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An improved acoustic Fourier boundary element method formulation using fast Fourier transform integration

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Effective use of the Fourier series boundary element method (FBEM) for everyday applications is hindered by the significant numerical problems that have to be overcome for its implementation. In the FBEM formulation for acoustics, some integrals over the angle of revolution arise, which need to be calculated for every Fourier term. These integrals were formerly treated for each Fourier term separately. In this paper a new method is proposed to calculate these integrals using fast Fourier transform techniques. The advantage of this integration method is that the integrals are simultaneously computed for all Fourier terms in the boundary element formulation. The improved efficiency of the method compared to a Gaussian quadrature based integration algorithm is illustrated by some example calculations. The proposed method is not only usable for acoustic problems in particular, but for Fourier BEM in general. © 1997 Acoustical Society of America. [S0001-4966(97)04908-4]

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INTRODUCTION

The use of the boundary element method (BEM) in acoustics, tailored to problems involving axisymmetric bodies with arbitrary boundary conditions, has received some attention in the literature the past decade. Conceptually, the acoustic BEM for axisymmetric bodies applies a Fourier series expansion of the angular dependency of the acoustic variables in the problem. As a result, the surface integral in the Kirchhoff–Helmholtz integral equation reduces to a line integral and an integral over the angle of revolution (circumferential integral). The advantages of this so-called Fourier BEM approach are evident. Discretization of the body requires only meshing of the generator of the body with line elements. Also, the cost of numerically solving the system of equations is reduced because of a substantial decrease in the number of unknowns. The drawback of this method is the increased complexity of both the mathematical formulation and numerical implementation of the method.

The computation of the circumferential integrals (around the symmetry axis of the body) causes considerable numerical problems for the implementation of the Fourier boundary element method. The integrand of these integrals can be singular and has an oscillatory nature. Numerical values for these integrals were obtained using Trapezium quadrature by Akyol.¹ This method provided accurate results, but it was pointed out that the efficiency of the integral computation needed further investigation. A different method for the computation of the integrals was proposed by Soenarko² and Juhl,³ who reformulated the integrand and employed an (unspecified) series of elliptic integrals for the singular part of the integral and Gaussian quadrature for the regular part.

The evaluation of the circumferential integrals is time consuming and has to be done for each Fourier mode number

that is present in the boundary conditions for the acoustic problem. This hampers the effective application of the Fourier boundary element method for practical acoustic applications. A faster evaluation method proposed in this article is based on fast Fourier transform (FFT). This method handles the calculation of the integrals both efficiently and accurately for all Fourier harmonics simultaneously. It was implemented and tested and it compared favorably to an integration method based on Gaussian quadrature.

I. FOURIER BOUNDARY INTEGRAL EQUATIONS FOR ACOUSTICS

The Kirchhoff–Helmholtz integral equation is a mathematical description for the acoustic radiation of structures. Consider a simple axisymmetric body B (Fig. 1). The Kirchhoff–Helmholtz integral equation for the pressure $p(\mathbf{x})$ at an observer point \mathbf{x} can be written as⁴

$$C(\mathbf{x}) \cdot p(\mathbf{x}) = \int_S \left[p(\mathbf{y}) \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \nu} - G(\mathbf{x}, \mathbf{y}) \frac{\partial p(\mathbf{y})}{\partial \nu} \right] dS(\mathbf{y}), \quad (1)$$

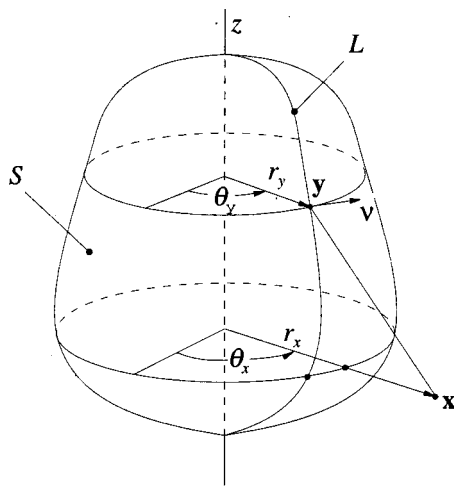
with

$$G(\mathbf{x}, \mathbf{y}) = \frac{e^{-ikR(\mathbf{x}, \mathbf{y})}}{4\pi R(\mathbf{x}, \mathbf{y})} \quad (2)$$

as the three-dimensional free-space Green's function, and $C(\mathbf{x})$ as a coefficient depending on the position of \mathbf{x} :

$$C(\mathbf{x}) = \begin{cases} 0, & \text{for } \mathbf{x} \text{ outside the acoustic medium,} \\ 1, & \text{for } \mathbf{x} \text{ inside the acoustic medium,} \\ \frac{1}{2}, & \text{for } \mathbf{x} \text{ on the smooth surface } S \text{ of the} \\ & \text{acoustic medium.} \end{cases} \quad (3)$$

