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AN EXTENSION TO THE RENEWAL THEOREM AND AN APPLICATION TO RISK THEORY

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In applied probability one is often interested in the asymptotic behavior of a certain quantity. If a regenerative phenomenon can be imbedded, then one has the problem that the event of interest may have occurred but cannot be observed at the renewal points. In this paper an extension to the renewal theorem is proved which shows that the quantity of interest converges. As an illustration an open problem in risk theory is solved.

1. Introduction. In applied probability one is often interested in the probability that some event $A(u)$ occurs, for instance, the ruin probability in risk theory or the tail of the stationary distribution in queueing theory. In nice cases it is possible to obtain exponential bounds of the form

$$\underline{C}e^{-Ru} \leq P[A(u)] \leq \bar{C}e^{-Ru} \tag{1}$$

for some $R \geq 0$. If all the stochastic changes in the system occur according to a renewal process, then it is often also possible to obtain the asymptotic behavior of the event $A(u)$:

$$\lim_{u\to\infty} P[A(u)]e^{Ru} = C \tag{2}$$

via a (defective) renewal equation. In the case of a risk process, the ruin probability, the probability that the surplus with initial capital u ever becomes negative, is asymptotically

$$\psi(u) \sim Ce^{-Ru},$$

provided the claims occur according to a renewal process. The result can be found in Feller [(1971), page 377], for Poisson arrivals or in Grandell (1991) for renewal arrivals.

In problems considered in recent research, changes in the system occur at time points which are not regenerative. One often can, however, imbed a regenerative phenomenon into the process. However, information is lost as a result of observing the process only at the renewal epochs. Let $A_1(u)$ denote the event that $A(u)$ can be detected by observation of the process at the renewal epochs only. Then obviously $P[A(u)] \geq P[A_1(u)]$ and a lower bound can be obtained. Typically an upper bound can be obtained so (1) remains true. Some examples from risk theory can be found in Björk and Grandell (1988), Asmussen (1996), Embrechts, Grandell and Schmidli (1993), Grige-

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lionis (1993) and Schmidli (1995, 1996). Note that these results also apply to one-server queues by a well-known duality argument; see Asmussen and Petersen (1989).

One can try to approach the problem of obtaining the asymptotic behavior (2) via a renewal approach as in Feller (1971), page 377. Because $A(u)$ can occur between renewal epochs, however, one has to look back. Thus a renewal equation will have the form

$$Z(u) = \int_0^u Z(u-y)(1-p(u,y))\,dB(y) + z(u), \tag{3}$$

where $B(y)$ is the distribution of the renewals, for instance the ladder height distribution, and $p(u,x)$ is the probability that $A(u)$ occurs between the step from u to $u-x$ given the state of the process at the renewal epochs. Typically $B(u)$ is a defective measure which can be changed to a proper measure by multiplying with e^{Ru} for some constant R. If we can show that the solution $Z(u)e^{Ru} = P[A(u)]e^{Ru}$ converges as $u \to \infty$, then we have found that the limit in (2) exists. The result, however, will not give an explicit expression for the constant C. Anyway an explicit expression would require explicit expressions for $p(u,x)$ and $B(y)$. One can only hope to find expressions for these quantities in special cases. An example in the case of an ordinary renewal equation can be found in Feller (1971), page 377.

The motivation of the present work was the open problem in risk theory to prove the (intuitively clear) asymptotic behavior of the ruin probability in a doubly stochastic risk model. We will restrict ourselves to the case of a so-called Björk–Grandell model (see Section 3) and prove the following result.

THEOREM 1. *Let (C_t) be a Björk–Grandell model and denote its ruin probability by $\psi(u)$. Then, under Assumptions* 2 *and* 3, *there exists a constant $0 < C < \infty$ such that*

$$\lim_{u\to\infty} \psi(u)e^{Ru} = C.$$

However, the approach also works for the more general model introduced by Schmidli (1996) provided regeneration points can be found.

For the rest of this paper, we assume that a probability space $(\Omega, \mathscr{F})$ is given which contains all stochastic objects defined. If nothing else is said we use a probability measure P on $(\Omega, \mathscr{F})$. We will, however, also consider a progressively equivalent measure Q.

The outline of the paper is as follows. First we prove a renewal theorem for (3) (Section 2). Assuming that $B(x)$ is a proper probability distribution, we will show that there exists a unique solution to (3). The main result will be that the solution $Z(u)$ converges as $u \to \infty$. Then we apply the result in order to prove Theorem 1 (Section 3).

