

AN EXTREMAL PROBLEM AND INEQUALITIES FOR ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

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ABSTRACT. We study two variations of the classical one-delta problem for entire functions of exponential type, known also as the Carathéodory–Fejér–Turán problem. The first variation imposes the additional requirement that the function is radially decreasing while the second one is a generalization which involves derivatives of the entire function. Various interesting inequalities, inspired by results due to Duffin and Schaeffer, Landau, and Hardy and Littlewood, are also established.

1. INTRODUCTION

In the present note we study some extremal problems concerning certain quantities over specific families of entire functions of exponential type. For $\Delta > 0$, we say that an entire function $G : \mathbb{C} \rightarrow \mathbb{C}$ has exponential type at most $2\pi\Delta$ if, for all $\varepsilon > 0$, there exists a positive constant C_ε such that

$$|G(z)| \leq C_\varepsilon e^{(2\pi\Delta + \varepsilon)|z|}, \quad \text{for all } z \in \mathbb{C}.$$

We adopt the usual convention that an entire function $f : \mathbb{C} \rightarrow \mathbb{C}$ is said to be real if its restriction to \mathbb{R} is real-valued, as well as, that the function $g^*(z)$ is defined by $g^*(z) = \overline{g(\bar{z})}$. For $f, g \in L^1(\mathbb{R})$ we denote by $f * g$ their convolution, which is defined by $(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x - y) dy$.

1.1. The one-delta problem. The classical *one-delta problem* is to determine the infimum

$$\mathcal{A} = \inf_{\substack{f \in \mathcal{F} \\ f(0) \neq 0}} \frac{\|f\|_1}{f(0)},$$

where the family \mathcal{F} consists of real entire functions $f : \mathbb{C} \rightarrow \mathbb{C}$ of exponential type at most 2π such that $f \in L^1(\mathbb{R})$, and $f(x) \geq 0$ for all $x \in \mathbb{R}$. This is a classical problem, and several of its variations are named after Carathéodory, Fejér and Turán. We refer to [9, 11, 14, 16] for comprehensive information about its history and for some recent contributions. It is known that $\mathcal{A} = 1$, and the unique extremal solution of the one-delta problem is the Fejér kernel, given by

$$K(z) = \left(\frac{\sin \pi z}{\pi z} \right)^2. \tag{1.1}$$

To obtain an equivalent formulation of this problem, we may consider a decomposition result due to Krein [1, p. 154]. It states that if $f : \mathbb{C} \rightarrow \mathbb{C}$ is an entire function of exponential type at most 2π such that $f \in L^1(\mathbb{R})$ and $f(x) \geq 0$ for all $x \in \mathbb{R}$, then there exists an entire function $g : \mathbb{C} \rightarrow \mathbb{C}$ in the Paley-Wiener

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space PW^2 such that $f(z) = g(z)g^*(z)$. Here, PW^2 is the subspace of $L^2(\mathbb{R})$ consisting of entire functions of exponential type at most π . Therefore, the one-delta problem can also be stated as finding

$$\mathcal{B} = \inf_{\substack{g \in PW^2 \\ g(0) \neq 0}} \frac{\|g\|_2}{|g(0)|}, \quad (1.2)$$

and clearly $\mathcal{B} = 1$, too. Other L^p -variations of this problem have also been studied in [4, 6, 13]. Note that (1.2) can be stated in yet another alternative way as follows: the inequality

$$1 \leq \int_{-\infty}^{\infty} |g(x)|^2 dx, \quad (1.3)$$

holds for every $g \in PW^2$ such that $g(0) = 1$, and (1.3) reduces to an equality if and only if

$$g(z) = \frac{\sin \pi z}{\pi z}.$$

Our main goal is to study some natural variations of each of the above versions of the one-delta problem.

1.2. Monotone-delta problem. The *monotone-delta problem* is to find

$$\mathcal{A}_1 = \inf_{\substack{f \in \mathcal{F}_1 \\ f(0) \neq 0}} \frac{\|f\|_1}{f(0)}, \quad (1.4)$$

where the family \mathcal{F}_1 consists of real entire functions $f : \mathbb{C} \rightarrow \mathbb{C}$ of exponential type at most 2π , such that $f \in L^1(\mathbb{R})$, $f(x) \geq 0$ for all $x \in \mathbb{R}$, and f is *radially decreasing*, that is, f is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. In the following theorem, we present some qualitative and quantitative information about this problem.

Theorem 1. *The following statements about the monotone-delta problem hold:*

- (a) *There exists an even function $F \in \mathcal{F}_1$ with $F(0) = 1$ that extremizes (1.4).*
- (b) *All the zeros of any even extremizer F lie in the set $S = \{z \in \mathbb{C} : |\operatorname{Re} z| > |\operatorname{Im} z| > 0\}$.*
- (c) *The constant \mathcal{A}_1 satisfies $1 < \mathcal{A}_1 \leq 1.2771\dots$*

We conjecture that the upper bound in part (c) is sharp in the sense that the first four significant digits of \mathcal{A}_1 are those shown above. One of the reasons for this claim is that our proof of part (c) of Theorem 1 is constructive. We construct concrete examples for which the value $1.2771\dots$ is attained. To obtain \mathcal{A}_1 , we first reformulate the monotone-delta problem (see Lemma 6 below) to the one of determining the infimum

$$\mathcal{A}_1 = \inf_{\substack{h \in \mathcal{F}_2 \\ h \neq 0}} \frac{2 \int_{-\infty}^{\infty} |x|^2 |h(x)|^2 dx}{\int_{-\infty}^{\infty} |x| |h(x)|^2 dx}, \quad (1.5)$$

where the family \mathcal{F}_2 consists of entire functions $h : \mathbb{C} \rightarrow \mathbb{C}$ of exponential type at most π such that $xh \in L^2(\mathbb{R})$ and $|h(x)| = |h(-x)|$. Then we find an explicit example of a function $h_0 \in \mathcal{F}_2$ (see (2.9)). For this h_0 , we compute explicitly the quotient in (1.5), which turns out to be $1.2771\dots$. Despite that h_0 is not the extremal function for (1.5), our conjecture is that the value $1.2771\dots$ is so close to the infimum \mathcal{A}_1 , that they differ only in the decimal digits after the fourth one. In Section 3.2 we give a deeper discussion of numerical issues, and a sharper conjecture for the value of \mathcal{A}_1 .

