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An Impact Parameter Representation of the Scattering Problem*)

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§ 3. The potential scattering

Hereafter, we shall restrict our discussion to the potential scattering of two spinless particles. The interparticle (optical) potential is assumed to be the central one, which is sectionary continuous, has at most r^{-1} singularity at the origin, and satisfies

$$\int^{\infty} dr |V(E, r)| < \infty . \tag{3·1}$$

In this section, we shall summarize the general formulation of potential scattering and show that $\mathrm{Im}\,\chi(E, \beta) \neq 0$ even in the region $E_{rc} > E > 0$.

According to the standard textbook, the scattering amplitude $f(E, 2ky)$ is expressed by

$$f(E, 2ky) = -\frac{1}{4\pi N^2}\int d^3r\, \varphi^*(\boldsymbol{r}, \boldsymbol{k}_f)\, U(E, r)\, \psi^{(+)}(\boldsymbol{r}, \boldsymbol{k}_i), \tag{3·2}$$

where N is a normalization factor $N = (2\pi)^{-3/2}$, the plane wave is

$$\varphi(\boldsymbol{r}, \boldsymbol{k}) = N \exp(i\boldsymbol{k}\cdot\boldsymbol{r}), \tag{3·3}$$

and the potential $V(E, r)$ is replaced by

$$U(E, r) \equiv 2\mu V(E, r). \tag{3·4}$$

The wave function $\psi^{(+)}$ is a solution of the Lippmann-Schwinger integral equation with the Green function corresponding to the outgoing wave,

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$$G_+(|\boldsymbol{r}_1-\boldsymbol{r}_2|)=\int d^3q\,\varphi^*(\boldsymbol{r}_1,\boldsymbol{q})\left[\frac{1}{k^2+i\varepsilon-q^2}\right]\varphi(\boldsymbol{r}_2,\boldsymbol{q}). \tag{3·5}$$

Similarly, the reaction matrix $R(E, 2ky)$ is defined by*)

$$R(E, 2ky)=-\frac{1}{4\pi N^2}\int d^3r\,\varphi^*(\boldsymbol{r},\boldsymbol{k}_f)\,U(E, r)\,\psi^{(P)}(\boldsymbol{r},\boldsymbol{k}_i), \tag{3·6}$$

where $\psi^{(P)}(\boldsymbol{r},\boldsymbol{k})$ expresses the standing wave, a solution of the Lippmann-Schwinger equation with another Green function,

$$G_P(|\boldsymbol{r}_1-\boldsymbol{r}_2|)=\mathrm{P}\int d^3q\,\varphi^*(\boldsymbol{r}_1,\boldsymbol{q})\left[\frac{1}{k^2-q^2}\right]\varphi(\boldsymbol{r}_2,\boldsymbol{q}), \tag{3·7}$$

P standing for the Cauchy principal value. It should be noted that these $f(E, 2ky)$ and $R(E, 2ky)$ are defined on the momentum shell $(|\boldsymbol{k}_f|=|\boldsymbol{k}_i|=k)$, and that $R(E, 2ky)$ is real, if the potential is real, i.e. $E_{rc}>E>0$.

As it is well known, these $f(E, 2ky)$ and $R(E, 2ky)$ are related by the Heitler integral equation,

$$f(E, 2ky_{fi})=R(E, 2ky_{fi})+i\frac{k}{4\pi}\int d\Omega_n R(E, 2ky_{fn})f(E, 2ky_{ni}). \tag{3·8}$$

We shall now introduce the restricted impact parameter reaction matrix $R(E,\beta)$ defined by

$$R(E,\beta)=2\int_0^1 y\,dy\,J_0(\beta y)\,R(E, 2ky). \tag{3·9}$$

According to the general theorem,[1),5)] the inversion of Eq. (3·9) is

$$R(E, 2ky)=\frac{1}{2}\int_0^\infty \beta\,d\beta\,J_0(\beta y)\,R(E,\beta)\,\theta(1-y), \tag{3·10}$$

and $R(E,\beta)$ is self-projected by $G_1(\alpha,\beta)$ as

$$R(E,\beta)=\int_0^\infty \alpha\,d\alpha\,G_1(\alpha\beta)\,R(E,\alpha). \tag{3·11}$$

Since the Heilter equation Eq. (3·8) has the mathematical character similar to the unitarity relation for $f(E, 2ky)$, it is reduced to the following form in terms of $A(E,\beta)$ and $R(E,\beta)$:

$$A(E,\beta)=R(E,\beta)+ik\int_0^\infty \alpha_1 d\alpha_1\int_0^\infty \alpha_2 d\alpha_2\,G_2(\beta\alpha_1\alpha_2)\,R(E,\alpha_1)\,A(E,\alpha_2). \tag{3·12}$$

In the case of the partial wave expansion, the corresponding Heitler equa-

*) The authors are indebted to Professor M. Kato at the University of Tokyo for suggesting them to introduce the reaction matrix in order to simplify their treatment.

tion becomes a simple algebraic equation, namely

$$A_l(E) = R_l(E)/[1 - ikR_l(E)], \qquad (3\cdot13)$$

where $R_l(E)$ is defined by

$$R_l(E) = 2\int_0^1 ydy\, P_l(1-2y^2)\, R(E, 2ky). \qquad (3\cdot14)$$

By expressing $A_l(E)$ in terms of the phase shifts $\delta_l(E)$, we find from Eq. (3·13) that

$$R_l(E) = \tan\delta_l/k\,. \qquad (3\cdot15)$$

We have seen that the Heitler equation for $A_l(E)$ is simple, but it is not so for $A(E, \beta)$. This difference comes mathematically from the fact that the Legendre function can be modified to be orthonormal in the physical region $1 \geq y \geq 0$ and has a simple addition theorem, while the Bessel function $J_0(\beta y)$ does not.

Let us solve Eq. (3·12) by iteration procedure by assuming that the series is convergent, c.f. reference 31). We find that

$$A(E, \beta) = \sum_{n=1}^{\infty} (ik)^{n-1} [\prod_{j=1}^{n} \int_0^\infty \alpha_j d\alpha_j R(E, \alpha_j)]\, G_n(\beta\alpha_1\cdots\alpha_n). \qquad (3\cdot16)$$

Here the first term $(n=1)$ in Eq. (3·16) is reduced to a simple term $R(E, \beta)$ in Eq. (3·12) by the relation (3·11).

In the above derivation, no restriction on potential is assumed. If the potential is real, then $R(E, \beta)$ is real. This is because $R(E, \beta)$ is also calculated by

$$R(E, \beta) = \sum_l 2(2l+1)\,[J_{2l+1}(\beta)/\beta]\, R_l(E), \qquad (3\cdot17)$$

and $R_l(E)$ is real in the region $E_{rc} > E > 0$. In this case, the decomposition of $A(E, \beta)$ into the real and imaginary parts is easy;

$$A(E, \beta) = (1/k)\,[A_{odd}(E, \beta) + iA_{even}(E, \beta)], \qquad (3\cdot18)$$

where

$$A_{odd}(E, \beta) = \sum_{m=0}^{\infty} (-1)^m k^{2m+1} [\prod_{j=1}^{2m+1} \int_0^\infty \alpha_j d\alpha_j R(E, \alpha_j)]\, G_{2m+1}(\beta\alpha_1\cdots\alpha_{2m+1}), \qquad (3\cdot19)$$

and

$$A_{even}(E, \beta) = \sum_{m=1}^{\infty} (-1)^{m+1} k^{2m} [\prod_{j=1}^{2m} \int_0^\infty \alpha_j d\alpha_j R(E, \alpha_j)]\, G_{2m}(\beta\alpha_1\cdots\alpha_{2m}). \qquad (3\cdot20)$$

Thus, we find that

$$\tan[\mathrm{Re}\,\chi(E,\beta)] = [(\xi-1)+2A_{\mathrm{even}}]/2A_{\mathrm{odd}}\,, \tag{3·21}$$

and

$$\xi(E,\beta) = [(2A_{\mathrm{odd}})^2+(1-2A_{\mathrm{even}})^2]^{1/2}. \tag{3·22}$$

We see that ξ is not equal to unity in general, namely $\mathrm{Im}\,\chi(E,\beta)\neq 0$.

§ 4. The first Born term and the MSW approximation

In this section, we shall consider the properties of the first Born term under the MSW approximation. The first Born term is expressed in the following form:

$$f_B(E,2ky) = -\frac{1}{4\pi}\int d^3r\,U(E,r)\exp[-i(\boldsymbol{k}_f-\boldsymbol{k}_i)\cdot\boldsymbol{r}] \tag{4·1}$$

$$= -\int_0^\infty rdr\,\frac{\sin(\Delta r)}{\Delta}\,U(E,r) \tag{4·2}$$

$$= -\int_0^\infty ada\,J_0(\Delta a)\int_a^\infty rdrU(E,r)\,[r^2-a^2]^{-1/2} \tag{4·3}$$

$$= 2k\int_0^\infty adaJ_0(\Delta a)\chi_e(E,2ka), \tag{4·4}$$

where χ_e is the eikonal phase defined in Eq. (1·21) and we have used the relation*)

$$\frac{\sin(\Delta r)}{\Delta} = \int_0^r adaJ_0(\Delta a)\,(r^2-a^2)^{-1/2}, \quad \text{for} \quad \Delta\geq 0. \tag{4·5}$$

It is an interesting fact that the first Born term can be expressed by the form of the impact parameter expansion itself. This character is utilized by Islam[32] to define an optical potential.

According to the definition of $A(E,\beta)$, Eq. (1·13), the first Born approximation of $A(E,\beta)$ is given in the form

$$A_B(E,\beta) = 4k\int_0^\infty adaG_1(\beta,2ka)\chi_e(E,2ka). \tag{4·6}$$

The order of magnitude of the upper value of the integral variable a is the characteristic length (R) of the potential, as shown by Eq. (4·3). Then, we shall rewrite the definition of $G_1(2kb, 2ka)$, Eq. (2·2), by introducing the new variable $z=2kRy$,

*) See p. 7 of reference 44).

$$G_1(2kb,\ 2ka) = \left(\frac{1}{2kR}\right)^2 \left\{\int_0^\infty - \int_{2kR}^\infty\right\} z dz\, J_0\left[\left(\frac{b}{R}\right)z\right] J_0\left[\left(\frac{a}{R}\right)z\right]. \tag{4·7}$$

In the short wavelength region $kR \gg 1$, we can neglect the second integral, if $\chi_e(E,\ 2ka)$ is not especially large for $a \ll R$. Under the conditions on our potential mentioned in § 3, the integral in Eq. (4·3),

$$\int_a^\infty r dr U(E,\ r)\,(r^2 - a^2)^{-1/2} \tag{4·8}$$

is convergent, i.e. $\chi_e(E,\ 2ka)$ has a finite value for all a. Thus, by using the property of Dirac's δ function[33]

$$\int_0^\infty x dx\, J_0(ax)\, J_0(bx) = \left(\frac{1}{ab}\right)^{1/2} \delta(a-b), \tag{4·9}$$

we get the following approximated form:

$$G_1(\beta,\ 2ka) \approx \frac{1}{(2k)^2}\frac{1}{a}\,\delta\left(a - \frac{\beta}{2k}\right) + O\left(\frac{1}{k^4}\right). \tag{4·10}$$

This is the simplest case of our MSW approximation, Eq. (2·40).

