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An Impact Parameter Formalism. II

——*High Energy Elastic Scattering*——

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The mathematical structure of the impact parameter formalism proposed by Kotani and Adachi is discussed. The formalism is regarded as a counterpart of the partial wave expansion of the scattering amplitude. An impact parameter amplitude is defined as a function of impact parameter and total energy. This amplitude has two characteristic features, corresponding to particle-like and wave-like pictures. The latter nature has not been taken into account in previous impact parameter formalisms.

In order to investigate the character of the impact parameter amplitude, our formalism is applied to high energy elastic scattering. A phenomenological analysis of the diffraction scattering is characterized by two parameters. Various expressions for the impact parameter amplitudes which are connected with the large angle proton-proton scattering are derived. The absorption coefficients for each partial waves are obtained.

The Lommel function of two variables is widely used in the diffraction scattering, just as it has been employed in the diffraction theory of classical light waves.

§ 1. Introduction

An impact parameter expansion of the scattering amplitude has been formulated exactly in a previous paper.[1] (Hereafter we refer to this paper as I.) An impact parameter amplitude is defined as an integral transform of the (relativistically invariant) scattering amplitude by using the Bessel function in place of the Legendre polynomials in the definition of a partial wave amplitude. The purpose of this paper is to discuss the character of our impact parameter amplitude, to clarify the mathematical structure of our formalism, and to offer examples of the application to the high energy elastic scattering of elementary particles.

It is shown that our impact parameter amplitude has an oscillating character as a function of the impact parameter (b) for large values of b. This oscillation, however, tends to disappear in general for increasing incident energies. This situation corresponds to the fact that, in quantum mechanics, we can construct a sufficiently localized wave packet at a very high energy. Thus, we may say that our impact parameter amplitude has two characteristic features. One of them corresponds to the semi-classical particle-like picture which is expected when the de Broglie wave length of the incident particle can be regarded as zero. Another is the wave-like picture which is expected from the general character of the wave function. Such an oscillating character with regard to the impact parameter is a special feature of our formalism.

Our formalism is constructed from several mathematical theorems which have been proved by mathematicians and by us. Within the mathematical framework, it is shown that our impact parameter formalism can be regarded as the counterpart of the partial wave expansion of the scattering amplitude. As a feature of our formalism, our impact parameter amplitude must satisfy the Kapteyn integral equation and some integrability conditions. It is, however, shown that these restrictions are not too stringent for the purpose of physical applications. On the other hand, the restrictions which are required for the full scattering amplitude in our formalism are also quite natural, namely holomorphy in the Lehmann ellipse.[2]

At the present stage, there is no complete dynamical theory of strong interactions. As an approach to the underlying dynamical theory of strong interactions, we shall try to investigate the theoretical significance of the experimental data along phenomenological lines and to observe what the data imply about the dynamics. When the behavior of the scattering amplitude at high energies is discussed, the impact parameter expansion has several advantages. The main advantage is that the concept of the impact parameter is convenient in formulating the intuitive semi-classical picture.[3] Making use of this characteristic feature, we may expect to get some notion of the interaction region and to find a clue to the underlying dynamical theory of strong interaction physics.

As a tool for our mathematical manipulations, it is useful to introduce the Lommel function of two variables.[4] It is interesting to note that the Lommel function of two variables has been widely discussed in the diffraction theory of classical light waves.[5]

In § 2, we shall give a summary of our formalism and discuss the mathematical structure. In order to make the character of our impact parameter amplitude clear, a phenomenological analysis of high energy diffraction scattering is made in §§ 3 and 4. As the first step of the application, only the linear term in the t dependence in the exponential form of the scattering amplitude is retained in § 3. Our analysis is characterized by two parameters, namely the ratio of the elastic to total cross sections and an interaction radius. In this case, the contribution from the oscillating part of the impact parameter amplitude is not large. In § 4, we discuss the various impact parameter amplitudes which correspond to the scattering amplitudes introduced by Foley et al.[6] and by Orear[7] for the analyses of large angle p-p scattering.

Qualitative discussion of the absorption coefficients in the partial wave analysis is given in § 5. We shall compare our results with the expressions obtained by Minami.[8]

In the Appendix, we discuss the properties of the third type of solution of the Kapteyn integral equation. It is shown that this solution is inappropriate for the purposes of physical applications.

§ 2. The impact parameter formalism

In this section, for completeness, we shall summarize the main results obtained in our previous paper, I, and discuss the structure of our formalism and the mathematical restrictions on the impact parameter amplitude and the full amplitude.

Consider the elastic scattering of two spinless particles with center-of-mass momentum p and scattering angle θ. Let $T(s, y)$ be a relativistically invariant scattering amplitude, where $(s)^{1/2}=W$ is the total energy in the center-of-mass system and $y=\sin(\theta/2)$. Throughout this paper, s is restricted to physical (real) values and can be treated as a simple parameter. The invariant momentum transfer can be expressed as $(-t)^{1/2}=2py$.

Our impact parameter amplitude $a(s, \beta)$ is defined by the relation

$$a(s,\beta)=\left(\frac{2p}{W}\right)\int_0^1 y\,dy\,J_0(\beta y)\,T(s,y), \tag{2·1}$$

where $J_0(\beta y)$ is the Bessel function of order zero. From this definition, the impact parameter amplitude must be an even function of the continuous parameter β. The impact parameter (b) is introduced by setting $\beta=2pb$ in our formalism. Hereafter we shall use the notation $a(s, b)$ in place of $a(s, \beta)$ for the discussion of physical problems.

The relation inverse to Eq. (2·1) can be expressed as

$$A(s,y)=\left(\frac{W}{2p}\right)\int_0^\infty \beta\,d\beta\,J_0(\beta y)\,a(s,\beta), \tag{2·2}$$

$$=\begin{cases} T(s,y), & \text{for } 0\leq y<1,\\ 0, & \text{for } 1<y<\infty.\end{cases}$$

The conditions on $T(s, y)$ and $a(s, \beta)$, by which the inverse relation is guaranteed, will be mentioned later.

On the other hand, by using the Legendre polynomials in place of the Bessel function in Eq. (2·1), the usual partial wave amplitude $a_l(s)$ is defined;

$$a_l(s)=\left(\frac{2p}{W}\right)\int_0^1 y\,dy\,P_l(1-2y^2)\,T(s,y), \tag{2·3}$$

and the inverse relation is

$$T(s,y)=\left(\frac{W}{p}\right)\sum_{l=0}^{\infty}(2l+1)\,P_l(1-2y^2)\,a_l(s). \tag{2·4}$$

As we shall prove later in Theorem-3, the relation between the impact parameter amplitude $a(s, \beta)$ and the partial wave amplitude $a_l(s)$ is easily obtained as follows:

$$a(s, \beta) = \sum_{l=0}^{\infty} 2(2l+1) \frac{J_{2l+1}(\beta)}{\beta} a_l(s), \tag{2·5}$$

and

$$a_l(s) = \int_0^{\infty} d\alpha\, J_{2l+1}(\alpha)\, a(s, \alpha). \tag{2·6}$$

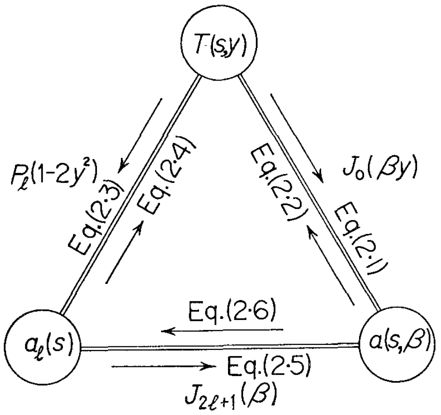

Fig. 1. The relations among three amplitudes $T(s, y)$, $a(s, \beta)$ and $a_l(s)$.

