ORIGINAL PAPER

AN EXTENSION RESULT ON SIMILARITY SOLUTIONS ARISING DURING MIXED CONVECTION

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Manuscript received: 01.01.2022; Accepted paper: 15.05.2022;
Published online: 30.06.2022.

Abstract. *In this paper, we are interested in the boundary value problem involving a third order autonomous ordinary nonlinear differential equation. Its solutions are the similarity solutions of a problem of boundary-layer theory dealing with mixed convection phenomena in a porous medium. We confirm our results by numerical illustrations using a shooting algorithm of Mathematica.*

Keywords: *boundary-layer; mixed convection; similarity solution; third order nonlinear differential equation.*

1. INTRODUCTION

The problems of fluid mechanics are usually governed by systems of partial differential equations. In the modeling of boundary-layer and in some cases, the system of partial differential equations reduces to a system involving a third order differential equation (see [1]). For example, in the study of the mixed convection phenomena created by a heated plate and embedded in a porous medium saturated with a fluid, we are led to consider in $\mathbb{R}_+ \mathrm{x} \mathbb{R}_+$, the following PDE system (see [2, 3]):

$$\begin{cases} \dfrac{\partial u}{\partial \bar{x}} + \dfrac{\partial v}{\partial \bar{y}} = 0 \\ u = -\dfrac{K}{\mu}\left(\dfrac{\partial \mathrm{p}}{\partial \bar{x}} + \rho g\right) \\ \nu = -\dfrac{K}{\mu}\dfrac{\partial \mathrm{p}}{\partial \bar{y}} \\ u\dfrac{\partial \mathrm{T}}{\partial \bar{x}} + \nu\dfrac{\partial \mathrm{T}}{\partial \bar{y}} = \alpha\left(\dfrac{\partial^2 T}{\partial \bar{x}^2} + \dfrac{\partial^2 T}{\partial \bar{y}^2}\right) \\ \rho = \rho_\infty[1 - \beta(\mathrm{T} - T_\infty)] \end{cases} \tag{1}$$

The unknowns are: the Darcy velocities of the fluid u and v in the directions of the $\bar{x}$ and $\bar{y}$ axes, the temperature of the fluid T , the pressure of the fluid p, the density of the fluid ρ.

The constants represent: the viscosity of the fluid μ. The thermal expansion coefficient of the fluid is β. The permeability of the saturated porous medium is K. The thermal diffusivity is α. The acceleration of the gravity is denoted by g. The density of the fluid far from the plate is designated by ρ_∞.

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In our system of coordinates, the boundary conditions along the plate are

$$v(\bar{x},0) = -\omega \bar{x}^{\frac{m-1}{2}} \text{ and } T(\bar{x},0) = T_\omega(\bar{x}) = T_\infty + A\bar{x}^m \ (A, m \in \mathbb{R}).$$

with $\omega \in \mathbb{R}$ is the mass transfer coefficient of the plate (see [4, 5]) ($\omega < 0$ corresponds to a fluid suction, $\omega = 0$ is for an impermeable wall and $\omega > 0$ corresponds to a fluid injection). The boundary conditions far from the plate are

$$u(\bar{x},\bar{y}) \to 0 \text{ and } T(\bar{x},\bar{y}) \to T_\infty \text{ as } \bar{y} \to +\infty$$

The first equation of the system (1), called the continuity equation, is automatically satisfied by introducing the stream function φ as (see [3]):

$$u = \frac{\partial \varphi}{\partial \bar{y}}$$

and

$$v = -\frac{\partial \varphi}{\partial \bar{x}}$$

We eliminate p from Equations in System (1) by differentiating the second equation with respect to $\bar{y}$ and the third equation with respect to $\bar{x}$. The equations of this system can be rewritten as (see [3]):

$$\frac{\partial^2 \varphi}{\partial \bar{x}^2} + \frac{\partial^2 \varphi}{\partial \bar{y}^2} = \frac{\rho_\infty \beta g k}{\mu} \frac{\partial T}{\partial \bar{y}} \tag{2}$$

and

$$\frac{\partial T}{\partial \bar{x}} \frac{\partial \varphi}{\partial \bar{y}} - \frac{\partial T}{\partial \bar{y}} \frac{\partial \varphi}{\partial \bar{x}} = \alpha \left(\frac{\partial^2 T}{\partial \bar{x}^2} + \frac{\partial^2 T}{\partial \bar{y}^2} \right) \tag{3}$$

Using the following non-dimensional quantities (see [5])

$$x = \frac{1}{L}\bar{x}, \qquad y = \frac{\sqrt{P_e}}{L}\bar{y}, \Psi(x,y) = \frac{1}{\alpha\sqrt{P_e}} \varphi(\bar{x},\bar{y})$$

and

$$\Theta(x,y) = \frac{T(\bar{x},\bar{y}) - T_\infty}{T_0 - T_\infty} \tag{4}$$

where L is an arbitrary length scale, $P_e = \frac{U_0 L}{\alpha}$ is the Péclet number (a dimensionless number used in the case of mixed convection), where $U_0 = BL^m$ is the reference velocity ($B > 0$) and T_0 is the reference temperature, and $T_0 = T_\infty + |A|L^m$, we obtain the following equations (see [5]):

$$\frac{1}{P_e} \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = \frac{R_a}{P_e} \frac{\partial \Theta}{\partial y} \tag{5}$$

and

$$\frac{\partial \Psi}{\partial y} \frac{\partial \Theta}{\partial x} - \frac{\partial \Psi}{\partial x} \frac{\partial \Theta}{\partial y} = \frac{1}{P_e} \frac{\partial^2 \Theta}{\partial x^2} + \frac{\partial^2 \Theta}{\partial y^2} \tag{6}$$

where R_a is the Rayleigh number, $R_a = \frac{gk\beta L(T_0 - T_\infty)}{\mu} = \frac{gk\beta|A|L^{m+1}}{\mu}$. Suppose that the Péclet number is very large, then the resulting temperature boundary-layer is analogous to that in classical boundary-layer theory. Therefore, by letting $P_e \rightarrow +\infty$ in Eqs. (5) and (6), we obtain the following boundary-layer approximation equations (see [5]):

$$\frac{\partial^2\Psi}{\partial y^2} = \varepsilon\frac{\partial\Theta}{\partial y} \tag{7}$$

and

$$\frac{\partial\Psi}{\partial y}\frac{\partial\Theta}{\partial x} - \frac{\partial\Psi}{\partial x}\frac{\partial\Theta}{\partial y} = \frac{\partial^2\Theta}{\partial y^2} \tag{8}$$

where $\varepsilon = \frac{R_a}{P_e}$ is the mixed convection parameter.