Under the MSW approximation, the first Born term $A_B(E,\ \beta)$ becomes

$$A_{B,SW}(E,\ \beta) = \chi_e(E,\ \beta)/k\,, \tag{4·11a}$$

$$= -\frac{1}{2E}\int_{-\infty}^0 dz\, V[E,\ (b^2+z^2)^{1/2}]. \tag{4·11b}$$

By recalling the discussion in § 2.7, we find from Eqs. (4·11) that

$$\chi_{SW}(E,\ \beta) = \chi_e(E,\ \beta), \quad \text{for} \quad kR \gg 1. \tag{4·12}$$

We see that, as it is expected, $\chi_e(E,\ \beta)$ has generally some singularities in the complex β plane, while the exact $\chi(E,\ \beta)$ has not. Furthermore, we find from Eq. (4·4) that we have

$$\chi_f(E,\ \beta) = \chi_e(E,\ \beta), \quad \text{for all energy.} \tag{4·13}$$

This is just Eq. (2·84), which we are interesting in finding it. It, however, should be noted that this relation $\chi_f = \chi_{SW}$ is obtained from the first term of the Born expansion, but this result does not mean that it guarantees this relation for the higher expansion terms. We shall show in § 5 that Eq. (4·12) can be proved for every higher terms, while Eq. (4·13) is not. In general, the first Born approximation is mathematically the 3-dimensional Fourier transform, which is equal to the ordinary Hankel transform if the potential is symmetric. However, the higher Born terms are not related with the Fourier (or the ordinary Hankel) transforms simply. In this sense, we understand that the result

$\chi_f = \chi_e$ Eq. (4·13) is a special character of the first Born term.

It is worthwhile to see how various relations in the previous section are simplified under the MSW approximation. The Heitler equation itself Eq. (3·12) becomes

$$A_{SW}(E, \beta) = R_{SW}(E, \beta)/[1 - ikR_{SW}(E, \beta)]. \tag{4·14}$$

If there is no opening reaction channel, the approximated phase function $\chi_{SW}(E, \beta)$ is related to $R_{SW}(E, \beta)$ in the simple form:

$$R_{SW}(E, \beta) = \tan[\mathrm{Re}\,\chi_{SW}(E, \beta)]/k, \tag{4·15}$$

$$\xi_{SW}(E, \beta) = 1, \text{ i.e. } \mathrm{Im}\,\chi_{SW}(E, \beta) = 0, \quad \text{for} \quad E_{re} > E > 0\,. \tag{4·16}$$

These are the same forms as those in the partial wave expansion, Eqs. (3·13), (3·15) and (2·47), respectively. The latter two are consistent with the solution of the Heitler equation obtained by the iteration method, because the solution Eq. (3·18) has a simple form

$$A_{\mathrm{odd},SW}(E, \beta) = kR_{SW}(E, \beta)/[1 + k^2R_{SW}^2(E, \beta)] \tag{4·17}$$

and

$$A_{\mathrm{even},SW}(E, \beta) = k^2R_{SW}^2(E, \beta)/[1 + k^2R_{SW}^2(E, \beta)]. \tag{4·18}$$

Since Eq. (4·14) is directly derived from the Heitler equation by assuming the MSW approximation only, the convergence condition for the iteration method is not serious.

Let us introduce the further approximation such as the use of the first Born approximation for the reaction matrix $R(E, 2ky)$. By the definition of $R(E, 2ky)$ Eq. (3·6), the first Born term of $R(E, \beta)$ is given by $A_B(E, \beta)$, Eq. (4·6). Combining this first Born term of $R(E, \beta)$ under the MSW approximation with Eq. (4·14), we get $A_{GB}(E, \beta)$ Eq. (1·25). In the case of the real potential, the total elastic cross section is expressed under these approximations as follows:

$$\sigma_{\mathrm{el}}^{(BG)}(E) = 8\pi \int_0^\infty bdb \left[\frac{\chi_e^2(E, 2kb)}{1 + \chi_e^2(E, 2kb)} \right]. \tag{4·19}$$

It should be noted that Blankenbecler and Goldberger's result[22] shows that

$$\chi_{SW}(E, \beta) = \tan^{-1}\chi_e(E, \beta), \tag{4·20}$$

by comparing $A_{SW}(E, \beta)$ Eq. (2·76) with $A_{BG}(E, \beta)$ Eq. (1·25). If we approximate the denominator of Eq. (4·19) by unity, we obtain the modified Born approximation, i.e. $A_{B,SW}(E, \beta)$ Eq. (4·11):

$$\sigma_{\mathrm{el}}^{(MB)}(E) = 8\pi \int_0^\infty bdb[\chi_e(E, 2kb)]^2. \tag{4·21}$$

This is essentially what has been obtained by Gaunt.[9] As it is well known in the general case of the reaction matrix, if $A(E,\beta)$ is calculated through $R(E,\beta)$, it satisfies the unitarity relation automatically, irrespective of any approximation of $R(E,\beta)$. Furthermore, the second Born term of $R(E,\beta)$ vanishes under the MSW approximation. In these sense, although the weak potential approximation Eq. (1·5) is employed, the former approximation Eq. (4·19) can be used as the next approximation in the lower energy region where the first Born approximation fails to reproduce the experimental results, c.f. § 8.

§ 5. The evaluation of the impact parameter amplitude, $A(E,\beta)$

Our problem is to find a simpler relation between $\chi_{SW}(E,\beta)$ and $V(E,r)$ under the MSW approximation. For this purpose, it is convenient to calculate $A(E,\beta)$ itself rather than $R(E,\beta)$. We first express the partial wave amplitude $A_l(E)$ by using the Born series expansion of $\psi^{(+)}(\boldsymbol{r},\boldsymbol{k})$, and then obtain $A(E,\beta)$ by Eq. (2·16).

Let us define the partial wave expansion of $\psi^{(+)}(\boldsymbol{r},\boldsymbol{k})$ as follows:

$$\psi^{(+)}(\boldsymbol{r},\boldsymbol{k})=4\pi N\sum_{l,m} Y_l^m(\hat{\boldsymbol{k}})Y_l^{m*}(\hat{\boldsymbol{r}})i^l\psi_l^{(+)}(E,r). \tag{5·1}$$

Similarly, the plane wave $\varphi(\boldsymbol{r},\boldsymbol{k})$ is expanded by using $j_l(kr)$ in the place of $\psi_l^{(+)}(E,r)$, where $j_l(z)$ is a spherical Bessel function of order l. The Lippmann-Schwinger equation becomes

$$\psi_l^{(+)}(E,r_1)=j_l(kr_1)+\int_0^\infty (dr_2)\frac{1}{k}g_l^{(+)}(r_1,r_2)\psi_l^{(+)}(E,r_2), \tag{5·2}$$

where

$$(dr_j)=kr_j^2U(E,r_j)\,dr_j\,, \tag{5·3}$$

and the Green function $G_+(|\boldsymbol{r}_1-\boldsymbol{r}_2|)$ Eq. (3·5) is expanded as

$$G_+(|\boldsymbol{r}_1-\boldsymbol{r}_2|)=\sum_{l,m} Y_l^m(\hat{\boldsymbol{r}}_1)Y_l^{m*}(\hat{\boldsymbol{r}}_2)g_l^{(+)}(r_1,r_2) \tag{5·4}$$

and

$$g_l^{(+)}(r_1,r_2)=\frac{2}{\pi}\int_0^\infty q^2dq\,j_l(qr_1)\frac{1}{(k^2+i\varepsilon)-q^2}j_l(qr_2), \tag{5·5}$$

$$=(-ik)\begin{bmatrix} j_l(kr_1)h_l^{(1)}(kr_2)\theta(r_2-r_1) \\ +h_l^{(1)}(kr_1)j_l(kr_2)\theta(r_1-r_2)\end{bmatrix}. \tag{5·6}$$

Here $h_l^{(1)}$ is the spherical Hankel function of the first kind and $\theta(x)$ is a step function. The solution $\psi_l^{(+)}(E,r_1)$ is obtained by the iteration method. By defining the Born series expansion in the form

$$\psi_l^{(+)}(E,r)=\sum_{n=1}^\infty \psi_{l,n}^{(+)}(E,r), \tag{5·7}$$

and assuming its convergence, we get[20]

$$\phi_{l,n}^{(+)}(E, r_1) = (-i)^{n-1}\int_0^\infty (dr_n) K_{l,n-1}(r_1, r_n) j_l(kr_n), \tag{5·8}$$

where

$$K_{l,n-1}(r_1, r_n) = \int_0^\infty (dr_{n-1}) K_{l,n-2}(r_1, r_{n-1}) K_{l,1}(r_{n-1}, r_n), \tag{5·9}$$

$$K_{l,1}(r_1, r_2) = g_l^{(+)}(r_1, r_2)/(-ik) \tag{5·10}$$

and

$$K_{l,0}(r_1, r_1') = \delta(r_1 - r_1')/kr_1^2 U(E, r_1). \tag{5·11}$$

Correspondingly, the partial wave amplitude $A_l(E)$ is expanded in the form

$$A_l(E) = \sum_{n=1}^\infty A_{l,n}(E), \tag{5·12}$$

where

$$A_{l,n}(E) = \left(-\frac{1}{k}\right)\int_0^\infty (dr_1) j_l(kr_1) \phi_{l,n}^{(+)}(E, r_1) \tag{5·13}$$

$$= \left(-\frac{1}{k}\right)(-i)^{n-1}\int_0^\infty (dr_1)\int_0^\infty (dr_n) j_l(kr_1) K_{l,n-1}(r_1, r_n) j_l(kr_n). \tag{5·14}$$

The impact parameter amplitude $A(E, \beta)$ is obtained by

$$A(E, \beta) = \sum_{n=1}^\infty A_n(E, \beta), \tag{5·15}$$

where

$$A_n(E, \beta) = \sum_{l=0}^\infty 2(2l+1)\left[\frac{J_{2l+1}(\beta)}{\beta}\right] A_{l,n}(E). \tag{5·16}$$

