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An Impact Parameter Formalism. III

——*Strong Absorption Model*——

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In order to understand the physical meaning of an impact parameter amplitude, the formalism proposed by us is applied to various strong absorption models. A simple black disc model with a sharp boundary is not self-consistent; that is, the total elastic cross section derived from the angular distribution is not equal to what is calculated from the impact parameter amplitude. This inconsistency is expressed as the violation of the unitarity relation. It disappears in the high energy limit, where the semi-classical particle picture is applicable. A general rule is given for getting consistent results even in the finite energy region. It is observed that the model proposed by Fernbach, Serber and Taylor contains the same inconsistency. On the other hand, a sharp cutoff model with a sharp cutoff angular momentum involves no such difficulty. The strong interaction region expressed by this model has a diffuse edge with respect to the impact parameter. Thus, the concept of a radius of this region is introduced. It is pointed out that the so-called high energy approximation to the familiar partial-wave expansion of the scattering amplitude for large angular momenta does not include any limitation on the scattering angle.

§ 1. Introduction

The high enery collisions of nuclear particles are one of the important methods of obtaining information on nuclear properties. They furnish information about the interactions of the colliding particles. The study of low-energy collisions ordinarily tells us only a certain measure of the strength of an interaction. At high energies, on the other hand, the shorter wave-length of the incident particles makes them sensitive probes of the region of interaction. When the wave-lengths are sufficiently short, the angular distribution of elastically scattered particles becomes, in a sense, a detailed map of the region of interaction. In nuclear physics, as the scattering involving complex nuclei represents a complicated quantum-mechanical many-body problem for any energy region, it is difficult to correlate the experimental data directly with the properties of fundamental nuclear interactions. It has therefore been necessary to derive simpler methods which serve as an intermediary between the data and basic nuclear theory. Such methods are desirable even to analyze nucleon-nucleon scattering. In the cases of high energy elementary particle scattering, complete angular distributions are being measured. Since the impact parameter formalism proposed in previous papers by the authors (I[1] and II[2]) contains no limitation on either incident energies or scattering angles, it is more useful than other impact parameter formalisms.[1] In this sense, our formalism plays the role of a counterpart to a partial wave treatment. The impact parameter expansion of the full scattering amplitude is sometimes more convenient than the partial wave expansion, because of its faster convergence.

In order to understand the physical meaning of an impact parameter amplitude, we would like to apply our formalism to various strong absorption models. The formalism in which the concept of the impact parameter is used, is old; in fact it is known as the eikonal method. The importance of this method was emphasized, for example by Moliere,[3] Fernbach, Serber and Taylor,[4] and Glauber.[5] Their formulae, for example, the cross sections, are based on some approximations which are closely related to the WKB approximation.[5],[6] On the other hand, in our formalism, all quantities are expressed without using any approximation. We shall show that the eikonal method is an approximation to our formalism in the high energy limit.

In many nuclear collision problems, the incident particle is strongly absorbed upon entering the target nucleus. It is customary and convenient to define absorption as any process by which particles are removed from the entrance channel. One of the main advantages of elastic scattering in the presence of strong absorption is that the experimental data can be described without any knowledge of the details of the absorption mechanism.

The concept of strong absorption is incorporated in the "phenomenological" models in such a way that the nucleus is assumed "black" or highly opaque to certain incident waves. This assumption can be expressed directly, for example, in terms of the imaginary parts of the complex phase shifts. On the other hand, in the "potential" models, absorption is described by adding an imaginary part to the average interaction potential.

Historically, the first phenomenological model was the diffraction theory of Bohr, Peierls and Placzek and of Placzek and Bethe[7] for high energy neutron scattering. Corrections due to the presence of the Coulomb interaction in charged particle scattering were derived by Akhiezer and Pomeranchuk.[8] The concept of strong absorption is introduced in these papers by a semi-classical argument. The strong interaction zone is represented by an opaque spherical region of radius R with a sharp boundary. In the classical particle picture, particles moving with impact parameters $b \leq R$ are partly absorbed (i.e. removed from the incident beam), and those with $b > R$ are, in the absence of further interaction (for example, the Coulomb interaction), completely transmitted. Here R is the sum of target and incident particle radii. In the wave picture, this situation can be approximated by localizing the wave packet describing the incident particle along the trajectory corresponding to the partial wave with

$l=pb$, where p is the c.m. momentum (or wave number).*) This implies the existence of a "cut-off angular momentum" L, which is defined as that for which the classical turning point is equal to R. In the case of incident neutral particles, we have

$$L=pR. \tag{1·1}$$

Thus, the scattering amplitude in Akhiezer-Pomeranchuk's work[8] is expressed by a finite summation from $l=0$ to $l=L$ over the partial-wave amplitudes and by assuming that all real phase shifts are equal to zero. Akhiezer and Pomeranchuk made a further approximation by using MacDonald's expansion of the Legendre polynomials in terms of the Bessel function.[9] Thus, their results are restricted to the small-angle parts of the elastic scattering. Blair proposed that, if MacDonald's expansion is not used, this model can be applied to the heavier charged particle scattering even at larger scattering angles.[10]

We have used two different concepts: a black disc with a sharp boundary $b\leq R$ and a sharp cutoff angular momentum $l\leq L$. We can construct two models corresponding to these concepts. They are distinguished as the black disc model, and the sharp cutoff model, respectively. We shall discuss both of these models from the viewpoint of our impact parameter formalism in §§ 2 to 4. As there is no restriction on scattering angles in our formalism, the total elastic scattering cross section can be obtained by integrating the absolute square of the impact parameter amplitude over b (i.e. by summing the absolute probability for the elastic scattering at the distance b). On the other hand, we can derive the angular distribution corresponding to this amplitude and calculate the total elastic cross section by integrating over angles. It will be pointed out in § 2 that the elastic cross sections obtained by these two different methods are not equal for the simple black disc model. This means that this model with a sharp boundary cannot be consistent with our formalism.

It is expected from the beginning that there is such an inconsistency in the black disc model, because such a sharp value of impact parameter, $b\leq R$, contradicts the uncertainty principle. In § 3, we shall consider a modified black disc model which is self-consistent. The general rule for getting models which are consistent with our formalism is also mentioned. On the other hand, the sharp cutoff model emphasized by Blair[10] is consistent with our formalism. It will be shown in § 4 that the concept of a sharp cutoff angular momentum corresponds to a strong interaction region with a diffuse edge rather than the sharp boundary with respect to the impact parameter. Therefore, a definition of the radius of the strong interaction region is introduced, which is similar to the root-mean-square radius.

Fernbach, Serber and Taylor[4] tried, on the other hand, to calculate the real and imaginary parts of the phase functions by using the eikonal approxima-

*) We shall use units $\hbar=c=1$ in this paper.

tion.[5),6)] A rough description is as follows: Consider a very high energy particle moving through a region of interaction with a force center. As long as we can construct a sufficiently localized wave packet, it is reasonable to speak of the particle passing at a certain distance b from the center. Assuming that the deflection is small, Fernbach, Serber and Taylor compute the approximate change in the phase of the wave function according to the given potential form (the so-called optical model potential) and, from this, they obtain the impact parameter amplitude, which is a function of b. It will be shown in § 5 that this impact parameter amplitude does not satisfy the unitarity relation, except in the high energy limit. There is an inconsistency with respect to the total elastic cross section. The origin of this inconsistency is the same as in the black disc model. From the mathematical point of view, these inconsistencies are due to the fact that the impact parameter amplitudes in these cases fail to satisfy not only the unitarity relation but also the Kapteyn equation. [See theorem 2 of II.]

