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An implicit algorithm for two finite families of nonexpansive maps in hyperbolic spaces

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Abstract

In this article, we propose and analyze an implicit algorithm for two finite families of nonexpansive maps in hyperbolic spaces. Results concerning Δ-convergence as well as strong convergence of the proposed algorithm are proved. Our results are refinement and generalization of several recent results in CAT(0) spaces and uniformly convex Banach spaces.

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1. Introduction

Most of the problems in various disciplines of science are nonlinear in nature. Therefore, translating linear version of a known problem into its equivalent nonlinear version is of paramount interest. Furthermore, investigation of numerous problems in spaces without linear structure has its own importance in pure and applied sciences. Several attempts have been made to introduce a convex structure on a metric space. One such convex structure is available in a hyperbolic space. Throughout the article, we work in the setting of hyperbolic spaces introduced by Kohlenbach [1], which is restrictive than the hyperbolic type introduced in [2] and more general than the concept of hyperbolic space in [3]. Spaces like CAT(0) and Banach are special cases of hyperbolic space. The class of hyperbolic spaces also contains Hadamard manifolds, Hilbert ball equipped with the hyperbolic metric [4], $\mathbb{R}$-trees and Cartesian products of Hilbert balls, as special cases.

Recent developments in fixed point theory reflect that the iterative construction of fixed points is vigorously proposed and analyzed for various classes of maps in different spaces. Implicit algorithms provide better approximation of fixed points than explicit algorithms. The number of steps of an algorithm also plays an important role in iterative approximation methods. The case of two maps has a direct link with the minimization problem [5].

The pioneering work of Xu and Ori [6] deals with weak convergence of one-step implicit algorithm for a finite family of nonexpansive maps. They also posed an open question about necessary and sufficient conditions required for strong convergence of the algorithm. Since then many articles have been published on weak and strong convergence of implicit algorithms (see [7-10] and references therein).

It is worth mentioning that introducing and analyzing a general iterative algorithm in a more general setup is a problem of interest in theoretical numerical analysis. Very recently, Khan et al. [11] proposed and analyzed a general algorithm for strong convergence results in $CAT(0)$ spaces. We do not know whether their work can be extended to hyperbolic spaces. The purpose of this article is to investigate Δ- convergence as well as strong convergence through a two-step implicit algorithm for two finite families of nonexpansive maps in the more general setup of hyperbolic spaces. Our results can be viewed as refinement and generalization of several well-known results in $CAT(0)$ spaces and uniformly convex Banach spaces.

2. Preliminaries and lemmas

Let (X, d) be a metric space and K be a nonempty subset of X. Let T be a selfmap on K. Denote by $F(T) = \{x \in K : T(x) = x\}$, the set of fixed points of T. A selfmap T on K is said to be nonexpansive if $d(Tx, Ty) \leq d(x, y)$. Takahashi [12] introduced a convex structure on a metric space to obtain nonlinear version of some known fixed point results on Banach spaces.

We now describe an other convex structure on a metric space.

A hyperbolic space [1] is a metric space (X, d) together with a map $W : X^2 \times [0, 1] \to X$ satisfying:

(1) $d(u, W(x, y, \alpha)) \leq (1-\alpha)d(u, x) + \alpha d(u, y)$

(2) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$

(3) $W(x, y, \alpha) = W(y, x, (1-\alpha))$

(4) $d(W(x, z, \alpha), W(y, w, \alpha)) \leq (1-\alpha)d(x, y) + \alpha d(z, w)$

for all $x, y, z, w \in X$ and $\alpha, \beta \in [0, 1]$. We denote the above defined hyperbolic space by (X, d, W); if it satisfies only (1), then it coincides with the convex metric space introduced by Takahashi [12]. A subset K of a hyperbolic space X is convex if $W(x, y, \alpha) \in K$ for all $x, y \in K$ and $\alpha \in [0, 1]$.

A hyperbolic space (X, d, W) is said to be:

(i) strictly convex [12] if for any $x, y \in X$ and $\lambda \in [0, 1]$, there exists a unique element $z \in X$ such that

$$d(z, x) = \lambda d(x, y) \quad \text{and} \quad d(z, y) = (1-\lambda)d(x, y);$$

(ii) uniformly convex [13] if for all $u, x, y \in X$, $r > 0$ and $\varepsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that

$$\left.\begin{array}{l} d(x,u) \leq r \\ d(y,u) \leq r \\ d(x,y) \geq \varepsilon r \end{array}\right\} \Rightarrow d\left(W(x, y, \frac{1}{2}), u\right) \leq (1-\delta)r.$$

A map $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ which provides such a $\delta = \eta(r, \varepsilon)$ for given $r > 0$ and $\varepsilon \in (0, 2]$, is called modulus of uniform convexity. We call η monotone if it decreases with r (for a fixed ε). A uniformly convex hyperbolic space is strictly convex (see [14]).