Let us consider the first few terms as examples. The first Born term is the well-known one,

$$A_{l,1}(E) = -\frac{1}{k}\int_0^\infty (dr_1) j_l(kr_1) j_l(kr_1) \tag{5·17}$$

$$= 2\int_0^\infty da_1 J_{2l+1}(2ka_1)\left[-\frac{1}{2k}\int_{a_1}^\infty \frac{r_1 dr_1 U(E, r_1)}{\sqrt{r_1^2 - a_1^2}}\right] \tag{5·18}$$

$$= 2\int_0^\infty da_1 J_{2l+1}(2ka_1) \chi_e(E, 2ka_1), \tag{5·19}$$

where we have used the definition of χ_e Eq. (1·21) and the relation*)

$$j_l(kr)j_l(kr)=\frac{1}{kr}\int_0^r\frac{da}{\sqrt{r^2-a^2}}J_{2l+1}(2ka). \tag{5·20}$$

The second Born term is

$$A_{l,2}(E)=\left(-\frac{1}{k}\right)(-2i)\int_0^\infty(dr_1)[j_l(kr_1)]^2\int_{r_1}^\infty(dr_2)j_l(kr_2)h_l^{(1)}(kr_2). \tag{5·21}$$

Since we have the following relation**)

$$j_l(kr)h_l^{(1)}(kr)=\left(\frac{1}{kr}\right)\left[\int_0^r\frac{1}{\sqrt{r^2-a^2}}-i\int_r^\infty\frac{1}{\sqrt{a^2-r^2}}\right]J_{2l+1}(2ka)da, \tag{5·22}$$

then we get

$$\begin{aligned}A_2(E,\beta)=\left(-\frac{1}{k}\right)(-i)\Bigg[&8k^2\int_0^\infty da_1\chi_e(E,2ka_1)\int_{a_1}^\infty da_2\chi_e(E,2ka_2)\\&-2i\int_0^\infty da_1\int_{a_1}^\infty da_2\int_{a_1}^{a_2}\frac{r_1dr_1U(E,r_1)}{\sqrt{r_1^2-a_1^2}}\int_{r_1}^{a_1}\frac{r_2dr_2U(E,r_2)}{\sqrt{a_2^2-r_2^2}}\Bigg]\\&\times G_2(\beta,2ka_1,2ka_2).\end{aligned} \tag{5·23}$$

When the MSW approximation is applied, the second term has no contribution because of $a_1=a_2=\beta/2k$. Thus, we find that

$$A_{2,SW}(E,\beta)=(i/k)[\chi_e(E,\beta)]^2, \tag{5·24}$$

where we have used the relation

$$\int_0^\infty dr_1f(r_1)\int_{r_1}^\infty dr_2f(r_2)\cdots\int_{r_{n-1}}^\infty dr_nf(r_n)=\frac{1}{n!}\left[\int_0^\infty drf(r)\right]^n. \tag{5·25}$$

We shall consider the third Born term, which offers the new type of combination, $[h_l^{(1)}(kr)]^2$. It is

$$A_{l,3}(E)=\left(-\frac{1}{k}\right)(-i)^2\left\{\begin{aligned}&4\int_0^\infty(dr_1)[j_l(\rho_1)]^2\int_{r_1}^\infty(dr_2)h_l^{(1)}(\rho_2)j_l(\rho_2)\int_{r_2}^\infty(dr_3)h_l^{(1)}(\rho_3)j_l(\rho_3)\\&+2\int_0^\infty(dr_1)[j_l(\rho_1)]^2\int_{r_1}^\infty(dr_2)[j_l(\rho_2)]^2\int_{r_2}^\infty(dr_3)[h_l^{(1)}(\rho_3)]^2\end{aligned}\right\}, \tag{5·26}$$

*) Equation (5·20) is given on p. 24 of reference 44). It is easy to confirm that Eq. (5·19) is consistent with Eq. (4·4) within the physical region ($1>y\geq0$). However, $f_B(E,2ky)$ Eq. (4·4) has a value even for $y>1$, while, if $f_B(E,2ky)$ is obtained from Eqs. (5·19) and (5·16), it is equal to be zero for $y>1$; see Eqs. (2·69) and (A·3). This special character for $y>1$ is true for every higher Born terms.

**) See pp. 24 and 102 of reference 44).

where

$$\rho_j = kr_j\,. \tag{5·27}$$

The function $[h_l^{(1)}(kr)]^2$ can be expressed in the form,*)

$$[h_l^{(1)}(kr)]^2 = \frac{-2i}{kr}\int_r^\infty \frac{da}{\sqrt{a^2-r^2}}\, H_{2l+1}^{(1)}(2ka)\,, \tag{5·28}$$

where $H_{2l+1}^{(1)}(z)$ is the ordinary Hankel function of the first kind. Thus, when we calculate $A_3(E,\beta)$ from $A_{l,3}(E)$, the first term of Eq. (5·26) gives the combination $G_3(\beta, 2ka_1\, 2ka_2, 2ka_3)$ similar to the previous cases, while the second term introduces $G_3^{(1)}(\beta, 2ka_1, 2ka_2, 2ka_3)$, which is defined as

$$G_n^{(m)}(\beta_0\beta_1\cdots\beta_n) = \sum_l 2(2l+1)\prod_{i=0}^{n-m}\left[\frac{J_{2l+1}(\beta_i)}{\beta_i}\right]\prod_{j=n-m+1}^{n}\left[\frac{H_{2l+1}^{(1)}(\beta_j)}{\beta_j}\right]. \tag{5·29}$$

It should be noted that this new type appears only for $n\geq 3$ and $n-2\geq m$, and that there is at least one $J_{2l+1}(\beta_i)$ whose argument β_i cannot be larger than the β_j's of $H_{2l+1}^{(1)}(\beta_j)$. Under the MSW approximation, we have

$$G_n^{(m)}(\beta_0\beta_1\cdots\beta_n) \simeq G_n(\beta_0\beta_1\cdots\beta_n), \quad \text{for} \quad kR\to\infty\,. \tag{5·30}$$

The proof will be given in Appendix B. By applying almost the same argument as for the second term of $A_2(E,\beta)$, we shall again omit the term which includes $[h_l^{(1)}(kr_m)]^2$.

In general, we can decompose $A_{l,n}(E)$ into two parts,

$$A_{l,n}(E) = \left(-\frac{1}{k}\right)(-i)^{n-1}\{J_{l,n}(E) + H_{l,n}(E)\}, \tag{5·31}$$

where $J_{l,n}(E)$ consists of combinations of both $[j_l(\rho_m)]^2$ and $j_l(\rho_m)h_l^{(1)}(\rho_m)$, and $H_{l,n}(E)$ includes at least one combination $[h_l^{(1)}(\rho_m)]^2$. When $A_n(E,\beta)$ is derived from $H_{l,n}(E)$, it includes $G_n^{(m)}(\beta, \alpha\text{'s})$. According to the similar procedure, we need not take into account the contribution from $H_{l,n}(E)$ under the MSW approximation.

The first term $J_{l,n}(E)$ is expressed in the form

$$J_{l,n}(E) = E_{l,n}(E) + \sum_{m=1}^{n-1}\int_0^\infty (dr_1)\, j_l(\rho_1)\, h_l^{(1)}(\rho_1)$$

$$\times\int_0^{r_1}(dr_2)\, j_l(\rho_2)\, h_l^{(1)}(\rho_2)\cdots\int_0^{r_{m-1}}(dr_m)\, j_l(\rho_m)\, h_l^{(1)}(\rho_m)\int_0^{r_m}(dr_{m+1})\,[j_l(\rho_{m+1})]^2$$

*) See Eqs. (5·20) and (5·22).

$$\times \int_{r_{m+1}}^{\infty} (dr_{m+2}) h_l^{(1)}(\rho_{m+2}) j_l(\rho_{m+2}) \cdots \int_{r_{n-1}}^{\infty} (dr_n) h_l^{(1)}(\rho_n) j_l(\rho_n), \tag{5·32}$$

where

$$E_{l,n}(E) = \int_0^{\infty} (dr_1) [j_l(\rho_1)]^2 \int_{r_1}^{\infty} (dr_2) j_l(\rho_2) h_l^{(1)}(\rho_2) \cdots \int_{r_{n-1}}^{\infty} (dr_n) j_l(\rho_n) h_l^{(1)}(\rho_n). \tag{5·33}$$

As it will be shown in Appendix C, we can prove that

$$J_{l,n}(E) = 2^{n-1} E_{l,n}(E). \tag{5·34}$$

By replacing $j_l(\rho_m) h_l^{(1)}(\rho_m)$ by the integral form, it is easy to confirm that the $j_l(\rho_m) n_l(\rho_m)$ term has no contribution to $J_{l,n}(E)$ under the MSW approximation. Thus, it is enough to consider the simple expression, in which the combination of $j_l(\rho_m) h_l^{(1)}(\rho_m)$ is substituted by $[j_l(\rho_m)]^2$. This part of $J_{l,n}(E)$ will be called $J_{l,n}^{(S)}(E)$. By using Eqs. (5·25) and (5·19), we find

$$J_{l,n}^{(S)}(E) = 2^{n-1} E_{n,l}^{(S)}(E) \tag{5·35}$$

$$= \frac{2^{n-1}}{n!} \left[\int_0^{\infty} (dr) j_l(kr) j_l(kr)\right]^n \tag{5·36}$$

$$= \frac{2^{n-1}}{n!} (-2k)^n \left[\int_0^{\infty} da\, J_{2l+1}(2ka) \chi_e(E, 2ka)\right]^n. \tag{5·37}$$

Finally we get

$$A_n^{(S)}(E, \beta) = \frac{(2i)^{n-1}}{n!\,k} (2k)^{2n}$$

$$\times \left[\prod_{j=1}^{n} \int_0^{\infty} a_j\, da_j\, \chi_e(E, 2ka_j)\right] G_n(\beta, 2ka_1, \cdots 2ka_n). \tag{5·38}$$

By applying the MSW approximation, we get

$$A_{SW}(E, \beta) = \frac{1}{k} \sum_{n=1}^{\infty} \frac{(2i)^{n-1}}{n!} [\chi_e(E, \beta)]^n \tag{5·39}$$

$$= [\exp(2i\chi_e) - 1]/2ik. \tag{5·40}$$

This is just the result obtained under the eikonal (or semiclassical) approximation, Eq. (1·20). By comparing it with Eq. (2·76), we conclude that the relation $\chi_{SW}(E, \beta) = \chi_e(E, \beta)$ Eq. (4·12) is correct for all higher Born terms.

