AN IMBEDDING PROBLEM

J. W. CANNON AND S. G. WAYMENT

ABSTRACT. If H is an uncountable collection of pairwise disjoint continua in E^n, each homeomorphic to M, then there exists a sequence from H converging homeomorphically to an element of H. In the present paper the authors show that if $\{M_i\}$ is a sequence of continua in E^n which converges homeomorphically to M_0 and such that for each i, M_i and M_0 are disjoint and equivalently imbedded, then there exists an uncountable collection H of pairwise disjoint continua in E^n, each homeomorphic to M. For $n=2$, 3, and $n\geqq 5$ it is shown that one cannot guarantee that the elements of H have the same imbedding as M_0.

Introduction. Let M be a continuum in E^n. It is well known that if H is an uncountable collection of pairwise disjoint continua in E^n, each homeomorphic to M, then there exists $M_0 \in H$ and a sequence $\{M_i\}$ from H such that the sequence $\{M_i\}$ converges homeomorphically to M_0, that is, for $\epsilon > 0$ there exists N such that $i \geqq N$ implies the existence of a homeomorphism h_i of M_0 onto M_i which moves no point more than ϵ.

The following question is immediately raised: suppose M_0 is a continuum in E^n and $\{M_i\}$ is a sequence of pairwise disjoint continua in E^n such that $\{M_i\}$ converges homeomorphically to M_0 and for each i, $M_i \cap M_0 = \varnothing$. Does there necessarily exist an uncountable collection H of disjoint continua in E^n such that each M' in H is homeomorphic to M_0? We remark that similar questions are discussed in [4].

The purpose of this note is to answer the above question in the affirmative under the additional condition that the M_i, $i=0, 1, 2, \cdots$, are equivalently imbedded in E^n and to note that the answer to the question is negative for $n=2$, 3, and $n\geqq 5$ under the condition that the elements of H have the same imbedding as M_0. Two continua M_1 and M_2 are said to be equivalently imbedded in E^n provided there exists a homeomorphism of E^n onto E^n which carries M_1 onto M_2. Finally, we note that if a nontopological imbedding property is imposed on the collection H in E^2, then the answer is again negative.

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The general problem in E^n.

THEOREM 1. *Suppose that the pairwise disjoint sequence of continua $\{M_i\}$ converges homeomorphically to M_0 in E^n and suppose that for each i, M_i and M_0 are disjoint and equivalently imbedded in E^n. Then there exists an uncountable collection H of pairwise disjoint continua in E^n, each of which is homeomorphic to M_0.*

PROOF. The proof consists of constructing uncountably many sequences of homeomorphisms of M_0, each of which converges to a homeomorphism. The techniques of proof were first used by Bing [2], [4].

We remark that if M and M_0 are equivalently imbedded, then there exists a homeomorphism h of E^n onto E^n such that $h(M_0)=M$ and consequently the sequence $h(M_i)$ converges homeomorphically to M. We conclude that if M and M_0 are equivalently imbedded, then for each $\epsilon>0$ there is a continuum M' disjoint from M and a homeomorphism h' such that $h'(M)=M'$, h' moves no point of M more than ϵ, and M' and M are equivalently imbedded. We shall refer to h' as a disjoint ϵ-homeomorphism of M onto M'.

There exists a homeomorphism f_1 from M_0 onto M_1 and we shall choose ϵ_1 so that $\rho(x, f_1(x))<\epsilon_1$ for all $x\in M_0$. Let f_0 be the identity homeomorphism on M_0. For notational convenience let $f_0(M_0)=M^0$ and $f_1(M_0)=M^1$. If E_i is the set of elements (x_1, x_2) from $M_0\times M_0$ such that $\rho(x_1, x_2)>1/i$, then it follows for any finite collection K of homeomorphisms on M_0 that

$$\inf_{k\in K}\left\{\inf_{(x_1,x_2)\in E_i}\left[\rho(k(x_1), k(x_2))\right]\right\} = \delta > 0.$$