2. An extension to the renewal theorem. Let us consider now (3) where $B(x)$ is a proper probability distribution with $B(0) = 0$. We make the following assumption.

ASSUMPTION 1. Assume in (3) that $0 \le p(u, y) \le 1$, that $p(u, x)$ is continuous in u and that

$$\int_0^u p(u, y)\, dB(y)$$

is directly Riemann integrable.

Recall that a function is called *cadlag* if it is right continuous and the left limits exist at all points.

LEMMA 1. *Let $z(u)$ be bounded. Then, under Assumption 1, there exists exactly one solution $Z(u)$ to (3) which is bounded on bounded intervals. Moreover, if $z(u) \ge 0$, then $Z(u) \ge 0$. If $z(u)$ is continuous, then $Z(u)$ is cadlag.*

PROOF. Let $f(u)$ and $Z(u)$ be two solutions to (3) which are bounded on bounded intervals. Then

$$\begin{aligned} |f(u) - Z(u)| &= \left| \int_0^u (f(u-y) - Z(u-y))(1 - p(u, y))\, dB(y) \right| \\ &\le \int_0^u |f(u-y) - Z(u-y)|(1 - p(u, y))\, dB(y) \\ &\le \int_0^u |f(u-y) - Z(u-y)|\, dB(y). \end{aligned}$$

By iteration,

$$|f(u) - Z(u)| \le \int_0^u |f(u-y) - Z(u-y)|\, dB^{*n}(y).$$

Letting $n \to \infty$ yields $f(u) = Z(u)$. Hence there exists at most one solution.

Let $f_0(u)$ be the solution to the ordinary renewal equation

$$f_0(u) = \int_0^u f_0(u-y)\, dB(y) + z(u); \tag{4}$$

see also Feller (1971), page 359. Define

$$f_n(u) = \int_0^u f_{n-1}(u-y)(1 - p(u, y))\, dB(y) + z(u).$$

Fix $x > 0$ and let $0 \le u \le x$. Observe that, for $n \ge 1$,

$$\begin{aligned} |f_{n+1}(u) - f_n(u)| &\le \int_0^u |f_n(u-y) - f_{n-1}(u-y)|(1 - p(u, y))\, dB(y) \\ &\le \int_0^u |f_n(u-y) - f_{n-1}(u-y)|\, dB(y). \end{aligned}$$

Let m be such that $B^{*m}(x) < 1$. Then

$$|f_{n+m}(u) - f_{n+m-1}(u)| \le B^{*m}(x) \sup_{0 \le y \le x} |f_n(y) - f_{n-1}(y)|$$

and thus the sequence $f_n(u)$ converges to a function $Z(u)$ uniformly on each bounded interval. Moreover, by the dominated convergence theorem, $Z(u)$ satisfies (3).

If $z(u) \geq 0$, then $f_0(u) \geq 0$. It follows by induction that $f_n(u) \geq 0$ for all $n \in \mathbb{N}$. Thus $Z(u) \geq 0$. If $z(u)$ is continuous, then $f_0(u)$ is cadlag. By induction, it follows that $f_n(u)$ is cadlag for all n. Because $f_n(u)$ converges uniformly on bounded intervals, $Z(u)$ is also cadlag.

LEMMA 2. *Assume that $z(u) \geq 0$ is bounded. Let $Z(u)$ be the solution to* (3) *and $f(u)$ be the solution to the ordinary renewal equation* (4). *Then, under Assumption* 1, $f(u) \geq Z(u) \geq 0$.

PROOF. We already proved that $Z(u) \geq 0$. From the renewal equations we obtain

$$f(u) - Z(u) = \int_0^u (f(u-y) - Z(u-y))\,dB(y) + \int_0^u Z(u-y)p(u,y)\,dB(y)$$
$$\geq \int_0^u (f(u-y) - Z(u-y))\,dB(y)\;.$$

By iteration, the right-hand side tends to 0. Thus $f(u) \geq Z(u)$.

Recall from classical renewal theory that a directly Riemann integrable function is bounded. Moreover, if $z(u)$ is continuous almost everywhere (with respect to the Lebesgue measure) and if there exists a directly Riemann integrable function $a(u)$ such that $0 \leq z(u) \leq a(u)$, then $z(u)$ is directly Riemann integrable. We can now prove the main theorem.

THEOREM 2. *Assume that $z(u)$ is directly Riemann integrable and that Assumption* 1 *is satisfied. Denote by $Z(u)$ the solution to* (3) *which is bounded on bounded intervals. Then the limit*

$$\lim_{u\to\infty} Z(u)$$

exists and is finite provided $B(u)$ is not arithmetic. If $B(u)$ is arithmetic with span γ, then

$$\lim_{n\to\infty} Z(x+n\gamma)$$

exists and is finite for all x fixed.