By using the non-dimensionality (4), the boundary conditions become (see [5]):

On the plate

$$\Psi(x,0) = 0, \quad \Theta(x,0) = x^m \ (A > 0) \text{ or } \Theta(x,0) = -x^m \ (A < 0),\ 0 < x < \infty,$$

and far from the plate

$$\frac{\partial\Psi}{\partial y}(x,y) \rightarrow x^m, \quad \Theta(x,y) \rightarrow 0, \text{ as } y \rightarrow +\infty,\ 0 < x < \infty$$

By eliminating Θ in Eqs. (7) and (8), we obtain $\varepsilon\Theta = \frac{\partial\Psi}{\partial y} - x^m$ and

$$\frac{\partial^3\Psi}{\partial y^3} + \frac{\partial\Psi}{\partial x}\frac{\partial^2\Psi}{\partial y^2} - \frac{\partial\Psi}{\partial y}\left(\frac{\partial^2\Psi}{\partial x\partial y} - mx^{m-1}\right) = 0 \tag{9}$$

with the boundary conditions

$$\frac{\partial\Psi}{\partial x}(x,0) = -\omega x^{\frac{m-1}{2}}$$
$$\frac{\partial\Psi}{\partial y}(x,0) = x^m \tag{10}$$

and

$$\frac{\partial\Psi}{\partial y}(x,+\infty) = 0$$

We are looking for similarity solutions of Equation (9), with the boundary conditions (10) by introducing the dimensionless similarity variables (see [5]):

$$\Psi(x,y) = \sqrt{2}x^{\frac{m+1}{2}}F(t)$$

with

$$t(x,y) = \frac{1}{\sqrt{2}}x^{\frac{m-1}{2}}y \tag{11}$$

and

$$\Theta(x,y) = x^m\theta(t)$$

where F is a function to be determined. The function Ψ is called a similarity solution of (9), and the variable t is called a similarity variable.

Therefore, we obtain the following third order autonomous nonlinear differential equation

$$F''' + (1+m)FF'' + 2m(1-F')F' = 0 \tag{12}$$

with the boundary conditions:

$$F(0) = a_0 \in \mathbb{R},\ F'(0) = b < 0 \text{ and } F'(+\infty) = \lambda \in \{0,1\}$$

Then Ψ is a solution of Eq. (9) if and only if F is a solution of Eq. (12).

Remark 1.

a. When $m > -1$, Eq. (12) is equivalent to the following equation

$$f''' + ff'' + \beta f'(f'-1) = 0 \tag{13}$$

with $f(t) = \sqrt{1+m}\mathrm{F}\left(\frac{t}{\sqrt{1+m}}\right)$ and β is a constant depending on m. In this case, if $m = 0$ i.e. $\beta = 0$, then Eq. (12) or Eq. (13) reduces to the Blasius equation (as it is we well-know).

b. For $m = -1$, Eq. (12) reduces to $F''' + 2F'(F'-1) = 0$. See the end of this article. This equation has a first integral given by

$$H_F = 3F''^2 + 2F'^2(2F'-3).$$

2. THE BOUNDARY VALUE PROBLEM ($P_{\beta;a,b,\lambda}$)

In this section, let $m > -1$. We are interested in studying solutions of Eq. (13). Associated with this equation we have the following boundary value problem:

$$(P_{\beta;a,b,\lambda}) \qquad \begin{cases} f''' + ff'' + \beta f'(f'-1) = 0, \\ f(0) = a = a_0\sqrt{1+m}, \\ f'(0) = b < 0, \\ \lim_{t\to+\infty} f'(t) = \lambda, \end{cases}$$

where $f = f(t)$ for $t \in [0,+\infty)$, $\lambda \in \{0,1\}$ and the parameter $\beta > 1$ is a temperature power-law profile and $b = 1 + \varepsilon$ is the mixed convection parameter. Note that if $\lambda \notin \{0,1\}$, then the problem ($P_{\beta;a,b,\lambda}$) does not have any solutions (see [6]).

In the study of different cases, the case where $a \in \mathbb{R}$, $b \in \mathbb{R}$, $0 < \beta < 1$ see [7-9]. The case $a \in \mathbb{R}$, $b \in \mathbb{R}$, $\beta < 0$ was treated in [10-12]. In [13] and [14], some theoretical results were found with $-2 < \beta < 0$, and $b < 0$, the method used by the authors allows them to prove the existence of a convex solution for the case $a = 0$, but for the case $a \neq 0$ it seems that the method used for case $a = 0$ is not applicable. In what follows, we denote by f_c a solution of initial value problem below and by $[0, T_c)$ the right maximal interval of its existence

$$(Q_{\beta;a,b,c}) \qquad \begin{cases} f''' + ff'' + \beta f'(f'-1) = 0, \\ f(0) = a, \\ f'(0) = b, \\ f''(0) = c. \end{cases}$$

The approach used to study the problem $(P_{\beta;a,b,\lambda})$ is a shooting method. It consists of finding the values of a real parameter c for which f_c exists on $[0,+\infty)$ i.e. $T_c = +\infty$ and such that $f_c' \to \lambda \in \{0,1\}$ as $t \to +\infty$.

2.1. ON BLASIUS INEQUALITIES

Let $I \subseteq \mathbb{R}$ be an interval and $f: I \to \mathbb{R}$ be a function.

Definition 1. ([11]) We say that f is a subsolution (resp. a supersolution) of the Blasius equation if f is of class C^3 and if $f''' + ff'' \leq 0$ on I (resp. $f''' + ff'' \geq 0$ on I).

Definition 2. ([11]) Let $\varepsilon > 0$. We say that f is a ε-subsolution (resp. a ε-supersolution) of the Blasius equation if f is of class C^3 and if $f''' + ff'' \leq \varepsilon$ on I (resp. $f''' + ff'' \geq \varepsilon$ on I).

Proposition 1. ([11]) Let $t_0 \in \mathbb{R}$. There does not exist nonpositive concave subsolution of the Blasius equation on the interval $[t_0, +\infty)$.

Proposition 2. ([11]) Let $t_0 \in \mathbb{R}$. There does not exist ε-subsolution of the Blasius equation on the interval $[t_0, +\infty)$.

2.2 *PRELIMINARY RESULTS*

Proposition 3. Let us suppose that f is a solution of Equation (13) on the maximal interval $I = (T_-, T_+)$, for all $t \in I$.
(1) Let $H_0 = f'' + f(f' - \beta)$. Then $H_0' = (1-\beta)f'^2$;
(2) Let $H_1 = f'' + f(f' - 1)$. Then $H_1' = (1-\beta)f'(f'-1)$;
(3) Let $H_2 = 3f''^2 + (2f' - \beta)f^2$. Then $H_2' = -6ff''^2$;
(4) Let $H_3 = 2ff'' - f'^2 + (2f' - 3)f(f'-1)$. Then $H_3' = 2(2-\beta)ff'^2$;
(5) Let $H_4 = f'' + ff'$. Then $H_4' = (1-\beta)f'^2 + \beta f'$;
(6) Let $H_5 = f' + \frac{1}{2}f^2$. Then $H_5' = H_4$.