Let

$$\delta_1 = \inf_{i=0,1}\left\{\inf_{(x_1,x_2)\in E_1}\left[\rho(f_i(x_1), f_i(x_2))\right]\right\},$$

let $\eta_1=\rho(M^0, M^1)$, and let $\epsilon_1' =\min\{\delta_1, \eta_1, \epsilon_1\}$. For $\epsilon_2<\epsilon_1'/3$ there exist disjoint ϵ_2-homeomorphisms f_1^0 and f_1^1 on M^0 and M^1 respectively. Let f_0^0 and f_0^1 be the identity homeomorphisms on M^0 and M^1 respectively, let $f_j^i(M_i)=M^{ij}$, for $i, j=0, 1$, and let $f_{ij}(M_0)=f_j^i(M_i)$ $=f_j^i(f_i(M_0))$ for $i, j=0, 1$. Denote

$$\inf_{i,j=0,1}\left\{\inf_{(x_1,x_2)\in E_2}\left[\rho(f_{ij}(x_1), f_{ij}(x_2))\right]\right\}$$

by δ_2 and let $\eta_2=\min_{(i_1,i_2)\neq(j_1,j_2)}\{\rho(M^{i_1,i_2}, M^{j_1,j_2})\}$ and let ϵ_2' $=\min(\delta_2, \eta_2, \epsilon_2)$. To further simplify notation, let $\alpha(n)$ represent a finite sequence on the first n positive integers into the set $\{0, 1\}$, and for $\beta=0$ or 1 let $\alpha(n)\beta$ represent a finite sequence on the first $n+1$ integers into $\{0, 1\}$ with β being the value on $n+1$. Also, let $\alpha(\infty)$ represent a sequence on the positive integers into $\{0, 1\}$. We next choose $\epsilon_3<\epsilon_2'/3$ and proceed to define $f_1^{\alpha(2)}$ to be disjoint ϵ-homeomorphisms of $M^{\alpha(2)}$ and define $f_0^{\alpha(2)}$ to be the identity on $M^{\alpha(2)}$. Finally define $f_{\alpha(3)}(M_0)=f_\beta^{\alpha(2)}(M^{\alpha(2)})$ for each $\alpha(2)$ and $\beta=0, 1$. We continue the process inductively to obtain for each integer n, the 2^n homeomorphisms $f_{\alpha(n)}$.

Let a be a number in $[0, 1)$. If $.a_1a_2 \cdots =\alpha(\infty)$ represents the binary expansion of a always chosen to not repeat ones infinitely and $\alpha(n)$ represents the nth truncated approximation, then the association of the sequence $\{f_{\alpha(n)}\}$ with a is a one-to-one map from the interval $[0, 1)$ into the collection of sequences previously constructed and defines an uncountable collection of sequences of homeomorphisms on M_0. The function of $f_a(x)=\lim_n f_{\alpha(n)}(x)$ is well defined since $\rho(f_a(x), x) \leqq \sum_{i=1}^\infty \epsilon_i<\epsilon_1\sum_{i=0}^\infty 1/3^i=\epsilon_1(3/2)$. Since $f_a(\cdot)$ is defined on a compact subset of E^n, it follows from the usual advanced calculus argument on uniform convergence that $f_a(\cdot)$ is continuous. If x_1 and x_2 are distinct elements of M_0, then $(x_1, x_2)\in E_n$ for some n. Then $\rho(f_{\alpha(n)}(x_1), f_{\alpha(n)}(x_2)) >\delta_n$. However,

$$\rho(f_{\alpha(n)}(x_i), f_a(x_i)) < \sum_{i=n+1}^{\infty} \epsilon_i < \delta_n \sum_{i=1}^{\infty} 1/3^i = \delta_n/2$$

for $k=1, 2$ and consequently, by the triangle inequality, $f_a(x_1)\neq f_a(x_2)$ and f_a is one-to-one. We have shown that f_a is a one-to-one continuous map from a compact metric space M_0 into a Hausdorff space E^n, and hence f_a is a homeomorphism. We have left only to show that $f_a(M_0)$ and $f_b(M_0)$ are disjoint whenever $a\neq b$. Let $.a_1a_2 \cdots$ and $.b_1b_2 \cdots$ be the binary expansions of a and b and let n be the first integer with $a_n\neq b_n$. Then $\rho(f_{\alpha(n)}(M_0), f_{\beta(n)}(M_0))\geqq\eta_n\geqq\epsilon_n'>3\epsilon_{n+1}$. However, if $x\in M_0$ then