PROOF. We will only prove the nonarithmetic case. The arithmetic case follows similarly. By splitting $z(u)$ into its positive and its negative parts, we can assume that $z(u) \geq 0$. Let us first assume that $z(u)$ is continuous. It follows from Lemma 1 that $Z(u)$ is cadlag and nonnegative. Let $f(u)$ denote the solution to the ordinary renewal equation (4). By the renewal theorem

$f(u)$ converges as $u \to \infty$. Thus $f(u)$ is bounded. By Lemma 2, $Z(u)$ is also bounded. Let

$$g(u) = \int_0^u Z(u-y)p(u,y)\,dB(y).$$

Because $Z(u)$ is cadlag and because of Assumption 1, $g(u)$ is cadlag, too. For any $\varepsilon > 0$ there are only a finite number of jumps larger than ε because $g(u)$ is bounded by a directly Riemann integrable function and is cadlag. Thus $g(u)$ is continuous almost everywhere and therefore directly Riemann integrable. Here $Z(u)$ fulfills the ordinary renewal equation

$$Z(u) = \int_0^u Z(u-y)\,dB(y) + (z(u) - g(u)).$$

Thus $Z(u)$ converges as $u \to \infty$.

Now let $z(u) \geq 0$ be arbitrary. Because $0 \leq Z(u) \leq f(u)$ we can assume that $\mu = \int_0^\infty (1 - B(u))\,du < \infty$. Otherwise $\lim Z(u) = \lim f(u) = 0$. Choose sequences $(z_n(u))$ and $(\bar{z}_n(u))$ of continuous directly Riemann integrable functions such that

$$0 \leq z_n(u) \leq z(u) \leq \bar{z}_n(u)$$

and

$$\lim_{n\to\infty} \int_0^\infty (\bar{z}_n(u) - z_n(u))\,du = 0.$$

Let, for instance,

$$\tilde{z}_n(u) = \sup\left\{ z(x) \colon \frac{[un]}{n} \leq x < \frac{[un+1]}{n} \right\}$$

be the upper Riemann bound with step width n^{-1}. The function $\bar{z}_n(u)$ can be constructed as follows. On the interval $[(m - m^{-2})/n, (m+1-(m+1)^{-2})/n]$ for $m \geq 1$, we distinguish the cases $\tilde{z}_n((m-1)/n) \geq \tilde{z}_n(m/n)$ and $\tilde{z}_n((m-1)/n) < \tilde{z}_n(m/n)$. In the first case let

$$\bar{z}_n(u) = (1 - (m+1)^2(un - m))\tilde{z}_n((m-1)/n) + (m+1)^2(un - m)\tilde{z}_n(m/n)$$

if $u \in [m/n, (m + (m+1)^{-2})/n]$ and $\bar{z}_n(u) = \tilde{z}_n(u)$ otherwise. In the second case let

$$\bar{z}_n(u) = m^2(m - un)\tilde{z}_n((m-1)/n) + (1 - m^2(m - un))\tilde{z}_n(m/n)$$

if $u \in [(m - m^{-2})/n, m/n]$ and $\bar{z}_n(u) = \tilde{z}_n(u)$ otherwise. Note that for n large enough the function $\tilde{z}_n(u)$ is integrable and therefore directly Riemann integrable. The upper Riemann sum of $\bar{z}_n(u)$ with step width n^{-1} is bounded by three times the upper Riemann sum of $\tilde{z}_n(u)$. Therefore $\bar{z}_n(u)$ is directly Riemann integrable. The functions $z_n(u)$ can be constructed in a similar way.

Let $f_n(u)$, $\bar{f}_n(u)$, $Z_n(u)$, $\bar{Z}_n(u)$ denote the solutions to the ordinary renewal equations and (3) corresponding to $z_n(u)$ and $\bar{z}_n(u)$, respectively. Then

$$0 \le \limsup_{u\to\infty} Z(u) - \liminf_{u\to\infty} Z(u) \le \lim_{u\to\infty} (\bar{Z}_n(u) - Z_n(u))$$

$$\le \lim_{u\to\infty} (\bar{f}_n(u) - f_n(u)) = \frac{1}{\mu}\int_0^\infty (\bar{z}_n(u) - z_n(u))\,du.$$

However, the latter can be made arbitrarily small. Because $0 \le \lim Z(u) \le \lim f(u) < \infty$, the limit of $Z(u)$ must be finite. □