§ 6. The dispersion relation

The analytic properties of the impact parameter amplitude $A(E, \beta)$ with respect to β for the fixed physical value of E have been discussed. In this section, we shall study the analyticity of $A(E, \beta)$ with respect to E for the fixed value of β. On the other hand, Blankenbecler and Goldberger have derived a dispersion relation for $H(E, b)$ for the fixed b. By combining the approximate unitarity relation, they have found that the solution of their dispersion relation is $A_{GB}(E, \beta)$ Eq. (1·25) rather than the eikonal form $A_{SW}(E, \beta)$ Eq. (5·40). In order to investigate this difference, we shall approximate our exact dispersion relation for the fixed β by applying the MSW approximation. In this section, the full scattering amplitude is expressed by $f(E, t)$ instead of $f(E, 2ky)$, where $t=-\Delta^2$.

In order to guarantee the dispersion relation for $f(E, t)$, our discussion will be limited to the potentials which satisfy the conditions

$$\int_M^\infty r^2 dr |V(E, r)| < \infty \quad \text{and} \quad \int_0^{M'} r dr |V(E, r)| < \infty, \tag{6·1}$$

where M and M' are finite positive numbers. We shall further assume, for simplicity, that there is no bound state and that the potential can be expressed by superposition of the Yukawa potential,

$$rV(E, r) = \int_{m_0}^\infty dm\, \sigma(E, m) e^{-mr}, \tag{6·2}$$

where m_0^{-1} is the range of the potential. We shall assume that $\sigma(E, m)$ has no singularity in the complex E plane.

Under these assumptions, it has been proved[34] that the Mandelstam representation for $f(E, t)$ can be written in the form

$$f(E, t) = f_B(E, t) + \int_0^\infty \frac{dE'}{\pi} \frac{\mathrm{Im} f(E', t)}{E'-E-i\varepsilon}, \tag{6·3}$$

where

$$f_B(E, t) = -2\mu \int_{m_0}^\infty dm\, \sigma(E, m) \frac{1}{m^2-t}, \tag{6·4}$$

$$\mathrm{Im} f(E, t) = \int_{4m_0^2}^\infty \frac{dt'}{\pi} \frac{\rho(E, t')}{t'-t}, \tag{6·5}$$

$\rho(E, t)$ being the spectral function.

In order to get the dispersion relation for $A(E, \beta)$, it is convenient to discuss the first Born term $f_B(E, t)$ separately. When the definition of $A(E, \beta)$ Eq. (1·13) is applied, we encounter an integral,[4]

$$M(E, m^2, \beta) = \int_0^1 y dy\, J_0(\beta y) \frac{1}{m^2 + 4k^2 y^2}. \tag{6·6}$$

The function $M(E, t, \beta)$ in our impact parameter expansion corresponds to the Legendre function of the second kind, $Q_l(1-2y^2)$, in the partial wave expansion. [See Eqs. (6·14) and (6·18).] We perform explicitly this integral and get the result,

$$M(E, m^2, \beta) = \frac{1}{8}\sum_{n=1}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{\beta}{2}\right)^{2n}\sum_{r=0}^{n-1}\frac{(-1)^r}{(n-r)}\left(\frac{m}{2}\right)^{2r}\left(\frac{1}{k^2}\right)^{r+1}$$

$$+\frac{1}{8k^2}J_0\left(\frac{im\beta}{2k}\right)\ln\left|\frac{4k^2+m^2}{m^2}\right|. \tag{6·7}$$

From this expression we may read off the analytic properties of $M(E, m^2, \beta)$ by regarding it as a function of the complex variable E. There is a cut along the negative real E-axis, which comes from the logarithmic term. In fact, the discontinuity of $M(E, m^2, \beta)$ is

$$\frac{1}{2i}\{M(E+i\varepsilon, m^2, \beta)-M(E-i\varepsilon, m^2, \beta)\}$$

$$=\frac{\pi}{8k^2}J_0\left[\frac{\beta m}{2\sqrt{-k^2}}\right]\theta(-k^2-m^2/4). \tag{6·8}$$

The cut from the Born term $f_B(E, t)$ runs from $[-2\mu(m_0/2)^2]$ to $(-\infty)$ in the complex E-plane. Furthermore, we see that $M(E, m^2, \beta)$ becomes to be zero at $|E|=\infty$. Concerning the second integral in Eq. (6·3), if we substitute t' instead of m^2 in Eq. (6·6), the similar procedure can be adapted. Thus, the second integral has a cut from $(-2\mu m_0^2)$ to $(-\infty)$ in the complex E-plane.

We may now write the dispersion relation for $A(E, \beta)$ from the above analytic properties in the complex E-plane;*)

$$A(E, \beta)=A_B(E, \beta)+\int_{-\infty}^{-2\mu m_0^2}\frac{dE'}{\pi}\frac{\Delta A(E', \beta)}{E'-E}+\int_0^{\infty}\frac{dE'}{\pi}\frac{\operatorname{Im} A(E', \beta)}{E'-E-i\varepsilon}, \tag{6·9}$$

where the Born term $A_B(E, \beta)$ is reexpressed in the form

$$A_B(E, \beta)=-\frac{\mu}{\pi}\int_{-\infty}^{-\mu m_0^2/2}\frac{dE'}{E'-E}\int_{m_0}^{2\sqrt{-2\mu E'}}dm\,\sigma(E', m)\left[\frac{\pi}{4\mu E'}J_0\left(\beta\frac{m}{2\sqrt{-2\mu E'}}\right)\right], \tag{6·10}$$

and the discontinuity across the left-hand cut is

$$\Delta A(E, \beta)=\frac{1}{8\mu E}\int_{4m_0^2}^{-8\mu E}dt'\,J_0\left(\beta\sqrt{\frac{t'}{-8\mu E}}\right)\int_0^{\infty}\frac{dE'}{\pi}\frac{\rho(E', t')}{E'-E-i\varepsilon}. \tag{6·11}$$

*) This form of dispersion relation, of course, can be derived by a very direct procedure, which is similar to Eq. (252) on p. 615 of reference 23).

The dispersion relation for $A(E,\beta)$ has the same structure as for the partial wave amplitude, $A_l(E)$. It, however, is unnecessary to introduce the kinematical factor in order to guarantee the threshold behavior for the fixed β, in contrast with the case of the partial wave amplitude. Since the unitarity relation for $A(E,\beta)$ has the same form as for $A_l(E)$ with the opened reaction channel, we may solve the dispersion relation Eq. (6·9) by using methods proposed by Froissart,[35] Fry and Warnock,[36] and Islam and Kang.[37),16] As it is well known, in order to solve this dispersion relation, we have to assume both the discontinuity across the left-hand cut and the values of ξ^2. We shall not go further to discuss it here.

In order to investigate the properties of the dispersion relation under the MSW approximation, it is convenient to start from another form of the Mandelstam representation,

$$f(E,t)=f_B(E,t)+\int_{4m_0^2}^{\infty}\frac{dt'}{\pi}\frac{A_t(t',E)}{t'-t}, \tag{6·12}$$

where

$$A_t(t,E)=\int_0^{\infty}\frac{dE'}{\pi}\frac{\rho(E',t)}{E'-E-i\varepsilon}. \tag{6·13}$$

As it was done by Blankenbecler and Goldberger,[22] we shall use the following relation:*)

$$\frac{1}{t'-t}=\int_0^{\infty}ada\,J_0(a\sqrt{-t})\,K_0(a\sqrt{t'}),\ \text{for Re}\ \sqrt{t'}>\text{Im}|\sqrt{-t}|, \tag{6·14}$$

where $K_0(z)$ is the modified Bessel function of order zero. When this relation is substituted into Eqs. (6·12) and (6·4), the dispersion relation for $A(E,\beta)$ can be reduced to the form

$$A(E,\beta)=\int_0^{\infty}ada\,G_1(\beta,2ka)$$
$$\times\left[-4\mu\int_{m_0}^{\infty}dm\,\sigma(E,m)\,K_0(ma)+\int_{4m_0^2}^{\infty}\frac{dt'}{\pi}A_t(t',E)\,K_0(a\sqrt{t'})\right], \tag{6·15}$$

where the first and second integrals are defined for $(m_0/2)>|\text{Im}\,k|$ and $m_0>|\text{Im}\,k|$, respectively. This equation means that $A(E,\beta)$ is to be projected from the quantity in the square bracket, which may correspond to $H_g(E,2ka,q>1)$ of

*) See p. 137 of reference 44).

Eq. (2·71).*)

If the MSW approximation is applied and the potential is real, independent of energy, then the approximate dispersion relation becomes a very simple form:

$$A_{BG}(E,\beta)=A_{B,SW}(E,\beta)+\int_0^\infty \frac{dE'}{\pi}\frac{\operatorname{Im} A_{BG}(E',\beta)}{E'-E-i\varepsilon}\,. \tag{6·21}$$

As we shall show it, the solution of this equation is $A_{BG}(E,\beta)$ Eq. (1·25). Owing to the δ-functional character of $G_1(\beta, 2ka)$ under the MSW approximation, the integral variable a in Eq. (6·15) has been replaced by $(\beta/2k)=b$. The Born term can be reexpressed in the form

$$A_{B,SW}(E,2kb)=-\frac{4\mu}{4k^2}\int_{m_0}^\infty dm\,\sigma(m)K_0(mb), \tag{6·22}$$

$$=-\frac{1}{2E}\int_{-\infty}^0 dz V[\sqrt{b^2+z^2}], \tag{6·23}$$

where we have used Eqs. (6·2), (1·21), and the relation,**)

$$K_0(mb)=\int_b^\infty dr\frac{e^{-mr}}{\sqrt{r^2-b^2}}\,,\quad \text{for}\quad \operatorname{Re} m>0\,. \tag{6·24}$$

Since $A_{B,SW}(E,\beta)$ is real for the physical values of E and β, the discontinuity

---

*) The consistency between Eqs. (6·9) and (6·15) is easily verified as follows: As long as $\operatorname{Re}(m/2k)>0$, we have, [See p. 23 of reference 44).]

$$K_0(am)=\int_0^\infty ydyJ_0(2kay)/[y^2+(m/2k)^2]. \tag{6·16}$$

Since we have a relation[5]

$$\int_0^\infty \alpha d\alpha G_1(\beta\alpha)J_0(\alpha y)=J_0(\beta y)\theta(1-y), \tag{6·17}$$

Eq. (6·6) can be expressed in the form

$$M(E,m^2,\beta)=\int_0^\infty adaG_1(\beta,2ka)K_0(am). \tag{6·18}$$

It should be noted that we have a relation[4]

$$M(E,m^2,2kb)=\frac{1}{4k^2}K_0(mb)-L(E,m^2,2kb), \tag{6·19}$$

where $L(E,\mathrm{m}^2,\beta)$ is the complementary function of $M(E,m^2,\beta)$, Eq. (6·6);

$$L(E,t,\beta)=\int_1^\infty ydyJ_0(\beta y)\frac{1}{t+(2ky)^2}\,. \tag{6·20}$$

These are a set of examples of $A(E,\beta)$, $H_f(E,\beta)$ and $B_f(E,\beta)$ in § 2.6.

