Research Article

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An extension of the method of brackets. Part 2

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Abstract: The method of brackets, developed in the context of evaluation of integrals coming from Feynman diagrams, is a procedure to evaluate definite integrals over the half-line. This method consists of a small number of operational rules devoted to convert the integral into a bracket series. A second small set of rules evaluates this bracket series and produces the result as a regular series. The work presented here combines this method with the classical Mellin transform to extend the class of integrands where the method of brackets can be applied. A selected number of examples are used to illustrate this procedure.

Keywords: definite integrals, method of brackets, Mellin transform, Bessel functions

MSC 2020: 33C99, 33C10

1 Introduction

The *method of brackets* is a collection of rules for the evaluation of a definite integral over the half-line $[0, \infty)$. It was developed in the calculation of integrals arising from Feynman diagrams, and its operational rules have appeared in [1–3]. These rules are described in Section 3. The method has been used in [4–7] to compute a variety of definite integrals appearing in [8].

The fundamental object is a *bracket series*, a formal expression of the form

$$S = \sum_{n=0}^{\infty} \phi_n f(n) \langle an + b \rangle,$$

where $\phi_n = (-1)^n/n!$ is called the *indicator* of S, the coefficients $\{f(n)\}$ form a sequence of complex numbers and $a, b \in \mathbb{C}$. The final term $\langle an + b \rangle$ is the *bracket*, defined by the divergent integral

$$\langle u \rangle = \int_0^{\infty} x^{u-1} \mathrm{d}x, \quad \text{for } u \in \mathbb{C}.$$

The operational rules for bracket series are described in Section 3. One of these rules associates a value with the sum S.

The goal of the work presented here is to connect the method of brackets with the Mellin transform. Section 4 shows how to produce a series for a function starting with an analytic expression for its Mellin transform. Section 5 then uses this procedure to evaluate a variety of definite integrals. Section 6 presents a two-dimensional integral to show that the method applies directly. Finally, Section 7 shows that the method yields an incorrect

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power series representation of the function e^{-x} but, in spite of this, the formal use of this series yields correct values of integrals. The explanation of this phenomenon is still an open question.

2 The method of brackets

This is a method that evaluates definite integrals over the half line $[0, \infty)$. The application of the method consists of a small number of rules, deduced in the heuristic form, some of which are placed on solid ground [9].

For $a \in \mathbb{C}$, the symbol

$$\langle a \rangle = \int_0^\infty x^{a-1} \mathrm{d}x$$

is the *bracket* associated with the (divergent) integral on the right. The symbol

$$\phi_n := \frac{(-1)^n}{\Gamma(n+1)}$$

is called the *indicator* associated with the index n. The notation $\phi_{i_1 i_2 \cdots i_r}$, or simply $\phi_{12\cdots r}$, denotes the product $\phi_{i_1}\phi_{i_2}\cdots\phi_{i_r}$.

2.1 Rules for the production of bracket series

The first part of the method is to associate with the integral

$$I(f) = \int_0^\infty f(x) \mathrm{d}x$$

a bracket series according to the following.

Rule P_1. Assume f has the expansion

$$f(x) = \sum_{n=0}^\infty \phi_n a_n x^{\alpha n + \beta - 1}.$$

Then $I(f)$ is assigned the bracket series

$$I(f) = \sum_{n \geq 0} a_n \langle \alpha n + \beta \rangle.$$

Rule P_2. For $\alpha \in \mathbb{C}$, the multinomial power $(a_1 + a_2 + \cdots + a_r)^\alpha$ is assigned the r-dimension bracket series

$$\sum_{n_1 \geq 0} \sum_{n_2 \geq 0} \cdots \sum_{n_r \geq 0} \phi_{n_1 n_2 \ldots n_r} a_1^{n_1} \ldots a_r^{n_r} \frac{\langle -\alpha + n_1 + \cdots + n_r \rangle}{\Gamma(-\alpha)}.$$

Rule P_3. Each representation of an integral by a bracket series has associated an *index of the representation* via

$$\text{index} = \text{number of sums} - \text{number of brackets}.$$

It is important to observe that the index is attached to a specific representation of the integral and not just to integral itself. The experience obtained by the authors using this method suggests that, among all representations of an integral as a bracket series, the one with *minimal index* should be chosen.