The monotone-delta problem has also been considered in \mathbb{R}^d , for $d \geq 2$. In [5], using techniques from the theory of de Branges spaces, the authors found the exact solution of the monotone-delta problem when d is

even. Nonetheless, the authors state that the case when d is odd seems more subtle and remains open in general.

Despite that Lemma 6 below provides an integral representation of any function in \mathcal{F}_1 , the first interesting explicit example of a function in this class we constructed was based on the classical method of Sonin, which was itself invented with the intention to obtain information about the monotonicity of the successive relative minima and maxima of certain oscillatory solutions of ordinary differential equations (see [18, Section 7.31]). If $g : \mathbb{C} \rightarrow \mathbb{C}$ is a real entire function in PW^2 and satisfies a second-order differential equation of the form $y'' + (B/x)y' + Cy = 0$, with $B, C > 0$, Sonin's method suggests to construct the function

$$f(z) = (g(z))^2 + \frac{(g'(z))^2}{C}. \quad (1.6)$$

It is clear that $f \in \mathcal{F}_1$. Moreover $f(x)$ is a “lid” of $g^2(x)$ in the sense that $f(x) \geq g^2(x)$ for every $x \in \mathbb{R}$ and f interpolates g^2 and possesses inflection points at its local maxima. Figure 1 shows Fejér's kernel $K(x)$ and its lid $f(x)$.

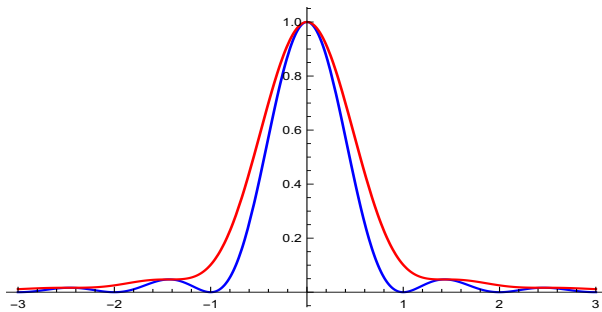


FIGURE 1. The Fejér kernel $K(x)$ defined in (1.1) and its lid $f(x)$.

1.3. The one-delta problem with derivatives. The function in (1.6) appears in a classical inequality for entire functions. Duffin and Schaeffer [8, p. 239] proved that if a real entire function $g : \mathbb{C} \rightarrow \mathbb{C}$ of exponential type at most π is such that $|g(x)| \leq 1$ for all $x \in \mathbb{R}$, then

$$(g(x))^2 + \frac{(g'(x))^2}{\pi^2} \leq 1, \quad \text{for all } x \in \mathbb{R}.$$

Inspired by this inequality, we prove that specific sums of the L^2 -norms of a function $g \in PW^2$, normalized by $g(0) = 1$, and its consecutive derivatives, are bounded from below. Our result may be considered a variation of the one-delta problem where one wishes to minimize sums of L^2 -norms of an entire function and of its derivatives, and reads as follows:

Theorem 2. *Let N be a nonnegative integer and the real polynomial*

$$\mathcal{P}(x) = \sum_{n=0}^N a_n x^n$$

be positive for every $x \in [0, 1]$. Then the inequality

$$\left(\int_0^1 \frac{1}{\mathcal{P}(t^2)} dt \right)^{-1} \leq \int_{-\infty}^{\infty} \sum_{n=0}^N \frac{a_n}{\pi^{2n}} |g^{(n)}(x)|^2 dx \quad (1.7)$$

holds for every $g \in PW^2$ which obeys the normalization $g(0) = 1$. Moreover, equality in (1.7) is attained if and only if

$$g(z) = \left(\int_0^1 \frac{1}{\mathcal{P}(t^2)} dt \right)^{-1} \int_0^1 \frac{\cos(\pi zt)}{\mathcal{P}(t^2)} dt. \quad (1.8)$$

Note that when $N = 0$ and $a_0 = 1$, we recover the inequality (1.3), which once again shows that the latter is a natural result in the spirit of the one-delta problem. Moreover, choosing the polynomial $\mathcal{P}(x) = 1 + a\pi^2 x$, we obtain the following corollary.

Corollary 3. *Fix $a > 0$. Then the inequality*

$$\frac{\pi\sqrt{a}}{\arctan(\pi\sqrt{a})} \leq \int_{-\infty}^{\infty} \left(|g(x)|^2 + a |g'(x)|^2 \right) dx, \quad (1.9)$$

holds for every $g \in PW^2$ with $g(0) = 1$ and the unique extremal function for which (1.9) reduces to an equality is

$$g(z) = \frac{\pi\sqrt{a}}{\arctan(\pi\sqrt{a})} \int_0^1 \frac{\cos(\pi zt)}{1 + a\pi^2 t^2} dt.$$

Observe that for $a = 1/\pi^2$ (1.9) reduces to the following lower bound for the integral of the function in (1.6):

$$\int_{-\infty}^{\infty} \left(|g(x)|^2 + \frac{|g'(x)|^2}{\pi^2} \right) dx \geq \frac{4}{\pi}, \quad g \in PW^2, \quad g(0) = 1.$$

Different choices of the polynomial $\mathcal{P}(x)$ allow us to obtain other interesting inequalities.