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S: surface
L: body generator
x: observer point with cylindrical coordinates (r_x, θ_x, z_x)
y: surface point with cylindrical coordinates (r_y, θ_y, z_y)
 ν : surface normal

FIG. 1. Simple axisymmetric body B .

For any other position of \mathbf{x} on S for which there is no unique surface normal, for example when \mathbf{x} is on an edge or a corner, the value of $C(\mathbf{x})$ is given by

$$C(\mathbf{x}) = \begin{cases} 1 + \frac{1}{4\pi} \int_S \frac{\partial}{\partial \nu} \left(\frac{1}{R(\mathbf{x}, \mathbf{y})} \right) dS(\mathbf{y}) \\ \text{for the exterior acoustic problem,} \\ -\frac{1}{4\pi} \int_S \frac{\partial}{\partial \nu} \left(\frac{1}{R(\mathbf{x}, \mathbf{y})} \right) dS(\mathbf{y}) \\ \text{for the interior acoustic problem.} \end{cases} \quad (4)$$

The distance R between the points \mathbf{x} and \mathbf{y} is defined as $R(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$. The surface normal direction ν is directed inward the acoustic medium (see Fig. 1).

The geometry of an axisymmetric body with arbitrary boundary conditions can be described using a cylindrical coordinate system (r, θ, z) . All variables then become functions of the cylindrical coordinates r , θ , and z , i.e.,

$$p(\mathbf{x}) = p(r_x, \theta_x, z_x),$$

$$p(\mathbf{y}) = p(r_y, \theta_y, z_y),$$

$$G(\mathbf{x}, \mathbf{y}) = \frac{e^{-ikR(\mathbf{x}, \mathbf{y})}}{4\pi R(\mathbf{x}, \mathbf{y})} = G(r_x, \theta_x, z_x; r_y, \theta_y, z_y),$$

$$dS(\mathbf{y}) = r_y d\theta_y dL_y,$$

with (r_x, θ_x, z_x) and (r_y, θ_y, z_y) as the coordinates of the observer point \mathbf{x} and the surface point \mathbf{y} , respectively, and dL_y as the differential length of the generator L of the body at \mathbf{y} . Because of the axisymmetric properties of the body B the variables can be expanded in Fourier series. This expansion was reported by Soenarko² as follows:

$$p(\mathbf{y}) = \sum_{n=0}^{\infty} [p_n^s \sin(n\theta_y) + p_n^c \cos(n\theta_y)], \quad (5)$$

$$p(\mathbf{x}) = \sum_{n=0}^{\infty} [p_n^{s*} \sin(n\theta_x) + p_n^{c*} \cos(n\theta_x)], \quad (6)$$

with superscript $*$ to discern the Fourier coefficients for the surface point \mathbf{y} and observer point \mathbf{x} . Note that the Fourier coefficients still depend on the coordinates r and z , but the dependence of pressure p on coordinate θ is expressed through the sine and cosine terms of the Fourier expansion. The other functions of Eq. (1) can be expanded likewise:

$$G(\mathbf{x}, \mathbf{y}) = \frac{e^{-ikR(\mathbf{x}, \mathbf{y})}}{4\pi R(\mathbf{x}, \mathbf{y})} = \sum_{n=0}^{\infty} [K_n^s \sin(n\theta_y) + K_n^c \cos(n\theta_y)], \quad (7)$$

$$\frac{\partial p}{\partial \nu}(\mathbf{y}) \equiv p'(\mathbf{y}) = \sum_{n=0}^{\infty} [p_n'^s \sin(n\theta_y) + p_n'^c \cos(n\theta_y)], \quad (8)$$

$$\frac{\partial G}{\partial \nu}(\mathbf{x}, \mathbf{y}) \equiv G'(\mathbf{x}, \mathbf{y}) = \sum_{n=0}^{\infty} [K_n'^s \sin(n\theta_y) + K_n'^c \cos(n\theta_y)]. \quad (9)$$

The Fourier coefficients in these equations are independent of θ_y but still dependent on the r_y and z_y . Observe that the Fourier coefficients of the expansions of the Green's function and its normal derivative (i.e., K_n^s , K_n^c , $K_n'^s$, and $K_n'^c$) are also dependent on all the cylindrical coordinates of point \mathbf{x} : r_x , θ_x , and z_x .