Hence, it is instructive to show the relations between the full scattering amplitude $T(s, y)$, the impact parameter amplitude $a(s, \beta)$ and the partial wave amplitude $a_l(s)$. These relations are illustrated by the triangle shown in Fig. 1.

In terms of the impact parameter amplitude, the various cross sections are given in the following exact forms,

$$\sigma_{tot}(s) = \frac{4\pi}{pW} \operatorname{Im} T(s, 0)$$

$$= 8\pi \int_0^{\infty} b\,db\, \operatorname{Im} a(s, b), \tag{2·7}$$

$$\sigma_{el}(s) = \frac{8\pi}{W^2} \int_0^1 y\, dy |T(s, y)|^2$$

$$= 8\pi \int_0^{\infty} b\,db |a(s, b)|^2, \tag{2·8}$$

$$\sigma_{re}(s) = \frac{4\pi}{pW} F(s, 0)$$

$$= 8\pi \int_0^{\infty} b\,db f(s, b), \tag{2·9}$$

where $\sigma_{tot}(s)$, $\sigma_{el}(s)$ and $\sigma_{re}(s)$ are the total, elastic and reaction cross sections, respectively (see § 4 of I).*) According to Van Hove,[9] in the expression for

*) The term "reaction cross section" is used, according to the usage of Blatt and Weisskopf.[10] It is called the inelastic cross section in our previous paper, I, and the absorption cross section in much of the literature on nuclear physics.[11]

$\sigma_{re}(s)$, the function $F(s, y)$ is an overlap function which expresses all contributions from inelastic channels, and the opacity function $f(s, b)$ is defined by the relation

$$f(s, b) = \left(\frac{2p}{W}\right)\int_0^1 y\,dy\,J_0(2pby)\,F(s, y). \tag{2·10}$$

In the same way as in Eq. (2·2), the inverse relation can be expressed in the following form

$$\left(\frac{W}{2p}\right)\int_0^\infty \beta\,d\beta\,J_0(\beta y) f(s, \beta) = \begin{cases} F(s, y), & \text{for } 0 \leq y < 1, \\ 0, & \text{for } 1 < y < \infty. \end{cases} \tag{2·11}$$

We shall now discuss the mathematical structure of our formalism. The theorem establishing the inverse relations (2·2) and (2·11) has been proved by MacRobert.[12] This theorem may be stated as follows:

[Theorem–1]. *If* $T(s, y)$ *is a holomorphic function in the region* $p<y<q$ *and if* $a_\nu(s, \beta)$ *be expressed in the form*

$$a_\nu(s, \beta) = \left(\frac{2p}{W}\right)\int_p^q y\,dy J_\nu(\beta y)\,T(s, y) \tag{2·12}$$

for $0<p<q<\infty$, *and* $\mathrm{Re}\,\nu > -1$, *then, we find*

$$A_\nu(s, y) = \left(\frac{W}{2p}\right)\int_0^\infty d\beta\beta\,J_\nu(\beta y)\,a_\nu(s, \beta)$$

$$= \begin{cases} T(s, y), & \textit{for } p<y<q, \\ 0, & \textit{for } 0<y<p,\ q<y<\infty. \end{cases} \tag{2·13}$$

In order to apply this theorem, it is required that the scattering amplitude $T(s, y)$ is holomorphic at least in a region involving the interval $0<y<1$ in the y plane and for physical values of s. Actually, it has been proved by Lehmann[2] that the scattering amplitude $T(s, y)$ is holomorphic inside an ellipse with foci -1 and $+1$ in the $\cos\theta$ plane. Since $y^2 = (1-\cos\theta)/2$, the inverse relation (2·2) can certainly be defined.

It should, however, be noted that the Bessel function $J_0(\beta y)$ with a weighting factor y does not form a complete orthonormal set for the region $0<y<1$ and $0<\beta<\infty$. We shall, therefore, expect to find some restrictions on $a(s, \beta)$, in addition to the simple one that $a(s, \beta)$ is an even function of β. Such restrictions are found by considering the possibility of the following integral equation which is obtained by substituting Eq. (2·2) into Eq. (2·1);

$$a(s,\beta)=\int_0^1 y\,dy\,J_0(\beta y)\int_0^\infty \alpha\,d\alpha\,J_0(\alpha y)\,a(s,\alpha). \tag{2·14}$$

The validity of this integral equation is established by the following theorem:
[Theorem–2]. *Let $a(s,\beta)$ satisfy the following two conditions;*

[I] *The even function $a(s,\beta)$ has a continuous derivative for all positive values of β, and the integral*

$$\int_0^\infty d\beta\, a(s,\beta)\,\beta^{1/2-\varepsilon} \tag{2·15}$$

exists and is absolutely convergent, where ε is a small positive number.

[II] *The function $a(s,\beta)$ should satisfy the Kapteyn integral equation*[4)]

$$a(s,\beta)+\beta\frac{da(s,\beta)}{d\beta} \tag{2·16}$$

$$=\frac{1}{2}\int_0^\infty d\alpha\,\frac{J_1(\alpha)}{\alpha}\{(\alpha+\beta)\,a(s,\alpha+\beta)+(\alpha-\beta)\,a(s,\alpha-\beta)\}.$$

These are necessary and sufficient conditions to establish the integral equation (2·14) for $a(s,\beta)$.

The proof of this theorem is given in I.*) From the condition [I], it is to be noted that a step function of β is not permissible for the amplitude $a(s,\beta)$ in our formalism. Moreover, the functional form of our impact parameter amplitude $a(s,\beta)$ must be restricted by the Kapteyn integral equation. It is, however, to be noted that there is no such serious restriction on the full scattering amplitude $T(s,y)$, except the analytic property of $T(s,y)$ in the region $0<y<1$ (see the theorem–4).

