

AN EXTREMAL PROBLEM IN GRAPH THEORY

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To Bernhard Hermann Neumann on his 60th birthday

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$G(n; l)$ will denote a graph of n vertices and l edges. Let $f_0(n, k)$ be the smallest integer such that there is a $G(n; f_0(n, k))$ in which for every set of k vertices there is a vertex joined to each of these. Thus for example $f_0(3, 2) = 3$ since in a triangle each pair of vertices is joined to a third. It can readily be checked that $f_0(4, 2) = 5$ (the extremal graph consists of a complete 4-gon with one edge removed). In general we will prove: Let $n > k$, and

$$(1) \quad f(n, k) = (k-1)n - \binom{k}{2} + \left\lceil \frac{n-k}{2} \right\rceil + 1;$$

then $f_0(n, k) = f(n, k)$.

It will be convenient to say that the vertices x_1, \dots, x_k of G are visible from x_{k+1} , if all the edges (x_i, x_{k+1}) , $i = 1, \dots, k$ occur in G . A graph is said to have property P_k if every set of k of its vertices is visible from another vertex. G_n will denote a graph of n vertices (the number of edges being unspecified) and $G(m)$ denotes a graph having m edges. Let $G_n^{(0)} = (G_n; f(n; k))$ be defined as follows: the vertices of $G_n^{(0)}$ are x_1, \dots, x_n . The vertices x_i , $i = 1, \dots, k-1$ are joined to every other vertex and our $G_n^{(0)}$ has $\lceil n-k+2/2 \rceil$ further edges which are as disjoint as possible. In other words if $n-k+1$ is even $G_n^{(0)}$ has the further edges (x_{k+2j}, x_{k+2j+1}) , $j = 0, \dots, \lceil n-k-1/2 \rceil$, if $n-k+1$ is odd the edges are (x_k, x_{k+1}) , (x_k, x_{k+2}) , (x_{k+j+1}, x_{k+j+2}) , $j = 1, \dots, \lceil n-k-2/2 \rceil$. It is easy to see that $G_n^{(0)}$ has property P_k . Now we prove

THEOREM 1. *A graph $G(n; f(n, k))$ has property P_k if and only if it is our graph $G_n^{(0)}$.*

Theorem 1 is vacuous for $n \leq k$ and it is trivial for $n = k+1$, thus we can assume $n \geq k+2$. Clearly Theorem 1 implies (1). To see this it suffices to observe that if a $G(n; f(n, k)-1)$ would have property P_k we could add to it a new edge so that the resulting $G(n; f(n, k))$ would not be a $G_n^{(0)}$.

Since $G_n^{(0)}$ has property P_k we only have to prove that a $G(n; f(n, k))$ has property P_k then it must be our $G_n^{(0)}$. Before we give the somewhat complicated proof we outline a simple proof of (1) for $k = 2$.

LEMMA. Let G_n have property P_k then every pair of its vertices is visible from at least $k - 1$ vertices.

Assume that the Lemma is false. Then say x_1 and x_2 are visible from only $y_1, \dots, y_l, l \leq k - 2$. But then the set of $l + 2 \leq k$ vertices $x_1, x_2, y_1, \dots, y_l$ would not be visible from any vertex of G_n , which contradicts our assumption.

Let now $x_i, i = 1, \dots, n$ be the vertices of G_n and assume that v_i is the valency of x_i (i.e. x_i is joined to v_i vertices of G). Our Lemma implies

$$(2) \quad \sum_{i=1}^n \binom{v_i}{2} \geq (k - 1) \binom{n}{2}$$

since the number of pairs of vertices visible from x_i is $\binom{v_i}{2}$.

From (2) it is easy to deduce (1) for $k = 2$. To see this observe that the number of edges of a graph is $\frac{1}{2} \sum_{i=1}^n v_i$.

By (2) $\sum_{i=1}^n \binom{v_i}{2} \geq \binom{n}{2}$ and thus by a simple argument $\frac{1}{2} \sum_{i=1}^n v_i$ will be at least as large as in the case that one v_i say v_1 is as large as possible i.e. $v_1 = n - 1$, and v_2, \dots, v_n are as small as is consistent with (2). Now it is easy to see that P_2 implies $v_i \geq 2$ for all i . Hence

$$(3) \quad \frac{1}{2} \sum_{i=1}^n v_i \geq \frac{1}{2}(n - 1 + 2(n - 1)) = \frac{3}{2}(n - 1)$$

which agrees with (1) for $k = 2$ if n is odd. If n is even a similar but somewhat more complicated argument proves (1).