Corollary 4. *Fix $0 < a < 1/\pi^2$. Then*

$$a \int_{-\infty}^{\infty} |g'(x)|^2 dx + \left(\frac{1}{2\pi\sqrt{a}} \log \left(\frac{1 + \sqrt{a}\pi}{1 - \sqrt{a}\pi} \right) \right)^{-1} \leq \int_{-\infty}^{\infty} |g(x)|^2 dx. \quad (1.10)$$

for every $g \in PW^2$ which obeys $g(0) = 1$.

In particular, letting $a \rightarrow 1/\pi^2$ in (1.10) we obtain

$$\int_{-\infty}^{\infty} |g'(x)|^2 dx \leq \pi^2 \int_{-\infty}^{\infty} |g(x)|^2 dx, \quad g \in PW^2, \quad g(0) = 1,$$

which is exactly the L^2 -version of the classical Bernstein inequality that holds for every L^p , $p \geq 1$ (see [2, Theorem 11.3.3]).

Observe that the Bernstein inequality follows from Theorem 2 if we set $\mathcal{P}(t) = 1 + \varepsilon - t$ and let $\varepsilon \rightarrow 0$. Applying the same reasoning with $\mathcal{P}(t) = (1 + \varepsilon - t)^N$, we obtain:

Corollary 5. *Let N be a nonnegative integer. Then the inequality*

$$\sum_{k=0}^N \frac{(-1)^k}{\sigma^{2k}} \binom{N}{k} \int_{-\infty}^{\infty} |f^{(k)}(x)|^2 dx \geq 0$$

holds for every function of exponential type at most σ such that $f \in L^2(\mathbb{R})$. In particular, for $N = 2$,

$$\int_{-\infty}^{\infty} |f'(x)|^2 dx \leq \frac{1}{2} \left(\sigma^2 \int_{-\infty}^{\infty} |f(x)|^2 dx + \frac{1}{\sigma^2} \int_{-\infty}^{\infty} |f''(x)|^2 dx \right).$$

The latter is a curious result that resembles some classical ones, due to Landau and Hardy and Littlewood. In 1913, Landau [15] proved that if f is a real function, $f \in C^2(\mathbb{R})$, and the inequalities $\|f\|_{\infty} \leq 1$ and $\|f''\|_{\infty} \leq 1$ for the uniform norms of f and f'' on the real line hold, so does $\|f'\|_{\infty} \leq \sqrt{2}$.

Hardy and Littlewood [12, Theorem 6] proved that, if y and y'' are in $L^2[0, \infty)$, then

$$\left(\int_0^\infty [y'(x)]^2 dx \right)^2 \leq 4 \int_0^\infty [y(x)]^2 dx \int_0^\infty [y''(x)]^2 dx.$$

Moreover, the constant 4 is the best possible. The equality is attained if and only if $y(x) = cY(ax)$, where c and a are real constants and

$$Y(x) = e^{-x/2} \sin \left(\frac{\sqrt{3}}{2} x - \frac{\pi}{3} \right).$$

Theorem 7 in [12] states that, under the same requirements, the inequality

$$\int_0^\infty (y^2(x) + [y''(x)]^2 - [y'(x)]^2) dx \geq 0$$

holds with equality as before, but with $a = 1$.

2. PROOF OF THEOREM 1

For $f \in L^1(\mathbb{R})$, we normalize the Fourier transform \widehat{f} of f as

$$\widehat{f}(\xi) = \int_{-\infty}^\infty f(x) e^{-2\pi i \xi x} dx.$$

2.1. Proof of a). Replacing $f(x)$ by $(f(x) + f(-x))/2$, we see that we may restrict our search for the infimum (1.4) to the even functions in \mathcal{F}_1 . To prove the existence of an extremizer, we follow an argument in [4], which consists in showing that a certain weak limit is a viable candidate. Consider an extremizing sequence $\{f_n\}_{n \geq 1} \subset \mathcal{F}_1$ such that f_n is even, $f_n(0) = \|f_n\|_\infty = 1$, and $\lim_{n \rightarrow \infty} \|f_n\|_1 = \mathcal{A}_1$. Hence, $\{f_n\}_{n \geq 1}$ is a bounded sequence in $L^2(\mathbb{R})$, which implies that there exists $F \in L^2(\mathbb{R})$ such that $\{f_n\}$, after passing to a subsequence, converges to F weakly in $L^2(\mathbb{R})$. Now the Paley-Wiener theorem [17, Theorem 4.1] and Mazur's lemma [3, Corollary 3.8] yield that $\text{supp}(\widehat{F}) \subset [-1, 1]$, and by Fourier inversion, F is an entire function of exponential type at most 2π . Moreover, using the weak convergence, we conclude that, for every $x \in \mathbb{R}$,

$$\begin{aligned} f_n(x) &= \int_{-1}^1 \widehat{f}_n(\xi) e^{2\pi i x \xi} d\xi = \int_{-\infty}^\infty f_n(y) \frac{\sin 2\pi(y-x)}{\pi(y-x)} dx \rightarrow \int_{-\infty}^\infty F(y) \frac{\sin 2\pi(y-x)}{\pi(y-x)} dx \\ &= \int_{-1}^1 \widehat{F}(\xi) e^{2\pi i x \xi} d\xi = F(x) \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Therefore, F is an even radially decreasing function such that $F(x) \geq 0$ and $F(0) = 1$. Finally, using Fatou's lemma, we conclude that $F \in L^1(\mathbb{R})$.