With the Fourier series description for the circumferential dependence of the acoustic variables, a modified form of the Kirchhoff-Helmholtz integral equation can be obtained. To that end, it is convenient to reformulate the Fourier coefficients of the Green's function and its derivative. The coefficients are determined by the standard Fourier transformation rules, for instance,

$$K_n^s = \frac{1}{\pi} \int_0^{2\pi} \frac{e^{-ikR(\mathbf{x}, \mathbf{y})}}{4\pi R(\mathbf{x}, \mathbf{y})} \sin(n\theta_y) d\theta_y, \quad n=0, 1, 2, \dots \quad (10)$$

By defining $\theta \equiv \theta_y - \theta_x$ such that $d\theta = d\theta_y$ Eq. (10) can be written as

$$K_n^s = \frac{1}{\pi} \int_0^{2\pi} \frac{e^{-ikR(\mathbf{x}, \mathbf{y})}}{4\pi R(\mathbf{x}, \mathbf{y})} \sin(n(\theta + \theta_x)) d\theta. \quad (11)$$

With the aid of a trigonometric identity, Eq. (11) can be rewritten as

$$K_n^s = \frac{1}{\pi} \int_0^{2\pi} \frac{e^{-ikR(\mathbf{x}, \mathbf{y})}}{4\pi R(\mathbf{x}, \mathbf{y})} \sin(n\theta) \cos(n\theta_x) d\theta + \frac{1}{\pi} \int_0^{2\pi} \frac{e^{-ikR(\mathbf{x}, \mathbf{y})}}{4\pi R(\mathbf{x}, \mathbf{y})} \cos(n\theta) \sin(n\theta_x) d\theta. \quad (12)$$

Because $\sin(n\theta)$ is an odd function of $\theta = \pi$ and the remainder of the integrand is symmetric around $\theta = \pi$ in the interval

$[0, 2\pi]$, the first integral of Eq. (12) vanishes. Introducing

$$H_n = \int_0^{2\pi} \frac{e^{-ikR(\mathbf{x}, \mathbf{y})}}{4\pi R(\mathbf{x}, \mathbf{y})} \cos(n\theta) d\theta, \quad (13)$$

Eq. (12) becomes

$$K_n^s = \frac{1}{\pi} H_n \sin(n\theta_x). \quad (14)$$

The cosine coefficients K_n^c of Eq. (7) can be derived in a similar manner:

$$K_n^c = \frac{1}{\pi} H_n \cos(n\theta_x). \quad (15)$$

The Fourier coefficients of Eq. (9) can also be determined analogously. Using

$$H'_n = \int_0^{2\pi} \frac{\partial}{\partial \nu} \left(\frac{e^{-ikR(\mathbf{x}, \mathbf{y})}}{4\pi R(\mathbf{x}, \mathbf{y})} \right) \cos(n\theta) d\theta, \quad (16)$$

they can be written as

$$K_n^{s'} = \frac{1}{\pi} H'_n \sin(\theta_x), \quad (17)$$

$$K_n^{c'} = \frac{1}{\pi} H'_n \cos(\theta_x). \quad (18)$$

The Fourier coefficients in Eqs. (5), (6), and (8) can be expressed similarly, for instance,

$$p_n^s(\mathbf{y}) = \frac{1}{\pi} \int_0^{2\pi} p(\mathbf{y}) \sin(n\theta_y) d\theta_y, \quad (19)$$

and similar expressions for $p_n^c(\mathbf{y})$, $p_n^{s*}(\mathbf{x})$, $p_n^{c*}(\mathbf{x})$, $p_n^{s'}(\mathbf{y})$, and $p_n^{c'}(\mathbf{y})$. By expanding all variables in Fourier series, Eq. (1) takes the form (where some of the summations take the index m for clarity)