**) See p. 138 of reference 43).

across the physical (positive) E-axis is defined by

$$\operatorname{Im} A_{BG}(E, \beta) = \frac{1}{4k^2} \int_{4m_0^2}^{\infty} \frac{dt'}{\pi} \rho(E, t') K_0(\beta \sqrt{t'}/2k). \tag{6·25}$$

Thus we see that the dispersion relation for $A_{BG}(E, \beta)$ has only the right-hand cut arising from $A_t(t, E)$. By assuming the approximate unitarity relation Eq. (2·75) for all energies, the dispersion relation Eq. (6·21) was solved by Blankenbecler and Goldberger. According to the standard method,[38),23)] we shall define

$$L(E, \beta) \equiv \frac{A_{B,SW}(E, \beta)}{A_{BG}(E, \beta)}, \tag{6·26}$$

which approaches unity as $|E| \to \infty$, and is a function analytic in the entire cut E-plane. Here we assume that $EA_{BG}(E, \beta)$ has not the zero. Any such function can be written as

$$L(E, \beta) = 1 + \int_0^{\infty} \frac{dE'}{\pi} \frac{\operatorname{Im} L(E', \beta)}{E' - E - i\varepsilon}, \tag{6·27}$$

where

$$\operatorname{Im} L(E, \beta) = -(2\mu E)^{1/2} A_{B,SW}(E, \beta). \tag{6·28}$$

Then, we find that

$$L(E, \beta) = 1 - i\chi_e(E, \beta). \tag{6·29}$$

Finally, we get the form of $A_{BG}(E, \beta)$ given in Eq. (1·25).

We have now a question: Why do we get $A_{BG}(E, \beta)$ rather than the eikonal form $A_{SW}(E, \beta)$ under the same MSW approximation? When we have derived $A_{SW}(E, \beta)$ in § 5, we first have obtained the exact solution of the Lippmann-Schwinger equation by the iteration method, and then applied the MSW approximation to the final solution of $A(E, \beta)$. As it was shown in § 4, the solution $A_{BG}(E, \beta)$ has been obtained by using the first Born approximation for the reaction matrix. It is the simplest solution satisfying the approximate unitarity relation Eq. (2·75), which includes even the contribution from the reaction channel. On the other hand, in the case of Blankenbecler and Goldberger's derivation, the MSW approximation is applied to the dispersion relation itself which is the starting point corresponding to the Lippmann-Schwinger equation. As a result, the analytic porperties have been changed appreciably when Eq. (6·15) is approximated by Eq. (6·21). It will be shown in the next section that $A_{SW}(E, \beta)$ gives the expected result for the scattering by the square well potential, while $A_{BG}(E, \beta)$ does not.

Finally, we shall mention the dispersion relation for the fixed b. In this case, $A(E, 2kb)$ have an additional singularity at $|E| = \infty$.[4),16)] This singularity arises because $J_0(2kby)$, which is an entire function of E, has such a singularity. This fact can be seen explicitly in the function, $M(E, m^2, \beta)$, Eqs. (6·6) and (6·7). Therefore, we need make more subtractions to get the dispersion relation for the fixed b case under the same conditions as for the fixed β case. At a glance, it seems that Blankenbecler and Goldberger have derived the dispersion relation for the fixed b. This is because, in the high energy limit, we may neglect a contribution from $\infty > \Delta > 2k$, namely the L-term of Eq. (6·19). This procedure corresponds to the MSW approximation, as it was discussed in § 2.7. If we neglect the L-term, we have not the singularity at $|E| = \infty$, and arrive at the same dispersion relation even for the fixed b case: We would rather have a new singularity at $|E| = \infty$ for the fixed β case, because of $K_0(m\beta/2k)$ of Eq. (6·19).

§ 7. Scattering by a square well potential*)

We shall apply our results to the scattering by the square well potential:

$$V(E, r) = V_0\theta(R-r). \tag{7·1}$$

As Glauber[13] has shown it, our eikonal phase is expressed in the form,

$$\chi_e(E, \beta) = -\alpha[1-(\beta/2kR)^2]^{1/2}\theta(2kR-\beta), \tag{7·2}$$

where α is defined by Eq. (1·2). We see that $A_{SW}(E, \beta)$ has a cut in the complex β plane with the physical value of E.

The scattering amplitude in this case is expressed as,

$$\mathrm{Im}\, f_{SW}(E, 2ky) = \frac{1}{4k}\int_0^{2kR} \beta d\beta\, J_0(\beta y)\left\{1-\cos\left[2\alpha\sqrt{1-\left(\frac{\beta}{2kR}\right)^2}\right]\right\} \tag{7·3a}$$

$$= (kR^2)\left[\frac{1}{z}J_1(z) - \int_0^1 xdxJ_0(zx)\cos(2\alpha\sqrt{1-x^2})\right], \tag{7·3b}$$

and

$$\mathrm{Re}\, f_{SW}(E, 2ky) = -(kR^2)\int_0^1 xdxJ_0(zx)\sin(2\alpha\sqrt{1-x^2}), \tag{7·4}$$

where

$$z = 2kRy. \tag{7·5}$$

The first term of $\mathrm{Im}\, f_{SW}(E, 2ky)$ is familiar in optics as characterizing the diffraction scattering from a black sphere. This "shadow" scattering, therefore, depends on R and k, but is independent of V_0, as it is expected. The remaining

*) The authors are indebted to Miss S. Okamoto and Miss M. Yoshii for discussing the contents of this section.

terms express the scattering by the potential, and are the complicated functions of V_0 and R. The angular distribution for this case will be discussed elsewhere in connection with our general conclusions in § 2.8.

We shall compare the total cross sections obtained by various approximations mentioned in §§ 2.7 and 4. Since the first Born term is real which does not satisfy the unitarity relation, we calculate the total elastic cross section, by Eq. (2·52),

$$\frac{\sigma_{\rm el}{}^{(B)}(E)}{\pi R^2}=2\alpha^2\left[1-\left(\frac{1}{2kR}\right)^2+\frac{\sin(4kR)}{(2kR)^3}-\frac{\sin^2(2kR)}{(2kR)^4}\right]. \tag{7·6}$$

By applying the MSW approximation, as we expect, Eq. (4·21) becomes

$$\sigma_{\rm el}{}^{(MB)}(E)=2\alpha^2(\pi R^2). \tag{7·7}$$

In the case of Blankenbecler and Goldberger where the impact parameter reaction matrix $R(E,\beta)$ is approximated by the first Born term, Eq. (4·19) becomes

$$\frac{\sigma^{(BG)}_{\rm total}(E)}{\pi R^2}=4\left[1-\frac{1}{\alpha^2}\ln(1+\alpha^2)\right] \tag{7·8}$$

$$=\begin{cases}2\alpha^2-\dfrac{4}{3}\alpha^4, & \text{for}\quad \alpha\ll 1\,,\\[2ex] 4, & \text{for}\quad \alpha\to\infty\,.\end{cases} \tag{7·9}$$

The eikonal form $A_{SW}(E,\beta)$ gives us Eq. (2·79);[13]

$$\frac{\sigma^{(SW)}_{\rm total}(E)}{\pi R^2}=2+\frac{1}{\alpha^2}-\frac{2\sin(2\alpha)}{\alpha}-\frac{\cos(2\alpha)}{\alpha^2} \tag{7·10}$$

$$=\begin{cases}2\alpha^2-\dfrac{4}{9}\alpha^4, & \text{for}\quad \alpha\ll 1\,,\\[2ex] 2, & \text{for}\quad \alpha\to\infty\,.\end{cases} \tag{7·11}$$

The numerical results for these cases are shown in Fig. 1. The order of the validity of the MSW approximation may be seen by comparing $\sigma_{\rm el}^{(MB)}(E)$ [case IV] with the exact first Born term $\sigma_{\rm el}^{(B)}(E)$ [case III] in Fig. 1, because they give the almost same numerical results even when $kR=2$. We see that the result obtained by using only the MSW approximation, $\sigma^{(SW)}_{\rm total}(E)$, leads to $2\pi R^2$, twice its geometrical cross-sectional area. This is the famous result known as the "extinction paradox",[12),11)] which is due to the wave character in both the classical and quantum physics.*)

Unfortunately, we cannot apply our formalism directly to the scattering by a hard sphere, because the Lippmann-Schwinger equation has been used from the biginning, so the potential should be finite, Eq. (3·1). Since our original

*) The authors would like to express their sincere thanks to Professor H. Wergeland at Institutt for Teoretisk Fysikk, Trondheim, for sending copies of his papers on this problem.

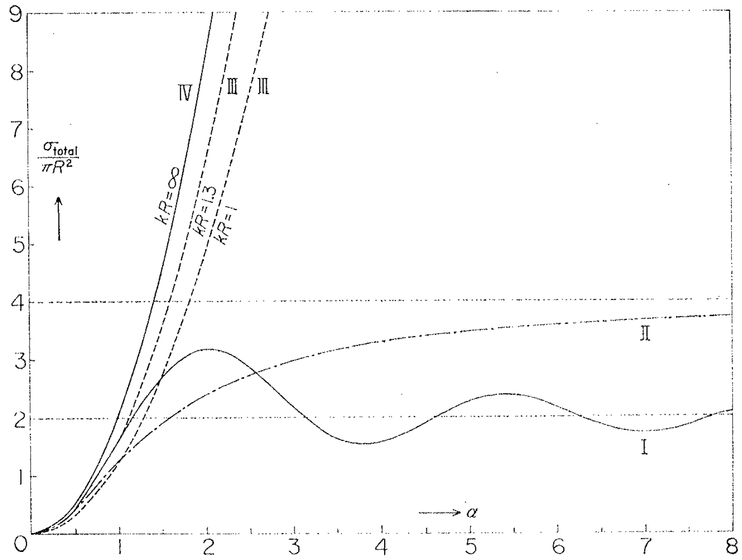

Fig. 1. Various approximate Ctotal ross sections as a function of $\alpha = RV_0\mu/k$. Curve I: $\sigma_{\text{total}}^{(SW)}(E)$; Curve II: $\sigma_{\text{total}}^{(BG)}(E)$: curve III: $\sigma_{\text{cl}}^{(B)}(E)$ with $kR=1$ and 1.3: Curve IV $\sigma_{\text{cl}}^{(MB)}(E)$. The unit of the cross section is πR^2.

formalism has no such restriction, this scattering can be treated by our formalism, in principle. Although we have not yet derived the final simple results, we can take out some interesting character and confirm our expected results. For example, we have the relations in the partial wave expansion,

$$\tan[\text{Re}\,\delta_l(E)] = j_l(kR)/n_l(kR) \tag{7·12}$$

and

$$\text{Im}\,\delta_l(E) = 0, \text{ i.e. } \eta_l(E) = 1\,, \tag{7·13}$$

where R is the range of the hard sphere. According to the relation between $\chi(E, \beta)$ and $\delta_l(E)$, we obtain

$$\zeta(E, \beta) = 1, \text{ for all the } \beta\text{'s and } E\text{'s}, \tag{7·14}$$

$$\xi \cos[2\,\text{Re}\,\chi] = \sum_l 2(2l+1)\left[\frac{J_{2l+1}(\beta)}{\beta}\right]\left[\frac{n_l^2 - j_l^2}{j_l^2 + n_l^2}\right], \tag{7·15}$$

$$\xi \sin[2\,\text{Re}\,\chi] = \sum_l 2(2l+1)\left[\frac{J_{2l+1}(\beta)}{\beta}\right]\left[\frac{2j_l n_l}{j_l^2 + n_l^2}\right], \tag{7·16}$$

and

$$\begin{aligned} 1-\xi^2 &= 4kK(E, \beta) \\ &= \sum_l \sum_m 4(2l+1)(2m+1) \\ &\times \left[\frac{J_{2l+1}(\beta)}{\beta}\right]\left[\frac{J_{2m+1}(\beta)}{\beta}\right]\left[\frac{2(j_l n_m - n_l j_m)^2}{(j_l^2+n_l^2)(j_m^2+n_m^2)}\right]. \end{aligned} \tag{7·17}$$

As it is required by the unitarity relation, it is easy to see the result of Eq. (2·35), i.e.

$$\int_0^\infty \beta d\beta (1-\xi^2) = 0\,, \tag{7·18}$$

because of Eq. (2·7).