2.2. Proof of b). Let F be an even extremizer with $F(0) = 1$. Clearly, it has no real zeros. Indeed, since F is real, nonnegative and decreasing on the positive real axis, if it vanishes at $x_0 > 0$, it does for all $x > x_0$ which is impossible because F is entire and $F(0) = 1$. A similar argument shows that F cannot vanish at a negative x_0 . Therefore, all the zeros of F satisfy $|\text{Im } z| > 0$. Now, assume that F has a zero at $z = ib$, for $b \in \mathbb{R}$. Since F is real-valued, it also has $z = -ib$ as a zero. Consider the entire function

$$G(z) = \frac{b^2 F(z)}{z^2 + b^2}$$

Note that $G(0) = 1$ and $G \in \mathcal{F}_1$. Since

$$\int_{-\infty}^\infty G(x) dx < \int_{-\infty}^\infty F(x) dx$$

we get a contradiction. Therefore, all the zeros of F satisfy $|\operatorname{Re} z| > 0$. Now, assume that $z = a + ib$ is a zero of F with $|b| \geq |a| > 0$. Since F is real-valued and even, we have that $z = a - ib$, $z = -a + ib$, and $z = -a - ib$ are also zeros. Note that all these zeros are different. Then, the entire function

$$H(z) = \frac{(a^2 + b^2)^2 F(z)}{((z - a)^2 + b^2)((z + a)^2 + b^2)}$$

is in \mathcal{F}_1 , and using that $|b| \geq |a|$, it is easy to see that

$$\int_{-\infty}^{\infty} H(x) dx < \int_{-\infty}^{\infty} F(x) dx$$

which gives a contradiction. We conclude that $|b| < |a|$.

2.3. Representation lemma. The following lemma gives a representation for any even function in \mathcal{F}_1 .

Lemma 6. *If $f \in \mathcal{F}_1$, then it can be represented in \mathbb{R} in the form*

$$f(x) = \int_{-\infty}^x -t |h(t)|^2 dt, \quad (2.1)$$

where $h : \mathbb{C} \rightarrow \mathbb{C}$ is an entire function of exponential type at most π such that $|h(x)| = |h(-x)|$ for all $x \in \mathbb{R}$, and $xh \in L^2(\mathbb{R})$. Conversely, if f is a function of the form (2.1), then it has an analytic extension to \mathbb{C} which is an even function in \mathcal{F}_1 .

Proof. Let $f \in \mathcal{F}_1$ be even. Then clearly $\lim_{x \rightarrow \pm\infty} x f(x) = 0$. Integration by parts yields

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^{\infty} -x f'(x) dx. \quad (2.2)$$

By the Plancherel-Pólya theorem, f' has exponential type 2π and so does $-zf'(z)$. The monotonicity requirement implies $-xf'(x) \geq 0$ for all $x \in \mathbb{R}$, and then (2.2) yields $-xf' \in L^1(\mathbb{R})$. From the Krein decomposition theorem [1, p. 154], it follows that $-zf'(z) = g(z)g^*(z)$ for some $g \in PW^2$. Moreover, since f attains its maximum at $x = 0$, then $f'(0) = 0$. Defining $h(z) = g(z)/z$, we rewrite the latter in the form

$$-zf'(z) = z^2 h(z)h^*(z), \quad (2.3)$$

where h is an entire function of exponential type at most π and $xh \in L^2(\mathbb{R})$. Since f' is odd, then $|h(x)| = |h(-x)|$ for $x \in \mathbb{R}$. Finally, integrating (2.3) appropriately, we arrive at (2.1). Conversely, assume the representation (2.1). Note that f has an analytic extension on \mathbb{C} (also denoted by f) of the form

$$f(z) = \int_{-\infty}^0 -t |h(t)|^2 dt + \int_{[0,z]} -s h(s)h^*(s) ds,$$

where $[0, z]$ denotes the straight segment connecting 0 and z . Since h is an entire function of exponential type at most π , f is an entire function of exponential type at most 2π . From (2.1) it follows that $\lim_{x \rightarrow -\infty} f(x) = 0$, and using the fact that $|h(x)| = |h(-x)|$, we conclude that f is also even and $\lim_{x \rightarrow \infty} f(x) = 0$. On the other hand, differentiating (2.1) we derive

$$f'(x) = -x |h(x)|^2 \quad \text{for } x \in \mathbb{R}, \quad (2.4)$$

which implies that f is radially decreasing and $f(x) \geq 0$. Moreover, (2.4) and $xh \in L^1(\mathbb{R})$ imply $\lim_{x \rightarrow \pm\infty} x f(x) = 0$. Integration by parts shows that

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^{\infty} x^2 |h(x)|^2 dx,$$

which yields $f \in L^1(\mathbb{R})$. □

2.4. Proof of c). Since $K(x)$ is the unique extremal solution for the one-delta problem, we have that $1 < \mathcal{A}_1$. On the other hand, as mentioned in the introduction, from Lemma 6, we can reformulate the monotone-delta problem as the one to determine

$$\mathcal{A}_1 = \inf_{\substack{h \in \mathcal{F}_2 \\ h \neq 0}} \frac{2 \int_{-\infty}^{\infty} |x|^2 |h(x)|^2 dx}{\int_{-\infty}^{\infty} |x| |h(x)|^2 dx}, \quad (2.5)$$

where the family \mathcal{F}_2 consists of those entire functions $h : \mathbb{C} \rightarrow \mathbb{C}$ of exponential type at most π such that $xh \in L^2(\mathbb{R})$ and $|h(x)| = |h(-x)|$.