Proof: Statements 1-5 follow immediately from Eq. (13) and for statement 6 we get it directly.

Proposition 4. Let f be a solution of the Eq. (13) on some maximal interval $I = (T_-, T_+)$ and $\beta > 1$.
(1) If F is any anti-derivative of f on I, then $(f''e^F)' = -\beta f'(f'-1)e^F$;
(2) If $T_+ = +\infty$ and if $f'(t) \to \lambda \in \mathbb{R}$ as $t \to +\infty$, and if, moreover, f is of constant sign at infinity, then $f''(t) \to 0$ as $t \to +\infty$;
(3) If $T_+ = +\infty$ and if $f'(t) \to \lambda \in \mathbb{R}$ as $t \to +\infty$, then $\lambda = 0$ or $\lambda = 1$;

(4) If $T_+ < +\infty$, then f'' and f' are unbounded near T_+ ;
(5) If there exists a point $t_0 \in I$ satisfying $f''(t_0) = 0$ and $f'(t) = \mu$, where $\mu = 0$ or 1 then for all $t_0 \in I$, we have $f(t) = \mu(t - t_0) + f(t_0)$;
(6) If $f'(t) \to 0$ as $t \to +\infty$, then $f(t)$ does not tend to $-\infty$ or $+\infty$ as $t \to +\infty$.

Proof: The first statement follows immediately from Eq. (13). For the proof of Statements 2-5, see [11]. For Statement 6, assume for the sake of contradiction that $f(t) \to -\infty$ as $t \to +\infty$. So, the function H_0 is nonincreasing for all $t \geq 0$, and hence $H_0(0) \geq H_0(0)$. Since $f'(t) \to 0$ as $t \to +\infty$, we deduce from Statement 2 of Proposition 4 that

$$H_0(0) = c + a(b - \beta) \geq -\beta f(t).$$

This leads to a contradiction. For $f(t) \to +\infty$ as $t \to +\infty$. There exists a point $t_0 > 0$ where the function H_1 is nondecreasing for $t \geq t_0$. Also from Statement 2 of Proposition 4, we obtain $H_1(t_0) \leq -f(t)$ as $t \to +\infty$ which is a contradiction.

2.3 *RESULTS AND DISCUSSION*

Consider the problem $(P_{\beta;a,b,\lambda})$, with $b < 0$, which represents the opposing mixed convection. We are interested in concave, convex and convex-concave solutions of this problem. As mentioned above, we have used the shooting method. Define the following sets:

$C_0 = \{c \leq 0;\ f_c'' \leq 0\ on\ [0, T_c)\}$

$C_1 = \{c \geq 0;\ f_c' \leq 0\ and\ f_c'' \geq 0\ on\ [0, T_c)\}$

$C_2 = \{c \geq 0;\ \exists t_c \in [0, T_c), \exists \varepsilon_c > 0\ s.t\ f_c' < 0\ on\ (0, t_c),\ f_c' > 0\ on\ (t_c, t_c + \varepsilon_c)\ and\ f_c'' > 0\ on\ [0, t_c + \varepsilon_c)\}$

$C_3 = \{c \geq 0;\ \exists s_c \in [0, T_c), \exists \varepsilon_c > 0\ s.t\ f_c'' > 0\ on\ [0, s_c),\ f_c'' < 0\ on\ (s_c, s_c + \varepsilon_c)\ and\ f_c' < 0\ on\ [0, s_c + \varepsilon_c)\}$

Lemma 1. Let $\beta > 1$. If $c \leq 0$. Then f_c is a concave and decreasing solution of $(P_{\beta;a,b,-\infty})$ on $[0, T_c)$, with $T_c < +\infty$.

Proof: If $c \leq 0$, then it follows from Proposition 4, Statement 1, that $f_c''(t) < 0$ and $f_c'(t) < 0$ for all $t \in [0, T_c)$, we have

$$f_c'''(t) + f_c'(t) f_c''(t) = -\beta f_c'(t)(f_c'(t) - 1) < 0$$

Therefore, for all $a \in \mathbb{R}$, f_c is decreasing and a nonpositive concave subsolution of the Blasius equation on $[t_0, T_c)$.Thanks to Proposition 1, we have $T_c < +\infty$, with $f_c'(t) \to -\infty$ as $t \to T_c$.

Remark 2. We note that C_0, C_1, C_2 and C_3 are disjoint nonempty subsets of $\mathbb{R}$, and we have $C_1 \cup C_2 \cup C_3 = (0, +\infty)$ (see [11]), and thanks to Lemma 1, we have if f_c starts concave, it remains concave, thus $C_0 = (-\infty, 0]$, and it is clear for all $a \in \mathbb{R}$, the problems $(P_{\beta;a,b,0})$ and $(P_{\beta;a,b,1})$ have no concave solution.

Lemma 2. ([11]) Let $\beta > 1$. Then f_c is a convex solution of the boundary value problem $(P_{\beta;a,b,0})$ on $(0, +\infty)$ if and only if $c \in C_1$.

Lemma 3. ([8]) Let $\beta > 1$. If $c \in C_3$, then f_c is a convex-concave, decreasing solution of $(P_{\beta;a,b,-\infty})$ on $[0, T_c)$ with $T_c < +\infty$ (see Fig. 1).

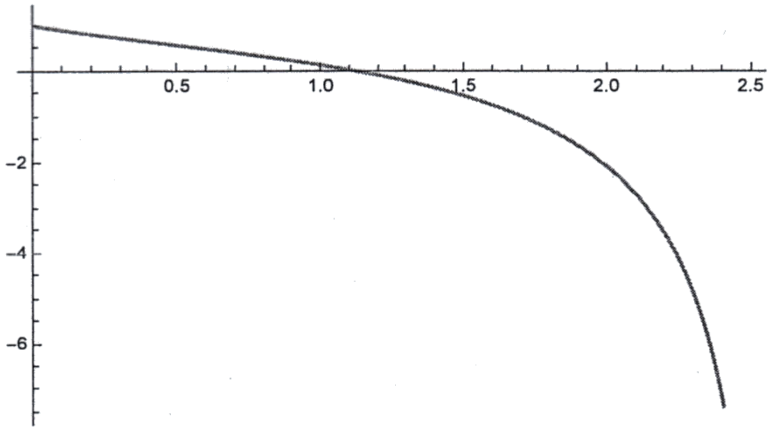

Figure 1. $\boldsymbol{\beta = 1.1, a = 1, b = -1, c = 1}$

Remark 3. From Proposition 4, (Statements 1,3 and 5), if $c \in C_2$, there are only three possibilities for the solution of the problem $(Q_{\beta;a,b,c})$. More precisely,
(1) f_c is convex and $f_c'(t) \to +\infty \ as \ t \to T_c$;
(2) there exists a point $t_0 \in [0, T_c)$ such that $f_c''(t_0) = 0$ and $f_c'(t_0) > 1$;
(3) f_c is a convex solution of $(P_{\beta;a,b,1})$.