Example 3.1. The evaluation of the elementary integral

$$\int_0^\infty e^{-x}\mathrm{d}x = 1 \tag{1}$$

leads to the bracket series $\sum_n \phi_n \langle n+1\rangle$, a representation of index 0. The evaluation of

$$\int_0^\infty \frac{\mathrm{d}x}{e^{px}+1} = \frac{\ln 2}{p} \tag{2}$$

can be obtained from $\sum_{n_1,n_2,n_3} \phi_{123}(-n_1)^{n_3} p^{n_3} \langle 1+n_1+n_2\rangle\langle 1+n_3\rangle$, a representation of index 1. Finally, the reader can verify that the evaluation of

$$\int_0^\infty e^{-x}x^{a+b}L_m^a(x)L_n(x)\mathrm{d}x = (-1)^{m+n}(a+b)!\binom{a+m}{n}\binom{b+n}{m}, \tag{3}$$

where $L_n^\lambda(z)$ is the associated Laguerre function, can be completed with a representation of index 2. This evaluation appears as entry 7.414.9 in [8].

2.2 Rules for the evaluation of a bracket series

Rule E$_1$. The one-dimensional bracket series is assigned the value

$$\sum_{n\geq 0} \phi_n C(n)\langle an+b\rangle = \frac{1}{|a|}C(n^*)\Gamma(-n^*),$$

where n^* is obtained from the vanishing of the bracket, that is, n^* solves $an+b=0$. This is precisely the Ramanujan Master Theorem.

The next rule provides a value for multi-dimensional bracket series of index 0, that is, the number of sums is equal to the number of brackets.

Rule E$_2$. Assuming the matrix $A=(a_{ij})$ is non-singular, then the assignment is

$$\sum_{n_1\geq 0}\cdots\sum_{n_r\geq 0} \phi_{n_1\ldots n_r} C(n_1,\ldots,n_r)\langle a_{11}n_1+\cdots+a_{1r}n_r+c_1\rangle\cdots\langle a_{r1}n_1+\cdots+a_{rr}n_r+c_r\rangle$$
$$= \frac{1}{|\det(A)|}C(n_1^*,\ldots,n_r^*)\Gamma(-n_1^*)\cdots\Gamma(-n_r^*),$$

where $\{n_i^*\}$ is the (unique) solution of the linear system obtained from the vanishing of the brackets. There is no assignment if A is singular.

Rule E$_3$. The value of a multi-dimensional bracket series of positive index is obtained by computing all the contributions of maximal rank by Rule E_2. These contributions to the integral appear as series in the free indices. Series converging in a common region are added, and divergent/null series are discarded. There is no assignment to a bracket series of negative index. If all the resulting series are discarded, then the method is not applicable.

3 The generation of series

This section describes how to obtain a series for a function $f(x)$ assuming the knowledge of its Mellin transform.

Theorem 4.1. *Let*

$$\varphi(s) = \int_0^\infty x^{s-1} f(x)\,dx \tag{4}$$

be the Mellin transform of a function $f(x)$*. Then, for any choice of* $\alpha, \beta \in \mathbb{C}$*, the function* f *admits an expansion of the form*

$$f(x) = \sum_{n=0}^{\infty} \phi_n C(n) x^{\alpha n + \beta}, \tag{5}$$

where the coefficient $C(n)$ *is given by*

$$C(n) = \frac{|\alpha| \varphi(-(\alpha n + \beta))}{\Gamma(-n)}. \tag{6}$$

Proof. Replace (5) in (4) to obtain

$$\varphi(s) = \sum_{n=0}^{\infty} \phi_n \int_0^\infty C(n) x^{\alpha n + \beta + s - 1} dx = \sum_n \phi_n C(n) \langle \alpha n + \beta + s \rangle.$$

Rule E1 now $\varphi(s) = \frac{1}{|\alpha|} \Gamma\left(\frac{\beta+s}{\alpha}\right) C\left(-\frac{\beta+s}{\alpha}\right)$, and (6) follows from here by making $n = -(\beta + s)/\alpha$. □

The result in Theorem 4.1 gives no information about the convergence of the series (5). In particular, examples of functions for which such series do not exist are discussed below. These include series where all the coefficients vanish (the so-called *null series*) and also those for which all the coefficients blow up (the *divergent series*). The use of these formal series in the process of integration has been presented in [10].