We now transform this optimization problem over \mathcal{F}_2 into another unrestricted, smooth optimization problem over \mathbb{R}^{d+1} , so that we may construct functions h in a systematic way with standard numerical optimization methods. For this purpose, we make a couple of helpful observations. First, note that if $h \in \mathcal{F}_2$ then $h \in L^1(\mathbb{R})$. In fact, by the Cauchy-Schwarz inequality, we have

$$\int_1^{\infty} |h(x)| dx = \int_1^{\infty} |x h(x)| \cdot \frac{1}{x} dx \leq \sqrt{\int_1^{\infty} x^2 |h(x)|^2 dx} \cdot \sqrt{\int_1^{\infty} \frac{1}{x^2} dx} < \infty.$$

Therefore, \widehat{h} is continuous in \mathbb{R} , and in particular $\widehat{h}(\pm 1/2) = 0$. Denoting $I = [-1/2, 1/2]$ we have that $\text{supp } \widehat{h} \subset I$. Therefore, by the Stone-Weierstrass theorem we may approximate \widehat{h} uniformly by a polynomial times χ_I , where χ_I denotes the characteristic function of the interval I .

With the previous observations in mind, we consider functions of the form

$$\widehat{h}(x) = \left(\frac{1}{4} - x^2\right) g(x) \chi_I(x), \quad (2.6)$$

where

$$g(x) = \sum_{i=0}^d a_i x^i \in \mathbb{R}[x]$$

is a polynomial of degree d . Note that the factor $(\frac{1}{4} - x^2)$ means that $\widehat{h}(\pm 1/2) = 0$. Denoting $\mathbf{a} = (a_0, a_1, \dots, a_d) \in \mathbb{R}^{d+1}$, the infimum in (2.5), restricted to this class, becomes

$$\mathcal{A}_{1,d} := \min_{\mathbf{a} \in \mathbb{R}^{d+1} \setminus \{0\}} \frac{2\mathbf{a} \cdot N\mathbf{a}}{\mathbf{a} \cdot D\mathbf{a}}, \quad (2.7)$$

where $N, D \in \mathbb{R}^{(d+1) \times (d+1)}$ are defined by

$$N_{ij} = \int_{-\infty}^{\infty} |x|^2 f_i(x) \overline{f_j(x)} dx; \quad D_{ij} = \int_{-\infty}^{\infty} |x| f_i(x) \overline{f_j(x)} dx; \quad f_i(x) = \left[\left(\frac{1}{4} - y^2\right) y^i \chi_I \right]^\wedge(-x).$$

For all $d \leq 20$ and $0 \leq i \leq d$, it is easy to see by direct computation of f_i that $x f_i \in L^2(\mathbb{R})$, so that $h = \mathbf{a} \cdot (f_0, \dots, f_d) \in \mathcal{F}_2$ for all $\mathbf{a} \in \mathbb{R}^{d+1}$. The matrices N and D may be computed explicitly for a given d , and this is then a smooth optimization problem over \mathbb{R}^{d+1} . Solving it numerically for $d = 2$, we find

$$\widehat{h}_0(x) = \left(\frac{1}{4} - x^2\right) \left(1 - \frac{9}{5}x^2\right) \chi_I(x), \quad (2.8)$$

which yields

$$h_0(x) = \frac{(108 - 25\pi^2 x^2) \sin(\pi x) - \pi x (11\pi^2 x^2 + 108) \cos(\pi x)}{40\pi^5 x^5}. \quad (2.9)$$

By direct computation in exact rational arithmetic, this gives

$$\mathcal{A}_1 \leq \frac{49484}{38745} = 1.27717\dots$$

This proves part (c). Moreover, using the representation (2.1) we obtain the function in \mathcal{F}_1

$$f_0(x) = \frac{P(\pi x) + Q(\pi x) \sin(2\pi x) + R(\pi x) \cos(2\pi x)}{738\pi^8 x^8}, \quad (2.10)$$

where

$$\begin{aligned} P(x) &= 242x^6 + 3001x^4 + 4176, \\ Q(x) &= -242x^5 - 576x^3 - 11664x, \\ R(x) &= 1463x^4 + 7488x^2 - 5832; \end{aligned}$$

see Figure 2.

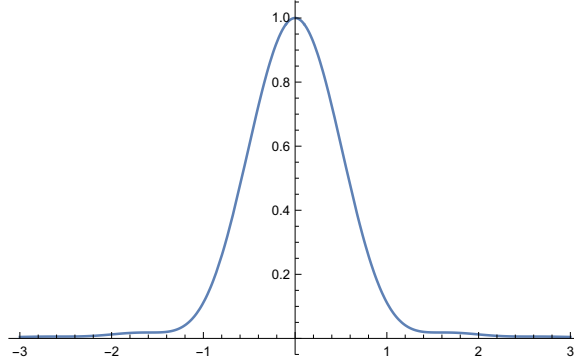


FIGURE 2. The function $f_0(x)$ defined in equation (2.10).

Additionally, we solve (2.7) for all $d \leq 20$ and observe that, as the degree d increases, the sequence $\mathcal{A}_{1,d}$ decreases very slowly, showing only a tiny improvement from 1.27717... only in the fifth decimal digit. More precisely, we obtain $\mathcal{A}_1 \leq 1.27713505\dots$ with those much more detailed calculations performed with large degree d of the polynomials g . In Section 3.2, we will show some tables with the results of these computations (see Table 1), and compare the results with those of another numerical approach. In Figure 3 and Figure 4,

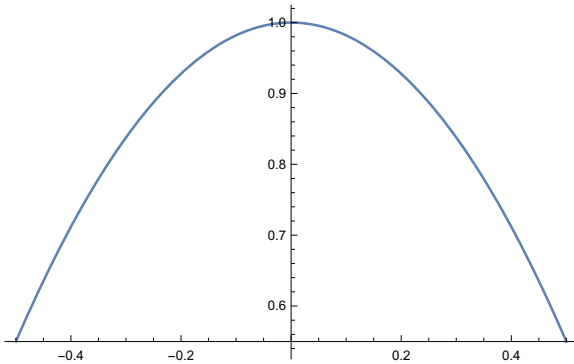


FIGURE 3. The function $\widehat{h}_0(x)/\widehat{h}_0(0)$ defined in equation (2.8).