Hence, it is worthwhile to consider general solutions of the Kapteyn integral equation (2·16). The Kapteyn equation appears originally in the Webb-Kapteyn theory of the Neumann series.[4)] Therefore, we have the following theorem:

*) In course of the proof of Theorem-2, we have used the Lommel integral

$$L_{\beta\alpha}=\beta\alpha\int_0^1 y\,dy\,J_0(\beta y)J_0(\alpha y), \tag{2·17}$$

and the summation

$$L_{\beta\alpha}'=\sum_{l=0}^\infty 2(2l+1)J_{2l+1}(\beta)J_{2l+1}(\alpha). \tag{2·18}$$

In I, it was proved that $L_{\beta\alpha}=L_{\beta\alpha}'$. After completion of the previous work, we discovered that this relation had been proved by Wilkins[13)] as part of a more general form. Furthermore, it was found that Bateman had proved it in 1907.[14)] The author would like to express his thanks to Professor K. Watanabe for making a copy of Bateman's paper available to him.

[Theorem–3]. *In order that $a(s, \beta)$ should satisfy the Kapteyn integral equation, it is necessary and sufficient that $a(s, \beta)$ should be expressible for all real values of β by the Neumann series,*

$$a(s, \beta) = \sum_{l=0}^{\infty} 2(2l+1) \frac{J_{2l+1}(\beta)}{\beta} c_{2l+1} \tag{2·19}$$

with the coefficient

$$c_{2l+1} = \int_0^\infty d\alpha\, J_{2l+1}(\alpha) a(s, \alpha). \tag{2·20}$$

Further, this theorem is true provided only that the integral

$$\int_0^\infty d\alpha\, \alpha\, a(s, \alpha)\, (1+\alpha)^{-3/2} \tag{2·21}$$

is absolutely convergent.

The proof has been given, for example, in Titchmarsh's book (p. 354).[15),16),4)] It is easy to prove that the coefficient $c_{2l+1}(s)$ is equivalent to the partial wave amplitude $a_l(s)$, Eq. (2·3). This equivalence is derived by the use of various relations among $J_{2l+1}(\beta)$, $J_0(\beta y)$ and $P_l(1-2y^2)$ given in the Appendix of I.

Let us consider another type of solution of the Kapteyn equation. From the definition of the total elastic cross section $\sigma_{el}(s)$, Eq. (2·8), it is natural to require an integrability condition

$$\int_0^\infty \beta |a(s, \beta)|^2 d\beta < \infty. \tag{2·22}$$

Under this requirement, one type of solution is derived by Hardy and Titchmarsh.[16)] This type is our definition of $a(s, \beta)$, Eq. (2·1) itself.

[Theorem–4]. *Suppose that $\beta|a(s, \beta)|^2$ is integrable over $(0, \infty)$. Then, in order that $a(s, \beta)$ should be a solution of the Kapteyn integral equation* (2·16), *it is necessary and sufficient that it should be of the form*

$$a(s, \beta) = \left(\frac{2p}{W}\right) \int_0^1 y\, dy\, J_0(\beta y)\, \lambda(s, y), \tag{2·23}$$

where $y|\lambda(s, y)|^2$ is integrable over $0<y<1$.

The proof of this theorem is given in Hardy and Titchmarsh's paper.[16)] In the course of the proof of this theorem, it is shown that $a(s, \beta)$ is independent of the values of $\lambda(s, y)$ for $y>1$. Comparing our definition of $a(s, \beta)$, we may choose $\lambda(s, y) = T(s, y)$. The integrability condition

$$\int_0^1 y\, dy |T(s, y)|^2 < \infty, \tag{2·24}$$

is also the integral which appears in the definition of the total elastic cross section, Eq. (2·8). Thus, $T(s, y)$ is restricted only by a condition quite natural for the purpose of physical applications. Let us suppose that the form of $a(s, \beta)$ is given first and it satisfies the Kapteyn integral equation. Then we can calculate $A(s, y)$ by using Eq. (2·2). According to Theorem–1, the corresponding amplitude $T(s, y)$ for $0<y<1$ is obtained by putting $T(s, y)=A(s, y)$, but the expression for $T(s, y)$ for $1<y$ is not derived directly, because $A(s, y)=0$ for $1<y$. We may, however, define $T(s, y)$ even in the region $y>1$ by the analytical continuation of $T(s, y)$ defined for $y<1$, because the value of $a(s, \beta)$ is independent of the value of $T(s, y)$ for $y>1$, as stated in Theorem–4.

In order to understand these theorems, let us consider an example such that the scattering amplitude $T(s, y)$ is given by

$$T(s, y)=\left(\frac{W}{2p}\right)P_l(1-2y^2), \tag{2·25}$$

where $P_l(z)$ is the Legendre polynomials. In this case, our impact parameter amplitude $a(s, \beta)$ is given by

$$a(s, \beta)=\frac{1}{\beta}J_{2l+1}(\beta). \tag{2·26}$$

It is easy to show that this $a(s, \beta)$ satisfies the conditions which are required in our formalism (see the Appendix of I). It should be noted that

$$T(s, y)\neq 0, \quad \text{for } y\rightarrow(1-\varepsilon). \tag{2·27}$$

In this example, we see that our impact parameter amplitude oscillates in general as a function of the parameter β. From the point of view of the general character of the wave function in quantum mechanics, such an oscillating behavior will be expected. Another type of solution of the Kapteyn equation will be discussed in the Appendix.

By returning to our original transformation formulae and substituting Eq. (2·1) into Eq. (2·2), we get an integral equation for $T(s, y)$

$$T(s, y)=\int_0^\infty \beta\, d\beta\, J_0(\beta y)\int_0^1 y'\, dy'\, J_0(\beta y)\, T(s, y') \quad \text{for } 0\leq y<1. \tag{2·28}$$

The condition for the existence of the last integral is

$$\int_0^1 y\, dy|T(s, y)|<\infty. \tag{2·29}$$

After interchanging the order of integrations, we have a convergent integral in the sense of Dirac's δ function.

Thus, we can conclude that, in order to establish our impact parameter formalism, e.g. our transformation formulae (2·1) and (2·2), the impact parameter amplitnde $a(s, \beta)$ should satisfy the integrability condition (2·22) which is the strongest one among the conditions (2·15), (2·21) and (2·22). Also, the full scattering amplitude $T(s, y)$ should be analytic in the region involving $0<y<1$. The integrability conditions for $T(s, y)$, Eqs. (2·24) and (2·29), are satisfied by this analyticity requirement.