Example 4.2. Entry 3.761.4 in [8] gives the Mellin transform of $g(x) = \sin x$ as

$$\varphi(s) = \int_0^\infty x^{s-1} \sin x \, dx = \Gamma(s) \sin\left(\frac{\pi s}{2}\right).$$

Take $\alpha = 1$ and $\beta = 0$ in Theorem 4.1 to obtain

$$C(n) = \frac{\varphi(-n)}{\Gamma(-n)} = \sin\left(-\frac{\pi n}{2}\right) = \begin{cases} 0 & \text{if } n \text{ is even}, \\ (-1)^{k+1} & \text{if } n \text{ is odd}, n = 2k+1. \end{cases}$$

Then (5) reproduces the Taylor series for $f(x) = \sin x$. The Taylor series for $\cos x$ is obtained by the same procedure.

Example 4.3. Example 4.2 shows that one can recover the Taylor series of elementary functions from Theorem 4.1. This example shows how to recover the series for the Bessel function

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(\nu + n + 1)} \left(\frac{x}{2}\right)^{2k}. \tag{7}$$

The Mellin transform

$$\varphi(s) = \int_0^\infty x^{s-1} J_\nu(x) dx = 2^{s-1} \frac{\Gamma\left(\frac{\nu+s}{2}\right)}{\Gamma\left(\frac{2+\nu-s}{2}\right)}$$

appears as entry 6.561.14 in [8]. The procedure described here gives

$$C(n) = \frac{|\alpha|}{\Gamma(-n)} 2^{-\alpha n-\beta-1} \frac{\Gamma\left(\frac{\nu-\alpha n-\beta}{2}\right)}{\Gamma\left(\frac{2+\nu+\alpha n+\beta}{2}\right)}. \tag{8}$$

In order to cancel the term $\Gamma(-n)$ in the denominator it is convenient to choose $\beta = \nu$ and $\alpha = 2$. Then (8) reduces to

$$C(n) = \frac{1}{2^{2n+\nu}\Gamma(\nu+n+1)},$$

and this establishes (7).

Example 4.4. The Mellin transform of the Bessel function $K_\nu(x)$ is given as entry 6.561.16 in [8]:

$$\int_0^\infty x^{s-1} K_\nu(x)\, dx = 2^{s-2}\Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right).$$

The usual process now gives, choosing $\alpha = 2$ and $\beta = \nu$,

$$C(n) = \frac{1}{2^{2n+\nu+1}}\Gamma(-n-\nu),$$

and this yields the series

$$g(x) = \sum_{n=0}^\infty \frac{(-1)^n \Gamma(-n-\nu)}{2^{2n+\nu+1} n!} x^{2n+\nu}. \tag{9}$$

Naturally $g(x)$ cannot be $K_\nu(x)$, since the Bessel function K_ν does not have a power series expansion at $x = 0$. Indeed, the definition

$$I_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{k=0}^\infty \frac{\left(\frac{x}{2}\right)^{2k}}{k!\Gamma(\nu+k+1)}$$

gives the expansion

$$\begin{aligned} K_n(x) &= \frac{1}{2}\left(\frac{x}{2}\right)^{-n} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}\left(-\frac{x}{2}\right)^{2k} + (-1)^{n+1} I_n(z) \ln\left(\frac{x}{2}\right) \\ &+ \frac{(-1)^n}{2}\left(\frac{x}{2}\right)^n \sum_{k=0}^\infty [\psi(k+1) + \psi(k+n+1)] \frac{\left(\frac{x}{2}\right)^{2k}}{k!(n+k)!}, \end{aligned}$$

which shows that $K_n(x)$ has a logarithmic singularity at $x = 0$, and hence it is not analytic there.

Example 4.5. The function Ei, called the exponential integral, has the bracket series

$$\text{Ei}(x) = \sum_n \phi_n \frac{1}{n} x^n.$$

This is obtained from the Mellin transform

$$\int_0^\infty x^{\mu-1} \text{Ei}(-x)\, dx = -\frac{\Gamma(\mu)}{\mu}$$

appearing as entry 6.223 in [8] and the choice $\alpha = 1$ and $\beta = 0$.