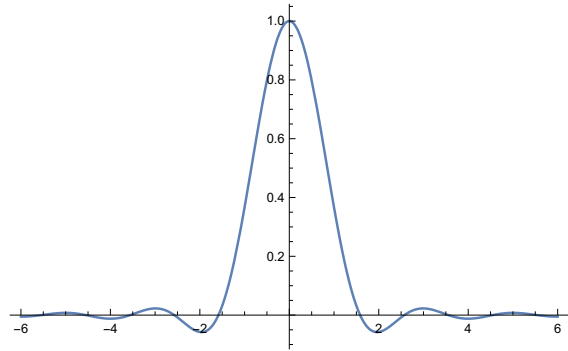


FIGURE 4. The function $h_0(x)/h_0(0)$ defined in equation (2.9).

we plot the functions $4\widehat{h}_0$ and $\frac{600}{91}h_0$, respectively, where, since $h_0(0) = \frac{91}{600}$ and $\widehat{h}_0(0) = \frac{1}{4}$, we renormalized the plots accordingly.

3. SOME FUNCTIONS IN \mathcal{F}_1

3.1. The lid function. In this subsection, we apply Sonin's method to construct a nice sequence of functions in \mathcal{F}_1 . For any positive real numbers B and C , consider the differential equation

$$y'' + \frac{B}{x}y' + Cy = 0. \quad (3.1)$$

Let $y = g$, $g : \mathbb{R} \rightarrow \mathbb{R}$ be a solution of the equation (3.1). The *lid* of g^2 is the function defined by

$$f(x) = (g(x))^2 + \frac{(g'(x))^2}{C}. \quad (3.2)$$

Note that $f(x) \geq 0$ for all $x \in \mathbb{R}$, and

$$f'(x) = -\frac{2B(g'(x))^2}{xC}.$$

This implies that f is radially decreasing. Moreover, if we suppose that the solution g has an analytic extension on \mathbb{C} of exponential type at most π , and $g \in L^2(\mathbb{R})$, we conclude that $f \in \mathcal{F}_1$.

Let us show some examples of lids. For $\alpha > 0$, consider the Bessel function of the first kind of order α , which is defined by

$$J_\alpha(z) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu (z/2)^{\alpha+2\nu}}{\nu! \Gamma(\nu + \alpha + 1)}.$$

Let us remark some properties of the Bessel functions mentioned in [18, Section 1.71]. It is known (see [18, Equation 1.71.3]) that J_α satisfies the differential equation

$$y'' + x^{-1}y' + (1 - \alpha^2 x^{-2})y = 0. \quad (3.3)$$

Now, define the function

$$g_\alpha(z) = \frac{J_\alpha(\pi z)}{(\pi z)^\alpha}.$$

A straightforward change of variables in (3.3) shows that g_α satisfies the differential equation

$$y'' + \frac{2\alpha + 1}{x}y' + \pi^2 y = 0.$$

The function g_α is an even entire function of exponential type π . Moreover, using the decay of J_α (see [18, Equations 1.71.10 and 1.71.11]) we see that $g_\alpha \in L^2(\mathbb{R})$. Therefore, inserting g_α in (3.2) we actually construct the lid of g_α^2 , with $B = 2\alpha + 1$ and $C = \pi^2$. In the particular case $\alpha = 1/2$ we know that

$$g_{1/2}(x) = \frac{\sin(\pi x)}{\pi x},$$

and therefore

$$f_{1/2}(x) = (g_{1/2}(x))^2 + \frac{(g'_{1/2}(x))^2}{\pi^2}$$

is the lid of $K(x)$. Straightforward calculations show that the Fourier transform of $f_{1/2}$ is

$$\widehat{f_{1/2}}(\xi) = \max\{1 - |\xi|, 0\} + \frac{1}{\pi^2} \widehat{(g'_{1/2})^2}(\xi). \quad (3.4)$$

Then the Fourier transform - convolution de Margan type law yields

$$\widehat{(g'_{1/2})^2}(\xi) = \widehat{(g'_{1/2} * g'_{1/2})}(\xi) = ((2\pi i x \widehat{g_{1/2}}) * (2\pi i x \widehat{g_{1/2}}))(\xi) = -4\pi^2 \int_{-\infty}^{\infty} x \widehat{g_{1/2}}(x)(\xi - x) \widehat{g_{1/2}}(\xi - x) dx,$$

where we used the fact that $\widehat{g_{1/2}}(x) = \chi_I(x)$. This, together with (3.4), implies

$$\widehat{f_{1/2}}(\xi) = \begin{cases} \frac{2}{3} (1 - |\xi|)^2 (|\xi| + 2), & \text{if } |\xi| \leq 1; \\ 0, & \text{if } |\xi| > 1. \end{cases}$$

In particular, this example allows us to obtain the bound $\mathcal{A}_1 \leq 1.333\dots$. In fact, one can repeat the same argument for the function f_α , for any $\alpha > 0$. The Fourier transform of f_α can be computed using [18, Equation 1.71.6]. Finally, we minimize the ratio $\widehat{f_\alpha}(0)/f_\alpha(0)$ with respect to α , and obtain that it is attained for $\alpha_0 = 0.787\dots$ and $\widehat{f_{\alpha_0}}(0)/f_{\alpha_0}(0) = 1.284\dots$. Hence $\mathcal{A}_1 \leq 1.284\dots$.