It has been pointed out by many authors[4),5),6),23)] that the eikonal approximation is closely related to the familiar partial wave expansion of the scattering amplitude in the case that the number of angular momenta l is so large that the sum over l may be replaced by an integral. The impact parameter is introduced as

$$b=\left(l+\frac{1}{2}\right)/p. \tag{1·2}$$

This approximation has been regarded as a good one at small angles. It will be pointed out in § 6 that the only important assumption in this procedure is the high energy approximation, and, mathematically, there is no restriction on the applicable range of scattering angles. We shall call this approximation the high energy approximation.

In order to simplify our discussion in this paper, we shall consider only the case of a spinless neutral projectile incident on a spin-zero spherical target nucleus.

§ 2. The black disc model

The black disc model is the limiting case of a "black" nucleus whose radius R is much larger than the wave length $\lambda\!\!\!^{-}=(1/p)$. In this paper, however, in order to make the comparison with other models easier, we assume that particles that strike the nucleus are partly but uniformly absorbed by it. This is the case of an absorptive (i.e. negative imaginary) potential which is confined to a disc of radius R and absorbs effectively enough that the disc may be considered almost opaque.

In our impact parameter formalism, this model seems, at first glance, to correspond to the following assumptions concerning the impact parameter amplitude

$a(s, b)$, where s is the square of the total energy in the center of mass system:

$$\mathrm{Re}\, a(s, b) = 0,$$

$$\mathrm{Im}\, a(s, b) = \begin{cases} 1/2(1-\xi) & \text{for } b<R, \\ 0 & \text{for } b>R. \end{cases} \tag{2·1}$$

Here, ξ is the amplitude of a transmitted wave of unit initial amplitude and is assumed to be constant (uniform absorption), though ξ is generally a function of b and s. There are only two parameters (R and ξ) in this simple diffraction theory. In the case where $b<R$ and $\xi=0$, there is no outgoing wave and the incident wave is completely absorbed. For $b>R$ and $\xi=0$, the amplitude of the outgoing wave is the same as for the ingoing wave; hence there is no absorption and no scattering. The only (elastic) scattering in this case ($\xi=0$) is shadow scattering. The assumption $\mathrm{Re}\, a(s, b)=0$ means that the real parts of all partial wave phase shifts are equal to zero.

As stated in § 1, the use of a sharp value of the impact parameter b in the assumption (Eq. (2·1)) contradicts the uncertainty principle. Therefore, an inconsistency should be expected for this model in the finite energy region, but this inconsistency may disappear in the high energy limit. This is because the semi-classical particle picture can be adopted in this limit, where the de Broglie wave-length of the incident particles is regarded as zero. In this section, these matters will be discussed quantitatively.

In our formalism, the definitions of a total cross section (σ_{tot}) and elastic (scattering) cross section (σ_{el}) are expressed in the following forms [see Eqs. (2·7) and (2·8) of II],

$$\sigma_{\mathrm{tot}}(s) = 8\pi \int_0^\infty bdb\, \mathrm{Im}\, a(s, b) \tag{2·2}$$

and

$$\sigma_{\mathrm{el}}(s) = 8\pi \int_0^\infty bdb |a(s, b)|^2. \tag{2·3}$$

It should be noted that these definitions are derived exactly, without using any approximation, as proved in § 4 of I. With our assumptions (Eq. (2·1)), we find the well-known results for these cross sections,

$$\sigma_{\mathrm{tot}}(s) = 2\pi R^2 (1-\xi), \tag{2·4}$$

$$\sigma_{\mathrm{el}}(s) = \pi R^2 (1-\xi)^2. \tag{2·5}$$

The reaction cross section,*) of course, is derived as follows:

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*) In our first paper (I),[1] the reaction cross section (σ_{re}), the impact parameter opacity function $f(s, b)$, and the partial-wave opacity function $f_l(s)$, were called the inelastic cross section (σ_{in}), the impact parameter inelastic amplitude $f(s, b)$, and the partial wave inelastic amplitude $f_l(s)$, respectively.

$$\sigma_{re}(s)=\sigma_{tot}-\sigma_{el},$$
$$=\pi R^2(1-\xi^2). \qquad (2\cdot6)$$

In our previous paper (I), this σ_{re} is also defined exactly by using an impact parameter opacity function $f(s, b)$ as follows:

$$\sigma_{re}(s)=8\pi\int_0^\infty bdbf(s, b). \qquad (2\cdot7)$$

On comparing the formal definition of σ_{re} (Eq. (2·7)) with the result (2·6), it is natural to assume that

$$f(s, b)=\begin{cases}(1-\xi^2)/4, & \text{for } b<R,\\ 0, & \text{for } b>R.\end{cases} \qquad (2\cdot8)$$

We shall show that these assumptions for $a(s, b)$ and $f(s, b)$, Eqs. (2·1) and (2·8), do not satisfy the unitarity relation, except in the high energy limit. Before considering this point in detail, we shall calculate the differential cross section.

The full scattering amplitude $T(s, y)$ is expressed in the following form:*) [See Eq. (2·2) of II.]

$$T(s, y)=(2pW)\int_0^\infty bdbJ_0(2pby)a(s, b), \quad \text{for } 0\leq y<1, \qquad (2\cdot10)$$

where $y=\sin(\theta/2)$ and $W=(s)^{1/2}$, θ and p being the scattering angle and the momentum in the c.m. system, respectively. The function $J_0(2pby)$ is the Bessel function of zeroth order. According to our assumptions (Eq. (2·1)), we find that

$$\mathrm{Re}\ T(s, y)=0, \qquad (2\cdot11)$$

$$\mathrm{Im}\ T(s, y)=(2pW)\int_0^R b\,db\,J_0(2pby)\,(1-\xi)/2,$$
$$=RW(1-\xi)\,J_1(2pRy)/2y. \qquad (2\cdot12)$$

The differential cross section for the elastic scattering is, therefore,

$$\frac{d\sigma_{el}}{d\Omega}=\frac{1}{W^2}|T(s, y)|^2,$$
$$=\pi R^2(1-\xi)^2\left(\frac{1}{4\pi}\right)\left[\frac{J_1(2pRy)}{y}\right]^2, \qquad (2\cdot13)$$

*) This invariant amplitude $T(s, y)$ is related to the ordinary definition of the scattering amplitude $f(\theta)$ as follows:

$$T(s, y)=Wf(\theta). \qquad (2\cdot9)$$

which is familiar in optics as characterizing, the diffraction scattering from a black sphere.[7] The total cross section is obtained by integration;

$$\sigma_{\mathrm{el}}(s) = 2\pi R^2 (1-\xi)^2 \int_0^1 y dy [J_1(2pRy)/y]^2, \tag{2·14}$$

$$= \pi R^2 (1-\xi)^2 [1 - J_0^2(2pR) - J_1^2(2pR)], \tag{2·15}$$

$$\approx \pi R^2 (1-\xi)^2 \left[1 - \frac{1}{\pi pR}\right], \qquad \text{for } pR \gg 1. \tag{2·16}$$