Furthermore, since $A_l(E)$ is given in the following form as $E\to 0$:

$$A_l(E) = (-R + ikR^2)\delta_{l,0} + O(k^2R^3)\,, \tag{7·19}$$

the total cross section, $\sigma_{\text{total}}(E)$ [or $\sigma_{\text{el}}(E)$], is expressed as

$$\sigma_{\text{total}}(E) = \frac{2\pi}{k}\int_0^\infty \beta d\beta \left[\sum_l 2(2l+1)J_{2l+1}(\beta)\frac{1}{\beta}\ \text{Im}\ A_l(E)\right] \tag{7·20}$$

$$= 4\pi R^2 \left[\int_0^\infty d\beta J_1(\beta) + O(k^2R^2)\right] \tag{7·21}$$

$$= 4\pi R^2 \left[(2k)\int_0^\infty db J_1(2kb) + O(k^2R^2)\right] \tag{7·22}$$

$$= 4\pi R^2 [1 + O(k^2R^2)]\,, \tag{7·23}$$

where we have used Eq. (2·8). This well-known example indicates that, when we treat the low energy phenomena ($k\to 0$) by the impact parameter formalism we may have some confusion if we separate β by $2kb$ in the earlier stage, as shown in Eq. (7·22). If β itself is kept as an integral variable like Eq. (7·21), we can avoid such a confusion in the complicated cases.

§ 8. Discussion

We have introduced the MSW approximation in which the condition $kR \gg 1$ is modified by the approximation of $G_n(\beta_j)$ in the form of the δ-function. In the case of potential scattering, we have obtained $A_{SW}(E, \beta)$ which has been derived previously under the eikonal approximation. In our derivation, we did not make any special assumptions about θ and V_0 explicitly. As we have seen in § 2.7, the approximate total cross section $\sigma_{\text{total}}^{(SW)}(E)$ should give a very reliable result, if $kR \gg 1$.

As an example, let us consider the charge exchange scattering of protons by hydrogen atoms. The classical maximum angular momentum is

$$L = R_H k \simeq 1.8\times 10^2 \sqrt{E_L}\,,\quad \text{for}\quad E_L \text{ in keV}\,, \tag{8·1}$$

where R_H is assumed to be the Bohr radius and E_L is the kinetic energy (in keV) of the incident proton in the laboratory system. Thus, many angular momentum states have to be taken into account even below $E_L=100$ eV, so that the treatment by the partial wave expansion may not be adequate. The first Born approximation may explain the experimental result on $\sigma_{\text{total}}(E)$ above 10 keV, in contrast with the case of the electron scattering by hydrogen atoms.[39] In the latter case, the first Born term can explain the experiments in the much lower energy region than the former. These facts can be understood qualitatively: The electron velocity is greater than the proton velocity at the same laboratory energy, so that $\alpha=R_H V_0/v$ is smaller in the electron case than in the proton case. As we have seen in Fig. 1, the first Born approximation gives a larger value than our approximated results, $\sigma_{\text{total}}^{(SW)}(E)$. This expected behavior agrees with the discrepancy between the experimental results and the calculated $\sigma_{\text{total}}^{(B)}(E)$.*) Thus, we may imagine that our approximate result, $\sigma^{(SW)}(E)$ or even $\sigma^{(BG)}(E)$, can explain the data on proton-hydrogen charge exchange scattering. The quantitative comparison will be discussed in another paper of this journal.

The impact parameter amplitude $A(E,\beta)$ is an entire function of complex β for the fixed E. On the other hand, the corresponding partial wave amplitude $A_l(E)$ has the Regge (moving) pole in the complex l-plane.[41] If we interpolate the discrete l-values of the Legendre polynomials in the standard manner and define $A_l(E)$ for nonintegral values of l by Eq. (2·13), the $A(E,l)$ so defined cannot contain any poles.[40] The definition of the right interpolation to get the Regge pole was done via the Schrödinger equation.[41),40] In this sense, it is desirable to derive the Schrödinger equation in the impact parameter representation.

The solution of the Schrödinger equation,

$$\boldsymbol{\varDelta}\psi(\boldsymbol{r},\boldsymbol{k})+[k^2-U(E,r)]\psi(\boldsymbol{r},\boldsymbol{k})=0\,, \tag{8·2}$$

is expanded by assuming the spherical symmetric potential:

$$\psi(r,y,k)=\frac{N}{2}\int_0^\infty \beta d\beta J_0(\beta y)\,\varphi(r,\beta,k)\,, \tag{8·3}$$

where the $\hat{\boldsymbol{k}}$-direction is chosen as the z-axis and N is a normalization factor. If the Laplacian $\boldsymbol{\varDelta}$ is expressed by the polar coordinates,

$$\boldsymbol{\varDelta}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\Lambda\,, \tag{8·4}$$

then we find that

$$\Lambda J_0(\beta y)=\frac{1}{4}\left[-\beta^2\frac{\partial^2 J_0(\beta y)}{\partial\beta^2}-3\beta\frac{\partial J_0(\beta y)}{\partial\beta}-\beta^2 J_0(\beta y)\right]. \tag{8·5}$$

Since $\varphi(r,\beta,k)$ in Eq. (8·3) corresponds to H_f in Eq. (2·61c), it can be de-

*) See Fig. (5) of reference 39).

fined by the inversion formula Eq. (2·60a) with an arbitrary q. Thus, we find the Schrödinger equation for $\varphi(r, \beta, k)$ in the form

$$\frac{\partial^2\varphi}{\partial r^2}+\frac{2}{r}\frac{\partial\varphi}{\partial r}+[k^2-U(E, r)]\varphi=\frac{1}{4r^2}\left[\beta^2\varphi+3\beta\frac{\partial\varphi}{\partial\beta}+\beta^2\frac{\partial^2\varphi}{\partial\beta^2}\right]. \tag{8·6}$$

It should be noted that the function φ is restricted by the integrability condition

$$\int_0^\infty \beta d\beta|\varphi(r, \beta, k)|^2<\infty \tag{8·7}$$

for the fixed values of r and k. This corresponds to Eq. (2·62). If the angular variable y in $\psi(r, y, k)$ is restricted to the physical value of y, the function φ should satisfy four conditions similar to for $A(E, \beta)$.

We have not yet succeeded in discussing the solutions of this equation in general, especially in connection with $A(E, \beta)$, as it was established in the case of the partial wave expansion. It may, however, be worthwhile to show some of the typical solutions, by which mathematical character in our impact parameter representation can be understood. Since this equation is a separable type, we shall assume the following type of solution:

$$\varphi(r, \beta, k)=u(r, k, \lambda)v(\beta, \lambda)/\beta\,. \tag{8·8}$$

Then, we get a couple of equations

$$\frac{d^2u}{dr^2}+\frac{2}{r}\frac{du}{dr}+\left(k^2-U-\frac{\lambda}{r^2}\right)u=0 \tag{8·9}$$

and

$$\frac{d^2v}{d\beta^2}+\frac{1}{\beta}\frac{dv}{d\beta}+\left(1-\frac{4\lambda+1}{\beta}\right)v=0\,. \tag{8·10}$$

If we require that $v(\beta, \lambda)$ is an even and entire function with respect to β, then we have

$$\lambda=l(l+1), \quad \text{for} \quad l=0, 1, \cdots, \tag{8·11}$$

and

$$v(\beta, l)=J_{2l+1}(\beta), \tag{8·12}$$

by considering the integrability condition, Eq. (8·7). Under this condition, Eq. (8·9) is just the equation for the radial part in the partial wave expansion. Thus we can write a whole solution in the form

$$\varphi(r, \beta, k)=\sum_{l=0}^\infty 2(2l+1)\left[\frac{J_{2l+1}(\beta)}{\beta}\right]u_l(r, k). \tag{8·13}$$

This is the form in the Webb-Kapteyn theory of the Neumann series*) and

*) See §16.4 of reference 42) and §3 of reference 1).

corresponds to Eq. (2·16) for $A(E, \beta)$.

It should be noted that we never require the property that the function $\varphi(r, \beta, k)$ is an exponential type 1. In order to see this property explicitly, let us consider the case of a free particle as an example. In this case, the solution $u_l(r, k)$ is expressed in terms of the spherical Bessel functions $z_l(\rho)$ where $\rho = kr$. Let us define functions:

$$h^{(1)}(\rho, \beta) = \sum_{l=0}^{\infty} 2(2l+1)\left[\frac{J_{2l+1}(\beta)}{\beta}\right][i^l h_l^{(1)}(\rho)] \tag{8·14}$$

$$= 4\int_0^{\infty} y dy\, J_0(\beta y) \exp[i\rho(1-2y^2)] \tag{8·15}$$

$$= \left(\frac{1}{\rho i}\right) \exp\left[i\rho + i\frac{\beta^2}{8\rho}\right]; \tag{8·16}$$

$$h^{(2)}(\rho, \beta) = \sum_l 2(2l+1)[J_{2l+1}(\beta)/\beta][i^l h_l^{(2)}(\rho)] \tag{8·17}$$

$$= -4\int_1^{\infty} y dy\, J_0(\beta y) \exp[i\rho(1-2y^2)] \tag{8·18}$$

$$= \left(\frac{i}{\rho}\right) e^{-i\rho} \sum_{k=0}^{\infty} \left(\frac{i\beta}{4\rho}\right)^k J_k(\beta); \tag{8·19}$$

$$j(\rho, \beta) = \tfrac{1}{2}[h^{(1)}(\rho, \beta) + h^{(2)}(\rho, \beta)] \tag{8·20}$$

$$= \sum_l 2(2l+1)[J_{2l+1}(\beta)/\beta][i^l j_l(\rho)] \tag{8·21}$$

$$= 2\int_0^1 y dy\, J_0(\beta y) \exp[i\rho(1-2y^2)] \tag{8·22}$$

$$= \left(\frac{1}{2\rho i}\right) e^{-i\rho} \sum_{k=0}^{\infty} \left(\frac{4\rho i}{\beta}\right)^{k+1} J_{k+1}(\beta). \tag{8·23}$$

Only the function $j(\rho, \beta)$ is an exponential type 1. This is because $j(\rho, \beta)$ is an inverse transform of the plane wave, as shown by Eq. (8·22).