$$\begin{aligned} C(\mathbf{x}) & \left\{ \sum_{m=0}^{\infty} [p_m^{s*} \sin(m\theta_x) + p_m^{c*} \cos(m\theta_x)] \right\} \\ &= \int_L \int_0^{2\pi} \left\{ \sum_{n=0}^{\infty} [p_n^s \sin(n\theta_y) + p_n^c \cos(n\theta_y)] \right\} \times \left\{ \sum_{m=0}^{\infty} \frac{1}{\pi} H'_m [\sin(m\theta_x) \sin(m\theta_y) + \cos(m\theta_x) \cos(m\theta_y)] \right\} r_y d\theta_y dL_y \\ & - \int_L \int_0^{2\pi} \left\{ \sum_{n=0}^{\infty} [p_n^{s'} \sin(n\theta_y) + p_n^{c'} \cos(n\theta_y)] \right\} \times \left\{ \sum_{m=0}^{\infty} \frac{1}{\pi} H_m [\sin(m\theta_x) \sin(m\theta_y) + \cos(m\theta_x) \cos(m\theta_y)] \right\} r_y d\theta_y dL_y. \end{aligned} \quad (20)$$

Matching the terms on the left- and right-hand sides of Eq. (20) and using the orthogonality properties for integrals involving $\sin(m\theta_y)\sin(n\theta_y)$, $\sin(m\theta_y)\cos(n\theta_y)$, and $\cos(m\theta_y)\cos(n\theta_y)$, the following expressions can be obtained after integration over $d\theta_y$:

$$C(\mathbf{x})p_n^{s*}(\mathbf{x}) = \int_L [p_n^s H'_n - p_n^{s'} H_n] r_y dL_y, \quad (21)$$

$$C(\mathbf{x})p_n^{c*}(\mathbf{x}) = \int_L [p_n^c H'_n - p_n^{c'} H_n] r_y dL_y. \quad (22)$$

All functions in Eqs. (21) and (22) are no longer explicitly dependent on angle θ . When the Fourier coefficients of surface pressure p_n^s , p_n^c and its normal derivative $p_n^{s'}$, $p_n^{c'}$ are known, the acoustic pressure at any observer point \mathbf{x} inside, outside or on body B can be expressed as [see Eq. (6)]

$$\begin{aligned} C(\mathbf{x})p^*(\mathbf{x}) &= \int_L \left\{ \sum_{n=0}^{\infty} [p_n^s H'_n - p_n^{s'} H_n] \sin(n\theta_x) \right. \\ & \quad \left. + \sum_{n=0}^{\infty} [p_n^c H'_n - p_n^{c'} H_n] \cos(n\theta_x) \right\} r_y dL_y. \end{aligned} \quad (23)$$

When observer point \mathbf{x} is on the surface of body B , Eqs. (21) and (22) can be rewritten as

$$C(\mathbf{x})p_n^s(\mathbf{x}) = \int_L [p_n^s H'_n - p_n^{s'} H_n] r_y dL_y, \quad (24)$$

$$C(\mathbf{x})p_n^c(\mathbf{x}) = \int_L [p_n^c H'_n - p_n^{c'} H_n] r_y dL_y \quad (25)$$

(where the superscript $*$ has disappeared). These modified Kirchhoff–Helmholtz integral equations form an implicit expression for the surface pressure and its normal derivative. It can be used to determine the boundary values of p when p' is known and vice versa.²

II. COMPUTATIONAL ASPECTS OF FOURIER BEM

The solution of the Kirchhoff–Helmholtz equation for axisymmetric structures [Eq. (23)] can be obtained numerically by solving Eqs. (24) and (25) using standard boundary element procedures. The generator L of the axisymmetric body is discretized and the geometry and acoustic variables p and p' are assumed to vary according to isoparametric shape functions on the surface of the body. The discretization of the body involves only line elements. The evaluation of the integrals (4), (13), and (16) over the angular coordinate θ has to be sufficiently accurate and efficient, to use this method effectively.¹

The possible singularities in the circumferential integrands of Eqs. (13) and (16) are not the only difficulties in their computation. The cosine function in the expressions for H_n and H'_n causes the total integrand to oscillate rapidly for high Fourier mode numbers n . Moreover, the $R(\mathbf{x}, \mathbf{y})^{-1}$ function in these integrals causes a steep slope of the integrand near $\theta=0$ and $\theta=2\pi$, when the distance between \mathbf{x} and \mathbf{y} is relatively small. Therefore, special attention should be paid to the evaluation of these integrals.