This result, Eq. (2·15), is not equal to the previous one, Eq. (2·5), which is derived from the integration over the impact parameter. This discrepancy disappears in the high energy limit ($p \to \infty$), as shown in Eq. (2·16). This may be understood more naturally in the following form; by changing the variable y into $x=2py$, we have

$$\sigma_{\mathrm{el}} = 2\pi R^2 (1-\xi)^2 \int_0^{2p} \frac{dx}{x} J_1(xR) J_1(xR), \tag{2·17}$$

and when the upper limit of this integration is assumed to be infinity, we get Eq. (2·5), because of the following property [p. 533 of Watson[9]],

$$\int_0^\infty \frac{dx}{x} J_{2l+1}(x) J_{2m+1}(x) = \frac{1}{2(2l+1)} \delta_{l,m}. \tag{2·18}$$

The mathematical reason why such a discrepancy appears except in the high energy limit is as follows: Our assumptions for the impact parameter amplitude $a(s, b)$, Eq. (2·1), and the impact parameter opacity function $f(s, b)$, Eq. (2·8), do not satisfy the unitarity relation within the finite energy region. According to the general theory in § 5 of I, $f(s, b)$ should be related to $a(s, b)$ through the following unitarity relation:

$$\mathrm{Im}\, a(s, b) = |a(s, b)|^2 + f(s, b) + K(s, b), \tag{2·19}$$

where the correction factor $K(s, b)$ should have the property,

$$\int_0^\infty b db K(s, b) = 0. \tag{2·20}$$

The explicit form of $K(s, b)$ is given in § 5 of I. It was proved generally that, in the high energy limit $[p \to \infty]$,[1]

$$\lim_{s \to \infty} K(s, b) = 0. \tag{2·21}$$

We shall call this assumption $[K(s, b)=0]$ the high energy approximation. It is easy to confirm that our assumptions, Eqs. (2·1) and (2·8), are consistent with the unitarity relation in the high energy limit, but inconsistent in the finite energy region, where $K(s, b) \neq 0$. [c.f. Eqs. (5·1) and (5·2) of this paper.]

The above inconsistency is clearly due to our assumptions for $a(s, b)$ and $f(s, b)$. Let us consider another additional requirement on them. According to the general theory of our impact parameter formalism, the impact parameter amplitude $a(s, b)$ is defined by the following transformation of the full amplitude $T(s, y)$ [Eq. (2·1) of II],

$$a(s, b)=\left(\frac{2p}{W}\right)\int_0^1 y\,dy\,J_0(2pby)\,T(s, y). \tag{2·22}$$

In order that this definition be consistent with the inversion formula, Eq. (2·10), the impact parameter amplitude $a(s, b)$ should be a continuous function of b and satisfy the Kapteyn equation [Theorem 2 of II]. The assumption $\mathrm{Im}\,a(s, b)=1/2\cdot(1-\xi)\theta(R-b)$, Eq. (2·1), does not satisfy either of requirements, because of the character, of its sharp cutoff on b. [Here $\theta(x)$ is a step function; $\theta(x)=1$ for $x>0$ and $\theta(x)=0$ for $x<0$.]

In the high energy limit, however, the situation is rather different. In order to see this situation directly, it is convenient to use the variable $x=2py$ again, instead of y itself; in fact, the definition of $a(s, b)$, Eq. (2·22), can be reexpressed in the form

$$a(s, b)=\left(\frac{1}{2pW}\right)\int_0^{2p} x\,dx\,J_0(bx)\,T(s, x). \tag{2·23}$$

The appearance of x in the full amplitude $T(s, y)$ is rather natural, because $T(s, y)$ is a relativistic-invariant quantity and should therefore be a function of the invariant squared energy (s) and the invariant momentum transfer [$t=-(2py)^2=-x^2$], instead of y alone. Now, let us first assume the form of $T(s, y)$ given by Eqs. (2·11) and (2·12). Then, we shall get a new impact parameter amplitude; namely

$$\mathrm{Re}\,a(s, b)=0,$$
$$\mathrm{Im}\,a(s, b)=\frac{1}{2}R(1-\xi)\int_0^{2p} dx\,J_0(xb)\,J_1(xR). \tag{2·24}$$

This new $\mathrm{Im}\,a(s, b)$ has no sharp cut such as the old $\mathrm{Im}\,a(s, b)$ has, but in the high energy limit, the new one reduces to the old one, which is to be expected from the semi-classical particle picture. The latter result is proved on the basis of the following formula [p. 406 of Watson's book[9)]];

$$\int_0^\infty dx\,J_0(xb)\,J_1(xR)=\begin{cases}\dfrac{1}{R}, & \text{for } 0\leq b<R,\\[2mm] 0, & \text{for } R<b.\end{cases} \tag{2·25}$$

This is due to the fact that, in the high energy limit (Eq. (6·2) of II), the definition of $a(s, b)$, Eq. (2·23), is mathematically the Hankel transform of $T(s, y)$, and the inversion, Eq. (2·10), is just the inverse Hankel transform. [This limiting case is another impact parameter formalism considered by Blankenbecler and Goldberger.[11]] As is well known, in the Hankel transform, the amplitude $a(s, b)$ need not satisfy the complicated Kapteyn equation, but it is required that $a(s, b)$ and $T(s, y)$ satisfy a simple integrability condition, which is also, of course, required for our general case. [See § 2 of II.]

On the other hand, it is worthwhile to note that there is no such discrepancy for the absorption cross section, although our impact parameter opacity function $f(s, b)$ does not satisfy the Kepteyn equation either: This $f(s, b)$ is related to the overlap function $F(s, y)$ by a transformation similar to that for $a(s, b)$ and $T(s, y)$. [See Eqs. (2·10) and (2·11) of II.] For example, we have

$$F(s, y) = 2pW\int_0^\infty b\, db\, J_0(2pby) f(s, b). \tag{2·26}$$

By using this $F(s, y)$, the reaction cross section is defined as follows [Eq. (2·9) of II]:

$$\sigma_{\rm re}(s) = \frac{4\pi}{pW} F(s, 0). \tag{2·27}$$

It is easy to see that this $\sigma_{\rm re}$ due to $F(s, 0)$ is equal to the previous definition of $\sigma_{\rm re}$ due to the integration over b, Eq. (2·7).

A modification of this simple black disc model has been used by Cork, Wenzel and Causey[12] to analyze the high energy proton-proton scattering data. They assume the real and imaginary parts of $a(s, b)$ as functions of b. Their assumed forms still satisfy neither the unitarity relation nor the Kapteyn equation. Their assumed forms, however, may be used to analyze the data for small scattering angles as the original simple black disc model does in the sense of the high energy approximation, which will be discussed in § 6.