Lemma 2.1. [15]*Let (X, d, W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. For $r > 0$, $\varepsilon \in (0, 2]$, $a, x, y \in X$, the inequalities*

$$d(x,\ a) \leq r, d(y,\ a) \leq r \text{ and } d(x,\ y) \geq \varepsilon r$$

imply

$$d(W(x,\ y,\ \lambda),\ a) \leq (1 - 2\lambda(1-\lambda)\eta(s,\ \varepsilon))r,$$

where $\lambda \in [0, 1]$ *and* $s \geq r$.

The concept of Δ-convergence in a metric space was introduced by Lim [16] and its analogue in CAT(0) spaces has been investigated by Dhompongsa and Panyanak [17]. In this article, we continue the investigation of Δ-convergence in the general setup of hyperbolic spaces.

For this, we collect some basic concepts.

Let $\{x_n\}$ be a bounded sequence in a hyperbolic space X. For $x \in X$, define a continuous functional $r(., \{x_n\}): X \to [0, \infty)$ by:

$$r(x,\ \{x_n\}) = \limsup_{n\to\infty} d(x,\ x_n).$$

The asymptotic radius $\rho = r(\{x_n\})$ of $\{x_n\}$ is given by:

$$\rho = \inf\{r(x,\ \{x_n\}) : x \in X\}.$$

The asymptotic center of a bounded sequence $\{x_n\}$ with respect to a subset K of X is defined as follows:

$$A_K(\{x_n\}) = \{x \in X : r(x, \{x_n\}) \leq r(y, \{x_n\}) \text{ for any } y \in K\}.$$

If the asymptotic center is taken with respect to X, then it is simply denoted by $A(\{x_n\})$. It is known that uniformly convex Banach spaces and even CAT(0) spaces enjoy the property that "bounded sequences have unique asymptotic centers with respect to closed convex subsets". The following lemma is due to Leustean [15] and ensures that this property also holds in a complete uniformly convex hyperbolic space.

Lemma 2.2. [15]*Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity. Then every bounded sequence* $\{x_n\}$ *in* × *has a unique asymptotic center with respect to any nonempty closed convex subset K of X.*
Recall that a sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write Δ - $\lim_n x_n = x$ and call x as Δ - limit of $\{x_n\}$.

Iterative construction by means of classical algorithms like:

(i) $x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n \geq 0$, [18],
(ii) $x_{n+1} = \alpha_n x_n + (1-\alpha_n)T(\beta_n x_n + (1-\beta_n)Tx_n,\ ), \quad n \geq 0$, [19],

are vigorously analyzed for approximation of fixed points of various maps under suitable conditions imposed on the control sequences. The algorithm (i) exhibits weak convergence even in the setting of Hilbert space. Moreover, Chidume and Mutangadura [20] constructed an example for Lipschitz pseudocontractive map with a unique fixed point for which the algorithm (i) fails to converge.

Kirk [21] proved a fixed point theorem using Browder's type implicit algorithm (i.e., $x_t = (1 - t)x + tT(x_t)$) in a complete CAT (0) space. More precisely, he proved the following result:

Theorem 2.3. [21] Let K be a bounded closed convex subset of a complete CAT(0) space X and $f : K \to K$ be a nonexpansive map. Fix $x \in K$ and for each $t \in [0, 1)$ let x_t be the unique fixed point such that

$$x_t \in [x, Tx_t] \quad \text{and} \quad d(x, x_t) = td(x, Tx_t).$$

Then $\{x_t\}$ converges as $t \to 1^-$ to the unique fixed point of f which is nearest to x.

Furthermore, he posed an open question: whether Theorem 2.3 can be extended to spaces of nonpositive curvature.

Denote the set $\{1, 2, 3, \ldots N\}$ by I.

In 2001, Xu and Ori [6] obtained weak convergence result using an implicit algorithm for a finite family of nonexpansive maps as follows:

Theorem 2.4. [6]*Let* $\{T_i : i \in I\}$ *be a family of nonexpansive selfmaps on a closed convex subset* C *of a Hilbert space with* $\cap_{i=1}^{N} F(T_i) \neq \emptyset$, *let* $x_0 \in C$ *and let* $\{\alpha_n\}$ *be a sequence in* $(0, 1)$ *such that* $\lim_{n\to\infty} a_n = 0$. *Then the sequence* $x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n$, *where* $n \geq 1$ *and* $T_n = T_{n(\text{mod } N)}$ (here the mod N function takes values in I), *converges weakly to a point in* F.