Calculation of the circumferential integrals consumes a major portion of the total amount of computation time for the Fourier boundary element method. The integrals need to be computed often, and the calculation itself is computationally expensive. In general, the line integrals from Eqs. (24) or (25) need to be computed numerically for each Fourier harmonic n , for a number of observer points \mathbf{x} . This requires a value for H_n and H'_n and thus two circumferential integral evaluations on each integration point of the line integral, for each Fourier mode number n . In addition, applying fixed point numerical integration for these circumferential integrals, like Gaussian or Trapezium quadrature, requires a large number of integration points in circumferential direction to obtain sufficiently accurate results.¹ This is particularly true for high Fourier mode numbers n where the integrand evinces an oscillatory behavior. The long computation times resulting from the application of fixed point integration methods as proposed by Akyol,¹ Soenarko,² and Juhl³ weaken the advantages of the Fourier BEM compared to the three-dimensional BEM. A solution to this problem will be presented in the next section.

A. Integral evaluation based on fast Fourier transform

For a sufficiently accurate and efficient evaluation of the integrals (13) and (16) for H_n and H'_n , respectively, a new method based on fast Fourier transform (FFT) was developed. The integrands in the expressions for H_n and H'_n consist of a reasonable smooth (but possibly singular) function multiplied by a cosine function. The integrand without the cosine function is an even periodic function around $\theta=0$ with a period equal to 2π . Given an even periodic function $f(x)$ with period $2L$, the Fourier coefficients F_n of this function are given by⁵

$$\begin{aligned} F_n &= \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi}{L} x\right) dx \\ &= \frac{1}{L} \int_0^{2L} f(x) \cos\left(\frac{n\pi}{L} x\right) dx. \end{aligned} \quad (26)$$

With $x = \theta$ and $L = \pi$, this shows that Eqs. (13) and (16) are valid expressions of the Fourier coefficients H_n and H'_n of the complex even functions:

$$h(\theta) = \frac{e^{-ikR(\theta)}}{4R(\theta)} \quad (27)$$

and

$$h'(\theta) = \frac{\partial}{\partial \nu} \left(\frac{e^{-ikR(\theta)}}{4R(\theta)} \right), \quad (28)$$

respectively, which are in fact slightly modified forms of the three-dimensional free-space Green's function from Eq. (2) and its normal derivative. Hence, to compute the integrals in the expressions for H_n and H'_n we can use the n th Fourier coefficient of the complex functions h and h' , respectively.

In numerical mathematics, the algorithm normally used for an efficient computation of the Fourier coefficients of a (complex) function is fast Fourier transform (FFT). FFT algorithms are optimized for speed while their accuracy is unaffected. Therefore, they are a good alternative for computing the integrals H_n and H'_n , but are generally more expensive than most fixed point numerical integration routines. However, a significant advantage of the proposed method is that by one FFT, the Fourier coefficients of many Fourier modes n are calculated, whereas the fixed point integration methods required an integral evaluation for every Fourier mode n .

An algorithm for the evaluation of the integrals in Eqs. (13) and (16) using FFT requires the following actions:

1. Determination of the number of samples n_{FFT} needed for computation of integrals H_n and H'_n with a desired accuracy.
2. Evaluation of the functions h [Eq. (27)] and h' [Eq. (28)] on n_{FFT} equidistant values of the parameter θ in the interval $[0, 2\pi]$.
3. Fast Fourier transformation of the n_{FFT} computed function values.
4. Selection of the n th terms of the calculated Fourier spectrums

which are numerical values for the integrals H_n and H'_n .