§ 3. The modified black disc model

We shall consider a simple modification of the black disc model, which gives rise to no such inconsistency as that mentioned in § 2. It is assumed again that the real part of the scattering amplitude is equal to zero. Let us assume the angular distribution characterizing the familiar diffraction theory, namely, Eq. (2·13). Then, the full amplitude $\mathrm{Im}\, T(s, y)$ and the impact parameter amplitude $\mathrm{Im}\, a(s, b)$ are determined by Eqs. (2·12) and (2·24), respectively. The latter can be expressed in the following alternative form,

$$\operatorname{Im} a(s,\ b) = \frac{1}{2}(1-\xi)\, g_1(2pR,\ b/R), \tag{3·1}$$

where

$$g_1(\beta, k) = \sum_{m=0}^{\infty}\left(\frac{1}{k}\right)^{m+1} J_{m+1}(\beta k)\, J_{m+1}(\beta). \tag{3·2}$$

This new impact parameter amplitude $\operatorname{Im} a(s, b)$ has an oscillating part with respect to the large impact parameter, except in the high energy limit. [See Eq. (A·14) of the Appendix.] This character is what to be expected from the general principles of quantum mechanics. An example of this type of oscillation will be shown in Fig. 1 for the sharp cutoff model in the next section.

At first glance, we might consider the following impact parameter opacity function,

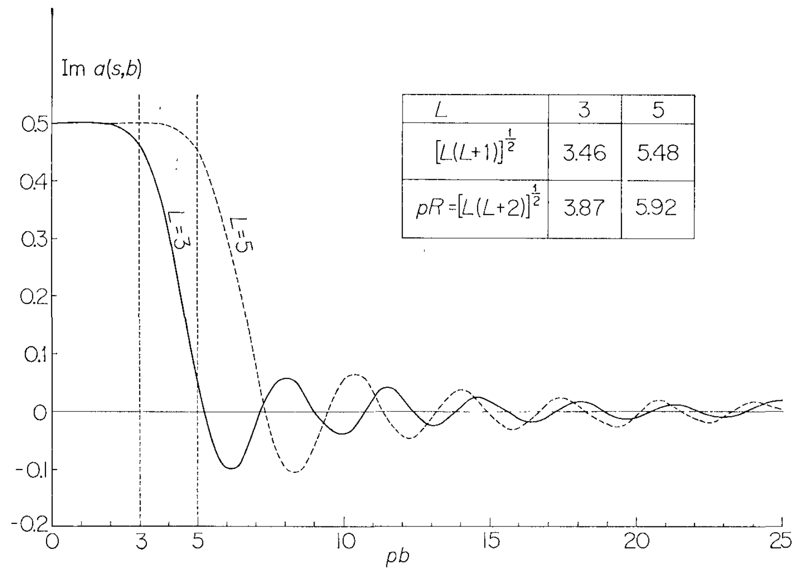

Fig. 1. The impact parameter amplitude $\operatorname{Im} a(s, b)$, Eq. (4·19), as a function of pb for the cases of $L=3$ and $L=5$. The numerical values are shown for the equivalent radius R times the c. m. momenta p. It is assumed that $\eta=0$.

$$f(s,\ b) = \frac{1}{4}R(1-\xi^2)\int_0^{2p} dx\, J_0(xb)\, J_1(xR). \tag{3·3}$$

This $f(s, b)$ is reduced to Eq. (2·8) in the high energy limit and satisfies the Kapteyn equation. However, the unitarity relation is not satisfied by these forms of $\operatorname{Im} a(s, b)$ and $f(s, b)$.

Let us consider the overlap function $F(s, y)$ which is consistent with the unitarity relation [Eq. (4·2) of I] for the full amplitude $T(s, y)$ given by Eqs. (2·11) and (2·12). We get the following form,

$$F(s,y) = (pW/2)R^2(1-\xi)(1/pRy)J_1(2pRy)$$
$$-(pW/4)R^2(1-\xi)^2 G_2(2pR, y), \tag{3·4}$$

where

$$G_2(\beta, y) = 4\sum_{l=0}^{\infty}(2l+1)P_l(1-2y^2)\left[\frac{2}{\beta}\sum_{m=0}^{\infty}J_{2l+2m+2}(\beta)\right]^2, \tag{3·5}$$

which is derived from Eq. (A·15) of the Appendix. The impact parameter opacity function is obtained easily,

$$f(s, b) = \frac{1}{2}(1-\xi)g_1(2pR, b/R)$$
$$-\frac{1}{4}(1-\xi)^2 g_2(2pR, b/R), \tag{3·6}$$

where g_1 is defined by Eq. (3·2) and

$$g_2(\beta, k) = 8\sum_l (2l+1)\left(\frac{1}{\beta k}\right) J_{2l+1}(\beta k)\,[\sum_m J_{2l+2m+2}(\beta)]^2. \tag{3·7}$$

This $f(s, b)$, of course, satisfies the Kapteyn equation.

According to the definitions of $\sigma_{re}(s)$, Eq. (2·7) or Eq. (2·27), we find

$$\sigma_{re} = \pi R^2(1-\xi^2) + \pi R^2(1-\xi)^2[J_0^2(2pR) + J_1^2(2pR)], \tag{3·8}$$
$$\approx \pi R^2(1-\xi^2) + \pi R^2(1-\xi)^2/\pi pR, \text{ for } pR \gg 1. \tag{3·9}$$

The total elastic cross section $\sigma_{el}(s)$ is derived from its definition (Eq. (2·3)) by substituting $\mathrm{Im}\, a(s, b)$ given in Eq. (3·1). It is not difficult to confirm that the expression calculated in this way is equal to $\sigma_{el}(s)$ obtained from the angular distribution, Eq. (2·15). The sum of these elastic and reaction cross sections is consistent with the total cross section derived from the optical theorem for the full scattering amplitude, Eq. (2·12);

$$\sigma_{tot} = (4\pi/pW)\,\mathrm{Im}\, T(s, 0), \tag{3·10}$$
$$= 2\pi R^2(1-\xi). \tag{3·11}$$

Thus, we have established a modification of the black disc model, which is consistent with our impact parameter formalism and gives the familiar angular distribution of diffraction scattering. It is clear that this modified black disc model is reduced to the simple black disc model in the high energy limit. [See Eqs. (A·14) and (A·22) of the Appendix.]

Finally, let us summarize a general rule for finding values of $a(s, b)$ and $f(s, b)$, which are consistent with our formalism. If the expression for the full scattering amplitude $T(s, y)$ or the partial wave amplitude $a_l(s)$ is given experimentally or theoretically, there exists, in principle, no problem in determining

$a(s, b)$, because of its definition, Eq. (2·22) or Eq. (4·16). If neither of them is known, let us first determine $a_c(s, b)$ from the viewpoint of the semi-classical particle picture, for example, the black disc model or the Fernbach-Serber-Taylor model mentioned in § 5. Of course, this $a_c(s, b)$ should satisfy the unitarity relation with $K(s, b) = 0$, but need not satisfy the Kapteyn equation in general. Then, according to the definition (Eq. (2·10)), we can calculate $T(s, y)$, which gives the angular distribution of the elastic scattering. An expression for $a(s, b)$ which satisfies the Kapteyn equation is obtained from the calculated $T(s, y)$ by using the defnition (Eq. (2·22)), as was proved in § 2 of II. This calculated $a(s, b)$ is reduced to $a_c(s, b)$ in the high energy limit. The opacity function $f(s, b)$ or the overlap function $F(s, y)$ can be calculated by using the unitarity relation. The function $f(s, b)$ obtained in this way is consistent with the Kapteyn equation. If the semi-classical opacity function $f_c(s, b)$ is given first, we must, of course, assume either $\mathrm{Re}\, a_c(s, b)$ or $\mathrm{Im}\, a_c(s, b)$. The remaining procedure is the same. It should be noted that we have had an example in this section of the fact that, even when both the impact parameter amplitude $a(s, b)$ and the opacity function $f(s, b)$ satisfy the Kapteyn equation, it is, of course, not guaranteed that they satisfy the unitarity relation.