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Appendix A

The MSW approximation of $G_n(\beta_0\beta_1\cdots\beta_n)$

We shall prove that the function $G_n(\beta_0\beta_1\cdots\beta_n)$ defined in Eq. (2·37) is reduced to the product of Dirac's δ-functions, as shown in Eq. (2·41). Our procedure of the proof is the generalization of $I_2(\alpha_1\alpha_2\,;\,y)$, Eqs. (2·29) and (2·94), and $G_2(\beta\alpha_1\alpha_2)$, Eq. (2·33). The function $G_n(\beta_0\beta_1\cdots\beta_n)$ is a symmetric function of all arguments β_j, where we shall use the abbreviations such that $G_n(\beta_j)=G_n(\beta_0\beta_1\cdots\beta_n)$, if there is no confusion. If $G_n(\beta_j)$ is decomposed into

$$G_n(\beta\beta_1\cdots\beta_n)=\int_0^1 y_{bn}dy_{bn}J_0(\beta y_{bn})I_n(\beta_1\cdots\beta_n\,,\,y_{bn}),\quad\text{for}\quad n\geq 1\,, \tag{A·1}$$

then we have the inversion of it in the form

$$I_n(\beta_1\cdots\beta_n\,;\,y_{bn})\theta(1-y_{bn})=\int_0^\infty \beta d\beta\, J_0(\beta y_{bn})G_n(\beta\beta_1\cdots\beta_n), \tag{A·2}$$

where we define*)

$$I_1(\beta\,;\,y)=J_0(\beta y) \tag{A·3}$$

$$=\sum_l 2(2l+1)P_l(1-2y^2)\,[J_{2l+1}(\beta)/\beta]\,\theta(1-y). \tag{A·4}$$

By substituting Eq. (2·38) into Eq. (A·2), we find that the generalized expression of $I_n(\beta_j\,;\,y)$ is

$$I_n(\beta_1\cdots\beta_n\,;\,y)=\sum_l 2(2l+1)P_l(1-2y^2)\prod_{j=1}^n\left[\frac{J_{2l+1}(\beta_j)}{\beta_j}\right]\theta(1-y). \tag{A·5}$$

Thus, we have the following recurrence formula

$$I_n(\beta_1\cdots\beta_n\,;\,y_{bn})=\frac{1}{8\pi}\int d\Omega_{n-1}I_{n-1}(\beta_1\cdots\beta_{n-1}\,;\,y_{bn-1})J_0(\beta_n y_{n-1,n}), \tag{A·6}$$

where we have used the famous relation for the Legendre function,

$$\int d\Omega_n P_l(1-2y_{2n}^2)P_m(1-2y_{n1}^2)=\frac{4\pi\delta_{lm}}{2l+1}P_l(1-2y_{21}^2). \tag{A·7}$$

By an iteration of Eq. (A·6), we have an integral such as

$$\frac{1}{8\pi}\int d\Omega_{n-1}J_0(\beta_{n-1}y_{n-2,n-1})J_0(\beta_n y_{n-1,n}). \tag{A·8}$$

This is just the same form as Eq. (2·29). By using the Neumann addition theorem,**) Eq. (2·29) can be reduced to Eq. (2·30). By continuing the iteration procedure and introducing a new integral variable $x_{ij}=2ky_{ij}$, we get the integral representation of $G_n(\beta_j)$ in the form***)

*) The definition of $I_1(\beta\,;y)$ is consistent with Eqs. (6·17) and (2·2). Equation (A·4) can be proved by substituting Eqs. (2·69), (2·3) and (6·17) into Eq. (A·4).

**) See p. 358 of reference 42).

***) This is the generalization of Eq. (3·3) of reference 22) and Eq. (A·7) of reference 1).

$$
\begin{aligned}
&G_n(2kb,\ 2kb_1,\ \cdots 2kb_n)\\
&\quad=\frac{1}{(2k)^{2n}}\int_0^{2k} x_{bn}dx_{bn}\int_0^{2k} x_{1n}dx_{1n}\cdots\int_0^{2k} x_{n-1,n}dx_{n-1,n}\\
&\quad\times J_0\{bx_{bn}\}J_0\{b_1x_{bn}[1-(x_{1n}/2k)^2]^{1/2}\}\\
&\quad\times J_0\{b_1x_{1n}[1-(x_{bn}/2k)^2]^{1/2}\}J_0\{b_2x_{1n}[1-(x_{2n}/2k)^2]^{1/2}\} \qquad (\mathrm{A}\cdot 9)\\
&\quad\cdots\\
&\quad\times J_0\{b_{n-2}x_{n-2,n}[1-(x_{n-3,n}/2k)^2]^{1/2}\}J_0\{b_{n-1}x_{n-2,n}[1-(x_{n-1,n}/2k)^2]^{1/2}\}\\
&\quad\times J_0\{b_{n-1}x_{n-1,n}[1-(x_{n-2,n}/2k)^2]^{1/2}\}J_0\{b_nx_{n-1,n}\}\,.
\end{aligned}
$$

By applying the short wavelength approximation Eq. (1·1) ($k\to\infty$) and using Eq. (4·9), we have the final expression Eq. (2·41) under the MSW approximation.

Appendix B

The MSW approximation of $G_n^{(m)}(\beta_0\beta_1\cdots\beta_n)$

We shall show that under the MSW approximation, the function $G_n^{(m)}(\beta_0\beta_1\cdots\beta_n)$, Eq. (5·29), behaves as $G_n(\beta_0\beta_1\cdots\beta_n)$.

We decompose the function $G_n^{(m)}(\beta_j)$ into two parts; one of them includes the $J_{2l+1}(\beta_j)$ part of $H_{2l+1}^{(1)}(\beta_j)$ and the other the $N_{2l+1}(\beta_j)$ part;

$$G_n^{(m)}(\beta_0\beta_1\cdots\beta_n)=G_n(\beta_0\beta_1\cdots\beta_n)+\sum_{q=1}^{m}\sum\nolimits_P i^q G_n^{(m,q)}(\beta_0\beta_1\cdots\beta_n)\,, \qquad (\mathrm{B}\cdot 1)$$

where $q\leq m$ is the number of $N_{2l+1}(\beta_j)$. In order to avoid a confusion, the argument of N_{2l+1} is written as α_j instead of β_j. The symbol $\sum_P$ means to take a sum over all possible permutation to interchange $(\alpha_1\cdots\alpha_q)$ with any of the arguments $(\beta_{n-m+1}\cdots\beta_n)$ which appear in the $H_{2l+1}^{(1)}(\beta_j)$. The second term becomes to be zero under the MSW approximation, although there are some conditions which will be discussed later in this appendix.

One of the typical terms among $G_n^{(m,q)}$ is written as

$$G_n^{(m,q)}(\beta_0\cdots\beta_{n-q}\alpha_1\cdots\alpha_q)=\sum\nolimits_l 2(2l+1)\prod_{i=0}^{n-q}\left[\frac{J_{2l+1}(\beta_i)}{\beta_i}\right]\prod_{j=1}^{q}\left[\frac{N_{2l+1}(\alpha_j)}{\alpha_j}\right]. \qquad (\mathrm{B}\cdot 2)$$

By taking Eq. (2·7) into consideration, this is expressed as

$$G_n^{(m,q)}(\beta_i\alpha_j)=\int_0^\infty \xi d\xi G_{n-q+1}(\beta_0\beta_1\cdots\beta_{n-q}\xi)\,S_q(\xi\alpha_1\cdots\alpha_q)\,, \qquad (\mathrm{B}\cdot 3)$$

where

$$S_q(\xi,\ \alpha_j)=\sum\nolimits_l 2(2l+1)\left[\frac{J_{2l+1}(\xi)}{\xi}\right]\prod_{j=1}^{q}\left[\frac{N_{2l+1}(\alpha_j)}{\alpha_j}\right], \qquad (\mathrm{B}\cdot 4)$$

$$=\prod_{j=1}^{q}\Big[\int_0^\infty \xi_j d\xi_j S_1(\xi_j,\alpha_j)\Big]G_q(\xi\xi_1\cdots\xi_q), \tag{B·5}$$

and

$$S_1(\xi\alpha)=\sum_l' 2(2l+1)J_{2l+1}(\xi)N_{2l+1}(\alpha)/\xi\alpha. \tag{B·6}$$

Thus, in order to investigate the property $G_n^{(m,q)}(\beta_i\alpha_j)$ under the MSW approximation, it is sufficient to know the behavior of $S_1(\xi\alpha)$, because of the known property of $G_n(\beta_i)$ in the previous appendix.

An integral form of $N_{2l+1}(\alpha)$ is known as

$$\frac{N_{2l+1}(\alpha)}{\alpha}=\Big[\int_0^1 y dy N_0(\alpha y)P_l(1-2y^2)\Big]-\frac{1}{\pi}\frac{S_{2l+1}(\alpha)}{\alpha}, \tag{B·7}$$

where $S_{2l+1}(\alpha)$ is Schläfli's polynomials.*) This corresponds to Eq. (2·15). By combining Eq. (A·4) with the relations,**)

$$\begin{aligned}&\sum_l' 2(2l+1)J_{2l+1}(\xi)S_{2l+1}(\alpha)/\pi\alpha\xi\\ &=(2/\pi)(\alpha^2-\xi^2)^{-1} \qquad \text{for} \quad \alpha>\xi, \qquad \text{(B·8)}\\ &=\int_0^\infty y dy N_0(\alpha y)J_0(\xi y) \qquad \text{for} \quad \alpha>\xi, \qquad \text{(B·9)}\end{aligned}$$

the final expression of $S_1(\xi\alpha)$ is

$$S_1(\xi\alpha)=-\int_1^\infty y dy N_0(\xi y)J_0(\alpha y) \qquad \text{for} \quad \alpha>\xi, \tag{B·10}$$

$$=-\int_{2kR}^\infty z dz\, N_0(\xi z/2kR)J_0(\alpha z/2kR) \qquad \text{for} \quad \alpha>\xi. \tag{B·11}$$

This last expression just corresponds to the second term of Eq. (4·7), which is neglected under the MSW approximation.