In 2007, Plubtieng et al. [9] generalized the algorithm of Xu and Ori [6] for two finite families $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ of nonexpansive maps and studied its weak and strong convergence. Given x_0 in K(a subset of Banach space), their algorithm reads as follows:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n [\beta_n x_n + (1 - \beta_n) S_n x_n] \tag{2.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in (0, 1).

Inspired and motivated by the work of Kirk [21], Xu and Ori [6] and Plubtieng et al. [9], we investigate Δ-convergence as well as strong convergence through a two-step implicit algorithm for two finite families of nonexpansive maps in the more general setup of hyperbolic spaces.

The two-step algorithm (2.1) can be defined in a hyperbolic space as:

$$\begin{aligned} x_n &= W(x_{n-1}, T_n y_n, \alpha_n), \\ y_n &= W(x_n, S_n x_n, \beta_n), \quad n \geq 1 \end{aligned} \tag{2.2}$$

where $T_n = T_{n(\text{mod } N)}$ and $S_n = S_{n(\text{mod } N)}$.

In order to establish that algorithm (2.2) exists, we define a map $G_1 : K \to K$ by: $G_1 x = W(x_0, T_1 W(x, S_1 x, \beta_i), \alpha_i)$. For a given $x_0 \in K$, the existence of $x_1 = W(x_0, T_1 W(x_1, S_1 x_1, \beta_1), \alpha_1)$ is guaranteed if G_1 has a fixed point. Now for any $u, v \in K$ and making use of (4), we have

$$\begin{aligned} d(G_1 u, G_1 v) &= d(W(x_0, T_1 W(u, S_1 u, \beta_1), \alpha_1), W(x_0, T_1 W(v, S_1 v, \beta_1), \alpha_1) \\ &\leq \alpha_1 d(T_1 W(u, S_1 u, \beta_1), T_1 W(v, S_1 v, \beta_1)) \\ &\leq \alpha_1 d(W(u, S_1 u, \beta_1), W(v, S_1 v, \beta_1)) \\ &\leq \alpha_1 [(1 - \beta_1) d(u, v) + \beta_1 d(S_1 u, S_1 v)] \\ &\leq \alpha_1 [(1 - \beta_1) d(u, v) + \beta_1 d(u, v)] \\ &\leq \alpha_1 d(u, v). \end{aligned}$$

Since $a_1 \in (0, 1)$, therefore G_1 is a contraction. By Banach contraction principle, G_1 has a unique fixed point. Thus the existence of x_1 is established. Continuing in this way, we can establish the existence of x_2, x_3 and so on. Thus the implicit algorithm (2.2) is well defined.

In 2010, Laowang and Panyanak [22] obtained a generalized version of Lemma 1.3 of Schu [23] in a uniformly convex hyperbolic space where the proof relies on the fact that modulus of uniform convexity η increases with r (for a fixed ε).

We prove the generalized version of Lemma 1.3 of Schu [23] in a uniformly convex hyperbolic space with monotone modulus of uniform convexity.

Lemma 2.5. *Let (X, d, W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{a_n\}$ be a sequence in $[b, c]$ for some $b, c \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n, x) \leq r$, $\limsup_{n\to\infty} d(y_n, x) \leq r$ and $\lim_{n\to\infty} d(W(x_n, y_n, \alpha_n), x) = r$ for some $r \geq 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

Proof. The case $r = 0$ is trivial. Suppose $r > 0$ and assume $\lim_{n\to\infty} d(x_n, y_n) \neq 0$. If $n_1 \in \mathbb{N}$, then $d(x_n, y_n) \geq \frac{\lambda}{2} > 0$ for some $\lambda > 0$ and for $n \geq n_1$. Since $\limsup_{n\to\infty} d(x_n, x) \leq r$ and $\limsup_{n\to\infty} d(y_n, x) \leq r$, we have:

(i) $d(x_n, x) \leq r + \frac{1}{n}$;

(ii) $d(y_n, x) \leq r + \frac{1}{n}$ for each $n \geq 1$.

Moreover, $d(x_n, y_n) \geq \frac{\lambda}{2} \geq \left(r + \frac{1}{n}\right) \frac{\lambda}{2(r+1)}$, where $\frac{\lambda}{2(r+1)} \leq 1$. So it follows from Lemma 2.1, that

$$\begin{aligned} d(W(x_n, y_n, \alpha_n), x) &\leq \left(1 - 2\alpha_n(1-\alpha_n)\eta\left(r + \frac{1}{n}, \frac{\lambda}{2(r+1)}\right)\right)\left(r + \frac{1}{n}\right) \\ &\leq \left(1 - 2\alpha_n(1-\alpha_n)\eta\left(r + 1, \frac{\lambda}{2(r+1)}\right)\right)\left(r + \frac{1}{n}\right) \\ &\leq \left(1 - 2b(1-c)\eta\left(r + 1, \frac{\lambda}{2(r+1)}\right)\right)\left(r + \frac{1}{n}\right). \end{aligned}$$

Thus, by letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} d(W(x_n, y_n, \alpha_n), x) \leq \left(1 - 2b(1-c)\eta\left(r + 1, \frac{\lambda}{2(r+1)}\right)\right) r < r,$$

a contradiction to the fact that $\lim_{n\to\infty} d(W(x_n, y_n, \alpha_n), x) = r$ for some $r \geq 0$. □

We now prove a metric version of a result due to Bose and Laskar [24] which plays a crucial role in proving Δ-convergence of the algorithm (2.2).