4 The evaluation of integrals with an integrand formed by the product of two terms

The goal of this section is to present a procedure to evaluate integrals of the form

$$I = \int_0^\infty x^{s-1} f_1(ax) f_2(bx)\,\mathrm{d}x \tag{10}$$

under the assumption that the function $f_1(x)$ admits the expansion of the form

$$f_1(x) = \sum_{k=0}^{\infty} \phi_k A_1(k) x^{\alpha_1 k + \beta_1}$$

and that the Mellin transform the second factor $f_2(x)$,

$$\varphi_2(s) = \mathcal{M}(f_2(x))(s) = \int_0^\infty x^{s-1} f_2(x)\,\mathrm{d}x \tag{11}$$

is a known function.

The procedure is described in a sequence of steps. The final expression for I is given in Theorem 5.1.

Step 1. Use the method developed in Section 4 to produce a series for $f_2(x)$ from $\varphi_2(s)$ in the form

$$f_2(x) = \sum_k \phi_k A_2(k) x^{\alpha_2 k + \beta_2}. \tag{12}$$

This is precisely the result given in Theorem 4.1.

Step 2. Replacing (12) in (11) gives

$$\varphi_2(s) = \sum_k \phi_k A_2(k) \langle s + \alpha_2 k + \beta_2 \rangle.$$

The bracket series on the right is now evaluated to obtain

$$\varphi_2(s) = \frac{1}{|\alpha_2|} \Gamma\left(\frac{s+\beta_2}{\alpha_2}\right) A_2\left(-\frac{s+\beta_2}{\alpha_2}\right).$$

This can be expressed as

$$A_2(k) = \frac{|\alpha_2| \varphi_2(-\beta_2 - k\alpha_2)}{\Gamma(-k)}.$$

Step 3. Replace the expansions

$$\begin{aligned} f_1(x) &= \sum_n \phi_n A_1(n) x^{\alpha_1 n + \beta_1} \\ f_2(x) &= \sum_k \phi_k \frac{|\alpha_2|}{\Gamma(-k)} \varphi_2(-\beta_2 - k\alpha_2) x^{\alpha_2 k + \beta_2} \end{aligned}$$

in the integral (10) and write the x-integral as a bracket to obtain

$$I = |\alpha_2| \sum_{n,k} \phi_{n,k} \frac{A_1(n) a^{\alpha_1 n + \beta_1} b^{\alpha_2 k + \beta_2}}{\Gamma(-k)} \varphi_2(-\beta_2 - k\alpha_2) \langle s + \beta_1 + \beta_2 + \alpha_1 n + \alpha_2 k \rangle.$$

This two-dimensional bracket series now yields solutions, depending on which index, n or k, is kept as the free one. The solutions are as follows.

***n* free.** This gives

$$\mathcal{I}_1 = \frac{a^{\beta_1}}{b^{\beta_1+s}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} A_1(n)\varphi_2(s+\beta_1+\alpha_1 n)\left(\frac{a}{b}\right)^{\alpha_1 n}.$$

***k* free.** This gives

$$\mathcal{I}_2 = \frac{|\alpha_2|}{|\alpha_1|}\frac{b^{\beta_2}}{a^{s+\beta_2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} A_1\left(-\frac{s+\beta_1+\beta_2+\alpha_2 k}{\alpha_1}\right)\Gamma\left(\frac{s+\beta_1+\beta_2+\alpha_2 k}{\alpha_1}\right)\frac{\varphi_2(-\beta_2-k\alpha_2)}{\Gamma(-k)}\left(\frac{b}{a}\right)^{\alpha_2 k}.$$

The results are now summarized as follows.

Theorem 5.1. *Assume f_1 admits an expansion of the form*

$$f_1(x) = \sum_{k=0}^{\infty} \phi_k A_1(k) x^{\alpha_1 k+\beta_1},$$

where $\phi_k = (-1)^k/k!$ and $\alpha_1, \beta_1 \in \mathbb{C}$. Moreover, assume that the Mellin transform of f_2

$$\varphi(s) = \int_0^{\infty} x^{s-1} f_2(x)\,\mathrm{d}x$$

is known. Then the integral I in (10) *is given in terms of the series $\mathcal{I}_1$ and $\mathcal{I}_2$ given above. The rules for brackets show that if both series converge in a common region, the value of the integral is given by the sum of $\mathcal{I}_1$ and $\mathcal{I}_2$.*

The remainder of the section contains examples that illustrate Theorem 5.1.