Thus, we conclude that all $G_n^{(m,q)}(\beta_i\alpha_j)$ can be neglected consistently under the MSW approximation, if $\alpha>\xi$. The problem, therefore, becomes whether or not this condition is satisfied in our case. As an example, let us consider the second term of the third Born approximation Eq. (5·26), which includes $G_3^{(1)}(\beta_j)$. This part of $A(E,\beta)$ is written as $A_3^{(1)}(E,\beta)$

$$A_3^{(1)}(E,\beta)=\Big(\frac{4}{k}\Big)(2ik)^3\int_0^\infty\frac{(dr_1)}{kr_1}\int_0^{r_1}\frac{a_1da_1}{(r_1^2-a_1^2)^{1/2}}\int_{r_1}^\infty\frac{(dr_2)}{kr_2}\int_0^{r_2}\frac{a_2da_2}{(r_2^2-a_2^2)^{1/2}}$$

*) See p. 108 of reference 44), p. 284 of reference 42) and p. 41 of reference 45).
**) See pp. 32 and 93 of reference 45).

$$\times\int_{r_2}^{\infty}\frac{(dr_3)}{kr_3}\int_{r_3}^{\infty}\frac{a_3da_3}{(a_3^2-r_3^2)^{1/2}}\,G_3^{(1)}(\beta,\,2ka_1,\,2ka_2,\,2ka_3)\,. \tag{B·12}$$

We see that Eq. (B·12) requires the inequalities $a_3>r_3>r_2>a_2$ and $r_2>r_1>a_1$. On the other hand, the function $G_n(\beta_i\xi_j)$ in Eqs. (B·3) and (B·5) becomes Dirac's δ-function under the MSW approximation. That is, in the case of $A_3^{(1)}(E,\beta)$, we have the following inequality in the notation of this appendix:

$$\alpha_1>\beta_1=\beta_2=\xi_1=\beta_0\,. \tag{B·13}$$

We see that our condition $\alpha_1>\xi_1$ is satisfied. As we have noticed below Eq. (5·29), we always have such inequalities in our potential scattering problem. This inequality means that the first term of $G_n^{(m)}(\beta_j)$ Eq. (B·1), also has no contribution. Thus, we can conclude that $H_{l,n}(E)$ which involves the combinations $[h_l^{(1)}(\rho_j)]^2$, has no contribution under the MSW approximation.

Appendix C

The simplification of $J_{l,n}(E)$

We have separated the Born series amplitude $A_{l,n}(E)$ into $J_{l,n}(E)$ and $H_{l,n}(E)$, Eq. (5·31). The function $J_{l,n}(E)$, Eq. (5·32), includes n terms of $J_{2l+1}(\beta_j)$, and is reduced to a very simple form, Eq. (5·34). We shall prove this procedure.

Let us introduce the abbreviated notations:

$$J(\rho_m)=j_l(kr_m)\,j_l(kr_m)\,, \tag{C·1}$$

$$M(\rho_m)=j_l(kr_m)\,h_l^{(1)}(kr_m)\,. \tag{C·2}$$

Also we define three functions by assuming two positive integers $p\geq m$:

$$I_0(m,\,p)=\int_0^{r_m}(dr_{m+1})\,M(\rho_{m+1})\,I_0(m+1,\,p)\,, \tag{C·3}$$

where

$$I_0(p+n,\,p)=1,\quad\text{for}\quad n=0,\,1,\,2\cdots, \tag{C·4}$$

$$I_0(p-1,\,p)=\int_0^{r_{p-1}}(dr_p)\,M(\rho_p)\,; \tag{C·5}$$

$$I^\infty(m,\,p)=\int_{r_m}^{\infty}(dr_{m+1})\,M(\rho_{m+1})\,I^\infty(m+1,\,p)\,, \tag{C·6}$$

where

$$I^\infty(p+n,\,p)=1,\quad\text{for}\quad n=0,\,1,\,2\cdots, \tag{C·7}$$

$$I^{\infty}(p-1,\ p)=\int_{r_{p-1}}^{\infty}(dr_p)\,M(\rho_p)\,; \tag{C·8}$$

$$I_q(m,\ p)=\int_{r_q}^{r_m}(dr_{m+1})\,M(\rho_{m+1})\,I_q(m+1,\ p)\,, \tag{C·9}$$

where

$$I_q(p+n,\ p)=1, \quad \text{for} \quad n=0,\ 1,\ 2,\ \cdots, \tag{C·10}$$

$$I_q(p-1,\ p)=\int_{r_q}^{r_{p-1}}(dr_p)\,M(\rho_p)\,; \tag{C·11}$$

$$I_q^{\infty}(m,\ p)=\int_{r_q}^{\infty}(dr_m)\,M(\rho_m)\,I_q(m,\ p)\,, \tag{C·12}$$

where

$$I_q^{\infty}(p+n,\ p)=1, \quad \text{for} \quad n=1,\ 2,\ \cdots. \tag{C·13}$$

By using these notations, $J_{l,n}(E)$ can be expressed as

$$\begin{aligned} &J_{l,n}(E)-E_{l,n}(E)\\ &=\sum_{m=1}^{n-1}\int_0^{\infty}(dr_1)\,M(\rho_1)\,I_0(1,\ m-1)\int_0^{r_{m-1}}(dr_m)\,M(\rho_m)\\ &\times\int_0^{r_m}(dr_{m+1})\,J(\rho_{m+1})\,I^{\infty}(m+1,\ n)\,. \end{aligned} \tag{C·14}$$

By interchanging the order of integrations over r_m and r_{m+1} and after repeating q times of interchanges, we have

$$\begin{aligned} &J_{l,n}(E)-E_{l,n}(E)\\ &=\sum_{m=1}^{n-1}\int_0^{\infty}(dr_1)\,M(\rho_1)\,I_0(1,\ m-q)\int_0^{r_{m-q}}(dr_{m+1})\,J(\rho_{m+1})\\ &\times I_{m+1}(m-q,\ m)\,I^{\infty}(m+1,\ n)\,, \end{aligned} \tag{C·15}$$

where we have used the relations

$$\int_0^a dx\int_0^x dy f(x,\ y)=\int_0^a dy\int_y^a dx f(x,\ y)\,, \tag{C·16}$$

$$\int_0^\infty dx \int_0^x dy f(x, y) = \int_0^\infty dy \int_y^\infty dx f(x, y). \tag{C·17}$$

By putting $q = m - 1$, we get

$$J_{l,n}(E) = E_{l,n}(E) + \sum_{m=1}^{n-1} \int_0^\infty (dr_{m+1}) J(\rho_{m+1}) I^\infty(m+1, n) I_{m+1}^{\infty}(1, m). \tag{C·18}$$

Now we rename the suffix of r in the cyclic order such as r_p by r_{p-m} and r_q by r_{n-m+q} if $q \leq m$. Since in our notation we have

$$E_{l,n}(E) = \int_0^\infty (dr_1) J(\rho_1) I^\infty(1, n), \tag{C·19}$$

we get a compact form

$$J_{l,n}(E) = \sum_{m=0}^{n-1} \int_0^\infty (dr_1) J(\rho_1) I^\infty(1, n-m) I_1^\infty(n-m+1, n). \tag{C·20}$$

From the last factor, we have

$$I_1^\infty(n-m+1, n) = \int_{r_1}^\infty (dr_{n-m+1}) M(\rho_{n-m+1}) I_1(n-m+1, n)$$

$$= \left[\int_{r_1}^{r_2} + \int_{r_2}^{r_3} + \cdots + \int_{r_{n-m-1}}^{r_{n-m}} + \int_{r_{n-m}}^\infty \right] (dr_{n-m+1}) M(\rho_{n-m+1}) I_1(n-m+1, n). \tag{C·21}$$

By using the relation,

$$\int_a^\infty dx \int_a^x dy f(x, y) = \int_a^\infty dy \int_y^\infty dx f(x, y), \tag{C·22}$$

and renaming the integral variables r_j in the order, we find that

$$J_{l,n}(E) = \sum_{m=0}^{1} \binom{n-1}{m} E_{l,n}$$

$$+ \sum_{m=2}^{n-1} \int_0^\infty (dr_1) J(\rho_1) I^\infty(1, n-m+1) \sum_{q=2}^{n-m+1} \int_{r_1}^{r_q} (dr_{n-m+2}) M(\rho_{n-m+2}) I_1(n-m+2, n). \tag{C·23}$$

Here we have used the special case $p=0$ of the following sum of the binomial coefficients,*)

$$\sum_{r=1}^{N}\binom{N+p-r}{p}r=\binom{N+p+1}{p+2}, \tag{C·24}$$

where N, p, q are positive integers.

Similarly, by using Eq. (C·24) with $p=1$, we proceed to the next stage, i.e.

$$J_{l,n}(E)=\sum_{m=0}^{2}\binom{n-1}{m}E_{l,n}(E)$$
$$+\sum_{m=3}^{n-1}\int_0^\infty(dr_1)J(\rho_1)I^\infty(1,\ n-m+2) \tag{C·27}$$
$$\times\sum_{q=2}^{n-m+1}\binom{n-m+2-q}{1}\int_{r_1}^{r_q}(dr_{n-m+3})M(\rho_{n-m+3})I_1(n-m+3,\ n),$$

where we have used the relation*)

$$\sum_{r=p}^{N+p}\binom{N+p+q-r}{q}=\binom{N+q+1}{q+1}. \tag{C·28}$$

By repeating p times of the similar procedures, we get the general expression

$$J_{l,n}(E)=\sum_{m=0}^{p}\binom{n-1}{m}E_{l,n}(E)$$
$$+\sum_{m=p+1}^{n-1}\int_0^\infty(dr_1)J(\rho_1)I^\infty(1,\ n-m+p) \tag{C·29}$$
$$\times\sum_{q=2}^{n-m+1}\binom{n-m+p-q}{p-1}\int_{r_1}^{r_q}(dr_{n-m+p+1})M(\rho_{n-m+p+1})I_1(n-m+p+1,\ n).$$

It is not difficult to see that we have

$$\sum_{m=0}^{n-1}\binom{n-1}{m}=2^{n-1}, \tag{C·30}$$

and we arrive at Eq. (5·34).

*) Equations (C·24) and (C·28) can be proved by using the following relations:

$$\binom{p}{q}=\binom{p+1}{q+1}-\binom{p}{q+1} \tag{C·25}$$

and

$$\binom{q-1}{q}=0. \tag{C·26}$$

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