Another example of this rule is the sharp cutoff model mentioned in the next section.

§ 4. The sharp cutoff Model

The sharp cutoff model is essentially based on the partial wave formalism. The usual partial wave expansion of the full amplitude $T(s, y)$ is

$$T(s, y) = \left(\frac{W}{p}\right) \sum_{l=0}^{\infty} (2l+1) P_l(1-2y^2)\, a_l(s). \tag{4·1}$$

The l-th partial wave amplitude is related to the corresponding scattering (complex) phase shift by the following relations [c.f., Eq. (2·3) of I],

$$\mathrm{Re}\, a_l(s) = \frac{1}{2} \eta_l \sin 2\alpha_l, \tag{4·2}$$

$$\mathrm{Im}\, a_l(s) = \frac{1}{2} (1 - \eta_l \cos 2\alpha_l). \tag{4·3}$$

Here α_l is the real part of the phase shift $\delta_l = \alpha_l + i\beta_l$ and the l-th partial wave absorption coefficient η_l is $\eta_l = \exp(-2\beta_l)$.

We first recall that the sharp cutoff model is expressed by the following assumptions,

$$\alpha_l(s) = 0, \qquad \text{for all } l, \tag{4·4}$$

and

$$\eta_l(s) = \begin{cases} \eta, & \text{for } l \leq L, \\ 1, & \text{for } l > L. \end{cases} \tag{4·5}$$

Here η is assumed to be constant and, in the case of $\eta=0$, the incident (spherical) waves with $l \leq L$ are completely absorbed. The corresponding partial wave opacity function $f_l(s)$ is given by

$$f_l(s) = \frac{1}{4}(1-\eta_l^2), \tag{4·6}$$

$$= \begin{cases} \frac{1}{4}(1-\eta^2), & \text{for } l \leq L, \\ 0, & \text{for } l > L, \end{cases} \tag{4·7}$$

because of the unitarity relation

$$\operatorname{Im} a_l(s) = |a_l(s)|^2 + f(s). \tag{4·8}$$

Then, in our sharp cutoff model, the cross sections and the full scattering amplitude can be obtained without using any further approximations:

$$\sigma_{\text{tot}} = 2\pi\bar{\lambda}^2(1-\eta)(L+1)^2, \tag{4·9}$$

$$\sigma_{\text{el}} = \pi\bar{\lambda}^2(1-\eta)^2(L+1)^2, \tag{4·10}$$

$$\sigma_{\text{re}} = \pi\bar{\lambda}^2(1-\eta^2)(L+1)^2, \tag{4·11}$$

$$\operatorname{Im} T(s,y)\,[(2p/W)(1-\eta)]^{-1}$$

$$= \sum_{l=0}^{L}(2l+1)P_l(1-2y^2), \tag{4·12}$$

$$= \left(-\frac{1}{4y}\right)\left[\frac{dP_L(1-2y^2)}{dy} + \frac{dP_{L+1}(1-2y^2)}{dy}\right], \tag{4·13}$$

$$= (L+1)^2 F(L+2, -L; 2; y^2), \tag{4·14}$$

$$\approx (L+1)^2\left[1 - \frac{L(L+2)}{2}y^2 + \frac{(L-1)L(L+2)(L+3)}{12}y^4 - \cdots\right], \quad \text{for } y \ll 1, \tag{4·15}$$

where $\bar{\lambda}$ is a wavelength ($\bar{\lambda}=p^{-1}$) and $F(a, b: c; x)$ are hypergeometric polynomials.

Our impact parameter amplitude $a(s,b)$ and the impact parameter opacity function $f(s,b)$ are related to the corresponding $a_l(s)$ and $f_l(s)$ in the following forms [Eq. (2·5) of II],

$$a(s,b) = \sum_{l=0}^{\infty}(2l+1)(2/\beta)J_{2l+1}(\beta)a_l(s), \tag{4·16}$$

$$f(s,b) = \sum_{l=0}^{\infty}(2l+1)(2/\beta)J_{2l+1}(\beta)f_l(s), \tag{4·17}$$

where

$$\beta = 2pb. \tag{4·18}$$

Therefore, in the sharp cutoff model, we have

$$\mathrm{Re}\, a(s,\ b) = 0,$$

$$\mathrm{Im}\, a(s,\ b) = \frac{1}{2}(1-\eta)\sum_{l=0}^{L}(2l+1)\left(\frac{2}{\beta}\right) J_{2l+1}(\beta), \tag{4·19}$$

$$f(s,\ b) = \frac{1}{4}(1-\eta^2)\sum_{l=0}^{L}(2l+1)\left(\frac{2}{\beta}\right) J_{2l+1}(\beta). \tag{4·20}$$

This $a(s, b)$ can, of course, be obtained from the other definition in terms of $T(s, y)$, Eq. (2·23), by substituting Eq. (4·12). It is not difficult to see that these forms of $a(s, b)$ and $f(s, b)$ each satisfy the Kapteyn equations.

We can easily prove that the cross sections obtained from the integrations over the impact parameter, Eqs. (2·2), (2·3) and (2·7), are the same as those cited above, Eqs. (4·9) to (4·11). These consistencies among various cross sections mean that the unitarity relation (2·19), is satisfied by our $a(s, b)$ and $f(s, b)$ obtained above. In order to see this situation directly, it is convenient in this case to write the expression for $K(s, b)$ in terms of the partial wave amplitudes, namely [Eq. (5·4) of I],

$$K(s,\ b) = \textstyle\sum_l (2l+1)\,(2/\beta)\, J_{2l+1}(\beta)\,|a_l(s)|^2 - |a(s,\ b)|^2. \tag{4·21}$$

In the case of our assumption (Eq. (4·5)), we have

$$K(s,\ b) = \frac{1}{4}(1-\eta)^2\sum_{l=0}^{L}(2l+1)\left(\frac{2}{\beta}\right)J_{2l+1}(\beta) - |a(s,\ b)|^2. \tag{4·22}$$

It is, therefore, clear that our expressions $a(s, b)$ and $f(s, b)$ satisfy the unitarity relation.

Thus, we have confirmed that the sharp cutoff model is consistent with our impact parameter formalism. Therefore, it is interesting to see the variation of $\mathrm{Im}\, a(s, b)$ as a function of b. This is shown in Fig. 1. The function $\mathrm{Im}\, a(s, b)$ shows a sharp decrease near $L+1 \approx pb$ and has an oscillating part. The transition from the almost constant part of $\mathrm{Im}\, a(s, b)$ to the oscillating part corresponds to a measure of the interaction region of non-uniform strength. Thus, we get some idea of the surface region in which the forces change with the increasing impact parameter, b.