Example 5.2. The elementary integral

$$I(a,b) = \int_0^{\infty} e^{-ax}\sin bx \;\;\mathrm{d}x \tag{13}$$

is now evaluated using Theorem 5.1. The condition $a > 0$ is imposed for convergence. Take

$$f_1(x) = e^{-x} = \sum_{k=0}^{\infty} \phi_k x^k$$

so that $\alpha_1 = 1$, $\beta_1 = 0$ and $A_1(k) = 1$. The Mellin transform of $f_2(x) = \sin x$ is given as entry 3.761.4 in [8]:

$$\varphi_2(s) = \int_0^{\infty} x^{s-1}\sin x \;\;\mathrm{d}x = \Gamma(s)\sin\left(\frac{\pi s}{2}\right) = \frac{\pi\Gamma(s)}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)}.$$

The series expansion for $f_2(x)$ produced in Step 2 is

$$f_2(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\frac{|\alpha_2|}{\Gamma(-k)}\Gamma(-\beta_2-\alpha_2 k)\sin\left(\tfrac{\pi}{2}(-\beta_2-k\alpha_2)\right)x^{\alpha_2 k+\beta_2},$$

where the parameters α_2, β_2 are arbitrary. Now choose $\alpha_2 = 1$ and $\beta_2 = 0$ to cancel the singular term $\Gamma(-k)$. This yields

$$f_2(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k!}\sin\left(\frac{\pi k}{2}\right)x^k. \tag{14}$$

Observe that the term $\sin(\pi k/2)$ vanishes for k even and (14) is nothing but the Taylor expansion of $\sin x$,

$$f_2(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1}.$$

The integral (13) has $s = 1$ and replacing all the parameters in the formula for $\mathcal{I}_1$ gives

$$\mathcal{I}_1 = \frac{1}{b}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\Gamma(n+1)\sin\left(\frac{\pi}{2}(n+1)\right)\left(\frac{a}{b}\right)^n,$$

and this series sums to

$$\mathcal{I}_1 = \frac{b}{a^2+b^2}$$

provided $|a| < |b|$. The formula for $\mathcal{I}_2$ also gives the same result, but now with the condition $|a| > |b|$.

Example 5.3. This example evaluates the integral

$$I = \int_0^{\infty} J_\mu(ax)J_\nu(bx)\,\mathrm{d}x, \tag{15}$$

which appears as entry 6.512.1 in [8]. Start with

$$f_1(x) = J_\mu(x) = \sum_{k=0}^{\infty}\frac{\phi_k}{\Gamma(k+\mu+1)2^{2k+\mu}}x^{2k+\mu},$$

so that $\alpha_1 = 2$, $\beta_1 = \mu$ and $A_1(k) = [\Gamma(k+\mu+1)2^{2k+\mu}]^{-1}$. The Mellin transform of the second factor in (15) is given as entry 6.561.14 in [8] by

$$\varphi_2(s) = \int_0^{\infty} x^{s-1}J_\nu(x)\,\mathrm{d}x = \frac{2^{s-1}\Gamma\left(\frac{\nu+s}{2}\right)}{\Gamma\left(1+\frac{\nu-s}{2}\right)}.$$

Now choose $\alpha_2 = 2$ and $\beta_2 = \nu$ and replace in the expression for $\mathcal{I}_1$ to obtain

$$\mathcal{I}_1 = \frac{a^\mu}{b^{\mu+1}}\frac{\Gamma\left(\frac{1+\nu+\mu}{2}\right)}{\Gamma(1+\mu)\Gamma\left(\frac{1+\nu-\mu}{2}\right)}\,{}_2F_1\left(\left.\begin{matrix}\frac{1+\mu+\nu}{2}\ \frac{1+\mu-\nu}{2}\\ \mu+1\end{matrix}\,\right|\,\frac{a^2}{b^2}\right).$$

The expression for $\mathcal{I}_2$ produces

$$\mathcal{I}_2 = \frac{b^\nu}{a^{\nu+1}}\frac{\Gamma\left(\frac{1+\mu+\nu}{2}\right)}{\Gamma(1+\nu)\Gamma\left(\frac{1+\mu-\nu}{2}\right)}\,{}_2F_1\left(\left.\begin{matrix}\frac{1+\nu+\mu}{2},\ \frac{1+\nu-\mu}{2}\\ \nu+1\end{matrix}\,\right|\,\frac{b^2}{a^2}\right).$$