3.2. An L^2 -computational approach. Another natural approach for constructing functions in \mathcal{F}_2 (and therefore in \mathcal{F}_1), and computationally solving (2.5), starts by finding an orthonormal system for the space $L^2(\mathbb{R}, x^2 dx)$. Note that \mathcal{F}_2 is a Hilbert space with the inner product

$$\langle f, g \rangle_{\mathcal{F}_2} = \langle xf, xg \rangle_{L^2(\mathbb{R})}.$$

For odd integers $k \geq 1$, we define the even functions

$$h_k(x) = \frac{4\sqrt{2} \cos \pi x}{\pi(k^2 - 4x^2)},$$

and note that $h_k \in \mathcal{F}_2$ for all odd integers $k \geq 1$. Gorbachev [10] previously considered this family of functions to obtain fine numerical estimates for other Fourier extremal problems, and the first and third authors [7] have also used this family for similar purposes in related extremal problems introduced by Carneiro, Milinovich, and Soundararajan [4]. Regarding this system, we can say the following:

Proposition 7. *The family $(h_k)_{\substack{k \geq 1 \\ k \text{ odd}}}$ is a complete orthonormal system in the closed subspace $\{h \in \mathcal{F}_2 : h \text{ is even}\}$.*

Proof. Note that, if $h \in \mathcal{F}_2$ is even, then $xh \in L^2(\mathbb{R})$ is odd. Furthermore, we have that

$$\widehat{(xh_k)}(t) = i(-1)^{\frac{k+1}{2}} \sqrt{2} \sin(\pi kt) \chi_I(t) =: s_k(t). \quad (3.5)$$

To see this, since $s_k \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, we may compute $\widehat{s_k}$ in a straightforward manner to verify that $\widehat{s_k}(x) = -xh_k(x)$, and then we conclude (3.5) by Fourier inversion in $L^2(\mathbb{R})$. Now consider the operator

$$T : \mathcal{F}_2 \rightarrow L^2(I)$$

defined by $Th(t) := \widehat{(xh)}(t)e^{\pi it}$. By Plancherel's theorem and the Paley-Wiener theorem, T is a linear isometry, that is, $\langle f, g \rangle_{\mathcal{F}_2} = \langle Tf, Tg \rangle_{L^2(I)}$. Therefore, for odd positive integers k and j , we find that

$$\langle h_k, h_j \rangle_{\mathcal{F}_2} = \langle s_k, s_j \rangle_{L^2(I)} = \delta_{kj},$$

where $\delta_{kj} = 1$ if $k = j$, and 0 otherwise. Here, to compute the inner product over $L^2(I)$, we may apply the identity $2 \sin(\pi kt) \sin(\pi jt) = \cos(\pi(k-j)t) - \cos(\pi(k+j)t)$ and use that $k \pm j$ is an even integer. This shows that h_k is orthonormal.

We now show that it is complete. Let $h \in \mathcal{F}_2$ be even, such that $\langle h, h_k \rangle_{\mathcal{F}_2} = 0$ for all odd positive integers k . We must show that $h \equiv 0$. First, denote $H(t) = \widehat{(xh)}(t)$, and note that, by Plancherel's theorem and

(3.5), the condition $\langle h, h_k \rangle_{\mathcal{F}_2} = 0$ implies that

$$\int_I H(t) \sin(\pi(2j-1)t) dt = 0 \quad (3.6)$$

for all positive integers j . Actually, since $\sin(-x) = -\sin x$, (3.6) holds for all integers j .

Now, since T is an isometry into $L^2(I)$, by the theory of Fourier series on $L^2(I)$, it is enough to show that $\langle Th, e_j \rangle_{L^2(I)} = 0$ for all integers j , where $e_j(t) = e^{2\pi i j t}$. In fact, for an integer j , we have

$$\begin{aligned} \langle Th, e_j \rangle_{L^2(I)} &= \int_I H(t) e^{\pi i t} e^{-2\pi i t j} dt \\ &= \int_I H(t) \cos(\pi(2j-1)t) dt - i \int_I H(t) \sin(\pi(2j-1)t) dt. \end{aligned}$$

The first integral in the last line is 0 since H is odd, and the second integral is 0 by (3.6). Therefore, $\langle Th, e_j \rangle_{L^2(I)} = 0$ for all integers j , and then $Th \equiv 0$ and $h \equiv 0$, as desired. \square

Once we have a complete orthonormal system, we proceed to obtain numerical examples as follows. For a positive integer d , let $\mathcal{F}_{2,d} = \text{span} \{h_{2j-1} : 1 \leq j \leq d\} \subset \mathcal{F}_2$. Let $Q \in \mathbb{R}^{d \times d}$ be the matrix defined by

$$Q_{ij} = \int_{-\infty}^0 -x h_{2i-1}(x) h_{2j-1}(x) dx.$$

Then, since h_k are orthonormal, one can see that the reciprocal of the infimum in (2.5), when taken over the space $\mathcal{F}_{2,d}$, satisfies

$$|\lambda_d| = \max_{\substack{h \in \mathcal{F}_{2,d} \\ h \neq 0}} \frac{\int_{-\infty}^0 -x |h(x)|^2 dx}{\int_{-\infty}^{\infty} |x|^2 |h(x)|^2 dx},$$

where λ_d is the largest eigenvalue (in absolute value) of Q , and the maximum is attained when

$$h = \mathbf{a} \cdot (h_1, h_3, \dots, h_{2d-1}), \quad (3.7)$$

for $\mathbf{a} \in \mathbb{R}^d$ an eigenvector of Q associated to λ_d . We calculate the eigensystems numerically for $d \leq 1000$. We find that $\lambda_{1000} = 0.783002554\dots$, giving a proof for the bound $\mathcal{A}_1 \leq 1/\lambda_{1000} = 1.277135042\dots$, which is only slightly smaller than our example (2.8) in the proof of Theorem 1 – again coinciding in the first four decimal digits. Moreover, this also coincides with the first seven decimal digits given by the polynomials of degree $d = 20$ that we constructed with the approach in Section 2.4.