In analogy to electron-nucleus scattering,[13] is expected that, at energies only just high enough to detect finite size effects, we shall be able to define some single parameter which will give an indication of the size of the nucleus. Let us define the following quantity :[*)]

*) The authors wish to express their thanks to Professor R. Glauber and Professor N. Nakanishi for kind comments on this part.

$$\langle b^2\rangle = 4\frac{[(\partial/\partial t)\ \mathrm{Im}\ T(s,y)]_{t=0}}{\mathrm{Im}\ T(s,0)},$$

$$= -\frac{1}{2p^2}\frac{[(1/y)(\partial/\partial y)\mathrm{Im}\ T(s,y)]_{y=0}}{\mathrm{Im}\ T(s,0)}. \tag{4·23}$$

For convenience, we shall express $\langle b^2\rangle$ by using the partial wave and impact parameter amplitudes,

$$\langle b^2\rangle = \left(\frac{1}{p^2}\right)\frac{\sum_l l(l+1)(2l+1)\mathrm{Im}\ a_l(s)}{\sum_l (2l+1)\mathrm{Im}\ a_l(s)}, \tag{4·24}$$

$$= \frac{\int_0^\infty b^3 db\ \mathrm{Im}\ a(s,b)}{\int_0^\infty b\ db\ \mathrm{Im}\ a(s,b)}. \tag{4·25}$$

These are similar to the definitions used by Baiquni[14] and Ida,[15] respectively. The last expression may suggest that we call $\langle b^2\rangle$ a "root-mean-square" impact parameter, although there are negative values of Im $a(s, b)$ for large values of b, as seen in Fig. 1.

Related to this r.m.s. impact parameter is the radius of the equivalent uniform absorption (R), which we shall call the equivalent radius.[16] This is the radius of the constant absorption distribution (the black disc model) which will give the same total cross section as the actual distribution in the high energy limit. It must therefore have the same r.m.s. radius, i.e. by substituting the assumptions (Eq. (2·1)) into the definition of $\langle b^2\rangle$,

$$\langle b^2\rangle = \frac{\int_0^R b^3\ db}{\int_0^R b\ db} = \frac{1}{2}R^2. \tag{2·26}$$

In the modified black disc model, we find, of course, that

$$\langle b^2\rangle = \frac{1}{2}R^2. \tag{4·27}$$

In the sharp cutoff model, the r.m.s. impact parameter is

$$\langle b^2\rangle = \frac{1}{2p^2}L(L+2). \tag{4·28}$$

According to these definitions, we can write, for example, in the sharp cutoff model,

$$\sigma_{\mathrm{tot}} = 2\pi R^2(1-\eta)\{1+[L(L+2)]^{-1}\}, \tag{4·29}$$

and

$$\frac{d\sigma_{\mathrm{el}}}{d\Omega} = \left(\frac{d\sigma_{\mathrm{el}}}{d\Omega}\right)_{\theta=0}[F(L+2,\ -L\,;\,2\,;\,y^2)]^2, \tag{4·30}$$

where

$$\left(\frac{d\sigma_{\mathrm{el}}}{d\Omega}\right)_{\theta=0} = \pi R^2(1-\eta)^2\left[1+\frac{1}{L(L+2)}\right]\frac{(L+1)^2}{4\pi}. \tag{4·31}$$

This sharp cutoff model itself, for example, has been applied by Matthews and Salam[17] and Simons[18] to the process of elementary particle scattering. Simons analyzed the data for the 2 GeV proton-proton scattering and his estimate of the radius of proton is smaller than what is obtained from Eq. $(4\cdot28)$. Also, the empirical partial wave analysis of $\pi+p$ elastic scattering above 1 Gev/c has been done by Perl and Corey using a similar idea.[19]

Generalizations of the sharp cutoff model were given by Greider and Glassgold[20] for high energy neutron scattering, by McIntyre, Wang and Becker[21] for heavier charged particle scattering, and by Frahn and Venter for both.[22] The essential point in their work is the assumption of smooth functions for $\eta_l(s)$ and $\alpha_l(s)$; namely, (i) a gradual, rather than sharp, transition of η_l from maximum to zero absorption and (ii) finite values for the real part of the scattering amplitude. From the viewpoint of our impact parameter formalism, the former generalization means that the diffuseness of the boundary of interaction region becomes large, and that the oscillating part of $\mathrm{Im}\, a(s,b)$ in Fig. 1 is smoothed, because of the decrease of contributions from the Bessel functions of higher order. But the oscillating part never disappears in the finite energy region. In their generalization, it was necessary to assume continuous values of l, instead of discrete values, for the purpose of making the analytical calculation possible and employing an approximation based on the Euler-McLaurin formula connecting summation to integration over l. On the other hand, our impact parameter amplitude $a(s,b)$ and opacity function $f(s,b)$ are continuous functions of b from the beginning. Therefore, the generalization of the simple sharp cutoff model should be easier in our impact parameter formalism.

§ 5. The Fernbach-Serber-Taylor model

In analogy to the eikonal approximation, Moliere[3] and Fernbach, Serber and Taylor[4] assume the following forms for $a(s,b)$ and $f(s,b)$:

$$a(s,b)=\frac{1}{2i}[\exp(2i\chi(s,b))-1], \tag{5·1}$$

$$=\frac{1}{2i}[\xi(s,b)\exp(2i\phi(s,b))-1], \tag{5·2}$$

$$f(s,b)=\frac{1}{4}[1-\xi^2(s,b)]. \tag{5·3}$$

Here ξ and ϕ are real functions of s and b. In general, these forms do not satisfy the unitarity relation, Eq. $(2\cdot19)$, except in the high energy limit [Eq. $(2\cdot21)$] or for the special case where the special form of the eikonal phase function $\chi(s,b)$ is chosen so that $K(s,b)=0$.

If we assume the forms of $a(s,b)$ and $f(s,b)$ given by Eqs. $(5\cdot1)$ and $(5\cdot3)$, we may get an inconsistency. One such example is the simple black disc model, as shown in § 2. We exhibit another example, which is the model considered by Fernbach, Serber and Taylor (the FST model).[4] In this FST model, a spherical nucleus with the radius R is assumed and the following forms of ξ and η are proposed,

$$\xi(s,b)=\begin{cases}\exp[-\kappa(R^2-b^2)^{1/2}], & \text{for } b<R,\\ 1, & \text{for } b>R,\end{cases} \tag{5·4}$$

and

$$\phi(s,b)=\begin{cases}k(R^2-b^2)^{1/2}, & \text{for } b<R,\\ 0, & \text{for } b>R.\end{cases} \tag{5·5}$$

Here the absorption coefficient κ is interpreted as the inverse of a mean free path inside the nucleus, and the propagation vectors of the waves outside and inside the nucleus are p and $p+k$, respectively. If our problem is considered as the scattering by a complex square-well potential whose radius is R and if the eikonal approximation is used, these κ and k can be related to the imaginary and real parts of the potential, respectively. [See p. 338 of reference 6).]

It can be shown that the impact parameter amplitude with these special choices satisfies neither the Kapteyn equation nor the unitarity relation. However, as this FST model is a more realistic modification of the black disc model, we shall consider this case and show that a discrepancy on the total elastic cross section similar to that met in the black disc model appears.