It does not seem easy to deduce (1) from (2) for $k > 2$. One could easily obtain

$$f(n, k) = (k - \frac{1}{2})n + O(1)$$

but a more precise estimation seems difficult. Hence to prove (1) and Theorem 1 we shall use a different method.

We say that $G(m)$ has property θ_t if it contains a set S of t vertices x_1, \dots, x_t each of which is joined to some vertex of $G(m)$ not in S . \bar{G} is the complementary graph of G i.e. two vertices are joined in \bar{G} if and only if they are not joined in G .

Put $n = k + t - 1$. Then

$$\binom{n}{2} - f(n, k) = \binom{t}{2} - \left\lceil \frac{t+1}{2} \right\rceil.$$

Now a simple argument shows that the fact that $G(n; f(n, k))$ does not have

property P_k is equivalent to $\tilde{G}(n; t(n, k)) = G\left(\binom{t}{2} - \lfloor (t+1)/2 \rfloor\right)$ having property θ_{t-1} . Thus Theorem 1 is equivalent to the following

THEOREM 2. *Every $G\left(\binom{t}{2} - \lfloor (t+1)/2 \rfloor\right)$ has property θ_{t-1} except if it is a $\tilde{G}_n^{(0)}$.*

Clearly our $\tilde{G}_n^{(0)}$ is a $G\left(t, \binom{t}{2} - \lfloor (t+1)/2 \rfloor\right)$ where the missing $\lfloor (t+1)/2 \rfloor$ edges are as disjoint as possible.

Theorem 2 is vacuous for $t < 2$ and trivial for $t \leq 3$. Henceforth assume $t \geq 4$.

To prove Theorem 2 let $G\left(\binom{t}{2} - \lfloor (t-1)/2 \rfloor\right) = G$ be any graphs which does not have property θ_{t-1} . We will show that it must be a $\tilde{G}_n^{(0)}$. First of all we can assume that all vertices of our G have valency $\leq t-2$. For if not then say x_1 is joined to y_1, \dots, y_{t-1} which shows that G has property θ_{t-1} which contradicts our assumption.

Assume next that G has a vertex x of valency $t-2$ (this will be the critical case). Denote by y_1, \dots, y_{t-2} the vertices joined to x and let z_1, \dots be the other vertices of G . Clearly no two z 's can be joined. For if (z_1, z_2) would be an edge of G then z_1, y_1, \dots, y_{t-2} are $t-1$ vertices each of them are joined to a vertex not in the set, or G has property θ_{t-1} . Also no y can be joined to two z 's. For if y_1 is joined to z_1 and z_2 then the $t-1$ vertices $z_1, z_2, y_2, \dots, y_{t-2}$ would show that G has property θ_{t-1} .

Next we show that at least $t-3$ y 's are joined to some z (as we know each y can be joined to at most one z). Assume that u y 's are joined to some z ($u < t-3$). Clearly ($v(G)$ denotes the number of edges of G)

$$(4) \quad v(G) = \binom{t}{2} - \left\lfloor \frac{t+1}{2} \right\rfloor = u + \binom{t-1}{2} - N \text{ or } u - N = \left\lfloor \frac{t}{2} \right\rfloor - 1,$$

where N is the number of the edges of the complete graph spanned by y_1, \dots, y_{t-2} which do not occur in G . Now clearly

$$(5) \quad N \geq \left\lfloor \frac{u+1}{2} \right\rfloor$$

since a y joined to a z cannot be joined to all the other y 's (since otherwise its valency would be $t-1$), hence a missing edge (i.e. an edge not in G) is incident to every y which is joined to a z and this proves (5). From (4) and (5) we have

$$(6) \quad \left\lfloor \frac{u}{2} \right\rfloor \geq \left\lfloor \frac{t}{2} \right\rfloor - 1$$

(6) clearly implies $u \geq t-3$ as stated.

Hence either $u = t-3$ or $u = t-2$. (4) and $u \leq t-2$ implies that we must have equality in (5) i.e. $N = \lfloor (u+1)/2 \rfloor$.

First we prove Theorem 2 if $u = t - 3$. (6) implies that if $u = t - 3$, t is odd and since $N = [(u + 1)/2] + [u/2] = [(t - 2)/2]$ and every y which is joined to a z must be adjacent to a missing edge we obtain that the $[u/2]$ missing edges must be isolated. In other words we can assume that our G contains all the edges of the complete graph spanned, by x, y_1, \dots, y_{t-2} with the exception of the edges $(y_{2i}, y_{2i+1}), i = 1, \dots, [(t - 2)/2]$. Further every $y_i, i = 2, \dots, t - 2$ is joined to exactly one z . If all these z 's coincide then G is spanned by $x, y_1, \dots, y_{t-2}, z$ and is clearly our $\bar{G}_n^{(0)}$ and Theorem 2 is proved in this case.