REMARK. One could try to follow the proof in Feller (1971). For any interval I, let $g(u, I)$ be the solution with $z(u) = \mathbb{1}_I(u)$, where $\mathbb{1}$ denotes the indicator function. By the selection theorem, there exists a sequence (u_n) tending to infinity and a measure V such that

$$g(u_n, dy) \to V(dy) \quad \text{as } n \to \infty.$$

In the ordinary renewal theorem, $g(u_n+x, dy) = g(u_n, dy-x)$ which can easily be seen from the ordinary renewal equation. In our case the perturbation term may be different because, in general, $p(u+x, y) \ne p(u, y)$ and therefore $g(u_n + x, dy) \ne g(u_n, dy - x)$. Thus V is not proportional to the Lebesgue measure. This is the reason that no explicit limit can be found.

3. The Björk-Grandell risk model. Let (L_i, σ_i) be a sequence of iid vectors with nonnegative first component and strictly positive second component and denote by (L, σ) a generic vector. Define $\Sigma_i = \sigma_1+\sigma_2+\cdots+\sigma_i$. Let $\lambda_t = L_i$ if $\Sigma_{i-1} \le t < \Sigma_i$, $\Lambda(t) = \int_0^t \lambda_s\,ds$ and define the point process

$$N_t = \tilde{N}_{\Lambda(t)},$$

where $(\tilde{N}_t)$ is a Poisson process with rate 1. The point process (N_t) is a special case of a so-called Cox process. Here (N_t) plays the role of the number of claims in the interval $(0, t]$, and λ_t is called the intensity at time t. Let (Y_i) be a sequence of iid random variables independent of $(N_t\colon t \ge 0)$ and Y_i the amount of the ith claim. Let G be the distribution function of the claims and $\hat{G}(r) = E[\exp\{rY\}]$ its moment generating function. We assume that the claims are strictly positive; that is, $G(0) = 0$. Then

$$C_t = u + ct - \sum_{i=1}^{N_t} Y_i$$

is the surplus of a collective insurance risk with initial capital u and linear premium income with rate c. This model was introduced by Björk and Grandell (1988) as a generalization of a model considered by Ammeter (1948).

Define the ruin probability

$$\psi(u) = P[\inf\{C_t\colon t \ge 0\} < 0 \mid C_0 = u]\,.$$

In order to avoid that $\psi(u) = 1$, we need the net profit condition

$$cE[\sigma] > E[L\sigma]E[Y_1].$$

This condition assures that $E[C_t] > 0$ and thus $C_t \to \infty$ as $t \to \infty$. As a consequence, $\inf C_t > -\infty$. Moreover, $\psi(u) < 1$ follows.

Let us consider the classical risk model, that is, $L = \lambda$ a.s. The following result goes back to Filip Lundberg and Harald Cramér.

PROPOSITION 1. *Let (C_t) be a classical risk model. Assume that the equation*

$$\lambda(\hat{G}(r) - 1) - cr = 0$$

has a strictly positive solution R. Assume also that there exists an $r > R$ such that $\hat{G}(r) < \infty$. Then

$$\lim_{u\to\infty} \psi(u)e^{Ru} = C$$

for some $0 < C < 1$.

A proof of the result can, for instance, be found in Grandell (1991). Note that if $\hat{G}(r) < \infty$, then

$$E[\exp\{-r(C_t - u) - (\lambda(\hat{G}(r) - 1) - cr)t\}] = 1; \tag{5}$$

in particular, $E[\exp\{-R(C_t - u)\}] = 1$.

Let us turn back to the Björk–Grandell model. In order to obtain the analogue of (5), consider the function

$$\phi(\vartheta, r) = E[\exp\{-\vartheta\sigma_1 - r(C_{\sigma_1} - u)\}].$$

An equation like (5) cannot hold for any t because $(C_{t+s} - C_t\colon s \geq 0)$ is not independent of $\mathscr{F}_t^C$ any more, where $(\mathscr{F}_t^C)$ denotes the natural filtration of (C_t). Thus we have to restrict to the times Σ_i. Note that if $\phi(\vartheta, r) < \infty$, then $\hat{G}(r) < \infty$. For a discussion of $\phi(\vartheta, r)$, see Embrechts, Grandell and Schmidli (1993). The following assumption is the natural analogue of the condition in Proposition 1.

ASSUMPTION 2. Assume that there exists a strictly positive solution R to the equation

$$\phi(0, r) = 1.$$

Assume also that there exists an $r > R$ such that $\phi(0, r) < \infty$.