The full scattering amplitude is expressed in the form

$$T(s,y)=ipW\int_0^R b\,db\,J_0(2pby)\,[1-\exp(-\kappa r+2ikr)], \tag{5·6}$$

where

$$r=(R^2-b^2)^{1/2}. \tag{5·7}$$

It is easily shown that the total cross section, $\sigma_{\rm tot}$, calculated from the optical theorem (3·6), is equal to the sum of $\sigma_{\rm el}^{(b)}$ and $\sigma_{\rm re}^{(b)}$ derived from the integration over the impact parameter, Eqs. (2·3) and (2·7), respectively. [See the paper of Fernbach, Serber and Taylor.[4]]

On the other hand, the elastic cross section derived from the integration over the angular distribution is

$$\sigma_{\rm el}^{(1)}=\left(\frac{8\pi}{W^2}\right)\int_0^1 ydy|T(s,y)|^2, \tag{5·8}$$

$$=\left(\frac{2\pi}{p^2W^2}\right)\int_0^{2p} x\,dx|T(s,x)|^2. \tag{5·9}$$

By performing an elementary but tedious calculation, it can be shown that

$$\sigma_{\text{el}}^{(b)} > \sigma_{\text{el}}^{(1)}. \tag{5·10}$$

However, in the high energy limit $[p \to \infty]$, we may define the following quantity,

$$\sigma_{\text{el}}^{(\infty)} = \left(\frac{2\pi}{p^2 W^2}\right) \int_0^\infty x\,dx |T(s, x)|^2, \tag{5·11}$$

and find that

$$\sigma_{\text{el}}^{(b)} = \sigma_{\text{el}}^{(\infty)}. \tag{5·12}$$

This situation is completely analogous to that met in the simple black disc model mentioned in § 2.

Thus, it is concluded that, if the angular distribution given by Eq. (5·6) is assumed, the total elastic cross section, $\sigma_{\text{el}}^{(b)}$, which is given by Eq. (6) of FST,[4] is overestimated and the reaction cross section, $\sigma_{\text{re}}^{(b)}$, Eq. (5) of FST,[4] is smaller than the correct value calculated through the unitarity relation.

A modification of the FST model which is self-consistent within the frame of our impact parameter formalism, can be derived on the basis of the rule mentioned at the end of § 3.

§ 6. Discussion

Let us consider the so-called high energy approximation to the familiar partial wave expansion of the scattering amplitude, Eq. (4·1), in the case that the angular momentum l is large. This approximation is based on the following four assumptions. The first is the MacDonald expansion [p. 157 of Watson's Bessel Function[9]],

$$P_l(\cos\theta) = J_0\left[\left(l + \frac{1}{2}\right)\theta\right] + O(\theta^2), \tag{6·1}$$

the second is the Euler-MacLaurin formula

$$\sum_{l=0}^{L} f_l = \int_{-1/2}^{L+1/2} dl\, f(l) + 0(f_0 - f_{1/2}). \tag{6·2}$$

The impact parameter is introduced by Eq. (1·2). Furthermore, the partial wave phase shift $\delta_l(s)$ is assumed to be equal to the eikonal phase function $\chi(s, b)$,

$$\chi(s, b) \simeq \delta_l(s), \qquad \text{for } l = pb - 1/2, \tag{6·3}$$

that is,*)

*) Cottingham and Peierls[24] have pointed out the following relation between the partial wave and impact parameter amplitudes on the basis of Eq. (2·6) of II:

$$a_l(s) = a(s, b) + O(1/p^2), \quad \text{for } b = (l + 1/2)/p. \tag{6·4}$$

In course of this derivation, it is assumed that $a(s, b)$ is a function of b defined on the range $O \leq b < \infty$, and belongs to a class of functions which may be expanded in a uniformly convergent series of Laguerre functions.[25]

$$a(s, b)) \simeq a_l(s), \qquad \text{for } l = pb - 1/2. \tag{6·5}$$

Thus, we get the expression

$$T(s, y) \simeq 2pW \int_0^\infty bdbJ_0(2pby) \left[\frac{1}{2i} (\exp(2i\chi) - 1) \right]. \tag{6·6}$$

It has been argued that this procedure is generally a good approximation in analyzing the small angle scattering data. In fact, only the first approximation (6·1) was discussed quantitatively. But, as was pointed out at the beginning of § 5, the last approximation, (6·5) apparently violates our unitarity relation, (2·19). It should be noted that the final expression for $T(s, y)$ in this procedure, Eq. (6·6), is the same as the impact parameter expansion, Eq. (2·10), except for the assumption $a(s, b) \simeq a_l(s)$ for $pb = l + 1/2$. This fact means that the only important assumption is the last, the high energy approximation (6·5) which corresponds to the asssumption $K(s, b) = 0$, Eq. (2·21). All other approximations can be bypassed by the rigorous mathematical method given in our formalism. The argument that the above procedure is only applicable to the small angle scattering, is not correct mathematically. This argument may be correct physically, if the crude strong absorption model is assumed in order to evaluate $a(s, b)$. The impact parameter amplitude used in the high energy approximation is $a_c(s, b)$, which is defined in § 3 as what corresponds to the semi-classical particle picture.

Since the oscillating part of our impact parameter amplitude is what is to be required to satisfy the unitarity relation and the Kapteyn equation, the contribution from this part is certainly related to the error due to the high energy approximation. In order to understand this situation, let us consider the sharp cutoff model. Since the partial-wave amplitude is defined by

$$\operatorname{Im} a_l(s) = \begin{cases} \dfrac{1}{2}(1-\eta), & \text{for } l \leq L, \\ 0, & \text{for } l > L, \end{cases} \tag{6·7}$$

the corresponding impact parameter amplitude in the high energy approximation due to Eq. (6·5) is

$$\operatorname{Im} a(s, b) \simeq \begin{cases} \dfrac{1}{2}(1-\eta), & \text{for } pb \leq L + \dfrac{1}{2}, \\ 0, & \text{for } pb > L + 1/2. \end{cases} \tag{6·8}$$

Comparing this $\operatorname{Im} a(s, b)$ with our previous result shown in Fig. 1, we can get some idea of the difference, though L is not large in our numerical calculation.

In the high energy approximation, the relation between the partial-wave and the impact parameter amplitudes is an approximate one, Eq. (6·5), while, in our formalism, it is given exactly by Eq. (4·16). In this sense, it may be interesting to investigate the connection between our impact parameter representation and the resonance theory.

A great deal of the work which has thus far been done on high energy nuclear collisions, is to be considered more or less as empirical studies of the "optical model". This model represents an attempt to deal with the problem of elastic scattering alone. Physically, in the relativistic energy region, the use of the concept of "potential" is questionable. Since our formalism is applicable to the relativistic energy region, it will be possible to discuss such high energy collision problems by considering the form of the impact parameter amplitude, instead of the assumed complex potential, for example, the Wood-Saxon (complex) potential.[26] The impact parameter amplitude emerges as a rather natural way of describing the specific results obtained for elastic scattering, and this enables us to give an explicit construction of the optical potential in the strong interaction region, if necessary. For example, inversely, at the present primitive stage of our formalism, it is interesting to find the form of the impact parameter amplitude corresponding to the Wood-Saxon-type potential, which has been used in extensively nuclear physics.[28] We hope that it will allow a very simple insight into the origin of the optical model. Many investigations to clarify the concept of the optical potential have so far been done on the basis of either the eikonal or the high energy approximations.[5),6)] The impact parameter amplitudes used in these works satisfy the unitarity relation only in the high energy limit, as discussed in this paper. Therefore, we shall have to check this point carefully.