Proposition 5. Let $\beta > 1$. There does not exist a convex solution of $(P_{\beta;a,b,+\infty})$.

Proof: Assume that f_c is convex solution of $(P_{\beta;a,b,+\infty})$, i.e. $f_c'(T_c) = +\infty$. There exists $t_0 \in [0, T_c)$ such that $f_c'(t) > 1$ for all $t \in [t_0, T_c)$. We have

$$f_c'''(t) + f_c(t)f_c''(t) = -\beta f_c'(t)(f_c'(t) - 1) < -\beta f_c'(t)(f_c'(t) - 1) = -\varepsilon$$

then f_c is a ε-subsolution of the Blasius equation on $[t_0, T_c)$. Therefore, from Proposition 2, we have $T_c < +\infty$. Furthermore, the function H_1 is decreasing for $t > t_0$. Hence for all $t \in [t_0, T_c)$, $H_1(t) \leq H_1(t_0)$. Then we have

$$f_c(t)(f_c'(t) - 1) < f_c''(t) + f_c(t)(f_c'(t) - 1) \leq H_1(t_0) < f_c''(t_0) + f_c(t_0)f_c'(t_0)$$

which is a contradiction with the fact that $f_c'(t) \to +\infty$ as $t \to T_c$.

Proposition 6. Let $\beta > 1$ and let f_c be a solution of (13) on some right maximal interval $[\tau; T+)$. If there exists $t_0 \in [\tau, T_c)$ such that $f_c'(t_0) = 0$ and $f_c''(t_0) < 0$, then for all $t > t_0$, $f_c''(t) < 0$ (see Fig. 1).

Proof. Let f_c be a solution of (13) on some right maximal interval of existence $[\tau, T+)$. Let $t_0 \in [\tau, T_c)$ be such that $f_c'(t_0) = 0$, $f_c''(t_0) < 0$, so t_0 is a maximum. We suppose that there exists $t_1 > t_0$ such that $f_c''(t_1) = 0$. It follows that $f_c' < 0$ on (t_0, t_1). Then we have

$$f_c'''(t_1) = -\beta f_c'(t_1)(f_c'(t_1) - 1) < 0,$$

which yields a contradiction.

2.3.1. The case $a \leq 0$.

Proposition 7. Let $\beta > 1$. The problem $(P_{\beta;a,b,1})$ has no convex solution.

Proof: Assume for the sake of contradiction that f is a convex solution of $(P_{\beta;a,b,1})$ on $[0,+\infty)$. Then there exists $t_0 \in (0,+\infty)$ such that $f(t_0) = 0$ and $0 < f'(t) < 1$ on $[t_0,+\infty)$. So the function H_1 is increasing for all $t > t_0$, i.e. $H_1(t) \geq H_1(t_0)$ for $t > t_0$. It then follows that $f''(t) - f''(t_0) \geq -f(t)(f'(t) - 1) > 0$, and we obtain a contradiction for t large enough since $f''(t) \to 0$, $f(t) > 0$ and $f'(t) < 1$.

Remark 4. Thanks to Propositions 5, 7 and Remark 3, if $c \in C_2$ there exists a point $t_0 \in [0, T_c)$ such that $f_c''(t_0)= 0$ and $f_c'(t_0) > 1$, i.e. we have only the second case of Remark 3 (see Figs. 2-3).

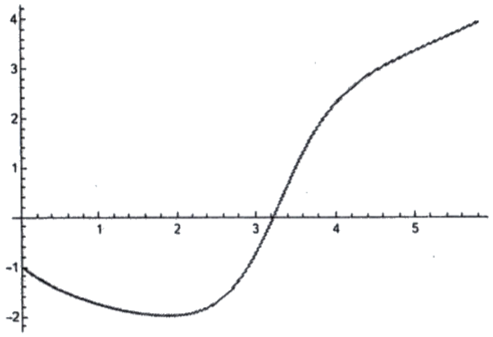

Figure 2. $\beta = 1.5, a = -1, b = -1.2, c = 1.4245$

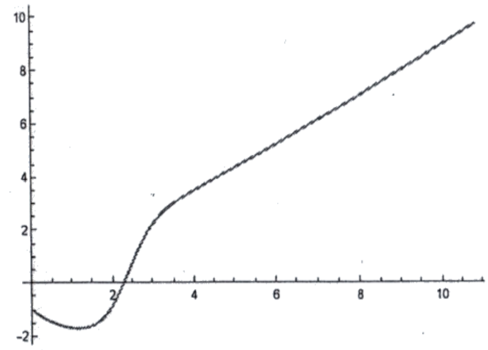

Figure 3. $\beta = 1.5, a = -1, b = -1.2, c = 1.5$

Lemma 4. Let $c \in C_2$ and $\beta > 1$. If there exists $t_1 \in [0, T_c)$ such that $f_c''(t_1) = 0$ and $f_c(t_1) < 0$, then $f_c'(t_1) > \frac{\beta}{\beta-1}$.

Proof: Let $c \in C_2$, then there exists $t_0 \in [0, T_c)$ which is the first point where $f_c'(t_0) = 0$, $f_c(t_0) < 0$. From Remark 3, there exists $t_1 > t_0$ such that $f_c''(t_1) = 0$. To reach a contradiction, suppose that $f_c(t_1) < 0$ and $f_c'(t_1) \leq \frac{\beta}{\beta-1}$. Thus the function H_4 is increasing on $[t_0, t_1]$. Therefore $f_c''(t_1) \leq f_c(t_1)f_c'(t_1)$, which is a contradiction.

Remark 5. Thanks to the previous Lemma, if $c \in C_2$ and if there exists $t_1 \in (t_0, T_c)$ such that $f_c'(t_1) \leq \frac{\beta}{\beta-1}$, then $f_c(t_1) > 0$. In this case, the problem $(P_{\beta;a,b,0})$ has no convex-concave solution on $[0,+\infty)$. Indeed, the function H_4 is increasing on $[t_1,+\infty)$, so we have $H_4(t_1) \leq H_4(t)$. By passing to the limit as $t \to +\infty$, thanks to Proposition 4 (Statements 2, 4 and 6), this leads to a contradiction, since $H_4(t_1) = f_c(t_1)\, f_c'(t_1) > 0$ and $H_4(+\infty) = 0$. Also in the same case the problem $(P_{\beta;a,b,-\infty})$ has no convex-concave solution on $[0, T_c)$, with $T_c < +\infty$, since there exists $t_* \in [t_1, T_c)$ such that $f_c'(t_*) = 0$. Thus the function H_4 is increasing on (t_1, t_*), hence $H_4(t_1) \leq H_4(t_*)$. It follows that $f_c(t_1)\, f_c'(t_1) \leq f_c''(t_*)$ and the contradiction occurs here too.