Lemma 2.6. *Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space and $\{x_n\}$ a bounded sequence in K such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in K such that*

$\lim_{m\to\infty} r(y_m, \{x_n\}) = \rho$, then $\lim_{m\to\infty} y_m = y$.

Proof. If $y_m \nrightarrow y$, then there exist a subsequence $\{y_{m_j}\}$ of $\{y_m\}$ and $M > 0$ such that

$$d(y_{m_j}, y) \geq \frac{M}{2} \quad \text{for all } j.$$

Observe that the inequality:

$$(\rho + \varepsilon)\left(1 - \eta\left(\rho + 1, \frac{M}{2(\rho+1)}\right)\right) < \rho \tag{2.3}$$

holds when $\varepsilon \to 0$, where $\varepsilon \in (0, 1]$ and ρ is the asymptotic radius of $\{x_n\}$.

Since $A(\{x_n\}) = \{y\}$, so for every $\varepsilon \in (0, 1]$ there exists an integer N_1 such that $d(y, x_n) \leq \rho+\varepsilon$, for all $n \geq N_1$. Since $\lim_{m\to\infty} r(y_m, \{x_n\}) = \rho = \lim_{j\to\infty} r(y_{m_j}, \{x_n\})$, so there exists an integer j^* such that $r(y_{m_j}, \{x_n\}) \leq \rho + \frac{\varepsilon}{2}$ for all $j \geq j^*$. Hence there exists an integer N_2 such that $d(y_{m_j}, x_n) \leq \rho + \varepsilon$ for all $n \geq N_2$.

That is,

$$d(y, x_n) \leq \rho + \varepsilon \leq \rho + 1 \quad \text{and} \quad d(y_{m_j}, x_n) \leq \rho + \varepsilon \leq \rho + 1,$$

for all $n \geq N = \max\{N_1, N_2\}$.

Using Lemma 2.1, we have

$$\begin{aligned} d\left(W\left(y, y_{m_j}, \frac{1}{2}\right), x_n\right) &\leq \left(1 - \eta\left(\rho + \varepsilon, \frac{M}{2(\rho+1)}\right)\right)(\rho + \varepsilon) \\ &\leq \left(1 - \eta\left(\rho + 1, \frac{M}{2(\rho+1)}\right)\right)(\rho + \varepsilon), \end{aligned}$$

so that letting $n \to \infty$, we have

$$r\left(W(y, y_{m_j}, \frac{1}{2}), \{x_n\}\right) \leq \left(1 - \eta\left(\rho + 1, \frac{M}{2(\rho+1)}\right)\right)(\rho + \varepsilon).$$

Now let $\varepsilon \to 0$ and use (2.3) to conclude that $r(W(y, y_{m_j}, \frac{1}{2}), \{x_n\}) < \rho$ which contradicts the fact that ρ is the asymptotic radius of $\{x_n\}$. Hence $\lim_{m\to\infty} y_m = y$. □

From now on for two finite families $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ of maps, we set $F = \cap_{i=1}^{N}(F(T_i) \cap F(S_i)) \neq \phi$

Lemma 2.7. *Let K be a nonempty closed convex subset of a hyperbolic space × and let $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ be two finite families of nonexpansive selfmaps on K such that $F \neq \varphi$. Then for the sequence $\{x_n\}$ defined implicitly in* (2.2), *we have* $\lim_{n\to\infty} d(x_n, p)$ *exists for each* $p \in F$.

Proof. For any $p \in F$, it follows from (2.2) that

$$\begin{aligned} d(x_n, p) &= d(W(x_{n-1}, T_n y_n, \alpha_n), p) \\ &\leq (1-\alpha_n)d(x_{n-1}, p) + \alpha_n d(T_n y_n, p) \\ &\leq (1-\alpha_n)d(x_{n-1}, p) + \alpha_n d(y_n, p) \\ &= (1-\alpha_n)d(x_{n-1}, p) + \alpha_n d(W(x_n, S_n x_n, \beta_n), p) \\ &\leq (1-\alpha_n)d(x_{n-1}, p) + \alpha_n(1-\beta_n)d(x_n, p) + \alpha_n\beta_n d(S_n x_n, p) \\ &\leq (1-\alpha_n)d(x_{n-1}, p) + \alpha_n(1-\beta_n)d(x_n, p) + \alpha_n\beta_n d(x_n, p) \\ &\leq (1-\alpha_n)d(x_{n-1}, p) + \alpha_n d(x_n, p). \end{aligned}$$