Finally, we shall give a brief summary of the unitarity condition on the impact parameter amplitude $a(s, b)$.*) From the equality of $\sigma_{\text{tot}}(s) = \sigma_{\text{el}}(s) + \sigma_{\text{re}}(s)$, we can deduce the relation

$$\operatorname{Im} a(s, b) = |a(s, b)|^2 + f(s, b) + K(s, b), \qquad (2\cdot30)$$

where the correction factor $K(s, b)$ must satisfy

$$\int_0^\infty b\, db\, K(s, b) = 0. \qquad (2\cdot31)$$

A detailed discussion of the properties of this correction factor $K(s, b)$ is given in I. For example, the correction factor $K(s, b)$ is given by

$$K(s, b) = \int_0^\infty d\beta_1\, \beta_1 \int_0^\infty d\beta_2\, \beta_2\, a^*(s, \beta_1)\, a(s, \beta_2)\, G(2pb\,;\, \beta_1, \beta_2\,;\, s), \qquad (2\cdot23)$$

where

$$G(\alpha\,;\,\beta, \gamma\,;\,s) = (2/\alpha\beta\gamma) \sum_{l=0}^{\infty} (2l+1)\, J_{2l+1}(\alpha)\, J_{2l+1}(\beta)\, J_{2l+1}(\gamma) - (1/\alpha^2\beta\gamma)\, L_{\alpha\beta}\, L_{\alpha\gamma}. \qquad (2\cdot33)$$

For special values of b, we have

$$\operatorname{Im} a(s, 0) = |a(s, 0)|^2 + f(s, 0), \qquad (2\cdot34)$$

$$\operatorname{Im} a(s, \infty) = |a(s, \infty)|^2 + f(s, \infty). \qquad (2\cdot35)$$

These relations are similar to the unitarity relation for the partial wave amplitude (c.f. Eq. (4·9) of I). It should be noted that, as is clear from Eq. (2·5), the zero impact parameter amplitude is equal to the s-wave amplitude,

$$a(s, 0) = a_0(s). \qquad (2\cdot36)$$

§ 3. Forward elastic scattering

We shall now apply our impact parameter formalism to the forward elastic

*) In I, we wrote the condition $1 \geq |a(s, b)|$ in Eq. (4·18a) of I. This upper limit seems to be too strong. What we can say now is that $a(s, b)$ is an entire function of b for physical values of s.

scattering. The invariant scattering amplitude $T(s, y)$ is, as is well known, a function of total energy squared s and momentum transfer $t=-4p^2y^2$. In order to discuss the diffraction scattering, let us consider the scattering amplitude $T(s, y)$ near the forward direction. The real part of the scattering amplitude and its imaginary part have in general different functional forms, so that we may assume that

$$T(s, y) = i \operatorname{Im} T(s, 0) e^{g_1(s)t} + \operatorname{Re} T(s, 0) e^{g_2(s)t}, \tag{3·1}$$

where $g_1(s)$ and $g_2(s)$ are real functions of s.

In order to find the form of the impact parameter amplitude for the scattering amplitude given above, it is sufficient to treat only the imaginary part, since we have assumed that the imaginary and real parts have similar dependences on the variable t. The spin and isotopic-spin dependences and the Coulomb correction to the scattering amplitude are not considered, because our purpose is to find the qualitative features of the high energy forward scattering.*)

The impact parameter amplitude $\operatorname{Im} a(s, b)$ corresponding to the imaginary part of the scattering amplitude (3·1) is expressed in the following form by substituting Eq. (3·1) into the definition (2·1);

$$\operatorname{Im} a(s, b) = \left(\frac{2p}{W}\right) \operatorname{Im} T(s, 0) \frac{e^{-\gamma(s)}}{2i\gamma(s)} [U_1(2i\gamma, 2pb) - iU_2(2i\gamma, 2pb)], \tag{3·2}$$

where $\gamma(s)$ is

$$\gamma(s) = 4p^2 g_1(s), \tag{3·3}$$

and the Lommel function $U_n(w, z)$ is defined as[4]

$$U_n(w, z) = \sum_{m=0}^{\infty} (-1)^m \left(\frac{w}{z}\right)^{n+2m} J_{n+2m}(z). \tag{3·4}$$

By using Eq. (3·4), we rewrite Eq. (3·2) in a different form:

$$\operatorname{Im} a(s, b) = \left(\frac{2p}{W}\right) \operatorname{Im} T(s, 0) \frac{e^{-\gamma}}{2} \sum_{l=0}^{\infty} \gamma^l \frac{1}{(pb)^{l+1}} J_{l+1}(2pb). \tag{3·5}$$

If we assume that this expression for the amplitude $a(s, \beta)$ is first given, the inverse of $\operatorname{Im} a(s, b)$ is given by $\operatorname{Im} A(s, y)$, Eq. (2·1), which is expressed as follows;

$$\operatorname{Im} A(s, y) = \begin{cases} \operatorname{Im} T(s, 0) e^{g_1(s)t}, & \text{for } 0<y<1, \\ (1/2) \operatorname{Im} T(s, 0) e^{g_1(s)t}, & \text{for } y=1, \\ 0 & , \quad \text{for } 1<y<\infty. \end{cases} \tag{3·6}$$

Here we have used the following formula (see p. 48 of Bateman's book[17]):

*) In case of proton-proton scattering, symmetrization of the angular dependence should be taken into account. We shall not consider it for brevity.

$$\int_0^\infty \beta^{\nu-\mu+1} J_\mu(a\beta)\, J_\nu(\beta y)\, d\beta = \begin{cases} \dfrac{2^{\nu-\mu+1} y}{\Gamma(\mu-\nu)\, a^\mu} (a-y^2)^{\mu-\nu-1}, & \\ & \text{for } 0<y<a, \\ 0, & \text{for } a<y<\infty, \end{cases} \tag{3·7}$$

where $a>0$ and $\mathrm{Re}\,\nu>\mathrm{Re}\,\mu>-1$.

It is instructive to express Eq. (3·2) in a somewhat familiar form. For this purpose we use the relations between the Lommel functions of two variables;

$$U_1(w, z) - V_1(w, z) = \sin\left(\frac{w}{2} + \frac{z^2}{2w}\right), \tag{3·8}$$

and

$$U_2(w, z) - V_0(w, z) = -\cos\left(\frac{w}{2} + \frac{z^2}{2w}\right), \tag{3·9}$$

where $V_n(w, z)$ is another Lommel function of two variables which is defined by a series[4]

$$V_n(w, z) = \sum_{m=0}^{\infty} (-1)^m \left(\frac{w}{2}\right)^{-n-2m} J_{-n-2m}(z). \tag{3·10}$$

It is now possible to rewrite Eq. (3·2), by using Eqs. (3·8) and (3·9), in the following form:

$$\mathrm{Im}\, a(s, b) = \frac{\mathrm{Im}\, T(s, 0)}{4pWg_1(s)} \left\{\exp\left[-\frac{b^2}{4g_1(s)}\right] - e^{-4p^2 g_1} \left[V_0(2i\gamma, \beta) + iV_1(2i\gamma, \beta)\right]\right\}. \tag{3·11}$$

From the definition of the V function (Eq. (3·10)) and the factor $\exp[-4p^2 g_1(s)]$, we can conclude that the second and third terms in Eq. (3·11) decrease rapidly and tend to zero in the high energy limit; so we can take only the first term for $\mathrm{Im}\, a(s, \beta)$ as a good approximation. This is equivalent to discussing only the part corresponding to the semi-classical particle picture of our impact parameter amplitude. The Gaussian distribution with respect to the impact parameter, the first term in Eq. (3·11), was previously obtained by Udgaonkar and Gell-Mann,[18] who used the vacuum-trajectory hypothesis in Regge pole theory,[19] and by Ida[20] and Gottfried and Jackson[28] in terms of different impact parameter formalisms. The real part of the impact parameter amplitude $a(s, b)$ is easily obtained in the same way.

