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AN EXTENSION OF THE FUGLEDE-PUTNAM THEOREM TO SUBNORMAL OPERATORS USING A HILBERT-SCHMIDT NORM INEQUALITY

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ABSTRACT. We prove that if A and B^* are subnormal operators acting on a Hilbert space, then for every bounded linear operator X, the Hilbert-Schmidt norm of $AX - XB$ is greater than or equal to the Hilbert-Schmidt norm of $A^*X - XB^*$. In particular, $AX = XB$ implies $A^*X = XB^*$. In addition, if we assume X is a Hilbert-Schmidt operator, we can relax the subnormality conditions to hyponormality and still retain the inequality.

1. In this paper an operator means a bounded linear operator on a separable infinite dimensional Hilbert space H. Let $B(H)$ and C_2 denote the class of all bounded linear operators acting on H and the Hilbert-Schmidt class in $B(H)$, respectively. It is known that C_2 forms a two-sided ideal in the algebra $B(H)$ and C_2 is itself a Hilbert space for the inner product

$$(X, Y) = \Sigma(Xe_j, Ye_j) = \mathrm{Tr}(Y^*X) = \mathrm{Tr}(XY^*)$$

where $\{e_j\}$ is any orthonormal basis of H and Tr( ) denotes the trace. In what follows, $\| \|_2$ denotes the Hilbert-Schmidt norm.

An operator T is called *subnormal* if T has a normal extension and *hyponormal* if $T^*T \geqslant TT^*$. The inclusion relation of these classes of nonnormal operators is as follows:

$$\text{Normal} \subsetneqq \text{Subnormal} \subsetneqq \text{Hyponormal}.$$

The above inclusions are all proper [5, Problem 160, p. 101].

THEOREM A [9]. *If A and B are normal, then*

$$\|AX - XB\|_2 = \|A^*X - XB^*\|_2$$

for every $X \in B(H)$.

THEOREM B [3]. *If A and B^* are subnormal operators and if X is an operator such that $AX = XB$, then $A^*X = XB^*$.*

In this paper we integrate Theorem A and Theorem B in order to prove a slightly stronger Theorem 1. Moreover in our Theorem 2 we have an extension of Weiss [8, Theorem 3] and Berberian [2, Theorem]. Finally we shall pose an open problem with respect to Theorem 1.

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THEOREM 1. *If A and B^* are subnormal, then the following inequality holds*:

$$\|AX - XB\|_2 \geq \|A^*X - XB^*\|_2 \qquad (*)$$

for every $X \in B(H)$. The equality holds for every $X \in B(H)$ when A and B are both normal.

PROOF. Since A is subnormal, there exists a normal extension $\tilde{N}_A$ of A on the Hilbert space $H \oplus H$ whose restriction to $H \oplus \{0\}$ is A [4], that is, $\tilde{N}_A$ is given by

$$\tilde{N}_A = \begin{pmatrix} A & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

on $H \oplus H$. Also a normal extension $\tilde{N}_{B^*}$ of B^* on $H \oplus H$ is given by

$$\tilde{N}_{B^*} = \begin{pmatrix} B^* & B_{12} \\ 0 & B_{22} \end{pmatrix}$$

on $H \oplus H$. Put $\tilde{X}$ on $H \oplus H$ as follows:

$$\tilde{X} = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}.$$

since $\tilde{N}^*_{B^*}$ is also normal, Theorem A easily implies

$$\|\tilde{N}_A\tilde{X} - \tilde{X}\tilde{N}^*_{B^*}\|_2 = \|\tilde{N}^*_A\tilde{X} - \tilde{X}\tilde{N}_{B^*}\|_2,$$

that is,

$$\left\|\begin{pmatrix} AX - XB & 0 \\ 0 & 0 \end{pmatrix}\right\|_2 = \left\|\begin{pmatrix} A^*X - XB^* & -XB_{12} \\ A^*_{12}X & 0 \end{pmatrix}\right\|_2$$

so that

$$\|AX - XB\|_2^2 = \|A^*X - XB^*\|_2^2 + \|A^*_{12}X\|_2^2 + \|XB_{12}\|_2^2. \qquad (1)$$

The equation (1) yields

$$\|AX - XB\|_2 \geq \|A^*X - XB^*\|_2 \qquad (*)$$

which is the desired norm inequality. When A and B are both normal, then $A_{12} = 0$ and $B_{12} = 0$ in (1), so that the equality holds in $(*)$, so the proof is complete.

The following corollary follows by Theorem 1.

COROLLARY 1 [3]. *If A and B^* are subnormal and X is an operator such that $AX = XB$, then $A^*X = XB^*$.*

Corollary 1 is some extension of the Fuglede-Putnam theorem [1], [5] and [7].

REMARK 1. As stated in the proof of the equality in Theorem 1, $\|A^*_{12}X\|_2^2 + \|XB_{12}\|_2^2$ in (1) is considered as the perturbed term of the difference between normality and subnormality.

3. In this section, we relax the hypotheses on A and B^* in Theorem 1 to hyponormality and strengthen the hypothesis on X to be in the Hilbert-Schmidt class.

THEOREM 2. *If A and B^* are hyponormal, then the following inequality holds:*

$$\|AX - XB\|_2 \geqslant \|A^*X - XB^*\|_2$$

for every X in Hilbert-Schmidt class. The equality holds when A and B are both normal.

PROOF. Define an operator $\mathfrak{T}$ on C_2 as follows: $\mathfrak{T}X = AX - XB$. Then, if we view C_2 as an underlying Hilbert space, then $\mathfrak{T}^*$ exists and is easily verified to be given by $\mathfrak{T}^*X = A^*X - XB^*$. Also

$$\begin{aligned}(\mathfrak{T}^*\mathfrak{T} - \mathfrak{T}\mathfrak{T}^*)X &= A^*(AX - XB) - (AX - XB)B^* \\ &\quad - \{A(A^*X - XB^*) - (A^*X - XB^*)B\} \\ &= (A^*A - AA^*)X + X(BB^* - B^*B). \end{aligned} \tag{2}$$

Left and right multiplication acting on C_2 as the Hilbert space by a positive operator is itself a positive operator. Since $\mathfrak{T}^*\mathfrak{T} - \mathfrak{T}\mathfrak{T}^*$ is the sum of two positive operators by the hyponormality of A and B^*, $\mathfrak{T}$ is hyponormal. Therefore $\|\mathfrak{T}X\|_2 \geqslant \|\mathfrak{T}^*X\|_2$ that is,

$$\|AX - XB\|_2 \geqslant \|A^*X - XB^*\|_2. \tag{3}$$

The proof of equality follows by (2) and (3).

REMARK 2. Berberian [**2**, Theorem] shows that if A and B^* are hyponormal, then $AX = XB$ implies $A^*X = XB^*$ for an operator X in Hilbert-Schmidt class and this is just the case of the equality for an operator X in Theorem 2. Weiss [**8**, Theorem 3] shows the case of the equality in Theorem 2 when $A = B$ is normal, by a different method.

REMARK 3. It is of interest to remark that Theorem 1, Theorem 2 and Corollary 1 do not involve symmetric hypotheses on A and B, but rather on A and B^*. In view of this point, it is natural and reasonable in Theorem A to interpret the hypothesis of normality of A and B as that of normality of A and B^*.

Open problem. Can the subnormality be relaxed by the hyponormality in Theorem 1? This is an open problem.

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