics Vol. **290**. Berlin, Heidelberg, New York: Springer 1987
12. Jackson, F. H.: Generalization of the differential operative symbol with an extended form of Boole's equation. Messenger Math. **38**, 57–61 (1909); q-Form of Taylor's Theorem. *ibid.* 62–64; On q-definite integrals. Quart. J. Pure Appl. Math. **41**, 193–203 (1910)
13. Jimbo, M.: A q-difference analogue of $U_q(g)$ and the Yang-Baxter equation. Lett. Math. Phys. **10**, 63–69 (1985)
14. Knapp, A. W.: Lie groups, Lie algebras, and cohomology, Mathematical Notes Vol. **34**, Princeton, NJ: Princeton University Press 1988
15. Koornwinder, T. H.: Representations of the twisted $SU(2)$ quantum group and some q-hypergeometric orthogonal polynomials. Nederl. Acad. Wetensch. Proc. Ser. **A92**, 97 (1989)
16. Kulish, P., Damashinsky, E.: On the q oscillator and the quantum algebra $su_q(1,1)$. J. Phys. **A23**, L415 (1990)
17. LeBlanc, R., Biedenharn, L. C.: Implementation of the $U(n)$ tensor operator calculus in a vector Bargmann Hilbert space. J. Phys. A: Math. Gen. **22**, 31–48 (1989)
18. Lusztig, G.: Quantum deformation of certain simple modules over enveloping algebras. Adv. Math. **70**, 237–249 (1988)
19. Macfarlane, A. J.: On q-analogues of the quantum harmonic oscillator and the quantum group $SU(2)_q$. J. Phys. A. Math. Gen. **22**, 4581 (1989)
20. Mackey, G. W.: The theory of unitary group representations. Chicago, IL: Univ. of Chicago Press 1976
21. Manin, Y.: Quantum groups and non-commutative geometry. Centre de Recherches Mathématiques, Université de Montréal 1988
22. Milne, S. C.: A q-analog of the Gauss summation theorem for hypergeometric series in $U(n)$. Adv. in Math. **72**, 59–131 (1988)
23. Perelomov, A.: Generalized coherent states and their applications, Texts and Monographs in Physics. Berlin, Heidelberg, New York: Springer 1986
24. Pressley, A., Segal, G.: Loop groups. Oxford: Clarendon Press 1986
25. Quesne, C.: Coherent states of the real symplectic group in a complex analytic parametrization. J. Math. Phys. **27**, 428 (1986)
26. Rosso, M.: Finite dimensional representations of the quantum analog of the enveloping algebra of a complex simple Lie algebra. Commun. Math. Phys. **117**, 581–593 (1988)
27. Rowe, D. J.: Coherent state theory of the noncompact symplectic group. J. Math. Phys. **25**, 2662 (1984)
28. Stone, M.: Supersymmetry and the Quantum Mechanics of Spin. Nucl. Phys. **B314**, 557–586 (1989)
29. Sun, C.-P., Fu, H.-C.: The q-deformed boson realisation of the quantum group $SU(n)_q$ and its representations. J. Phys. A. Math. Gen. **22**, L983 (1989)
30. Woronowicz, S. L.: Twisted $SU(2)$ group. An example of a non-commutative differential calculus. Pub. RIMS, Kyoto Univ. **28**, 117–181 (1987)
31. Yang, C. N., Ge, M. L., (eds.): Braid group, knot theory and statistical mechanics. Singapore, New Jersey, London, Hong Kong: World Scientific 1989

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