Proposition 8. The problem $(P_{\beta;a,b,0})$ has no negative convex- concave solution.

*Proof***:** Suppose for the sake of contradiction that f is a negative convex-concave solution of $(P_{\beta;a,b,0})$ on $[0,+\infty)$. Thanks to Lemma 1 and since $f'(+\infty) = 0$, there exists $t_0 \in [0,+\infty)$

such that $f_c'(t_0) = 0$. So the function H_2 is increasing for all $t_1 > t_0$. i.e. $H_2(t_0) \leq H_2(t)$. It follows that for all $t > t_0$

$$3f''^2(t_0) \leq 3f''^2(t) + \beta f'^2(t)(2f'(t) - 3),$$

and from Proposition 4 (Statements 2 and 4). This is a contradiction to the fact that $H_2(+\infty) = 0$, since $f'(t) \to 0$ and $f''(t) \to 0$ as $t \to +\infty$.

Remark 6. If the problem $(P_{\beta;a,b,0})$ has a convex-concave solution, then this solution changes its sign.

Lemma 5. If $c \in C_1 \cup C_3$, then there exists $c_* \leq -b\sqrt{\frac{\beta(3-2b)}{3}}$ such that $C_1 \cup C_3 \subset (0, c_*)$. Moreover, f_c is a negative solution.

Proof: Let $c \in C_1$. From Proposition 4 Statement 4, we know that the function H_2 is increasing on $[0, +\infty)$. Hence $H_2(0) \leq H_2(t)$. Thanks to Proposition 4, Statements 2 and 4, by passing to the limit as $t \to +\infty$, we have $H_2(+\infty) = 0$. So, we get $3c^2 + \beta b^2(2b - 3) \leq 0$, and so the result follows. Now, let $c \in C_3$. There exists $t_0 \in (0, T_c)$ such that $f_c''(t_0) = 0$. Hence H_2 is increasing on $[0, t_0)$, establishing the result. Next, f_c is a negative solution because $a \leq 0$ and $f' < 0$.

Remark 7. It follows from Lemma 5 that $[c_*, +\infty) \subset C_2$ and here the solution f_c changes convexity (See Fig. 4).

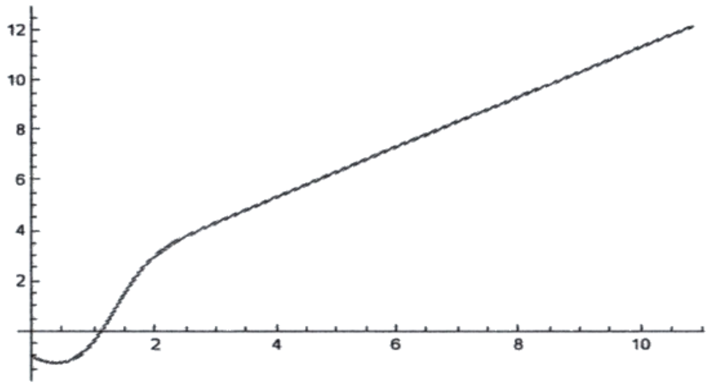

Figure 4. $\boldsymbol{\beta = 1.5, a = -1, b = -1.2, c = 3}$

Proposition 9. Let $1 < \beta \leq 2$. If $c \in C_2$, then there does not exist a nonpositive solution of the problem $(P_{\beta;a,b,-\infty})$.

Proof: Let $1 < \beta \leq 2$, $c \in C_2$ and let f_c be a nonpositive solution of the problem $(P_{\beta;a,b,-\infty})$. From Remark 4 and Propositions 6 and 8, there exist $t_0, t_1 \in [0, T_c)$ such that $t_0 < t_1$, $f_c''(t_0) > 0$ and $f_c''(t_1) < 0$. Thus, the funtion H_3 is decreasing on $[t_0, t_1]$. We have

$$-\beta f_c^2(t_0) > 2f_c(t_0) f_c''(t_0) - \beta f_c^2(t_0) \geq 2f_c(t_1) f_c''(t_1) - \beta f_c^2(t_1) > -\beta f_c^2(t_1),$$

and so $f_c(t_0) > f_c(t_1)$, which is a contradiction.

Corollary 1. Let $1 < \beta \leq 2$. The problem $(P_{\beta;a,b,\lambda})$ has infinitely many solutions which change the convexity and the sign at the same time (see Figs. 2-3).

Proof. The proof follows from Propositions 5, 7, 8, 9, and Remarks 3, 7.

Theorem 1. Let $\beta > 1, a \leq 0$ and $b < 0$.
(1) The problem $\left(P_{\beta;a,b,-\infty}\right)$ has infinitely many negative concave solutions on $[0, T_c)$, with $T_c < +\infty$.
(2) The problem $\left(P_{\beta;a,b,0}\right)$ has at least one negative convex solution and no negative convex-concave solution on $[0, +\infty)$.
(3) The problem $\left(P_{\beta;a,b,1}\right)$ has no convex solutions on $[0, +\infty)$.
(4) The problem $\left(P_{\beta;a,b,+\infty}\right)$ has no convex solutions on $[0, +\infty)$.

Proof: The first result follows from Lemma 1. The second follows Remark 2, Lemma 2 and Proposition 8. The third result follows from Proposition 7 while the last result follows from Proposition 5.

2.3.2 The case $a > 0$.

Let us divide the sets C_2 and C_3 into the following two subsets:

$$C_{2.1} = \{c \in C_2;\ f_c' > 0 \ on\ [t_c, T_c)\};$$
$$C_{2.2} = \{c \in C_2;\ \exists s_c > t_c\ s.t\ f_c' > 0\ on\ [t_c, T_c)\ and\ f_c'(s_c) = 0\};$$
$$C_{3.1} = \{c \in C_3;\ f_c(s_c) < 0\};$$
$$C_{3.2} = \{c \in C_3;\ f_c(s_c) > 0\}.$$

Lemma 6. If $c \in C_1 \cup C_2 \cup C_{3.1}$, then there exists $c_0 \geq -ab$, such that $c > c_0$.

Proof: From Proposition 4, Statement 4, if $c \in C_1$, then $T_c = +\infty$, $f_c'(t) < 0$ on $[0, +\infty)$ and $f_c'(t) \to 0$ as $t \to +\infty$. The function H_4 is decreasing on $[0, +\infty)$, and so we have $c + ab > 0$. If $c \in C_2 \cup C_{3.1}$, then there exists $t_0 \in [0, T_c)$ such that $f_c'(t_0) = 0$ or $f_c(t_0) = 0$. Since H_4 is decreasing on $[0, t_0)$, it follows that $c + ab \geq f_c''(t_0) > 0$.

