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AN ILL-POSED PROBLEM FOR A STRICTLY HYPERBOLIC EQUATION IN TWO UNKNOWNS NEAR A CORNER[1]

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In an earlier note [4] we gave a simple example of an ill-posed problem for a system of hyperbolic equations in a region whose boundary has a corner. The system was diagonal with coupling only at the boundary. Earlier we derived necessary and sufficient conditions for well-posedness [2] for a wide class of constant coefficient hyperbolic systems in such regions. In [3] we examined in some detail the phenomena which occur when these conditions are violated. The fundamental work for hyperbolic problems in regions with smooth boundaries was done by Kreiss [1].

It was pointed out by Sarason and Smoller [5] that the work of Strang [6] for the half-space problem implies that the corner problem is well posed for a strictly hyperbolic system in two unknowns iff the corresponding half-space problems are well posed. They constructed, using geometrical optics, a four dependent variable ill-posed example, where the half-space extensions were well posed.

In all the above-mentioned work, the boundary conditions imposed were local, i.e., of the form $Bu=f$ at $x_1=0$, where B is a matrix and u is the unknown vector on the boundary.

We have noticed that much of the theory can be extended to nonlocal pseudo-differential boundary conditions. In particular, conditions of the form

$$B(w_2, \cdots, w_n, s)\hat{u}(0, w_2, \cdots, w_n, s) = \hat{f}(w_2, \cdots, w_n, s),$$

where B is a matrix-valued function of the dual variables $x_i \to w_i$, $t \to s$, can be treated. Such boundary conditions are reasonable when nonlinear problems are linearized. We shall discuss this in detail in a future paper.

Our purpose here is to show that for such boundary conditions well-posedness of the two half-space problems does not imply well-posedness of the corner problem, even in the strictly hyperbolic two unknown variable case.

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We shall present necessary and sufficient algebraic conditions for well-posedness of this problem in the above-mentioned paper.

We consider the equation

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_x + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_y + \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \tag{1}$$

to be solved for the complex-valued functions u and v in the region $0<x$, y, t with initial conditions

$$u(x, y, 0) = v(x, y, 0) = 0. \tag{2}$$

Next, apply a Laplace transform in t, use the initial conditions (2), and call the dual variable $s=\eta+i\xi$, with $n>0$, ξ real. We have

$$s\hat{U} - A\hat{U}_x - B\hat{U}_y = \hat{F}, \tag{3}$$

where $\hat{U}$ and $\hat{F}$ are the transformed 2 vectors $\binom{\hat{u}}{\hat{v}}$ and $\binom{\hat{F}_1}{\hat{F}_2}$, respectively; A and B are defined in (1).

We impose boundary conditions for $\eta>1$:

$$\begin{aligned} &\text{(a)} \quad \hat{u}(0, y, s) = -\frac{1+\sqrt{(1-c^2)}}{c}\Phi_1(\xi)\hat{v}(0, y, s) + \hat{f}(y, s), \\ &\text{(b)} \quad \hat{u}(x, 0, s) = -\frac{1+\sqrt{(1-c^2)}}{c}\Phi_2(\xi)\hat{v}(x, 0, s) + \hat{g}(x, s), \end{aligned} \tag{4}$$

where c is any real number, $0<c<1$, $\Phi_1(\xi)$, $\Phi_2(\xi)$ are $C_0^\infty(-\infty, \infty)$ with $0\leqq\Phi_1\leqq 1$, $-1\leqq\Phi_2\leqq 1$, and $\Phi_1(\xi)$, $\Phi_2(\xi)\equiv 1$ if $-\frac{1}{2}<\xi<\frac{1}{2}$, $\Phi_1(\xi)\equiv 0$, $\Phi_2(\xi)\equiv -1$ if $|\xi|>1$.

The standard estimate for problems of this type is

$$(\eta-\eta_0)\,\|\hat{U}\|^2 + \|\hat{U}\|_B^2 \leqq K(\|\hat{F}\|^2 + \|\hat{f}\|_{B_1}^2 + \|\hat{g}\|_{B_2}^2) \tag{5}$$

uniformly in $s=\eta+i\xi$, for $\eta>\eta_0$, $\eta_0>0$ and fixed.

The norms are defined as

$$\begin{aligned} \|\hat{U}\|^2 &= \int_0^\infty\int_0^\infty (|\hat{u}(x, y, s)|^2 + |\hat{v}(x, y, s)|)^2\,dx\,dy, \\ \|\hat{U}\|_B^2 &= \int_0^\infty [|\hat{u}(0, y, s)|^2 + |\hat{v}(0, y, s)|^2 + |\hat{u}(y, 0, s)|^2 + |\hat{v}(y, 0, s)|^2]\,dy. \end{aligned} \tag{6}$$

$\|\ \|_{B_1}$ and $\|\ \|_{B_2}$ are defined analogously.