In order to discuss the p-p diffraction scattering, let us assume, for simplicity, that $\mathrm{Re}\, T(s, y) = 0$, i.e. the elastic scattering is entirely due to a shadow scattering. Then we may interpret the following ratio as a parameter representing the degree of opacity of the target,[21]

$$r(s) = \sigma_{\mathrm{el}}(s) / \sigma_{\mathrm{tot}}(s). \tag{3·12}$$

We shall hereafter call this ratio $r(s)$ the shadow parameter.

If we assume the form of $\operatorname{Im} T(s, y)$ given by Eq. (3·1), the elastic cross section can be easily obtained as follows:

$$\sigma_{\mathrm{el}}=\frac{[\sigma_{\mathrm{tot}}]^2}{32\pi g_1(s)}(1-\exp[-8p^2 g_1(s)]), \tag{3·13}$$

where we have used the optical theorem (2·7). Then the shadow parameter is represented by

$$r(s)=\frac{\sigma_{\mathrm{tot}}(s)}{32\pi g_1(s)}(1-\exp[-8p^2 g_1(s)]). \tag{3·14}$$

We note here that the shadow parameter is well represented by the zero impact parameter amplitude $a(s, 0)$ as noted by Ida[20] and Serber.[22] From Eq. (3·11), we have the following expression for $\operatorname{Im} a(s, 0)$

$$\operatorname{Im} a(s, 0)=\frac{\sigma_{\mathrm{tot}}(s)}{16\pi g_1(s)}(1-\exp[-4p^2 g_1(s)]), \tag{3·15}$$

where we have used the properties of the V function

$$V_0(w, 0)=1,\ \ V_1(w, 0)=0. \tag{3·16}$$

By combining Eq. (3·14) with Eq. (3·15), the shadow parameter is expressed as

$$r(s)=\frac{1}{4}\left[\frac{1-\exp[-8p^2 g_1(s)]}{1-\exp[-4p^2 g_1(s)]}\right]\cdot[2\operatorname{Im} a(s, 0)]. \tag{3·17}$$

On the vacuum-trajectory hypothesis, if the total cross section approaches a finite constant, the elastic cross section $\sigma_{\mathrm{el}}(s)$ decreases as $1/\ln s$ at high energies. In that case the shadow parameter and the zero impact parameter amplitude must vanish like $1/\ln s$ as $s\to\infty$. The experimental data on high energy p-p scattering may, however, show that the shadow parameter will approach a definite constant value ~ 0.22 in the high energy limit. It is possible within our approximations to explain this value if we assume that $\operatorname{Im} a(s, 0)$ approaches its maximum value 1/2 in the high energy limit, as noted previously by Frye.[23] As we see from the unitarity condition for zero impact parameter amplitude, Eq. (2·34), this assumption means that the opacity function $f(s, 0)$ attains the maximum value 1/4, namely a central ray is totally absorbed. Of course, these conclusions are slightly modified if we remove the assumption $\operatorname{Re} T(s, y)=0$.

We now investigate the meaning of the parameter $g_1(s)$. From the assumption on the scattering amplitude, Eq. (3·1), we have the relation

$$g_1(s)=\left[\frac{\partial}{\partial t}\operatorname{Im} T(s, t)\right]_{t=0}\Big/\operatorname{Im} T(s, 0). \tag{3·18}$$

If the shadow parameter and the total cross section approach constant values for increasing energies, the function $g_1(s)$ must approach a definite constant value in the high energy limit, as we see from Eq. (3·14). Then, we may consider its constant value as a measure of the radius of the interaction region. According to the detailed discussion of the interaction radius which will be given in a subsequent paper, the equivalent radius R can be expressed by the following relation,

$$R=[8g_1(s)]^{1/2}\approx\left[\frac{\sigma_{\mathrm{tot}}(s)}{4\pi r(s)}\right]^{1/2}. \tag{3·19}$$

For example, we have the value $R\approx 1.2\times 10^{-13}$cm for p-p scattering, if we assume $\sigma_{\mathrm{tot}}(\infty)\approx 40$ mb and $r(\infty)\approx 0.22$. It should be noted that the existence of the finite equivalent radius also contradicts the increasing radius predicted by the Regge pole theory.

Let us now derive the opacity function $f(s, b)$ for the amplitude (Eq. (3·1)) from the unitarity condition (2·31). In order to know the opacity function, we may calculate the correction factor (Eq. (2·32)) by using the impact parameter amplitude or the scattering amplitude and find

$$\begin{aligned} f(s, b) = &\frac{\sigma_{\mathrm{tot}}(s)}{16\pi g_1(s)}\left\{\exp\left[-\frac{b^2}{4g_1(s)}\right]-\exp[-4p^2g_1(s)]\,[V_0(2i\gamma, \beta)+iV_1(2i\gamma, \beta)]\right.\\ &-\frac{\sigma_{\mathrm{tot}}(s)}{16\pi g_1(s)}\exp[-8p^2g_1(s)]\sum_{l,m,n}\frac{J_{2l+1}(\beta)}{\beta}\\ &\left.\times\frac{2(2l+1)\gamma^{m+n+2}m!\,n!}{\Gamma(m-l+1)\Gamma(m+l+2)\Gamma(n-l+1)\Gamma(n+l+2)}\right\}. \end{aligned} \tag{3·20}$$

If we approximate the impact parameter amplitude by the first term in Eq. (3·11), the correction factor is equal to zero to a good approximation, as is proved generally in the high energy limit. In this approximation, the opacity function will be given in the form

$$f(s, b)\approx\frac{\sigma_{\mathrm{tot}}(s)}{2\pi R^2}\left\{\exp\left[-\frac{b^2}{R^2/2}\right]-\frac{\sigma_{\mathrm{tot}}(s)}{2\pi R^2}\exp\left[-\frac{2b^2}{R^2/2}\right]\right\}. \tag{3·21}$$

This form was previously suggested by Van Hove, who used the unitarity condition for partial wave amplitudes.[24] The first term of this expression corresponds to the Van Hove form for the uncorrelated jet model.[9] Therefore, it should be emphasized that the Van Hove form is obtained from the extremely high energy approximation in our formalism under the assumption Eq. (3·1).