That is

$$d(x_n, p) \leq d(x_{n-1}, p). \tag{2.4}$$

It follows from (2.4) that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$. Consequently, $\lim_{n\to\infty} d(x_n, F)$

exists. □

Lemma 2.8. *Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space × with monotone modulus of uniform convexity η and let $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ be two finite families of nonexpansive selfmaps of K such that $F \neq \varphi$. Then for the sequence $\{x_n\}$ defined implicitly in* (2.2), *we have*

$$\lim_{n\to\infty} d(x_n,\ T_l x_n) = \lim_{n\to\infty} d(x_n,\ S_l x_n) = 0 \quad \text{for each } l = 1, 2, \cdots, N.$$

Proof. It follows from Lemma 2.7 that, $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$. Assume that $\lim_{n\to\infty} d(x_n, p) = c$. The case $c = 0$ is trivial. Next, we deal with the case $c > 0$. Note that

$$\begin{aligned} d(y_n, p) &= d(W(x_n,\ S_n x_n,\ \beta_n), p) \\ &\le (1 - \beta_n) d(x_n, p) + \beta_n d(S_n x_n, p) \\ &\le d(x_n, p). \end{aligned}$$

Taking lim sup on both sides in the above estimate, we have

$$\limsup_{n\to\infty} d(y_n, p) \le c.$$

Since T_n is nonexpansive, so $\limsup_{n\to\infty} d(T_n y_n, p) \le c$. Further, $\limsup_{n\to\infty} d(x_{n-1}, p) \le c$.

Moreover,

$$\lim_{n\to\infty} d(x_n, p) = \lim_{n\to\infty} d(W(x_{n-1},\ T_n y_n,\ \alpha_n), p) = c.$$

So, by Lemma 2.5, we have

$$\lim_{n\to\infty} d(x_{n-1},\ T_n y_n) = 0. \tag{2.5}$$

Next, taking lim sup on both sides in the inequality

$$\begin{aligned} d(x_n,\ x_{n-1}) &= d(W(x_{n-1},\ T_n y_n,\ \alpha_n),\ x_{n-1}) \\ &\le \alpha_n d(T_n y_n,\ x_{n-1}), \end{aligned}$$

we have

$$\limsup_{n\to\infty} d(x_n,\ x_{n-1}) \le 0.$$

Hence,

$$\limsup_{n\to\infty} d(x_n,\ x_{n-1}) = 0. \tag{2.6}$$

Clearly,

$$d(x_n,\ x_{n+l}) \le d(x_n,\ x_{n+1}) + d(x_{n+1},\ x_{n+2}) + \cdots + d(x_{n+l-1},\ x_{n+l}).$$

Taking lim sup on both sides of the above inequality and using (2.6), we have

$$\lim_{n\to\infty} d(x_n,\ x_{n+l}) = 0 \quad \text{for } l < N.$$

Further, observe that

$$\begin{aligned} d(x_n, p) &\le (1 - \alpha_n) d(x_{n-1}, p) + \alpha_n d(T_n y_n, p) \\ &\le (1 - \alpha_n) d(x_{n-1},\ T_n y_n) + (1 - \alpha_n) d(T_n y_n, p) + \alpha_n d(y_n, p) \\ &\le (1 - \alpha_n) d(x_{n-1},\ T_n y_n) + (1 - \alpha_n) d(y_n, p) + \alpha_n d(y_n, p) \\ &\le (1 - \alpha_n) d(x_{n-1},\ T_n y_n) + d(y_n, p). \end{aligned}$$

Combining the inequalities after applying lim inf and lim sup on both sides in the above estimate and using (2.5), we get

$$c \leq \liminf_{n\to\infty} d(y_n, p) \leq \limsup_{n\to\infty} d(y_n, p) \leq c.$$

That is,

$$\lim_{n\to\infty} d(W(x_n,\ S_n x_n,\ \beta_n), p) = \lim_{n\to\infty} d(y_n, p) = c.$$