Remark 8. If $c \leq -ab$, then $c \in C_{3.2}$ and from Propositions 1 and 6, $T_c < +\infty$. Thus $C_{3.2} \neq \emptyset$, and the convex part of the solution f_c is positive (See Figs. 5-6).

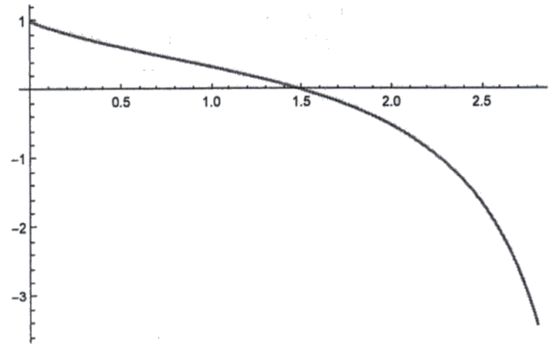

Figure 5. $\boldsymbol{\beta = 1.1, a = 1, b = -1, c = 1}$

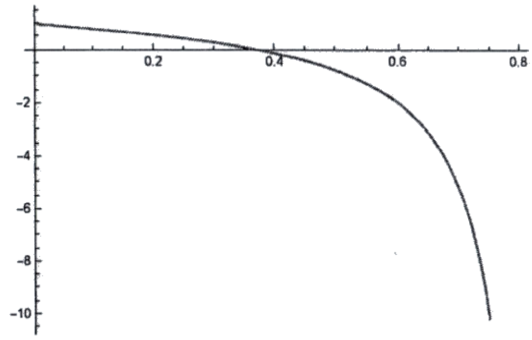

Figure 6. $\boldsymbol{\beta = 5.5, a = 1, b = -2, c = 1}$

Lemma 7. If $c \in C_1 \cup C_{2.1}$ and $b \geq -\frac{1}{2}a^2$, then the solution f_c is positive on $[0, +\infty)$, and there exists $c_* \geq -b\sqrt{\frac{\beta(3-2b)}{3}}$, for $c > c_*$.

Proof: Let $c \in C_1 \cup C_{2.1}$. By the definition of C_1 and $C_{2.1}$, thanks to Proposition 4, Statement 4 and Proposition 5, it follows that $T_c = +\infty$ and f_c' is bounded. If we suppose there exists

$t_0 \in [0, T_c)$ such that t_0 is the first point where $f_c(t_0) = 0$, $f_c'(t) < 0$ and $f_c''(t_0) > 0$. The function H_4 is decreasing on $[0, t_0]$. We have $H_4(t_0) = f_c''(t_0) > 0$. Therefore H_5 is strictly increasing on $[0, t_0]$ and so we obtain $b + \frac{1}{2}a^2 < f_c'(t_0) < 0$. This is a contradiction. On the other hand, since the solution is positive on $[0, +\infty)$, the function H_2 is strictly decreasing for $t \geq 0$. Thus, we obtain $3c^2 + \beta b^2(2b-3) > 0$, which implies that $c > -b\sqrt{\frac{\beta(3-2b)}{3}}$.

Remark 9. If $c \in C_{2.2}$ and $b \geq -\frac{1}{2}a^2$, the solution f_c is positive on $[0, t_1)$, where t_1 is the point such that $t_1 > s_c$, with $f_c(t_1) = 0$ and s_c is as in the definition of $C_{2.2}$.

Proposition 10. Let $c \in C_3$ and $b \geq -\frac{1}{2}a^2$. Then $C_{3.1} = \emptyset$.

Proof: Let $c \in C_3$ and $b \geq -\frac{1}{2}a^2$. Then there exists $t_0 \in [0, T_c)$ with $f_c(t_0) = 0$ and $f_c'(t_0) < 0$. Assume on the contrary that $f_c''(t_0) > 0$, so the function H_4 is strictly decreasing on $[0; t_0]$ and $H_4(t_0) = f_c''(t_0) > 0$. Then for all $t \in [0, t_0]$, $H_4 > 0$. It follows that H_5 is strictly increasing on $[0, t_0)$, which implies that $b + \frac{1}{2}a^2 < f_c'(t_0) < 0$. This is a contradiction, and the proof is complete.

Proposition 11. Let $1 < \beta \leq 2$ and $b \geq -\frac{1}{2}a^2$. Then $C_{2.2} = \emptyset$.

Proof: Let $1 < \beta \leq 2$, $b \geq -\frac{1}{2}a^2$ and $c \in C_{2.2}$. There exists $t_c \in [0, T_c)$ such that $t_c < s_c$ with $f_c(t_c) > 0$, $f_c'(t_c) = 0$, and $f_c''(t_c) > 0$, where t_c is as in definition of C_2 and s_c is as in the definition of $C_{2.2}$. Since the function H_3 is increasing on $[t_c, s_c]$, we have

$$-\beta f_c^2(t_c) < 2f_c(t_c)\, f_c''(t_c) - \beta f_c^2(t_c) \leq 2f_c(s_c)\, f_c''(s_c) - \beta f_c^2(s_c) < -\beta f_c^2(s_c).$$

This implies that $f_c(t_c) > f_c(s_c)$, and this is a contradiction (see Fig. 7 for $\beta > 2$).

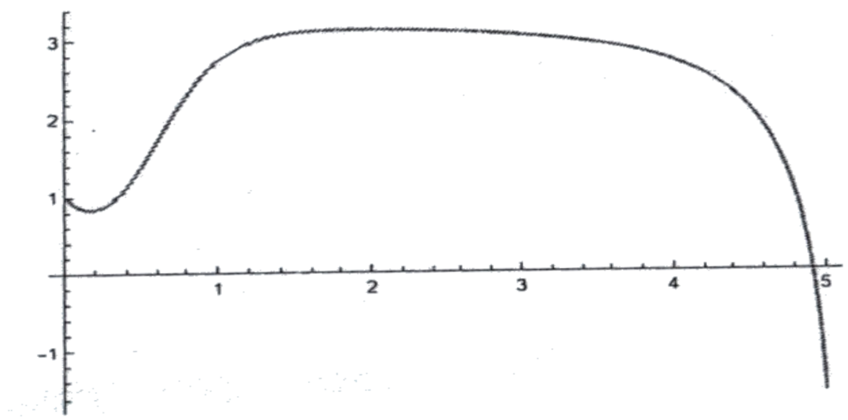

Figure 7. $\boldsymbol{\beta = 5.5}$, $\boldsymbol{a = 1, b = -1, c = 15}$

Remark 10. There exists $c_* > 0$, if $c < c_*$, and $b \geq -\frac{1}{2}a^2$, Then there exits $t_0 \in [0, T_c)$ such that $f_c(t_0) = 0$, $f_c'(t_0) < 0$ and $f_c''(t_0) < 0$, so $c \in C_{2.2} \cup C_{3.2}$. Since from Proposition 4, Statement 4 and Proposition 5, if $c \in C_{2.1}$ then $T_c = +\infty$. We can divide the set $C_{2.1}$ into the following two subsets:

$$C_{2.1.0} = \{c \in C_{2.1};\ f_c'(t) \to 0 \text{ as } t \to +\infty\};$$
$$C_{2.1.1} = \{c \in C_{2.1};\ f_c'(t) \to 1 \text{ as } t \to +\infty\}.$$

Proposition 12. Let $1 < \beta \leq 2$. If $b \geq -\frac{1}{2}a^2$, then $C_{2.1.0} = \emptyset$.