To complete our proof of the case $u = t - 3$ assume that y_1 is joined to z_i and y_j to $z_j, (z_i \neq z_j), 2 \leq i < j \leq t - 2$. But then the $t - 1$ vertices $x, z_i, z_j, \{y_l\} 1 \leq l \leq t - 2, l \neq i, l \neq j$ show that our G has property θ_{t-1} (x and z_i are joined to y_i, z_j is joined to y_j and every other $y_l, l \neq i, l \neq j$ is joined to y_i or y_j [since the missing edges were isolated]). This contradiction completes the proof of Theorem 2 if $u = t - 3$.

Assume next $u = t - 2$. Then each y is incident to at least one missing edge and since the number of missing edges is $[(u + 1)/2] = [(t - 1)/2]$ we obtain that for even t there are $(t - 2)/2$ isolated missing edges. Just as in the case $u = t - 3$ we see that all the $t - 2$ y 's must be joined to the same z . But then we again obtain our $\bar{G}_n^{(0)}$. This disposes of the case $u = t - 2, t$ even.

Assume next $u = t - 2, t$ odd. These are $[(t - 1)/2]$ missing edges and since each y is incident to one of them we can assume without loss of generality that the missing edges are $(y_1, y_2), (y_1, y_3), (y_{2l}, y_{2l+1}), l = 2, \dots, [(t - 2)/2]$. If all the y 's are joined to the same z we again get our $\bar{G}_n^{(0)}$. Thus we can assume that not all the y 's are joined to the same z . Now to complete our proof we have to distinguish two cases. Assume first that there is a z say z_i which is joined to only one y say y_i . This case can immediately be disposed of since the set of $t - 1$ vertices $x, z_i, \{y_l\}, 1 \leq l \leq t - 2, l \neq i$ shows that our G has property θ_{t-1} (x and z_i are joined to y_i and all other y 's are by our assumption joined to a z different from z_i). This contradiction proves Theorem 2 in this case.

Assume finally that every z is joined to more than one y and there are at least two z 's. Let, say, z_1 be joined to y_i and y_j and z_2 to y_r . Observe now that either every y is joined in G to one of the two vertices y_i and y_r or every y is joined to one of the two vertices y_j and y_r (this follows from the fact that the missing edges are either isolated or have at most one vertex of valency two). Assume thus that every y is joined either to y_i or to y_r . But then the set of $t - 1$ vertices $x, z_1, z_2, \{y_l\}, 1 \leq l \leq t - 2, l \neq i, l \neq r$ show that our G has property θ_{t-1} (x and z_1 are joined to y_i, y_2 to y_r and every $y_l, l \neq i, l \neq r$ is joined either to y_i or y_r). This contradiction completes the proof of Theorem 2 if G has a vertex of valency $\geq t - 2$.

Assume now that all vertices of $G = G\left(\binom{t}{2} - \lfloor (t+1)/2 \rfloor\right)$ have valency $< t-2$. We will show by induction with respect to i that then our G must have property θ_{t-1} and this will complete the proof of Theorems 2 and 1 and also of (1).

Assume that the maximum valency of a vertex of our G is $r < t-2$. Let x be joined to y_1, \dots, y_r . Denote as before by z_1, \dots the other vertices of G and let u be the largest number of z 's joined to a y . Assume that y_1 is joined to z_1, \dots, z_u . We evidently have

$$(7) \quad u \leq \min(t-r-1, r-1).$$

To prove (7) observe that $u \geq r$ would imply $v(y_1) > r$ and $u \geq t-r$ would imply that G satisfies θ_{t-1} (consider the vertices $y_2, \dots, y_r, z_1, \dots, z_u$).