Finally, by Lemma 2.5, we have

$$\lim_{n\to\infty} d(x_n,\ S_n x_n) = 0.$$

Moreover,

$$\begin{aligned} d(x_n,\ T_n x_n) &\leq d(x_n,\ T_n y_n) + d(T_n y_n,\ T_n x_{n-1}) + d(T_n x_{n-1},\ T_n x_n) \\ &\leq\ (1-\alpha_n) d(x_{n-1},\ T_n y_n) + d(x_{n-1},\ y_n) + d(x_{n-1},\ x_n) \\ &\leq\ (1-\alpha_n) d(x_{n-1},\ T_n y_n) + \beta_n d(x_n,\ S_n x_n) + 2d(x_{n-1},\ x_n) \end{aligned}$$

gives that

$$\lim_{n\to\infty} d(x_n,\ T_n x_n) = 0.$$

For each $l \in I$, we have

$$\begin{aligned} d(x_n,\ T_{n+l} x_n) &\leq d(x_n,\ x_{n+l}) + d(x_{n+l},\ T_{n+l} x_{n+l}) + d(T_{n+l} x_{n+l},\ T_{n+l} x_n) \\ &\leq 2d(x_n,\ x_{n+l}) + d(x_{n+l},\ T_{n+l} x_{n+l}). \end{aligned}$$

Therefore

$$\lim_{n\to\infty} d(x_n,\ T_{n+l} x_n) = 0 \quad \text{for each } l \in I.$$

Since for each $l \in I$, the sequence $\{d(x_n,\ T_l x_n)\}$ is a subsequence of $\cup_{i=1}^{N} \{d(x_n,\ T_{n+l} x_n)\}$ and $\lim_{n\to\infty} d(x_n,\ T_{n+l} x_n) = 0$ for each $l \in I$, therefore

$$\lim_{n\to\infty} d(x_n,\ T_l x_n) = 0 \quad \text{for each } l \in I.$$

Similarly, we have

$$\lim_{n\to\infty} d(x_n,\ S_{n+l} x_n) = 0 \quad \text{for each } l \in I,$$

and hence
$\lim_{n\to\infty} d(x_n,\ S_l x_n) = 0 \quad \text{for each } l \in I.$ □

3. Convergence in hyperbolic spaces

In this section, we establish Δ- convergence and strong convergence of the implicit algorithm (2.2).

Theorem 3.1. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space × with monotone modulus of uniform convexity η and let $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ be two finite families of nonexpansive selfmaps on K such that $F \neq \varphi$.*

Then the sequence $\{x_n\}$ defined implicitly in (2.2), *Δ-converges to a common fixed point of $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$.*

Proof. It follows from Lemma 2.7 that $\{x_n\}$ is bounded. Therefore by Lemma 2.2, $\{x_n\}$ has a unique asymptotic center, that is, $A(\{x_n\}) = \{x\}$. Let $\{u_n\}$ be any subsequence of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. Then by Lemma 2.8, we have $\lim_{n\to\infty} d(u_n,\ T_l u_n) = 0 =$

$\lim_{n\to\infty} d(u_n, S_l u_n)$ for each $l \in I$. We claim that u is the common fixed point of $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$.

Now, we define a sequence $\{z_m\}$ in K by $z_m = T_m u$ where $T_m = T_{m(\text{mod } N)}$.

Observe that

$$\begin{aligned} d(z_m, u_n) &\le d(T_m u, T_m u_n) + d(T_m u_n, T_{m-1} u_n) + \ldots + d(T u_n, u_n) \\ &\le d(u, u_n) + \sum_{i=1}^{m-1} d(u_n, T_i u_n). \end{aligned}$$

Therefore, we have

$$r(z_m, \{u_n\}) = \limsup_{n\to\infty} d(z_m, u_n) \le \limsup_{n\to\infty} d(u, u_n) = r(u, \{u_n\}).$$

This implies that $|r(z_m, \{u_n\}) - r(u, \{u_n\})| \to 0$ as $m \to \infty$. It follows from Lemma 2.6 that $T_{m(\text{mod } N)} u = u$. Hence u is the common fixed point of $\{T_i : i \in I\}$. Similarly, we can show that u is the common fixed point of $\{S_i : i \in I\}$. Therefore u is the common fixed point of $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$. Moreover, $\lim_{n\to\infty} d(x_n, u)$ exists by Lemma 2.7.

Suppose $x \ne u$. By the uniqueness of asymptotic centers,

$$\begin{aligned} \limsup_{n\to\infty} d(u_n, u) &< \limsup_{n\to\infty} d(u_n, x) \\ &\le \limsup_{n\to\infty} d(x_n, x) \\ &< \limsup_{n\to\infty} d(x_n, u) \\ &= \limsup_{n\to\infty} d(u_n, u), \end{aligned}$$

a contradiction. Hence $x = u$. Since $\{u_n\}$ is an arbitrary subsequence of $\{x_n\}$, therefore $A(\{u_n\}) = \{u\}$ for all subsequences $\{u_n\}$ of $\{x_n\}$. This proves that $\{x_n\}$ Δ-converges to a common fixed point of $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$. □

Recall that a sequence $\{x_n\}$ in a metric space X is said to be *Fejér monotone* with respect to K (a subset of X) if $d(x_{n+1}, p) \le d(x_n, p)$ for all $p \in K$ and for all $n \ge 1$. A map $T : K \to K$ is *semi-compact* if any bounded sequence $\{x_n\}$ satisfying $d(x_n, Tx_n) \to 0$ as $n \to \infty$, has a convergent subsequence.