§ 4. Various forms of impact parameter amplitudes

In the previous section, we mainly discussed the term linear in t in the exponential form of the scattering amplitude. However, the analysis of the experimental data show that the t^2 term or a much more complicated form of the scattering amplitude has to be taken into account. It will be useful to know in our formalism the forms of $a(s, b)$ for various scattering amplitudes, which correspond to expressions given by Foley et al.[6] or by Orear.[7]

Let us consider the case where the scattering amplitude is given by the expression

$$T(s, t) = T(s, 0) \exp[g(s)t + h(s)t^2], \tag{4·1}$$

where $g(s)$ and $h(s)$ are function of s. Substituting this expression into the definition of the impact parameter amplitude, Eq. (2·1), we have the expression

$$a(s, \beta) = \left(\frac{2p}{W}\right) T(s, 0) \exp[-\gamma(s) + \xi(s)/2] \left(\frac{1}{2i\gamma}\right)$$
$$\times \sum_{l=0}^{\infty} (-1)^l H_l[i(\gamma - \xi)/(2\xi)^{1/2}] (\xi/2\gamma^2)^{l/2} \left(\frac{2i\gamma}{\beta}\right)^{l+1} J_{l+1}(\beta), \tag{4·2}$$

where $\gamma(s)$ is given by the same definition as Eq. (3·3), the function $\xi(s)$ is

$$\xi(s) = 32p^4 h(s), \tag{4·3}$$

and $H_l(z)$ are the Hermite polynomials. It is easily shown that when $h(s) \to 0$, the expression (4·2) reduces to the expression (3·5), by using the following relation

$$\lim_{\xi \to 0} H_n[i(\gamma - \xi)/(2\xi)^{1/2}] (\xi/2)^{n/2} = (i)^n \gamma^n. \tag{4·4}$$

In terms of the Lommel function of two variables, the impact parameter amplitude (Eq. (4·2)) can be written as

$$a(s, \beta) = \left(\frac{2p}{W}\right) T(s, 0) \exp[-\gamma(s) + \xi(s)/2] \left(\frac{1}{2i\gamma}\right)$$
$$\times \sum_{l=0}^{\infty} (-1)^l H_l[i(\gamma - \xi)/(2\xi)^{1/2}] \left(\frac{\xi}{2\gamma^2}\right)^{l/2} [U_{l+1}(2i\gamma, \beta) + U_{l+3}(2i\gamma, \beta)], \tag{4·5}$$

where we have used the relation

$$U_\nu(w, z) + U_{\nu+2}(w, z) = \left(\frac{w}{z}\right)^\nu J_\nu(z). \tag{4·6}$$

From this expression, we can easily deduce the exponential form Eq. (3·11) for small $h(s)$, by using the following relation between the Lommel functions:

$$U_{l+1}(2i\gamma, \beta) - iU_{l+2}(2i\gamma, \beta) = \sum_{n=0}^{\infty} (-1)^n [U_{l+n+1}(2i\gamma, \beta) + U_{l+n+3}(2i\gamma, \beta)]. \tag{4·7}$$

Experiments on high energy p-p elastic scattering near the forward direction seem to show that the ratio $\gamma/(2\xi)^{1/2}$ is large. Accordingly, we can infer that the expression (4·5) is not different from the Gaussian function of b for practical physical situation.

We shall now find the expressions for the elastic cross section and the shadow parameter for the impact parameter amplitude (Eq. (4·2)). For this purpose, we shall again assume that $\mathrm{Re}\, T(s, y) = 0$. The elastic cross section is expressed in the following form:

$$\sigma_{\mathrm{el}}(s) = \frac{[\sigma_{\mathrm{tot}}(s)]^2}{32\pi g(s)} \cdot \frac{1}{1-8p^2 h(s)/g(s)}$$
$$\times \Big\{ 1 - \exp[-8p^2 g(s)(1-4p^2 h(s)/g(s))] \tag{4·8}$$
$$\times [1 - \sum_{n=1}^{\infty} (-i)^{n+1} \xi^{(n+1)/2} H_{n-1}[i(\gamma - \xi)/\xi^{1/2}] \frac{2n}{(n+1)!}]\Big\}.$$

The amplitude $a(s, 0)$ can be easily obtained;

$$\mathrm{Im}\, a(s, 0) = \frac{\sigma_{\mathrm{tot}}(s)}{16\pi g(s)} \cdot \frac{1}{1-8p^2 h/g} \Big\{ 1 - \exp[-4p^2 g + 16p^2 h]$$
$$\times \left[1 - \sum_{n=1}^{\infty} (-i)^{n+1} H_{n-1}[i(\gamma-\xi)/(2\xi)^{1/2}] \left(\frac{\xi}{2}\right)^{(n+1)/2} \frac{2n}{(n+1)!} \right] \Big\}. \tag{4·9}$$

In this approximation, the shadow parameter $r(s)$ is expressed as

$$r(s) = \frac{1}{4} F(s) [2\, \mathrm{Im}\, a(s, 0)], \tag{4·10}$$

where $F(s)$ is

$$F(s) = F_1(s)/F_2(s),$$
$$F_1(s) = 1 - \exp[-8p^2 g(s)(-4p^2 h(s)/g(s))]$$
$$\times \left[1 - \sum_{n=1}^{\infty} (-i)^{n+1} \xi^{(n+1)/2} H_{n-1}[i(\gamma-\xi)/\xi^{1/2}] \frac{2n}{(n+1)!} \right], \tag{4·11}$$
$$F_2(s) = 1 - \exp[-4p^2 g(s)(1-4p^2 h(s)/g(s))]$$
$$\times \left[1 - \sum_{n=1}^{\infty} (-i)^{n+1} \left(\frac{\xi}{2}\right)^{(n+1)/2} H_{n-1}[i(\gamma-\xi)/(2\xi)^{1/2}] \frac{2}{(n+1)!} \right].$$

It is clear that $F(s)$ tends to unity, as p increases.

In order to describe the entire angular region of p-p elastic scattering, an expression for the differential cross section has been introduced by Orear.[7] His expression is

$$\frac{d\sigma_{\mathrm{el}}}{d\Omega} = B^2 p^2 \exp\left[-\left(\frac{Rp_\perp}{2}\right)^2 \right] + \frac{A^2}{s} \exp[-ap_\perp], \tag{4·12}$$

where A and B are constant quantities, R is the equivalent radius and the transverse momentum $p_\perp$ is

$$p_\perp = [-t - t^2/4p^2]^{1/2}. \tag{4·13}$$

If we assume that the first and the second terms in this expression correspond to the imaginary and real parts of scattering amplitude, respectively, the scattering amplitude $T(s, y)$ can be expressed as

$$T(s, y) = iBpW \exp\left[-\frac{R^2 p_\perp^2}{8}\right] + A \exp\left[-\frac{ap_\perp}{2}\right]. \tag{4·14}$$

The imaginary part of the amplitude is given by using the following substitution

$$g(s) = R^2/8, \; h(s) = R^2/32p^2. \tag{4·15}$$

Therefore, the impact parameter amplitude corresponding to $\mathrm{Im}\, T(s, y)$ can be written in the form

$$\mathrm{Im}\, a(s, \beta) = \left(\frac{p^2}{W}\right) B$$
$$\times \sum_{l=0}^{\infty} (-i)^l H_l[-ipR/8^{1/2}] \left(\frac{p^2 R^2}{2}\right)^{l/2} \left(\frac{2}{\beta}\right)^{l+1} J_{l+1}(\beta). \tag{4·16}$$

On the other hand, the real part of the impact parameter amplitude is much more complicated; we have

$$\mathrm{Re}\, a(s, \beta) = \left(\frac{p}{W}\right) A \sum_{n=0}^{\infty} \frac{(-a)^n}{n!}$$
$$\times \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(n/2+1)\Gamma(n/2+m+1)}{\Gamma(n/2-m+1)} \left(\frac{2}{\beta}\right)^{m+n/2+1} J_{m+n/2+1}(\beta). \tag{4·17}$$

It is to be noted that these expressions for impact parameter amplitudes are expressed in terms of the Lommel function by using the formula (4·6). The experience obtained in this section will be useful in discussing the dynamical theory when deciding the form of impact parameter amplitudes. Whenever the angular dependence of the full scattering amplitude is expressed in the exponential form assumed in this paper, our impact parameter amplitude includes the Lommel function. This fact is clearly understood on the basis of the integral representation of Lommel's function.