Any desire to discuss the effective absorption in a quantitative way leads us back again to the question of treating inelastic processes. In the optical model, particles which undergo inelastic scatterings are removed by pretending that they have been absorbed within the nuclear target. Any mathematically comprehensive discussion of the optical model must therefore be based on a unified treament of elastic and inelastic transitions.[5),6)] For example, as regards high energy nucleon-nucleon scattering, we can investigate the inelastic part by claculating the overlap function, as done by Van Hove.[27] Our impact parameter formalism will be useful in this sense.

Furthermore, the spins of the target and the incident particle are assumed to be equal to zero at the present stage of our formalism. It is not difficult to take such spin effects into account, though the mathematical tools are very complicated. This will be shown in future work.[*]

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[*] After completing this paper, we received the following preprints: M. Luming and E. Predazzi, "An Alternate Way of summing the Partial Wave Series" (EFINS-65-85, the University of Chicago); M. Luming, "Impact Parameter Representation for Scattering Amplitudes involving Spins" (EFINS-65-81); E. Predazzi, "Integral Representation for Scattering Amplitudes, I and II", (EFINS-65-82 and -84); M. Luming and E. Predazzi, "On the Fourier-Bessel Representation for

(continued on p. 505)

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Appendix

We shall summarize some relations which are used to confirm the consistency of the modified black disc model within our formalism.

$$\int_0^\infty d\beta \beta^{-m} J_{m+1}(\beta) = \frac{1}{2^m m!}. \tag{A·1}$$

$$\sum_{m=0}^\infty \frac{1}{2^m m!} \beta^m J_{m+1}(\beta) = \frac{1}{2}\beta. \tag{A·2}$$

$$\beta \int_0^1 y\, dy\, J_0(\beta y) = J_1(\beta). \tag{A·3}$$

$$\int_0^1 dy\; J_{2l+1}(\beta y) = \int_0^1 dy\; J_1(\beta y) P_l(1-2y^2), \tag{A·4}$$

$$\int_0^1 dy\; J_{2l+1}(\beta y) = \left(\frac{2}{\beta}\right) \sum_{m=0}^\infty J_{2l+2m+2}(\beta). \tag{A·5}$$

$$\int_0^1 y\, dy (1-y^2)^m J_0(\beta y) = \frac{2^m m!}{\beta^{m+1}} J_{m+1}(\beta). \tag{A·6}$$

$$J_1(\beta y) = \sum_{m=0}^\infty \frac{1}{m!} y(1-y^2)^m \left(\frac{\beta}{2}\right)^m J_{m+1}(\beta). \tag{A·7}$$

$$\nu \int_0^z J_\nu^2(t) \frac{dt}{t} = \sum_{m=0}^\infty [J_{m+\nu}(z)]^2 - \frac{1}{2} J_\nu^2(z), \tag{A·8}$$

for $\mathrm{Re}\,\nu > 0$.

$$G_1(\beta) = 2\int_0^1 \frac{dy}{y} J_1(\beta y) J_1(\beta y), \tag{A·9}$$

Helicity Amplitudes" (EFINS-65-99); K. Fujimura, T. Kobayashi and M. Namiki, "Non-zero Elastic Amplitude Real Part Effects and Nucleon Recoil Effects on Relationship between Elastic Diffraction Scattering and Inelastic Collisions at High Energies" (Waseda University, Tokyo). In all these papers, the approximation of the high energy limit is employed, in the sense of the use of the Hankel transform.[11), 24)] The authors thank Professor Kunio Yamamoto for sending preprints of the first two papers to them. They also would like to express their thanks to Dr. M. Luming, Dr. E. Predazzi and Professor M. Namiki for sending these preprints before publication.

$$=4\sum_{l=0}^{\infty}(2l+1)\left[\int_0^1 dy\ J_{2l+1}(\beta y)\right]^2, \tag{A·10}$$

$$=1-J_0^2(\beta)-J_1^2(\beta). \tag{A·11}$$

$$g_1(\beta,k)=\sum_{m=0}^{\infty}\left(\frac{1}{k}\right)^{m+1}J_{m+1}(\beta k)\,J_{m+1}(\beta),$$

$$=\beta\int_0^1 dy\ J_0(k\beta y)\,J_1(\beta y). \tag{A·12}$$

$$g_1(\beta,k=1)=\frac{1}{2}[1-J_0^2(\beta)]. \tag{A·13}$$

$$\lim_{\beta\to\infty}g_1(\beta,k)=\int_0^{\infty}dx\ J_0(kx)\,J_1(x),$$

$$=\left\{\begin{array}{ll}1, & \text{for } k<1,\\ 1/2, & \text{for } k=1,\\ 0, & \text{for } k>1.\end{array}\right\} \tag{A·14}$$

$$G_2(\beta,y)=16\int_0^1 dy_1\int_0^1 dy_2\,J_1(\beta y_1)\,J_1(\beta y_2)\,I(y_1,y_2;y), \tag{A·15}$$

where $I(y_1, y_2; y)$ is defined by Eq. (4·3) of I.

$$G_2(\beta,y)=4\int_0^1 dy_1\int_0^1 dy_2 \tag{A·16}$$

$$\times\left[\sum\nolimits_l(2l+1)\,P_l(1-2y^2)\,J_{2l+1}(\beta y_1)\,J_{2l+1}(\beta y_2)\right],$$

$$=4\sum\nolimits_l(2l+1)\,P_l(1-2y^2)\left[\frac{2}{\beta}\sum_{m=0}^{\infty}J_{2l+2m+2}(\beta)\right]^2. \tag{A·17}$$

We should note the following relation,

$$G_1(\beta)=G_2(\beta,y=0). \tag{A·18}$$

$$g_2(\beta,k)=8\sum\nolimits_l(2l+1)\,(1/\beta k)\,J_{2l+1}(\beta k)\left[\sum\nolimits_m J_{2l+2m+2}(\beta)\right]^2, \tag{A·19}$$

$$=\frac{2\beta}{k}\int_0^1 dy_1\int_0^1 dy_2\sum\nolimits_l(2l+1)\,J_{2l+1}(\beta k)\,J_{2l+1}(\beta y_1)\,J_{2l+1}(\beta y_2), \tag{A·20}$$

$$=\beta^2\int_0^1 dy_1\int_0^1 dy_2\ J_1(\beta y_1)\,J_1(\beta y_2)$$

$$\times J_0[\beta k y_1(1-y_2^2)^{1/2}]\,J_0[\beta k y_2(1-y_1^2)^{1/2}]. \tag{A·21}$$

$$\lim_{\beta\to\infty} g_2(\beta, k) = \left\{ \begin{array}{lll} 1, & \text{for } k<1, \\ 1/4, & \text{for } k=1, \\ 0, & \text{for } k>1. \end{array} \right\} \tag{A·22}$$

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