Let $\theta(r)$ be the solution to the equation $\phi(\theta, r) = 1$ for any $r \in \mathbb{R}$. Such a solution exists if there is a ϑ such that $\phi(\vartheta, r) \geq 1$. It was shown in Embrechts, Grandell and Schmidli (1993) that $\theta(r)$ is convex. Thus if R exists, then it is unique because $\theta(0) = 0$. Because $\phi(0, r)$ is also convex, it follows that $\phi(0, r) > 1$ for all $r > R$ such that $\phi(0, r) < \infty$. Thus $\theta(r)$ exists for $r \geq R$ such that $\phi(0, r) < \infty$. For convenience, we set $\theta(r) = \infty$ if $\theta(r)$ does not exist.

Define $V_t = \Sigma_i - t$ if $\Sigma_{i-1} \le t < \Sigma_i$. Here V_t is the time remaining to the next jump of the intensity. This quantity is not observable. However, we need it in order to Markovize the risk process. It seems to be more natural to use the time since the last change of the intensity (which is not observable either), but then one would have to assume that (L, σ) has an absolutely continuous distribution. For a discussion, see Embrechts, Grandell and Schmidli (1993). Let $(\mathcal{F}_t)$ denote the natural filtration of the process (C_t, λ_t, V_t). The martingales and stopping times used later are always meant to be $(\mathcal{F}_t)$-martingales and $(\mathcal{F}_t)$-stopping times. The following lemma was proved in Embrechts, Grandell and Schmidli (1993).

LEMMA 3. *Let $\theta(r) < \infty$. Then, under Assumption 2, the process*

$$M_t^r = \exp\{-r(C_t - u) + (\lambda_t(\hat{G}(r) - 1) - cr - \theta(r))V_t - \theta(r)t\}$$

is a martingale.

The martingales defined above have mean value 1 and are strictly positive. Thus they can be used in order to change the measure. For a discussion of this method, see, for instance, Asmussen (1996) or Schmidli (1995, 1996). Define for any $A \in \mathcal{F}_t$ the measure

$$Q_r[A] = E_P[M_t^r; A].$$

Note that this is a proper definition because, for $s < t$ and $A \in \mathcal{F}_s \subset \mathcal{F}_t$,

$$E_P[M_t^r; A] = E_P[M_s^r; A]$$

by the martingale property. By the stopping theorem, the formula also holds for stopping times T and $A \in \mathcal{F}_T$ provided $A \subset \{T < \infty\}$. One can show, as in Schmidli (1995), that (C_t) is also a Björk–Grandell model under Q_r. However, under the new measure,

$$E_{Q_r}[C_{\sigma_1} - u] = -\theta'(r) E_{Q_r}[\sigma_1].$$

Thus the net profit condition is not fulfilled any more if, for instance, $r \ge R$. Under the new measures, ruin occurs almost surely.

As in Embrechts, Grandell and Schmidli (1993), we need the additional assumption that σ must fulfil a small tail condition. For a discussion of this assumption, see Embrechts, Grandell and Schmidli (1993).

ASSUMPTION 3. Assume that there exists an $r > R$ and a constant $B > 0$ such that

$$E_P\big[\exp\{(L(\hat{G}(r) - 1) - cr - \theta(r))(\sigma - v))\}\big|\sigma > v, L\big] \ge B \quad \text{a.s.}$$

for all $v \ge 0$. Moreover, assume that there exists an $\tilde{r} > r$ such that $\phi(0, \tilde{r}) < \infty$.

REMARK. Assumption 2 in Embrechts, Grandell and Schmidli (1993) implies the above assumption. Because in Embrechts, Grandell and Schmidli (1993) finite-time ruin probabilities were treated, a little bit stronger assumption had to be used.

Assumption 3 implies the corresponding condition under the measure Q_r.

LEMMA 4. *Let r and B be the constants given by Assumption 3. Then, under Assumption 3,*

$$E_{Q_r}\big[\exp\{-(L(\hat{G}(r)-1)-cr-\theta(r))(\sigma-v)\}\big|\sigma>v,L\big]\le B^{-1}\quad a.s.$$

for all $v\ge 0$.