Numerical problems can occur when \mathbf{x} coincides with \mathbf{y} because the functions h and h' cannot be evaluated due to the R^{-1} singularity in Eqs. (27) and (28). This problem can be circumvented by taking surface integration points \mathbf{y} that do not coincide with the observer point \mathbf{x} when the line integrals (24) and (25) are computed. Gauss-log integration⁶ should then be applied for the line integrals because the functions h and h' have logarithmic behavior near the singularity. Other methods proposed in literature to handle a singular integrand use a technique of subtracting and adding up the singular part of the integrand from the regular part, resulting in a regular and singular (surface) integral. Then, special (analytical) integration techniques^{7,8} are used for the singular integral, and ordinary (Gauss-Legendre) integration techniques are used for the regular part. To apply this technique here, a special integration technique should be developed for the integral over a ring-shaped surface of the R^{-1} singularity. The integration region can be split up in an integral over the circumference and an integral over a generator segment of the vibrating body. The circumferential integral yields elliptic integrals of the first and second kind,² but an analytical solution for the integration of these elliptic integrals over the generator segment is not available. Therefore, this regularization method cannot be used here. It should however be noted that this is not a consequence of the pro-

posed FFT method but a general result for all FBEM implementations.

The application of the method of computing the circumferential integrals using FFT techniques is not limited to acoustic problems governed by the Helmholtz equation. In like manner, the new method is usable in non-acoustic Fourier BEM applications. The only condition to be fulfilled for the kernel is that it has to be a periodic function of θ .

B. Efficiency of the FFT method

Regarding efficiency, it is preferable that the number of evaluations of the functions $h(\theta)$ and $h'(\theta)$ can be chosen as low as possible, because this number is directly related to the cost of the FFT algorithm in particular and the cost of the total method in general. The number of required evaluations of these functions is determined by the desired accuracy of the FFT process. Signal leakage and aliasing in the Fourier transform process should be taken into account. This means that precisely an integer number of periods of the periodic functions $h(\theta)$ and $h'(\theta)$ should be sampled, and that the sampling frequency should at least be twice the highest frequency present in the functions. The first requirement is easily met because we exactly know the period of the functions $h(\theta)$ and $h'(\theta)$ which is 2π for axisymmetric structures. To satisfy the requirements for aliasing, however, the frequency content of functions $h(\theta)$ and $h'(\theta)$ needs to be predicted because it is not known beforehand. This is the topic of the remaining part of this section. For numerical efficiency it is desirable that the number of Fourier transform points can be written as 2^f with f as a positive integer number.

A closer look at the functions h and h' that are Fourier transformed is illustrative to establish a reasonable expression for the minimum number of Fourier transform points n_{FFT} required for each integral evaluation. A representative picture of the function h is plotted in Fig. 2.

The steepness of the curve close to $\theta=0$ and $\theta=2\pi$ is determined by the ratio of the minimum and maximum R (distance between the points \mathbf{x} and \mathbf{y}), because of the factor R^{-1} in Eq. (27) for h . The ratio $R_{\text{max}}/R_{\text{min}}$ can serve as a dimensionless scale factor for the problem's geometry. The rapid change in steepness of the curve for large values of this ratio causes nonzero high-mode number components in the Fourier transform of the function h . Therefore a sufficiently high number of Fourier points has to be used. Hence, a criterion for the minimum number of Fourier points needed should be a function of a steepness parameter:

$$c_s = \frac{R_{\text{max}}}{R_{\text{min}}}. \quad (29)$$

The oscillations in the curves for the real and imaginary part are caused by the term $e^{-ikR(\theta)} = \cos(kR) - i \sin(kR)$ for large numbers of k and/or a large difference between the minimum and maximum value for R . Thus the criterion for the minimum number of Fourier points should also be a function of an oscillation parameter:

$$c_0 = k(R_{\text{max}} - R_{\text{min}}). \quad (30)$$

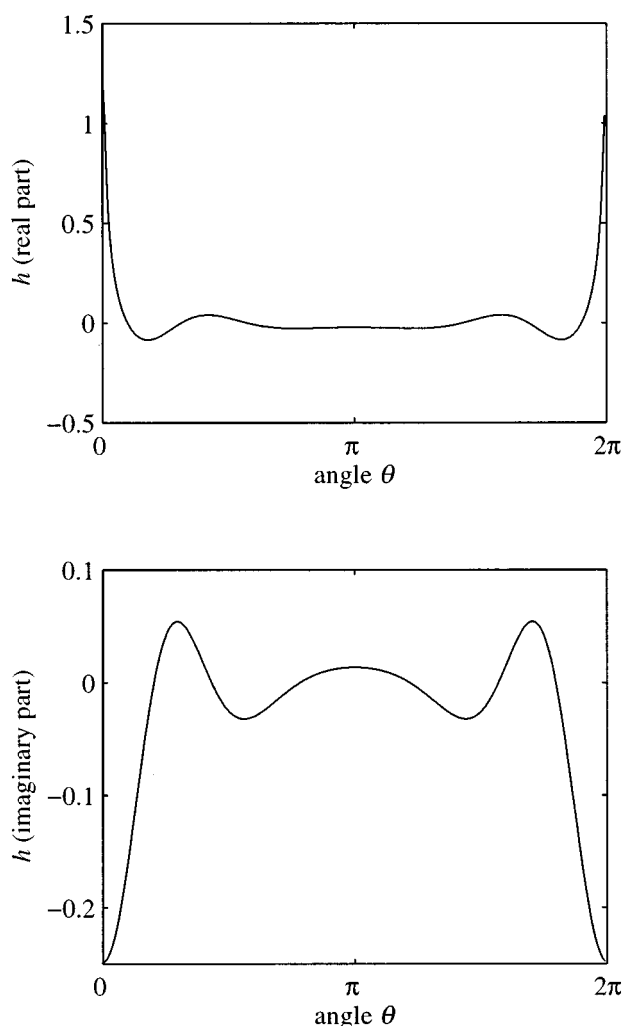


FIG. 2. Function h for $X=(5,0,0)$, $Y=(5,0,0.2)$ and $k=1$.

The same considerations can be made for the Fourier transform of h'_n , leading to the same parameters c_s and c_0 .

An expression for the minimum number of Fourier points is dependent on the characteristics of the integrands h_n and h'_n which can be described by the parameters c_s and c_0 . So an expression for n_{FFT} can be expressed as a function of those parameters:

$$n_{\text{FFT}} = n_{\text{FFT}}(c_s, c_0) = n_{\text{FFT}}(R_{\text{max}}, R_{\text{min}}, k) = n_{\text{FFT}}(\mathbf{x}, \mathbf{y}, k). \quad (31)$$

For an efficient application of the FFT method, an expression for n_{FFT} can be developed for a desired accuracy. The expression that can be derived is generally applicable for efficiently computing integrals in Eqs. (13) and (16) with FFT integration. For efficiency, it is also important to obtain a relatively simple expression n_{FFT} , because it must be used for each FFT based integral evaluation separately. Fortunately, a simple expression can be derived for practical application, as is illustrated in Sec. III.

C. Discretization process

The boundary element method is applied for the discretization of the modified Kirchhoff-Helmholtz integral equations (24) and (25). Assume that the generator L of body B

can be discretized with n_e line elements and that each line element i_e has n_n nodes. The total number of nodes is denoted as n_{nd} . Thus the coordinates r and z can be expressed in terms of the coordinates r_{i_n} and z_{i_n} of element node i_n using a piecewise polynomial approximation:

$$r(\xi) = \sum_{i_n=1}^{n_n} N_{i_n}^{n_n}(\xi) r_{i_n}, \quad (32)$$

$$z(\xi) = \sum_{i_n=1}^{n_n} N_{i_n}^{n_n}(\xi) z_{i_n}, \quad (33)$$

where $N_{i_n}^{n_n}(\xi)$ are the (n_n-1) th order isoparametric shape functions, ξ is the local element coordinate and i_n is the local node number. The boundary variables p_0 , p'_0 , p_n^s , p_n^c , $p_n'^s$, and $p_n'^c$ are also approximated using the same isoparametric shape functions as for the coordinates, thus on element i_e ,

$$\psi_{i_e}(\xi) = \sum_{i_n=1}^{n_n} N_{i_n}^{n_n}(\xi) \psi_{i_e i_n}, \quad (34)$$

where any of the boundary variables can be substituted for $\psi_{i_e}(\xi)$, and $\psi_{i_e i_n}$ is the value of the corresponding variable on local node i_n of element i_e . Using this approximation in, for instance, Eq. (24), yields

$$C(\mathbf{x}) p_n^s(\mathbf{x}) = \sum_{i_e=1}^{n_e} \left\{ \sum_{i_n=1}^{n_n} p_{ni_e i_n}^s \int_{-1}^1 N_{i_n}^{n_n}(\xi) H_n'(\xi) r(\xi) J_{i_e}(\xi) d\xi - \sum_{i_n=1}^{n_n} p_{ni_e i_n}'^s \int_{-1}^1 N_{i_n}^{n_n}(\xi) H_n(\xi) r(\xi) J_{i_e}(\xi) d\xi \right\}, \quad (35)$$

where $p_{ni_e i_n}^s$ is the value of p_n^s at local node i_n of element i_e and $J_{i_e}(\xi)$ the Jacobian of the transformation given by Eqs. (32) and (33), for element i_e :

$$J_{i_e}(\xi) = \left[\left(\frac{dr}{d\xi} \right)^2 + \left(\frac{dz}{d\xi} \right)^2 \right]^{1/2}. \quad (36)$$

Expressions similar to Eq. (35) can be obtained for the other boundary values using Eq. (25).