This confirms entry 6.512.1 in [8]:

$$\int_0^{\infty} J_\mu(ax)J_\nu(bx)\,\mathrm{d}x = \frac{b^\nu}{a^{\nu+1}}\frac{\Gamma\left(\frac{1+\mu+\nu}{2}\right)}{\Gamma(\nu+1)\Gamma\left(\frac{\mu-\nu+1}{2}\right)}\,{}_2F_1\left(\left.\begin{matrix}\frac{\mu+\nu+1}{2}\ \frac{\nu-\mu+1}{2}\\ \nu+1\end{matrix}\,\right|\,\frac{b^2}{a^2}\right).$$

The same procedure used in the previous example gives entry 6.574.1 in [8]:

$$\int_0^{\infty} x^{s-1}J_\mu(ax)J_\nu(bx)\,\mathrm{d}x = \frac{b^\nu}{2^{1-s}a^{\nu+s}}\frac{\Gamma\left(\frac{\mu+\nu+s}{2}\right)}{\Gamma(\nu+1)\Gamma\left(\frac{\mu-\nu+2-s}{2}\right)}\,{}_2F_1\left(\left.\begin{matrix}\frac{\nu+\mu+s}{2}\ \frac{\nu-\mu+s}{2}\\ \nu+1\end{matrix}\,\right|\,\frac{b^2}{a^2}\right).$$

The special case $\mu = \nu$ and $a = b = 1$ gives

$$\int_0^{\infty} x^{s-1}J_\mu(x)^2\,\mathrm{d}x = \frac{\Gamma\left(\mu+\frac{s}{2}\right)}{2^{1-s}\Gamma(\mu+1)\Gamma\left(1-\frac{s}{2}\right)}\,{}_2F_1\left(\left.\begin{matrix}\mu+\frac{s}{2}\ \ \frac{s}{2}\\ \mu+1\end{matrix}\,\right|\,1\right) = \frac{\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\mu+\frac{s}{2}\right)}{2\sqrt{\pi}\,\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(\mu+1-\frac{s}{2}\right)},$$

which for $\mu = 0$ produces

$$\int_0^\infty x^{s-1} J_0(x)^2 dx = \frac{\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{s}{2}\right)}{2\sqrt{\pi}\,\Gamma^2\left(1-\frac{s}{2}\right)}. \tag{16}$$

Example 5.4. The next integral evaluated is entry 6.574.1 in [8]

$$\int_0^\infty J_0^2(ax) J_1(bx)\, dx = \begin{cases} \dfrac{1}{b} & \text{if } b > 2a, \\ \dfrac{2}{\pi b} \sin^{-1}\left(\dfrac{b}{2a}\right) & \text{if } 0 < b < 2a. \end{cases}$$

An interesting point appears in this example. It will be shown that the method of brackets succeeds in the case $b > 2a$, and it fails to produce a value when $0 < b < 2a$. This problem will be discussed in a future publication.

Take $f_1(x) = J_1(x)$ so that

$$\alpha_1 = 2, \beta_1 = 1, \quad \text{and} \quad A_1(k) = \frac{1}{\Gamma(k+2)2^{2k+1}}.$$

On the other hand, (16) shows that $f_2(x) = J_0^2(x)$ has the Mellin transform

$$\varphi_2(s) = \frac{\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{s}{2}\right)}{2\sqrt{\pi}\,\Gamma^2\left(1-\frac{s}{2}\right)} = \frac{\sin^2(\pi s/2)}{2\pi^{5/2}}\Gamma\left(\frac{1-s}{2}\right)\Gamma^2\left(\frac{s}{2}\right).$$

The coefficient in the expansion of $f_2(x)$ is given by

$$C_2(n) = \frac{|\alpha_2|}{\Gamma(-n)} \frac{\Gamma\left(\frac{1+\alpha_2 n+\beta_2}{2}\right)\Gamma\left(-\frac{\alpha_2 n+\beta_2}{2}\right)}{2\sqrt{\pi}\,\Gamma^2\left(\frac{2+\alpha_2 n+\beta_2}{2}\right)}.$$

The parameters α_2, β_2 are arbitrary. These are chosen here as $\alpha_2 = 2$ and $\beta_2 = 0$, in order to cancel the singular term $\Gamma(-n)$. This yields

$$C_2(n) = \frac{\Gamma(1/2+n)}{\sqrt{\pi}\,\Gamma^2(n+1)}.$$

The series I_1 in Theorem 5.1 is

$$I_1 = \frac{b}{4\sqrt{\pi}\,a^2} \sum_{n=0}^\infty \phi_n \frac{\Gamma(n+1)\Gamma\left(-\frac{1}{2}-n\right)}{\Gamma^2(-n)\Gamma(n+2)} \left(\frac{b}{2a}\right)^{2n}.$$

This is a null series, in the sense of [10], where every coefficient vanishes.