PROOF. For a Borel set A and $v\ge 0$ such that $P[L_1\in A,\sigma_1>v]>0$,

$$\begin{aligned}
&E_{Q_r}\big[\exp\{-(L_1(\hat{G}(r)-1)-cr-\theta(r))(\sigma_1-v)\}\big|\sigma_1>v,L_1\in A\big]\\
&\quad=\frac{E_{Q_r}[\exp\{-(L_1(\hat{G}(r)-1)-cr-\theta(r))(\sigma_1-v)\};\sigma_1>v,L_1\in A]}{Q_r[L_1\in A,\sigma_1>v]}\\
&\quad=\frac{E_P[M_0^r\exp\{-(L_1(\hat{G}(r)-1)-cr-\theta(r))(\sigma_1-v)\};\sigma_1>v,L_1\in A]}{E_P[M_0^r;L_1\in A,\sigma_1>v]}\\
&\quad=\frac{E_P[\exp\{(L_1(\hat{G}(r)-1)-cr-\theta(r))v\};\sigma_1>v,L_1\in A]}{E_P[\exp\{(L_1(\hat{G}(r)-1)-cr-\theta(r))\sigma_1\};\sigma_1>v,L_1\in A]}\\
&\quad=\frac{E_P[\exp\{(L_1(\hat{G}(r)-1)-cr-\theta(r))v\}|\sigma_1>v,L_1\in A]}{E_P[\exp\{(L_1(\hat{G}(r)-1)-cr-\theta(r))\sigma_1\}|\sigma_1>v,L_1\in A]}.
\end{aligned}$$

Because A is arbitrary,

$$\begin{aligned}
&E_{Q_r}\big[\exp\{-(L_1(\hat{G}(r)-1)-cr-\theta(r))(\sigma_1-v)\}\big|\sigma_1>v,L_1\big]\\
&\quad=\frac{E_P\big[\exp\{(L_1(\hat{G}(r)-1)-cr-\theta(r))v\}\big|\sigma_1>v,L_1\big]}{E_P\big[\exp\{(L_1(\hat{G}(r)-1)-cr-\theta(r))\sigma_1\}\big|\sigma_1>v,L_1\big]}\\
&\quad=\frac{1}{E_P\big[\exp\{(L_1(\hat{G}(r)-1)-cr-\theta(r))(\sigma_1-v)\}\big|\sigma_1>v,L_1\big]}\le\frac{1}{B}.\qquad\square
\end{aligned}$$

Let $\tau=\inf\{t>0: C_t<0\}$ denote the epoch of ruin. As mentioned in the introduction, the claim times are not regenerative points. We therefore also consider the random walk (C_{Σ_i}). Let $\tau_1=\Sigma_i$ if $i=\inf\{j>0: C_{\Sigma_j}<u\}$ denotes the first ladder epoch. If no first ladder epoch exists, then we set $\tau_1=\infty$. Note that for $r\ge R$ we have

$$Q_r[\tau<\infty]=Q_r[\tau_1<\infty]=1$$

because the net profit condition is not fulfilled. For later use we need the following lemma.

LEMMA 5. *Under Assumptions 2 and 3, there exists an $r>R$ such that*

$$P[\tau\le\tau_1,\tau<\infty\mid C_0=u]\,e^{ru}$$

is uniformly bounded.

PROOF. Choose r as in Assumption 3. It is well known that then $E_P[\exp\{-r(C_{\tau_1}-u)\};\tau_1<\infty]<\infty$ because the moment generating functions of $(u-C_{\tau_1})$ and $(u-C_{\sigma_1})$ have the same abscissa of convergence which is larger than r. In the following computations we will suppress the conditioning on $\{C_0=u\}$. Then, because $\{\tau\le(\tau_1\wedge t)\}\in\mathscr{F}_\tau$,

$$\begin{aligned}
&P[\tau\le(\tau_1\wedge t)]\exp(ru)\\
&\quad=E_{Q_r}[(M_\tau^r)^{-1}\mathbb{1}_{\{\tau\le(\tau_1\wedge t)\}}]\exp(ru)\\
&\quad\le E_{Q_r}\big[\exp\{\theta(r)\tau\}\exp\{-(\lambda_\tau(\hat G(r)-1)-cr-\theta(r))V_\tau\}\mathbb{1}_{\{\tau\le\tau_1\}}\big],
\end{aligned}$$

where we used that $C_\tau<0$. Let τ^* denote the epoch of the last change of the intensity. Observe that

$$\begin{aligned}
&E_{Q_r}\big[\exp\{\theta(r)\tau\}\exp\{-(\lambda_\tau(\hat G(r)-1)-cr-\theta(r))V_\tau\}\mathbb{1}_{\{\tau\le\tau_1\}}\big]\\
&\quad=E_{Q_r}\big[\exp\{\theta(r)\tau\}E_{Q_r}\big[\exp\{-(\lambda_\tau(\hat G(r)-1)-cr-\theta(r))(V_{\tau^*}-(\tau-\tau^*))\}\\
&\qquad\qquad\big|\tau,\tau^*,\lambda_\tau,C_{\Sigma_1},C_{\Sigma_2},\dots,C_{\tau^*}\big]\mathbb{1}_{\{\tau\le\tau_1\}}\big].
\end{aligned}$$