$$\rho(f_{\alpha(n)}(x), f_a(x)) < \sum_{i=n+1}^{\infty} \epsilon_i < \epsilon_{n+1} \sum_{i=0}^{\infty} 1/3^i = \epsilon_{n+1}(3/2)$$

and similarly $\rho(f_{\beta(n)}(y), f_b(y))<\epsilon_{n+1}(3/2)$. Hence, again by the triangle inequality, $f_a(M_0)\cap f_b(M_0)=\varnothing$.

The problem in E^3. A homeomorphic image of the unit sphere is said to be wild or wildly imbedded in E^3 provided there is no self-homeomorphism of E^3 which takes S onto the unit sphere. The Fox-Artin sphere S_0 (a wild sphere) has the property that there exists a sequence of disjoint tame spheres $\{S_i\}$ such that $\{S_i\}$ converges homeomorphically to S_0. It follows easily that there exists a sequence of disjoint Fox-Artin spheres $\{S_i\}$, each imbedded like S_0, such that $\{S_i\}$ converges homeomorphically to S_0 and such that $S_i \subset \text{int}(S_0)$. Bing has shown [1], [3] that there do not exist uncountably many wild spheres in E^3. Thus if we consider the set H of all spheres in E^3 imbedded equivalently with the Fox-Artin sphere, then there exists a disjoint sequence of elements from H converging homeomorphically to an element from H but no subset H' of H can be composed of uncountably many pairwise disjoint elements. It is interesting to note that the existence of uncountably many disjoint spheres in E^3 follows from the existence of a sequence of Fox-Artin spheres converging to a Fox-Artin sphere by employing Theorem 1. Of course the result also follows trivially by considering the set of all spheres of radius r about the origin for r in $(0, 1]$.

The problem in E^2. A homeomorphic image $h(J)$ of the unit circle J is said to be thick or thickly imbedded in E^2 provided the Lebesgue measure $\mu(h(J)) > 0$. A simple closed curve J_0 with positive Lebesgue measure is easily constructed and one can construct a sequence of disjoint thick homeomorphic images J_i converging to J_0.

Let H be the collection of all thick homeomorphic images of the unit circle J. Suppose H contains an uncountable subset H' of pairwise disjoint elements. For each i, let D_i represent the closed disk with radius i centered at the origin. Then for some n, D_n intersects each of uncountably many elements of H' in a set of positive measure. Hence for some $\epsilon > 0$, D_n intersects each of infinitely many elements of H' in a set of measure greater than ϵ, which is impossible since the elements of H' are disjoint and the measure of D_n is finite. Hence H contains no uncountable subset of pairwise disjoint elements.

REMARK 1. An examination of the preceding technique shows that if H is any collection of pairwise disjoint measurable sets in a σ-finite measure space, then at most a countable number of the elements from H can have positive measure.

We note also that any homeomorphism of E^3 onto E^3 carries the set of wild spheres one-to-one and onto the set of wild spheres, while homeomorphisms of E^2 onto E^2 need not carry the set of thick Jordan curves onto the set of thick Jordan curves. This leads to a further sharpening of the question raised in the introduction.

QUESTION. Let M be a continuum in E^n for $n \neq 1$ and let H be the set of all homeomorphic images $h(M)$ of M into E^n such that $h(M)$ and M are equivalently imbedded. Does the existence of a disjoint sequence of elements $\{M_i\}$ from H such that $\{M_i\}$ converges homeomorphically to M imply that there exists an uncountable pairwise disjoint subcollection H' of H?

It is known [5] that there does not exist an uncountable collection of n-cells in E^n, $n \geqq 5$, whose boundaries are pairwise disjoint and not flat. Using this fact and an argument similar to that given in the discussion of the problem in E^3 in this paper, one can answer the above question in the negative for $n=3$ and $n \geqq 5$. Examples 3 and 4 in [4] give a negative answer for $n=2$. The question remains unsettled for $n=4$.

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University of Wisconsin, Madison, Wisconsin 53706 and
Utah State University, Logan, Utah 84321