§ 5. Absorption coefficients

A partial wave analysis of p-p diffraction scattering have been made by Minami[8] by assuming a similar form of the scattering amplitude to Eq. (3·1), and by Krish[25] by using a similar form to Eq. (4·11). Both of them assumed that $\mathrm{Re}\, T(s, y) = 0$. In this section, we shall compare their results qualitatively with our results obtained through use of the impact parameter amplitude.

In the partial wave expansion of the scattering amplitude, the absorption coefficient $\eta_l(s)$ is defined by the relation

$$a_l(s) = (\eta_l(s) \exp[2i\alpha_l(s)] - 1)/2i \tag{5·1}$$

and

$$0 \leq \eta_l(s) \leq 1, \tag{5·2}$$

where $\alpha_l(s)$ is the real part of the phase shift.

We shall assume again that $\alpha_l(s)=0$ in Eq. (5·1) for the diffraction scattering. In this case, $1-\eta_l(s)$ can be expressed as

$$1-\eta_l(s) = 2\int_0^\infty d\beta J_{2l+1}(\beta)\,[\mathrm{Im}\, a(s, \beta)], \tag{5·3}$$

by using Eqs. (2·6) and (5·1).

We can now calculate the absorption coefficient for various impact parameter amplitudes for the diffraction scattering. For example, if we consider the impact parameter amplitude (Eq. (3·2)), we get

$$1-\eta_l(s) = \left(\frac{2p}{W}\right) \mathrm{Im}\, T(s, 0) \exp[-\gamma]\, S_l, \tag{5·4}$$

where S_l is defined by

$$S_l = 2\sum_{n=0}^\infty \gamma^n \frac{n\,!}{\Gamma(n-l+1)\,\Gamma(n+l+2)}, \tag{5·5}$$

and we have used the following formula:

$$\int_0^\infty d\alpha \frac{J_{n+1}(\alpha)\,J_{2l+1}(\alpha)}{\alpha^{n+1}} = \frac{n\,!}{2^{n+1}\Gamma(n-l+1)\,\Gamma(n+l+2)}. \tag{5·6}$$

In the particular cases $l=0$ and $l=1$, we have

$$S_0 = \frac{2}{\gamma}(\exp[\gamma]-1),$$

$$S_1 = \frac{2}{\gamma}(\exp[\gamma]+1) - \frac{4}{\gamma^2}(\exp[\gamma]-1). \tag{5·7}$$

As Minami pointed out, the following recurrence formula can be proved for S_l:

$$S_{l+1} = S_{l-1} - [2(2l+1)/\gamma]\, S_l. \tag{5·8}$$

The expression (5·4) corresponds to the ξ_l' defined by Minami.[8)]

The absorption coefficient for the scattering amplitude (Eq. (4·2)) can be expressed in terms of a combined series of known functions. We shall, for simplicity, assume that $T(s, y)$ is purely imaginary, and we get

$$1-\eta_l(s) = \left(\frac{2p^2}{W}\right) B \sum_{n=0}^\infty (-i)^n H_n[-ipR/8^{1/2}] \left(\frac{p^2R^2}{2}\right)^{n/2} S_{n,l}, \tag{5·9}$$

where $S_{n,l}$ is defined by the relation

$$S_{n,l}=\frac{2n\,!}{\Gamma(n-l+1)\,\Gamma(n+l+2)}\,. \tag{5·10}$$

The recurrence formula for $S_{n,l}$ has a similar property to Eq. (5·8), i.e.

$$S_{n,l-1}-S_{n,l+1}=2\,(2l+1)\,S_{n+1,l}\,. \tag{5·11}$$

§ 6. Discussion

In many discussions of the behavior of the scattering amplitudes on the basis of the concept of impact parameter, the oscillating character of the impact parameter amplitude has never been discussed quantitatively. This oscillating part is characteristic of our formalism and will be discussed in detail in a subsequent paper. Let us consider it briefly here.

Our definition of $a(s, b)$, Eq. (2·1), is reduced to the following form by changing the variable y into $x=2py$;

$$a(s, b)=\frac{1}{2pW}\int_0^{2p} x dx J_0(bx)\, T(s, x)\,. \tag{6·1}$$

This expression is just the same as the definition used by Cottingham and Peierls.[26] When the upper limit of this integration is simply extended to infinity, the following amplitude $H(s, b^2)$ is defined:

$$H(s, b^2)=\frac{1}{2pW}\int_0^{\infty} x dx J_0(bx)\, T(s, x)\,. \tag{6·2}$$

This definition is essentially what is introduced by Blankenbecler and Goldberger,[27] except for the kinematical factor. This expression is the Hankel-transform of $T(s, y)$, and the inversion is just the inverse Hankel transform. Therefore, $H(s, b^2)$ need not satisfy the Kapteyn integral equation. Only the integrability condition is required, as in the usual Hankel transform. Generally $H(s, b^2)$ need not have an oscillating part. As we discussed in § 3, the oscillating part of our impact parameter amplitude tends to zero for increasing energies. It has been shown in § 3 that there is a contribution from the oscillating part at finite energy, although it is not large. The definition of $H(s, b^2)$, Eq. (6·2), seems to be unsuitable for a rigorous theoretical discussion of scattering problems at finite energy.

Our formalism is not limited to forward scattering angles. As the first step towards discussion of the large angle scattering, we have tried to find the forms of $a(s, b)$ for various assumptions on $T(s, y)$. On the basis of this experience, we find that the impact parameter amplitude in our formalism can generally be expressed by a combination of the Lommel functions of two variables and other known functions. These properties may offer us a clue as to how to construct a dynamical theory of strong interaction physics. The large angle p-p scattering was discussed by Cottingham and Peierls[26] who used a similar impact parameter formalism. They tried to fit the data by assuming the Van Hove form of opacity function $f(s, b)$ and adding a repulsive real part of $a(s, b)$. In our formalism, such a treatment can be performed more precisely, because the correction factor appearing in the unitarity condition is given in a more compact form and an estimate of the contribution from the oscillating part of $a(s, b)$ and $f(s, b)$ can be made by a method which will be mentioned in § 3 of the following paper.

It is necessary to find a theory by which the dynamical calculation of the impact parameter amplitude can be performed. Such an approach was made by Blankenbecler and Goldberger by using a dispersion relation.[27] A discussion of the analyticity of our impact parameter amplitude will be given in the forthcoming paper.