Proof: Let $1 < \beta \leq 2$, $b \geq -\frac{1}{2}a^2$ and $c \in C_{2.1.0}$. We deduce from Proposition 11 that the function H_3 is increasing on $[t_c, +\infty)$, where t_c is as in the definition of C_2, we then have for $t > t_c$,

$$-\beta f_c^2(t_c) < 2f_c(t_c)\ f_c''(t_c)\ -\beta f_c^2(t_c) \leq 2f_c(t)\ f_c''(t) - f_c'^2(t) + (2f_c'(t) - \beta) f_c^2(t).$$

From Proposition 4, Items 2, 4 and 6, it follows that $f_c(t) \to l < +\infty$ as $t \to +\infty$, which implies that $f_c(t_c) > l$ as $t \to +\infty$, which is a contradiction.

Lemma 8. Let $1 < \beta \leq 2$. *If* $c \in C_1 \cup C_{2.2} \cup C_{2.1.0}$. Then there exists $c^* \leq \frac{b^2 + (\beta - 2b)a^2}{2a}$, such that $c < c^*$.

Proof: Let $1 < \beta \leq 2$. If $c \in C_1 \cup C_{2.2} \cup C_{2.1.0}$. As we have seen above from Proposition 4 Statements 2 and 4, either there exists $t_0 \in [0, T_c)$ such that $f_c(t_0) = 0$ or $f_c'(t_0) = 0$ if $T_c < +\infty$. And if $T_c = +\infty$, we have $f_c'(t) \to 0$ as $t \to +\infty$. From Proposition 4, Statement 6, it follows that the function H_3 is strictly increasing on $[0, t_0)$ or on $[0, +\infty)$. We then get $2ac - b^2 + (\beta - 2b)a^2 < 0$, which implies that $c < \frac{b^2 + (\beta - 2b)a^2}{2a}$.

Remark 11. From the previous proposition, we deduce that there exists $c^* > 0$, such that $[c^*, +\infty) \subset C_{2.1.1}$. Thus $C_{2.1.1} \neq \emptyset$ (See Figs. 8-9).

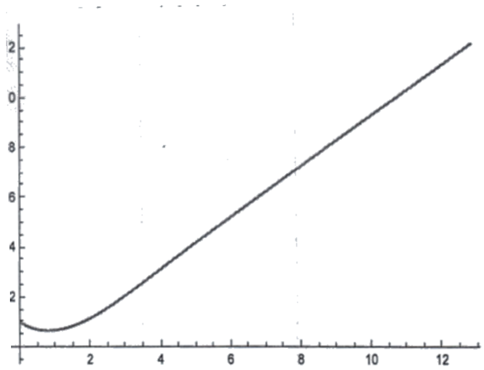

Figure 8. $\boldsymbol{\beta = 1.1, a = 1, b = -1, c = 2.2}$

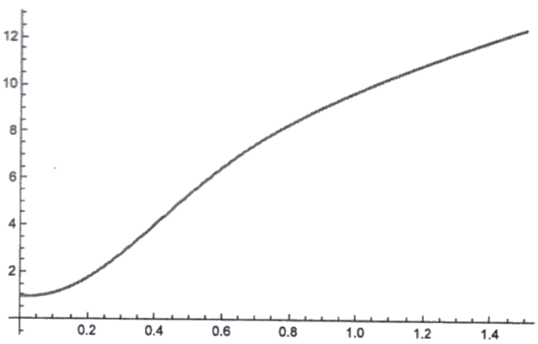

Figure 9. $\boldsymbol{\beta = 1.1, a = 1, b = -1, c = 55}$

Theorem 2. Let $\beta > 1, a > 0$ and $b < 0$.
(1) The problem $(P_{\beta;a,b,-\infty})$ has infinitely many concave and convex-concave solutions on $[0, T_c)$, with $T_c < +\infty$. If, in addition $b \geq -\frac{1}{2}a^2$, then the convex part of these solutions will be non-negative.
(2) The problem $(P_{\beta;a,b,0})$ has at least one convex solution on $[0, +\infty)$. If, in addition, $b \geq -\frac{1}{2}a^2$, then this solution becomes non-negative.
(3) If $\beta \leq 2$, the problem $(P_{\beta;a,b,1})$ has infinitely many positive solutions on $[0, +\infty)$.
(4) The problem $(P_{\beta;a,b,+\infty})$ has no solution on $[0, T_c)$.

Proof: The first statement follows from Lemma 1 and Remark 8. The second one follows from Remark 2, Lemma 2 and Lemma 7, while the third result follows from Propositions 5, 11, 12 and Remark 11. The last result follows from Proposition 5.

3. THE BOUNDARY VALUE PROBLEM $(P_{a_0,b,\lambda})$

In this section, where $m = -1$, as we observed in Remark 1, Eq. (12) reduces to the following equation

$$F''' + 2F'(F' - 1) = 0 \tag{14}$$

and we consider the boundary value problem:

$$(P_{a_0,b,\lambda}) \qquad \begin{cases} F''' + 2F'(F' - 1) = 0, \\ F(0) = a_0 \in \mathbb{R}, \\ F'(0) = b < 0, \\ F'(+\infty) = \lambda \in \{0,1\} \end{cases}$$

Statements 2-5 of Proposition 4 remain valid in this case. For any function F twice differentiable over an interval I, let

$$H_F = 3F''^2 + 2F'^2(2F' - 3).$$

Lemma 9. If F is a solution of Eq. (14) on an interval I, then the function H_F is constant on I. In other words, there exists μ ∈ ℝ such that

$$\forall t \in \mathbb{R};\ 3F''^2 + 2F'^2(2F' - 3) = \mu \tag{15}$$

Proof. We have

$$\begin{aligned} H'_F &= 6F''F''' + 4F'F''(2F' - 3) \\ &= 6F''[2F'(1 - F')] + 8F'^2F'' - 12F'F'' + 4F'^2F'' = 0 \end{aligned}$$

Remark 12. Let C_0, C_1, C_2 and C_3 be the sets defined in the second section and let us denote by F_c the solution of Eq. (14) satisfying the initial conditions $F_c(0) = a_0$, $F_c'(0) = b < 0$ and $F_c''(0) = c$. From Eq. (15), there exists $c_{**} = -b\sqrt{\frac{\beta(3-2b)}{3}}$ such that $C_3 = (0, c_{**})$, $C_1 = c_{**}$ and $C_2 = (c_{**}, +\infty)$.