Let f be a nondecreasing selfmap on $[0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$ and let $d(x, H) = \inf\{d(x, y): y \in H\}$. Then a family $\{T_i : i \in I\}$ of selfmaps on K with $F_1 = \cap_{i=1}^{N} F(T_i) \ne \phi$, satisfies condition (A) if

$$d(x, Tx) \ge f(d(x, F_1)) \quad \text{for all } x \in K,$$

holds for at least one $T \in \{T_i : i \in I\}$ or

$$\max_{i\in I} d(x, T_i x) \ge f(d(x, F_1)) \quad \text{for all } x \in K,$$

holds.

Different modifications of the condition (A) for two finite families of selfmaps have been made recently in the literature [25], [9] as follows:

Let $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ be two finite families of nonexpansive selfmaps on K with $F \ne \varphi$. Then the two families are said to satisfy:

(i) condition (B) on K if

$$d(x, Tx) \geq f(d(x, F)) \text{ or } d(x, Sx) \geq f(d(x, F)) \quad \text{for all } x \in K,$$

holds for at least one $T \in \{T_i : i \in I\}$ or one $S \in \{S_i : i \in I\}$;

(ii) condition (C) on K if

$$\frac{1}{2}\{d(x, T_i x) + d(x, S_i x)\} \geq f(d(x, F)) \quad \text{for all } x \in K.$$

Note that the condition (B) and the condition (C) are equivalent to the condition (A) if $T_i = S_i$ for all $i \in I$. We shall use condition (B) to study strong convergence of the algorithm (2.2).

For further development, we need the following technical result.

Lemma 3.2. [26]*Let K be a nonempty closed subset of a complete metric space (X, d) and $\{x_n\}$ be Fejér monotone with respect to K. Then $\{x_n\}$ converges to some $p \in K$ if and only if* $\lim_{n\to\infty} d(x_n, K) = 0$.

Lemma 3.3. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space × with monotone modulus of uniform convexity η and let $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ be two finite families of nonexpansive selfmaps on K such that $F \neq \varphi$. Then the sequence $\{x_n\}$ defined implicitly in* (2.2) *converges strongly to $p \in F$ if and only if* $\lim_{n\to\infty} d(x_n, F) = 0$.

Proof. It follows from (2.4) that $\{x_n\}$ is Fejér monotone with respect to F and $\lim_{n\to\infty} d(x_n, F)$ exists. Hence, the result follows from Lemma 3.2. □

We now establish strong convergence of the algorithm (2.2) based on Lemma 3.3.

Theorem 3.4. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space × with monotone modulus of uniform convexity η and let $\{T_i : i \in I\}$ and $\{S_i: i \in I\}$ be two finite families of nonexpansive selfmaps on K such that $F \neq \varphi$. Suppose that a pair of maps T and S in $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$, respectively, satisfies condition (B). Then the sequence $\{x_n\}$ defined implicitly in* (2.2) *converges strongly to $p \in F$.*

Proof. It follows from Lemma 2.7 that $\lim_{n\to\infty} d(x_n, F)$ exists. Moreover, Lemma 2.8 implies that $\lim_{n\to\infty} d(x_n, T_l x_n) = d(x_n, S_l x_n) = 0$ for each $l \in I$. So condition (B) guarantees that $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since f is nondecreasing with $f(0) = 0$, it follows that $\lim_{n\to\infty} d(x_n, F) = 0$. Therefore, Lemma 3.3 implies that $\{x_n\}$ converges strongly to a point p in F. □

Theorem 3.5. *Let K be a nonempty closed convex subset of a complete uniformly convex*

hyperbolic space × with monotone modulus of uniform convexity η and let $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ be two finite families of nonexpansive selfmaps on K such that $F \neq \varphi$. Suppose that one of the map in $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ is semi-compact. Then the sequence $\{x_n\}$ defined implicitly in (2.2) *converges strongly to $p \in F$.*

Proof. Use Lemma 2.8 and the line of action given in the proof of Theorem 3.4 in [9]. □

Remark 3.6. (1) Theorem 3.1 sets analogue of [17, Theorem 3.3], for two finite families

of nonexpansive maps on unbounded domain in a uniformly convex hyperbolic space X;

(2) Lemma 3.3 improves [8, Theorem 1] and [10, Theorem 3.1] for two finite families of nonexpansive maps on X;

(3) Theorem 3.4 extends and improves Theorem 3.3 (Theorem 3.4) of [9] from uniformly convex Banach space setting to the general setup of uniformly convex hyperbolic space;

(4) Theorem 3.5 improves and extends [8, Theorem 2] for two finite families of nonexpansive maps on X.