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Appendix

In this appendix, we shall discuss the third type of solutions of the Kapteyn integral equation, which was also found by Hardy and Titchmarsh[16] and was mentioned in the previous paper I (Eq. (3·13) of I).

[Theorm–5]. *Suppose that* $\beta^2|a(s,\beta)|^2$ *is integrable over* $(0, \infty)$. *Then, in order that* $a(s,\beta)$ *should be a solution of the Kapteyn integral equation, it is necessary and sufficient that* $a(s,\beta)$ *should be the form*

$$\beta a(s, \beta) = \int_0^1 du \sin(\beta u)\phi(u), \tag{A·1}$$

where

$$\int_0^1 du|\phi(u)|^2 < \infty. \tag{A·2}$$

The proof of this theorem is given, for example, in Titchmarsh's book (p. 355).[15),16] This third type of solution is more restricted at infinity ($\beta \to \infty$) than the second type mentioned in Eq. (2·23), while the second one is more restricted at $\beta = 0$.

We can show that this third type is an inappropriate solution for physical applications, although we mentioned it in I.*) In order to see this feature, let

*) Professor T. Kotani and the author would like to express their thanks to Professor R. Utiyama for pointing out this error in I.

us substitute Eq. (A·1) into Eq. (2·2). Then we find

$$T(s, y)=\left(\frac{W}{2p}\right)\int_y^1 du \frac{\phi(u)}{(u^2-y^2)^{1/2}} . \qquad (\mathrm{A}\cdot 3)$$

In general, if $T(s, y)$ expresses the physical scattering amplitude, we may expect $T(s, y) \neq 0$ for $y=1$, namely, non-zero backward scattering. In order to guarantee this property $T(s, y=+1) \neq 0$, the behavior of $\phi(u)$ near $u=1$ should be

$$\lim_{u\to 1-\varepsilon} \phi(u) \sim \varepsilon^{-1/2}. \qquad (\mathrm{A}\cdot 4)$$

This behavior of $\phi(u)$ is inconsistent with the integrability Eq. (A·2) which is a result of theorem–5. Inversely, if this requirement is assumed, we get

$$\lim_{y\to 1-0} T(s, y)=0. \qquad (\mathrm{A}\cdot 5)$$

Thus, we can conclude that, if we do not consider the case where there is no backward scattering, the impact parameter amplitude $a(s, \beta)$ should have the property that the integral

$$\int_0^\infty \beta^2 |a(s, \beta)|^2 d\beta \qquad (\mathrm{A}\cdot 6)$$

does not exist.

Let us first consider some examples such that the function $\phi(u)$ is given by a power series of u. In this case, it is sufficient to consider the following integral with $n \geq 0$.

$$\int_0^1 du\, u^n \sin(\beta u) = \frac{n!}{\beta^{n+1}} [U_{n+1}(2\beta, 0) \sin\beta - U_{n+2}(2\beta, 0) \cos\beta], \qquad (\mathrm{A}\cdot 7)$$

where $U_n(2\beta, 0)$ is a special case of the Lommel function of two variables.[4] The function $U_n(2\beta, 0)$ is defined by a series

$$U_n(2\beta, 0)=\sum_{m=0}^{\infty} \frac{(-1)^m \beta^{n+2m}}{\Gamma(n+2m+1)}, \qquad (\mathrm{A}\cdot 8)$$

and especially

$$U_{2n}(2\beta, 0)=(-1)^n\left[\cos\beta-\sum_{m=0}^{n-1} \frac{(-1)^m \beta^{2m}}{(2m)!}\right], \qquad (\mathrm{A}\cdot 9)$$

$$U_{2n+1}(2\beta, 0)=(-1)^n\left[\sin\beta-\sum_{m=0}^{n-1} \frac{(-1)^m \beta^{2m+1}}{(2m+1)!}\right]. \qquad (\mathrm{A}\cdot 10)$$

In order to study the properties of $a(s, \beta)$ and $T(s, y)$ in this example, let us consider the cases where $n=0$ and $n=1$ in Eq. (A·7). Then we have

$$a(s, \beta)=(1-\cos\beta)/\beta^2, \text{ for } n=0, \qquad (\mathrm{A}\cdot 11)$$

and

$$a(s,\beta)=j_1(\beta)/\beta, \text{ for } n=1, \tag{A·12}$$

respectively, where $j_1(\beta)$ is a spherical Bessel function of order one. Clearly, the integral, Eq. (A·6), does exist for these examples. If we substitute Eqs. (A·11) and (A·12) into Eq. (2·2), the forms of $A(s,y)$ will be given in the following forms;

$$A(s,y)=\left(\frac{W}{2p}\right)\begin{cases} \cosh^{-1}(1/y), & \text{for } 0<y<1, \\ 0 \quad , & \text{for } y=1, \\ 0 \quad , & \text{for } 1<y<\infty, \end{cases} \tag{A·13}$$

and

$$A(s,y)=\left(\frac{W}{2p}\right)\begin{cases} (2y^3)^{-1/2}(1-y^2)^{1/2}, & \text{for } 0<y<1, \\ 0 \quad , & \text{for } y=1, \\ 0 \quad , & \text{for } 1<y<\infty. \end{cases} \tag{A·14}$$

These properties are what is to be expected from the above general consideration.

We shall now seek some examples such that the function $\phi(u)$ behaves like $1/\sqrt{\varepsilon}$ as $u\to 1-\varepsilon$. As a simple example with the property $T(s,y=1)\neq 0$, we shall consider the example given by Eq. (2·25). In this case, the amplitude $A(s,y)$ is given in the form

$$A(s,y)=\left(\frac{W}{2p}\right)\begin{cases} P_l(1-2y^2), & \text{for } 0<y<1, \\ (-1)^l/2, & \text{for } y=1, \\ 0 \quad , & \text{for } 1<y<\infty. \end{cases} \tag{A·15}$$

The function $\phi(u)$ for the amplitude $a(s,\beta)$, Eq. (2·26), is easily obtained:

$$\phi(u)=\frac{2}{\pi^{1/2}}\frac{1}{(1-u^2)^{1/2}}\sin[(2l+1)\sin^{-1}u]. \tag{A·16}$$

As another example, we choose the amplitude $T(s,y)$ which was discussed in § 3 in detail. The function $\phi(u)$ for the amplitude $a(s,\beta)$, Eq. (3·5), can be expressed in the following form

$$\phi(u)=\frac{\exp[-\gamma]}{\pi^{1/2}}\sum_{n=0}^{\infty}\frac{\gamma^n}{\Gamma(n+1/2)}u(1-u^2)^{n-1/2}. \tag{A·17}$$

It is easily shown that these examples do not satisfy the integrability condition (A·2) in Theorem-5. Accordingly, these examples are not contained in the third type of solution. Thus, as we mentioned before in this appendix, the third type of solution is probably inappropriate for physical problems.

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