It follows that

$$\begin{aligned}
&E_{Q_r}\big[\exp\{-(\lambda_\tau(\hat G(r)-1)-cr-\theta(r))(V_{\tau^*}-(\tau-\tau^*))\}\\
&\qquad\qquad\big|\tau,\tau^*,\lambda_\tau,C_{\Sigma_1},C_{\Sigma_2},\dots,C_{\tau^*}\big]\\
&\quad=E_{Q_r}\big[\exp\{-(\lambda_\tau(\hat G(r)-1)-cr-\theta(r))(V_{\tau^*}-(\tau-\tau^*))\}\\
&\qquad\qquad\big|\tau,\tau^*,\lambda_\tau,V_{\tau^*}>\tau-\tau^*\big]\le B^{-1}
\end{aligned}$$

by Lemma 4. Thus

$$\begin{aligned}
P[\tau\le(\tau_1\wedge t)]\exp(ru)&\le B^{-1}E_{Q_r}[\exp\{\theta(r)\tau\}\mathbb{1}_{\{\tau\le\tau_1\}}]\\
&\le B^{-1}E_{Q_r}[\exp\{\theta(r)\tau_1\}]\\
&=B^{-1}E_P[\exp\{-r(C_{\tau_1}-u)\};\tau_1<\infty]<\infty.
\end{aligned}$$

The latter expression is independent of t and u. Thus the assertion follows by letting t tend to ∞. □

We are now able to prove Theorem 1.

PROOF OF THEOREM 1. Let $B(x)=P[\tau_1<\infty,u-C_{\tau_1}\le x]$ denote the defective ladder height distribution function. Let $p(u,x)=P[\tau\le\tau_1\mid\tau_1<\infty,C_0=u,C_{\tau_1}=u-x]$ for $0\le x\le u$. Then, by conditioning on the first ladder height,

$$\psi(u)=\int_0^u\psi(u-x)(1-p(u,x))\,dB(x)+P[\tau\le\tau_1,\tau<\infty\mid C_0=u].$$

This is not a renewal equation of the form (3) because the distribution function $B(x)$ is defective. Observe that

$$E_P[\exp\{R(u-C_{\tau_1})\};\tau_1<\infty]=Q_R[\tau_1<\infty]=1.$$

Thus we get a renewal equation of the form (3) by multiplying by e^{Ru}:

$$\begin{aligned}\psi(u)e^{Ru} &= \int_0^u \psi(u-x)\exp\{R(u-x)\}(1-p(u,x))\exp(Rx)\,dB(x) \\ &\quad + P[\tau \le \tau_1, \tau < \infty \mid C_0 = u]\exp(Ru).\end{aligned} \tag{6}$$

We first have to check Assumption 1. It is well known [see Embrechts, Grandell and Schmidli (1993)] that the minimum of (C_t) has an absolutely continuous distribution except at the starting point. Intuitively, this follows from the fact that, given the number of claims in a time interval with the same intensity, the claim epochs are uniformly distributed. Conditioning on $C_0 = u$ and $C_0 - C_{\tau_1} = x$, it follows that $p(u,x) = P[\inf\{C_t\colon 0 \le t \le \tau_1\} < -u \mid C_0 = 0, C_{\tau_1} = -x]$. Similar arguments as above show that $p(u,x)$ is continuous in u. By Lemma 5 there exists a constant K and an $r > R$ such that $P[\tau \le \tau_1, \tau < \infty \mid C_0 = u] \le Ke^{-ru}$. Then

$$\begin{aligned}\int_0^u p(u,x)\exp(Rx)\,dB(x) &\le \int_0^u p(u,x)\,e^{Ru}\,dB(x) \\ &= P[\tau \le \tau_1 < \infty \mid C_0 = u]\exp(Ru) \\ &\le K\exp\{-(r-R)u\}\end{aligned}$$

and Assumption 1 is satisfied because $\exp\{-(r-R)u\}$ is directly Riemann integrable and $p(u,x)$ is continuous in u. Because $P[\tau \le \tau_1, \tau < \infty \mid C_0 = u]$ is continuous in u and $K\exp\{-(r-R)u\}$ is directly Riemann integrable, it follows that

$$P[\tau \le \tau_1, \tau < \infty \mid C_0 = u]e^{Ru}$$

is directly Riemann integrable. By Theorem 2 the solution to (6) converges to some value $C < \infty$. That $C > 0$ follows from Theorem 3 of Schmidli (1996). □

REMARKS. (i) Theorem 1 generalizes Proposition 12 of Björk and Grandell (1988).