Lemma 10. For all $a_0 \in \mathbb{R}$, we have $C_0 = (-\infty, 0]$ and $F_c'(t) \to -\infty$ as $t \to T_c$.

Proof: Let $c \in C_0$. Assume for the sake of contradiction that there exists $t_0 \in [0, T_c)$ such that $F_c''(t) \geq 0$ for all $t \geq t_0$ with $F_c'(t_0) < 0$. From Eq. (14), we have $F_c'''(t_0) = -2F_c'(t_0)(F_c'(t_0) - 1) < 0$. This is a contradiction, and so F_c is a concave decreasing solution and $F'(t) \to -\infty$, as $t \to T_c$.

Proposition 13. Let F_c be a solution of Eq. (14) on some right maximal interval $[\tau, T_+)$. If there exists $t_0 \in [\tau, T_+)$ such that $F_c'(t_0) = 0$ and $F_c''(t_0) < 0$, then for all $t > t_0$, $F_c''(t) < 0$.

Proof: Let F_c be a solution of (14) on some right maximal interval $[\tau, T_+)$. Let $t_0 \in [\tau, T_+)$ be such that $F_c'(t_0) = 0$ and $F_c''(t_0) < 0$. We suppose that there exists $t_1 > t_0$, such that t_1 is the first point where $F_c''(t_1) = 0$, it follows that $F_c' < 0$ on $(t_0, t_1]$ since we have $H_F(t_0) = H_F(t_1)$ and hence $3F_c''^2(t_0) \leq 2F_c'^2(t_1)(2F_c'(t_1) - 3)$, and this a contradiction.

Proposition 14. The problem $(P_{a_0,b,0})$ has a unique solution.

Proof: If F is a solution of $(P_{a_0,b,0})$, by integrating Equation (15) between 0 and $+\infty$, we obtain $F''(0) = -b\sqrt{\frac{\beta(3-2b)}{3}} = c_{**}$. Hence the problem $(P_{a_0,b,\lambda})$ has at most one solution. On the other hand, if F_c is the solution of the initial value problem $(Q_{a_0,b,c})$ on its right maximal interval of existence $[0, T_c)$. Multiplying Eq. (14) by F'' and integrating between 0 and t yields

$$\forall t \in [0, T_c), \qquad \frac{F''}{F'\sqrt{3-2F'}} = -\sqrt{\frac{2}{3}}$$

Integrating this equation, letting $X = \sqrt{3-2F'}$ and using the fact that $F'(0) = b$, we obtain

$$\forall t \in [0, T_c), \qquad \frac{\sqrt{3-2F'}-\sqrt{3}}{\sqrt{3-2F'}+\sqrt{3}} = ke^{-\sqrt{2}t},$$

so that

$$F'(t) = \frac{3}{2}\left[1 - \left(\frac{1+ke^{-\sqrt{2}t}}{1-ke^{-\sqrt{2}t}}\right)^2\right] = \frac{-6ke^{-\sqrt{2}t}}{(1-ke^{-\sqrt{2}t})^2}$$

where k is a constant which depends on b and such that $k = \frac{\sqrt{3-2b}-\sqrt{3}}{\sqrt{3-2b}+\sqrt{3}}$. It is clear that $0 < k < 1$. Integrating again, we finally deduce that, for $m = -1$, the problem $(P_{a_0,b,0})$ has exactly one solution convex given by

$$F(t) = a_0 - \frac{3\sqrt{2}}{1-k} + \frac{3\sqrt{2}}{1-ke^{-\sqrt{2}t}}$$

Proposition 15. For all $a_0 \in \mathbb{R}$, the problem $(P_{a_0,b,+\infty})$ has no solution.

Proof: Let F_c be a solution of the problem $(P_{a_0,b,+\infty})$. Then there exists $t_0 \in [0, T_c)$ such that $F_c'(t_0) = 0$, since H_F is constant, hence $H_F(0) = H_F(t_0) = H_F(t)$. By passing to the limit as $t \to T_c$ and thanks to Proposition 4, Statements 2 and 4, we have $3F_c''^2(t_0) = +\infty$. This is again a contradiction.

Proposition 16. For all $a_0 \in \mathbb{R}$, the problem $(P_{a_0,b,1})$ has no solution and the problem $(P_{a_0,b,0})$ has no convex-concave solution.

Proof: Let F be a solution of the problem $(P_{a_0,b,1})$. Then, there exists $t_0 \in [0, +\infty)$ such that $F_c'(t_0) = 0$, since H_F is constant. Hence $H_F(0) = H_F(t_0) = H_F(t)$. By passing to the limit as $t \to +\infty$ and thanks to Proposition 4, Items 2 and 4, we have $3F''^2(t_0) = -2$. This is a contradiction. Also assume that F is a solution convex-concave of the problem $(P_{a_0,b,0})$.

From Proposition 16 and as above, $H_F(t_0) = H_F(t)$. Hence, we have $3F''^2(t_0) = 0$, this is a contradiction, since $H_F(+\infty) = 0$.

Lemma 11. If $c \in C_0$, then there exists $t_0 \in [0, T_c)$ such that $F_c'(t_0) = 0$ and $F_c''(t_0) < 0$. Moreover, $F_c'(t) \to -\infty$ as $t \to T_c$.

Proof: This follows from Propositions 15 and 16.

Theorem 3. Let $a_0 \in \mathbb{R}$ and $b < 0$.
(1) The problem $(P_{a_0,b,-\infty})$ has infinitely many solutions.
(2) The problem $(P_{a_0,b,0})$ has only one convex solution and no convex-concave solution.
(3) The problem $(P_{a_0,b,1})$ has no solution.
(4) The problem $(P_{a_0,b,+\infty})$ has no solution.

Proof: The first result follows from Lemma 9 and the second follows from Propositions 15 and 16. The third statement follows from Proposition 16 and the last one follows from Proposition 15.

4. CONCLUSION

In this work, we have presented a numerous new and important results for a problem which arises when looking for similarity solutions to problem of boundary-layer theory . We have studied the existence, uniqueness and the sign of concave, convex and convex-concave solutions to the autonomous third order nonlinear differential equation

$$f''' + \alpha f f'' + g(f') = 0,$$

where α ($\alpha \geq 0$) depends on m, and g is a given continuous function. Associated with the above equation, we have the following boundary conditions

$$f(0) = a \in \mathbb{R}, \quad f'(0) = b < 0 \text{ and } f'(+\infty) = \lambda \in \{0,1\},$$

according to the values of the real parameter m, when $m > -1$, we have the problem $(P_{\beta;a,b,\lambda})$ and for $m = -1$ we have the problem $(P_{a_0,b,\lambda})$, using the shooting method.

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