In Table 1, we compare the speed of convergence of the two numerical approaches we have presented. The first approach is described in Section 2.4, with functions h defined as in (2.6), via polynomials g of some degree d . The second approach is the L^2 -approach described in the present section, with functions h defined by (3.7). In both cases, the parameter d is the number of degrees of freedom in the construction of the function h . In both cases, the bounds for \mathcal{A}_1 appear to quickly converge to the first few decimal digits, yet we observe that in the polynomial approach, the bound for \mathcal{A}_1 seems to converge much faster to more decimal digits with small values of d . Together, all of this gives evidence to the conjecture that the sharp value of \mathcal{A}_1 , up to its first 8 significant digits, is

$$\mathcal{A}_1 = 1.27713504\dots \quad (3.8)$$

Furthermore, the normalized plot of the function h we constructed by using (3.7) with $d = 1000$ is almost indistinguishable from the plot of h_0 shown in Figure 4. Since the explicit function h_0 defined in (2.9) already

agrees with our conjecture (3.8) to four significant digits, we might expect it to behave close to an extremizer for \mathcal{A}_1 . Indeed, in Table 2, we compare the first 10 zeros of the functions h_0 in (2.9) and h in (3.7) (the latter with $d = 1000$). Note that there is a good agreement up to the second decimal digit. We remark that the latter do not change with respect to the values with $d = 500$, up to the digits shown, except for a minor change in the last digit of $x_{10} = 10.5240\dots$ (for $d = 500$).

d	\mathcal{A}_1 (polynomials)	d	\mathcal{A}_1 (L^2)
2	1.277171240	10	1.277199350
4	1.277148060	50	1.277136017
6	1.277137688	100	1.277135195
8	1.277135865	150	1.277135093
10	1.277135348	200	1.277135065
12	1.277135173	300	1.277135050
14	1.277135104	400	1.277135046
16	1.277135074	500	1.277135044
20	1.277135052	1000	1.277135042

TABLE 1. Comparison of the numerical bounds for \mathcal{A}_1 in the polynomial construction of Section 2.4 and in the L^2 -construction of Section 3.2, as the corresponding parameter d grows.

\cdot	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
Pol	1.5839	2.5715	3.5573	4.5470	5.5395	6.5340	7.5297	8.5264	9.5238	10.5220
L^2	1.5866	2.5648	3.5525	4.5444	5.5387	6.5344	7.5311	8.5284	9.5261	10.5243

TABLE 2. First positive zeros of the function h_0 via polynomials of degree 2 given in (2.9), and via the L^2 -approach as in (3.7) with $d = 1000$.

4. PROOF OF THEOREM 2

Let $g \in PW^2$. By Paley-Wiener's theorem, \hat{g} has compact support in $[-\frac{1}{2}, \frac{1}{2}]$, and using Plancherel's theorem, we obtain

$$\int_{-\infty}^{\infty} \sum_{n=0}^N \frac{a_n}{\pi^{2n}} |g^{(n)}(x)|^2 dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{n=0}^N a_n (4t^2)^n \right) |\hat{g}(t)|^2 dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathcal{P}(4t^2) |\hat{g}(t)|^2 dt. \quad (4.1)$$

Since $g \in PW^2$, then

$$g(z) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{g}(t) e^{2\pi izt} dt. \quad (4.2)$$

Then the fact that $g(0) = 1$, the positivity of $\mathcal{P}(x)$, the Cauchy-Schwarz inequality and (4.1) yield

$$1 = \left| \int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{g}(t) dt \right|^2 = \left| \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{\mathcal{P}(4t^2)} \hat{g}(t) \cdot \frac{1}{\sqrt{\mathcal{P}(4t^2)}} dt \right|^2 \leq \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathcal{P}(4t^2) |\hat{g}(t)|^2 dt \right) \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\mathcal{P}(4t^2)} dt \right), \quad (4.3)$$

which implies (1.7). Note that equality in (4.3) holds if and only if there is $\lambda \in \mathbb{C}$, such that

$$\widehat{g}(t) = \frac{\lambda}{\mathcal{P}(4t^2)}$$

almost everywhere in $[-\frac{1}{2}, \frac{1}{2}]$. Hence, from (4.2) we conclude that

$$g(z) = \lambda \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{e^{2\pi izt}}{\mathcal{P}(4t^2)} dt = \lambda \int_0^1 \frac{\cos(\pi zt)}{\mathcal{P}(t^2)} dt.$$

Since $g(0) = 1$, then the extremal function is unique and it is given by (1.8).

Remark 8. Since $\mathcal{P}(x) > 0$ for all $x \in [0, 1]$, the expression in (4.1) is nonnegative. Thus we obtain a norm in PW^2 , defined by

$$\|g\|_{\mathcal{P}} = \left(\int_{-\infty}^{\infty} \sum_{n=0}^N \frac{a_n}{\pi^{2n}} |g^{(n)}(x)|^2 dx \right)^{1/2},$$

which can be viewed as a Sobolev-type norm.

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