Denote by u_i the number of z 's joined to y_i ($u_1 = u$) and by w_i the number of y 's joined to y_i . By (7) $v(y_i) = 1 + u_i + w_i \leq r-1$. Thus by (7) the number E of edges incident to the vertices x, y_1, \dots, y_r equals

$$(8) \quad E = r + \sum_{i=1}^r (u_i + \frac{1}{2}w_i) \leq r(u+1) + \frac{r(r-u-1)}{2} = \frac{r(r+u+1)}{2} \leq r^2.$$

(8) follows from the fact that E is evidently maximal if all the u_i are $u = r-1$ (i.e. they are all as large as possible) and if $w_i = r-u-1 = 0$. From (7) we have (G_1 is the graph spanned by the z 's)

$$(9) \quad v(G_1) \geq \binom{t}{2} - \left\lfloor \frac{t+1}{2} \right\rfloor - r^2.$$

Assume first $r \leq t/2$. Then we obtain from (9)

$$(10) \quad v(G_1) > \binom{t-r}{2} - \left\lfloor \frac{t-r+1}{2} \right\rfloor$$

Hence by our induction assumption G_1 has property θ_{t-r-1} i.e. it contains a set of vertices z_1, \dots, z_{t-r-1} each of which is joined to some $z_j, j > t-r-1$. But then the $t-1$ vertices $z_1, \dots, z_{t-r-1}, y_1, \dots, y_r$ show that G has property θ_{t-1} , which proves Theorem 2 if $r \leq t/2$.

Assume next $t/2 < r \leq t-3$. From (7) we have $u_i \leq t-r-1$ and by (8) E is maximal if all the u_i are $t-r-1$ and $w_i = r-1-u_i = 2r-t$. But then by (8)

$$(11) \quad E \leq r+r(t-r-1) + \frac{r}{2}(2r-t) = \frac{rt}{2}.$$

From (11) we have

$$v(G_1) \geq \binom{t}{2} - \left\lfloor \frac{t+1}{2} \right\rfloor - \frac{rt}{2} > \binom{t-r}{2} - \left\lfloor \frac{t-r+1}{2} \right\rfloor$$

Thus the proof can be completed as in the previous case, and the proof of Theorem 2 is complete.

Denote by $f_0(n, k, r)$ the smallest integer for which there is a $G(n; f_0(n, k, r))$ in which every set of k vertices are visible from at least r vertices. We say that a graph has property $P_{k,r}$ if every set of k of its vertices is visible from at least r vertices. Just as in our Lemma we can show that if G_n has property $P_{k,r}$ then every pair of its vertices is visible from at least $k+r-2$ vertices (our old property P_k is $P_{k,1}$).

Thus we obtain as in (2) that if G_n has property $P_{k,r}$ then if $k > 1$

$$(2') \quad \sum_{i=1}^n \binom{v_i}{2} \geq (k+r-2) \binom{n}{2}.$$

From (2') we can deduce that if $n < n_0(k, r)$ then

$$(12) \quad f_0(n, k, r) = f_0(n, k+r-1) = f(n, k+r-1).$$

(12) certainly does not hold for every n, k and r . It is easy to see that $f_0(10, 2, 6) = 40$ but $f(10, 7) = 41$. Our Theorem 1 states that (12) always holds for $r = 1$ and perhaps it always holds for $r = 2$ also if $k > 1$. For $k = 1$ every G_n each vertex of which has valency $\geq r$ clearly has property $P_{1,r}$, thus $f_0(n, 1, r) = \lceil (rn+1)/2 \rceil$, in other words if $k = 1, r > 1$ then (12) is not true. We hope to return to these questions on another occasion.

Finally we can ask the following question: Denote by $F(n, k)$ the smallest integer for which there exists a directed graph $G(n; F(n, k))$ so that to every k vertices x_1, \dots, x_k of our G there is a vertex y of G so that all the edges $(y, x_i) i = 1, \dots, k$ occur in G and are directed away from y . It is easy to see that for $n \geq 3, F(n, 1) = n$ (for $n \leq 2$ there clearly is no solution). It is not hard to show that for $n \geq 7, F(n, 2) = 3n$ and for $n < 7$ there is no solution. For $k \geq 3$, we only have crude inequalities for $F(n, k)$. We say that G_n has property S_k (after Schütte who posed the problem) if for every set of k nodes (x_1, \dots, x_k) there is at least one node y in G_n so that all the edges $(y, x_i), i = 1, \dots, k$ occur in G and are directed away from y . Denote by $f(k)$ the smallest value of n for which an S_k -graph of n vertices exists. We have

$$(13) \quad (k-1)2^k + 3 \leq f(k) < ck^2 2^k.$$

(13) is due to P. Erdős, E. Szekeres and G. Szekeres (Math. Gazette 47 p. 220 and 49 p. 290). We can show that for $n > n_0(k)$

$$(14) \quad f(k-1) \cdot n \leq F(n, k) \leq f(k) \cdot n.$$

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