We have the following

THEOREM. *No estimate of type* (5) *is possible for problem* (3), (4)(a), (b). *However, problem* (3), (4)(a) *in the region* $0<x$, $-\infty<y<\infty$, *and*

(3), (4)(b), *in the region* $-\infty<x<\infty$, $0<y$ *both obey estimates of type* (5), *where the norms are modified in an obvious fashion, to be integrals over half- rather than quarter-space.*

PROOF. For the quarter-space problem, we consider

$$(7) \qquad \hat{U} = \exp(-csy - sx\sqrt{(1-c^2)})\begin{bmatrix} 1 \\ -c/(1+\sqrt{(1-c^2)}) \end{bmatrix}$$

for $s=\eta+i\xi$, $0<\eta$, $-\frac{1}{2}<\xi<\frac{1}{2}$. This function satisfies the homogeneous equations (3), (4)(a), (b). Moreover, the norms on the left side of (5) are finite.

For the right half-plane problem we can easily obtain the estimate

$$(8) \qquad (\eta - \eta_0)\,\|\hat{U}\|^2 \leqq K_1(\|\hat{F}\|^2 + \|\hat{U}\|_B^2)$$

independently of the boundary conditions (4)(a). (See e.g., Osher [2].)

We need only to obtain

$$(9) \qquad \|\hat{U}\|_B^2 \leqq K_2(\|\hat{F}\|^2 + \|\hat{f}\|_{B_1}^2 + \|\hat{U}\|^2).$$

Moreover, in a standard fashion, we can assume $\hat{F}\equiv 0$. (See, e.g., Osher [2].)

We can solve equation (3) for $F\equiv 0$, with boundary conditions (4)(a). Fourier transform (3) in y, then multiply by A^{-1}. We have

$$(10)(a) \qquad \tilde{U}_x - A^{-1}(s - Biw)\tilde{U} = 0,$$

where w is the dual variable, $\tilde{U}=\mathscr{F}\hat{U}$,

$$(b) \quad \tilde{u}(0, w, s) = -\frac{1+\sqrt{(1-c^2)}}{c}\Phi_1(\xi)\tilde{v}(0, w, s) + \tilde{f}(w, s).$$

Let

$$(11) \qquad \tilde{U} = T_1(w, s)\tilde{V}.$$

$T_1(w, s)$ is a unitary matrix-valued measurable function of w, s such that

$$(12) \qquad T_1^*(A^{-1}(s - Biw))T_1 = \begin{pmatrix} -K_+ & m_{12}(w, s) \\ 0 & K_+ \end{pmatrix}$$

where $K_+=\sqrt{(s^2+w^2)}$, $\operatorname{Re} K_+>0$.

Thus, the general solution to (10)(a) which does not grow exponentially as $x\to+\infty$, is

$$(13) \qquad \tilde{V} = \begin{pmatrix} \exp(-K_+x) \\ 0 \end{pmatrix} b_1(w, s),$$

or

$$\tilde{U}(0, w, s) = T_1(w, s)\begin{pmatrix} b_1(w, s) \\ 0 \end{pmatrix}. \tag{14}$$

Apply the boundary condition (10)(b). Thus

$$b_1(w, s) = \frac{\tilde{f}(w, s)[|s + K_+|^2 + |w|^2]^{1/2}}{\left[(s + K_+) + \dfrac{1 + \sqrt{(1 - c^2)}}{c}\, iw\Phi_1\right]}. \tag{15}$$

It is easy to show that the quantity multiplying $\tilde{f}(w, s)$ is uniformly bounded in w, s, if $\eta > 1$. Thus, this half-space problem is well posed.

We can do the analogous thing for the upper half-plane problem, arriving at

$$\tilde{U}(w, 0, s) = T_2(w, s)\begin{pmatrix} b_2(w, s) \\ 0 \end{pmatrix}. \tag{16}$$

Applying the boundary condition at $y=0$ leads to

$$b_2(w, s) = \frac{\tilde{g}(w, s)(|s - iw| + |s + iw|)^{1/2}}{\left[\sqrt{(s - iw)} - \dfrac{1 + \sqrt{(1 - c^2)}}{c}\sqrt{(s + iw)}\Phi_2(\xi)\right]}, \tag{17}$$

where each square root has positive real part Again we have $|b_2(w, s)| \leqq K_3\, |\tilde{g}(w, s)|$ if $\eta > 1$. Thus the half-space problem is well posed. Q.E.D.

Bibliography

1. H.-O. Kreiss, *Initial boundary value problems for hyperbolic systems*, Comm. Pure Appl. Math. **23** (1970), 277–298.

2. S. Osher, *Initial boundary value problems for hyperbolic systems in regions with corners*. I, Trans. Amer. Math. Soc. **176** (1973), 141–164.

3. ———, *Initial boundary value problems for hyperbolic systems in regions with corners*, II, Trans. Amer. Math. Soc. (to appear).

4. ———, *An ill-posed problem for a hyperbolic equation near a corner*, Bull. Amer. Math. Soc. **79** (1973), 143–144.

5. L. Sarason and J. Smoller, *Geometrical optics and the corner problem*, Arch Rational Mech. Anal. (to appear).

6. G. Strang, *Hyperbolic initial-boundary value problems in two unknowns*, J. Differential Equations **6** (1969), 161–171. MR **39** #633.

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