The series I_2 has the value

$$I_2 = \frac{1}{b\sqrt{\pi}} \sum_{k=0}^\infty \phi_k \frac{\Gamma\left(\frac{1}{2}+k\right)}{\Gamma(1-k)\Gamma(1+k)} \left(\frac{2a}{b}\right)^{2k}.$$

The presence of the factor $\Gamma(1-k)$ shows that the sum reduces to the value for $k = 0$, that is,

$$I_2 = \frac{1}{b}.$$

It is curious that none of the techniques developed for the method of brackets is able to produce the value of this integral for the case $0 < b < 2a$.

Example 5.5. Example 4.4 shows that it is possible to use the method of brackets to evaluate integrals involving functions that do not have power series representations. Another example of such a function is the *exponential integral* function, which is defined by

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{e^t}{t} \mathrm{d}t$$

for $x < 0$. In the case $x > 0$, we use the Cauchy principal value

$$\mathrm{Ei}(x) = -\lim_{\epsilon \to 0^+} \left[\int_{-x}^{-\epsilon} \frac{e^{-t}}{t} \mathrm{d}t + \int_{\epsilon}^{\infty} \frac{e^{-t}}{t} \mathrm{d}t \right].$$

This appears as entry 3.351.6 in [8]. The expansion

$$\mathrm{Ei}(x) = \gamma + \ln x + \sum_{n=1}^{\infty} \frac{x^n}{n n!},$$

for $x > 0$, shows the singular behavior of this function. A collection of integrals involving the exponential integral function appeared in [11].

The evaluation of the integral

$$I(a) = \int_0^{\infty} e^{-x} \mathrm{Ei}(-ax) \mathrm{d}x$$

is now performed by the method of brackets.

The Mellin transform of Ei yields, as shown in Example 4.5, the bracket series

$$\mathrm{Ei}(-ax) = \sum_n \phi_n \frac{a^n x^n}{n}.$$

The integral is now obtained from

$$I(a) = \sum_{n,k} \phi_{n,k} \frac{a^n}{n} \langle n + k + 1 \rangle.$$

The usual procedure now shows that the series $I_1(a)$, corresponding to n as a free index, has to be discarded and the one for k free yields

$$I_2(a) = \sum_{k=0}^{\infty} \phi_k \frac{a^n}{n} \Gamma(-n) \Big|_{n=-k-1} = -\sum_{k=0}^{\infty} \phi_k \frac{a^{-k-1}\Gamma(k+1)}{k+1} = -\frac{1}{a} \sum_{k=0}^{\infty} \frac{(-a)^{-k}}{k+1} = -\ln\left(1 + \frac{1}{a}\right).$$

Therefore,

$$I(a) = \int_0^{\infty} e^{-x} \mathrm{Ei}(-ax) \mathrm{d}x = -\ln\left(1 + \frac{1}{a}\right).$$

5 A two-dimensional problem

The method described here also applies to multidimensional integrals. An example illustrating this is presented next. Consider the integral

$$I(\alpha, \beta) = \int_0^{\infty} \int_0^{\infty} x^{\alpha-1} y^{\beta-1} \mathrm{Ei}(-x^2 y) K_1\left(\frac{x}{y}\right) \mathrm{d}x \ \mathrm{d}y.$$

The function Ei is the exponential integral with bracket series

$$\mathrm{Ei}(x) = \sum_n \phi_n \frac{1}{n} x^n.$$

This is obtained from the methods of Section 4 using the Mellin transform

$$\int_0^\infty x^{\mu-1}\mathrm{Ei}(-\beta x)\,\mathrm{d}x = -\frac{\Gamma(\mu)}{\mu\beta^\mu}$$

appearing as entry 6.223 in [8]. The series representation for $K_1(x)$ is

$$K_1(x) = \sum_k \phi_k 2^{-2k-2}\Gamma(-k-1)x^{2k+1}$$

as given in (9). The usual procedure now yields a two-dimensional bracket series

$$I(\alpha,\beta) = \sum_{n,k} \phi_{n,k} \frac{\Gamma(-k-1)}{n2^{2k+2}} \langle \alpha + 2n + 2k + 1\rangle \langle \beta + n - 2k - 1\rangle.$$