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The authors have contributed in this work on an equal basis. All authors read and approved the final manuscript.

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References

1. Kohlenbach, U: Some logical metatheorems with applications in functional analysis. Trans Am Math Soc. **357**, 89–128 (2005)
2. Goebel, K, Kirk, WA: Iteration processes for nonexpansive mappings Topological Methods in Nonlinear Functional Analysis. In: Singh SP, Thomeier S, Watson B (eds.) Contemp Math Am Math Soc AMS, Providence, RI. **21**, 115–123 (1983)
3. Reich, S, Shafrir, I: Nonexpansive iterations in hyperbolic spaces. Nonlinear Anal. **15**, 537–558 (1990)
4. Goebel, K, Reich, S: Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings. Marcel Dekker, New York. (1984)
5. Takahashi, W: Iterative methods for approximation of fixed points and thier applications. J Oper Res Soc Jpn. **43**(1):87–108 (2000)
6. Xu, HK, Ori, RG: An implicit iteration process for nonexpansive mappings. Num Funct Anal Optim. **22**(5-6):767–773 (2001)
7. Fukhar-ud-din, H, Khan, AR: Convergence of implicit iterates with errors for mappings with unbounded domain in Banach spaces. Int J Math Math Sci. **10**, 1643–1653 (2005)
8. Liu, JA: Some convergence theorems of implicit iterative process for nonexpansive mappings in Banach spaces. Math Commun. **7**, 113–118 (2002)
9. Plubtieng, S, Ungchittrakool, K, Wangkeeree, R: Implicit iteration of two finite families for nonexpansive mappings in Banach spaces. Numer Funct Anal Optim. **28**(56):737–749 (2007)
10. Sun, ZH, He, C, Ni, YQ: Strong convergence of an implicit iteration process for nonexpansive mappings in Banach space. Nonlinear Funct Anal Appl. **8**(4):595–602 (2003)
11. Khan, AR, Khamsi, MA, Fukhar-ud-din, H: Strong convergence of a general iteration scheme in CAT(0)-spaces. Nonlinear Anal. **74**, 783–791 (2011)
12. Takahashi, W: A convexity in metric spaces and nonexpansive mappings. Kodai Math Sem Rep. **22**, 142–149 (1970)
13. Shimizu, T, Takahashi, W: Fixed points of multivalued mappings in certain convex metric spaces. Topol Methods Nonlinear Anal. **8**, 197–203 (1996)
14. Leustean, L: A quadratic rate of asymptotic regularity for CAT(0)-spaces. J Math Anal Appl. **325**, 386–399 (2007)
15. Leustean, L: Nonexpansive iterations in uniformly convex *W*-hyperbolic spaces. In: Leizarowitz A, Mordukhovich BS, Shafrir I, Zaslavski A (eds.) Contemp Math Am Math Soc AMS **513**, 193–209 (2010). Nonlinear Analysis and Optimization I: Nonlinear Analysis
16. Lim, TC: Remarks on some FIxed point theorems. Proc Am Math Soc. **60**, 179–182 (1976)
17. Dhompongsa, S, Panyanak, B: On Δ-convergence theorems in CAT(0)-spaces. Comp Math Appl. **56**(10):2572–2579 (2008)
18. Mann, WR: Mean value methods in iteration. Proc Am Math Soc. **4**, 506–510 (1953)
19. Ishikawa, S: Fixed points by a new iteration method. Proc Am Math Soc. **44**, 147–150 (1974)
20. Chidume, CE, Mutangadura, SA: An example on the Mann iteration method for Lipschits pseudo-contarctions. Proc Am Math Soc. **129**, 2359–2363 (2001)
21. Kirk, WA: Geodesic geometry and FIxed point theory Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003). pp. 195–225. Univ Sevilla Secr Publ, Seville (2003)
22. Laowang, W, Panyanak, B: Approximating fixed points of nonexpansive nonself mappings in CAT(0) spaces. Fixed Point Theory Appl **2010** (2010). 367274, 11

23. Schu, J: Weak and strong convergence to fixed points of asymptotically nonexpansive mappings. Bull Aust Math Soc. **43**, 153–159 (1991)
24. Bose, SC, Laskar, SK: Fixed point theorems for certain class of mappings. J Math Phys Sci. **19**, 503–509 (1985)
25. Khan, SH, Fukhar-ud-din, H: Weak and strong convergence of a scheme for two nonexpansive mappings. Nonlinear Anal. **8**, 1295–1301 (2005)
26. Bauschke, HH, Combettes, PL: Convex Analysis and Monotone Operator Theory in Hilbert Spaces. Springer-Verlag, New York (2011)

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