(ii) In Schmidli (1995) the perturbed risk model $(C_t + \eta W_t)$ was considered, where (W_t) is a standard Brownian motion. Using the same methods as here, a limit

$$\lim_{u\to\infty} \tilde{\psi}(u)e^{Ru} = \tilde{C}$$

can be shown for the ruin probability $\tilde{\psi}(u)$ of the perturbed risk model and thus Theorem 3 of Schmidli (1995) can be generalized.

So far we only considered the case where the risk process starts at a point where the intensity changes. Now let (L_1, σ_1) have an arbitrary distribution. We denote the corresponding measures by P^1 and Q_r^1. The only additional assumption needed will be that

$$E_{P^1}[M_0^R] = E_{P^1}[\exp\{-R(C_{\sigma_1} - u)\}] < \infty. \tag{7}$$

This assumption is trivially satisfied if the last change of the intensity before 0 occurs at time $\Sigma_0 < 0$, where Σ_0 is deterministic or Σ_0 is distributed such that (λ_t) becomes stationary; see Grandell (1991), page 96. In this case we can express the asymptotic behavior in terms of the ordinary model.

COROLLARY 1. *Assume that* (7) *and Assumptions* 2 *and* 3 *are fulfilled. Then*

$$\lim_{u\to\infty} \psi^1(u)e^{Ru} = C\, E_{P^1}[M_0^R],$$

where C *is the constant obtained from Theorem* 1.

PROOF. Note first that

$$f(u) = \psi(u)e^{Ru} = E_{Q_R}\big[\exp\{RC_\tau - (\lambda_\tau(\hat{G}(R)-1) - cR)V_\tau\}\big|C_0 = u\big]$$

converges to C as $u \to \infty$. With the measure Q_R^1 we obtain

$$\psi^1(u)\exp(Ru) = E_{P^1}[M_0^R]E_{Q_R^1}\big[\exp\{RC_\tau - (\lambda_\tau(\hat{G}(R)-1) - cR)V_\tau\}\big|C_0 = u\big].$$

Let

$$\tilde{f}(u) = E_{Q_R^1}\big[\exp\{RC_\tau - (\lambda_\tau(\hat{G}(R)-1) - cR)V_\tau\}\big|C_0 = u\big].$$

We have to show that

$$\lim_{u\to\infty} \tilde{f}(u) = C.$$

Denote by H the distribution function of $u - C_{\sigma_1}$ under Q_R^1. Then

$$\begin{aligned}\tilde{f}(u) = \int_{-\infty}^{u} & f(u-x)(1-\tilde{p}(u,x))\,dH(x)\\ &+ E_{Q_R^1}\big[\exp\{RC_\tau - (\lambda_\tau(\hat{G}(R)-1) - cR)V_\tau\}\,\mathbb{1}_{\tau\le\sigma_1}\big|C_0 = u\big],\end{aligned}$$

where

$$\tilde{p}(u,x) = Q_R^1[\tau \le \sigma_1 | C_0 = u, C_{\sigma_1} = u - x].$$

Moreover,

$$\begin{aligned}&\exp\{RC_\tau - (\lambda_\tau(\hat{G}(R)-1) - cR)V_\tau\}\,\mathbb{1}_{\tau\le\sigma_1}\\ &\qquad\le \big(1 + \exp\{-(\lambda_\tau(\hat{G}(R)-1) - cR)V_0\}\big)\,\mathbb{1}_{\tau\le\sigma_1} \le 1 + (M_0^R)^{-1}.\end{aligned}$$

The latter expression is $E_{Q_R^1}$-integrable and therefore

$$E_{Q_R^1}\big[\exp\{RC_\tau - (\lambda_\tau(\hat{G}(R)-1) - cR)V_\tau\}\,\mathbb{1}_{\tau\le\sigma_1}\big|C_0 = u\big]$$

converges to 0 as $u \to \infty$. Note that

$$\int_{-\infty}^{u} \tilde{p}(u,x)\,dH(x) \le Q_R^1[\tau \le \sigma_1 | C_0 = u]$$

also tends to 0 as u tends to ∞. Because $f(u)$ is bounded, it follows that

$$\lim_{u\to\infty} \tilde{f}(u) = \lim_{u\to\infty} \int_{-\infty}^{u} f(u-x)\,dH(x)$$
$$= \int_{-\infty}^{\infty} C\,dH(x) = C.$$

This completes the proof. □

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