The vanishing of the brackets gives a linear system with solutions

$$n = -\frac{a+b}{3} \quad \text{and} \quad k = -\frac{a-2b+3}{6}.$$

The method of brackets now yields

$$I(\alpha,\beta) = -\frac{2^{(\alpha-2\beta-6)/3}}{\alpha+\beta}\Gamma\left(\frac{\alpha+\beta}{3}\right)\Gamma\left(\frac{\alpha-2\beta+3}{6}\right)\Gamma\left(\frac{\alpha-2\beta-3}{6}\right).$$

6 A curious series expansion

The process described in Section 4 produces a series for a function $f(x)$ in terms of the scale $x^{\alpha n+\beta}$, with arbitrary parameters α, β. This is now applied to the function $f(x) = e^{-x}$ and the parameters $\alpha = 2$ and $\beta = 0$. The expansion in Theorem 4.1 gives the coefficients $C(n)$ in

$$f(x) = \sum_{n=0}^\infty \phi_n C(n) x^{2n}$$

in terms of the Mellin transform of f, denoted by $\varphi(s)$, by formula (6)

$$C(n) = \frac{|\alpha|\varphi(-(\alpha n+\beta))}{\Gamma(-n)}.$$

In this case,

$$\varphi(s) = \int_0^\infty x^{s-1}e^{-x}\mathrm{d}x = \Gamma(s)$$

and the choice $\alpha = 2$ and $\beta = 0$ yields

$$C(n) = \frac{2\Gamma(-2n)}{\Gamma(-n)}.$$

Now write

$$\frac{\Gamma(-2n)}{\Gamma(-n)} = \frac{\Gamma(1+n)}{\Gamma(1+2n)}\frac{\sin(\pi n)}{\sin(2\pi n)} = \frac{(-1)^n n!}{2(2n)!},$$

to produce

$$C(n) = (-1)^n \frac{n!}{(2n)!}.$$

Then

$$f(x) = \sum_{n=0}^{\infty} \phi_n (-1)^n \frac{n!}{(2n)!} x^{2n} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}.$$

This is clearly incorrect, since the last series is e^{x^2}.

On the other hand, consider the evaluation of the integral

$$I(a) = \int_0^{\infty} e^{-ax} \cos x \ \, dx.$$

To compute it, use the representation obtained above

$$e^{-ax} = \sum_n \phi_n C(n)(ax)^{2n}$$

and the classical Taylor series for $\cos x$ written as

$$\cos x = {}_0F_1\left(\begin{matrix} - \\ \frac{1}{2} \end{matrix} \middle| -\frac{x^2}{4} \right) = \sqrt{\pi} \sum_k \phi_k \frac{1}{2^{2k}\Gamma\left(k+\frac{1}{2}\right)} x^{2k}$$

to produce

$$I(a) = \sum_{n,k} \phi_{nk} \frac{2\sqrt{\pi}\,\Gamma(-2n)\, a^{2n}}{\Gamma(-n) 2^{2k} \Gamma\left(k+\frac{1}{2}\right)} \langle 2n+2k+1 \rangle.$$

The usual procedure gives two series for $I(a)$:

$$I_1(a) = 2\sqrt{\pi} \sum_{k=0}^{\infty} \phi_k \frac{\Gamma\left(\frac{1}{2}+k\right)\Gamma(-2k)(4a^2)^k}{\Gamma(-k)^2},$$

a null series, so it is discarded. The second series is

$$I_2(a) = \frac{\sqrt{\pi}}{a} \sum_{n=0}^{\infty} \frac{\Gamma(1+2n)}{\Gamma\left(\frac{1}{2}+n\right) n!} \left(-\frac{1}{4a^2}\right)^n = \frac{1}{a}\, {}_1F_0\left(\begin{matrix} 1 \\ - \end{matrix} \middle| -\frac{1}{a^2} \right) = \frac{a}{a^2+1}.$$

The expression for $I_2(a)$ is the correct answer.

The question is why does this algorithm produces the value for the integral $I(a)$ is the subject of current research. An important part of the current investigation deals with the comparison of the method of brackets and other automatic algorithms for integration, such as those developed by C. Koutschan [12]. The authors expect to report on this in a future publication.

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