For the solution process, a collocation scheme is applied. The observer points \mathbf{x} on the boundary are chosen successively to coincide with each global node i_{nd} and \mathbf{y} is the (surface) point of integration, now explicitly a function of ξ through Eqs. (32) and (33). This collocation method results in a set of n_{nd} linear algebraic equations in terms of the unknown p_n , when p_n' is given on each node, and vice versa. The resulting equations may be written in the following matrix form:

$$\mathbf{A}_n p_n^s = \mathbf{B}_n p_n'^s, \quad (37)$$

where p_n^s and $p_n'^s$ are the column vectors containing the n_{nd} nodal values of p_n^s and p_n^0 , respectively. \mathbf{A}_n and \mathbf{B}_n are square matrices with the various integrals as in Eq. (35) as their elements. For the cosine terms of the Fourier series, a similar matrix equation can be derived:

$$\mathbf{A}_n p_n^c = \mathbf{B}_n p_n'^c, \quad (38)$$

where p_n^c and $p_n'^c$ are the column vectors containing the n_{nd} nodal values of p_n^c and $p_n'^c$. Thus, two matrix equations result, relating the terms of the Fourier expansion of the unknown variables to the Fourier coefficients of the boundary conditions. The matrix equations that describe the acoustic radiation have to be formed for every Fourier mode number n that is present in the Fourier expansion of the boundary conditions.

When the solution for all boundary values is computed, the value of the Fourier terms of the pressure and its derivative for any surface or exterior point can easily be obtained by applying an equation similar to Eq. (35) for p_n^{s*} and p_n^{c*} , substituting the calculated values for $p_{ni_e i_n}^s$, $p_{ni_e i_n}^c$, $p_{ni_e i_n}'^s$, and $p_{ni_e i_n}'^c$. The resulting equation is an explicit relationship between the acoustic variables on the surface of the radiating body and the acoustic variables at any other position in the acoustic medium.

III. APPLICATION OF THE FFT METHOD

The FFT based integration method was implemented in the in-house acoustic Fourier BEM code BARD to assess its efficiency compared to a Gaussian quadrature based method that was previously proposed. For the FFT method, an expression for the minimum number of Fourier points n_{FFT} was developed. For a large number of parameters values c_s in the range $[1, 2000]$ and c_0 in the range $[0, 100]$, the integrals (13) and (16) were computed with the proposed method until convergence was achieved. Each parameter pair c_s and c_0 has a specific minimum number of Fourier points n_{FFT} for the integrals to converge where the relative error in the computed value for the integrals did not exceed 10^{-3} . These pairs of c_s , c_0 and related n_{FFT} were then used in a curve-fitting procedure to obtain the relationship

$$n_{\text{FFT}}(c_s, c_0) = 14 \left(c_s + \frac{c_0}{2\pi} \right)^{0.9}.$$

In the BEM code the expression for n_{FFT} was implemented and its value is computed for each circumferential integral evaluation separately based on values for c_s and c_0 for that integral. Its value is rounded to the nearest subsequent power of 2, to enable the use of a fast radix-2 FFT algorithm. An efficient integration algorithm based on Gaussian quadrature was also implemented such that the maximum relative error with that method was 10^{-3} .

As a check for the accuracy of the developed FBEM code BARD, the response of an oscillating sphere with radius $a=1$ was computed. To model the sphere, a mesh with 10 quadratic line elements was used and the response was calculated for a dimensionless wavenumber range $ka=0, \dots, 5$. The relative errors in the computed surface pressure obtained with the FBEM compared with the analytically known solution for this problem^{9,10} were smaller than 1%.

To assess the efficiency of the FFT based integration method, the acoustic responses of two model vibrating bodies, a sphere and cylinder, were computed. The sphere and

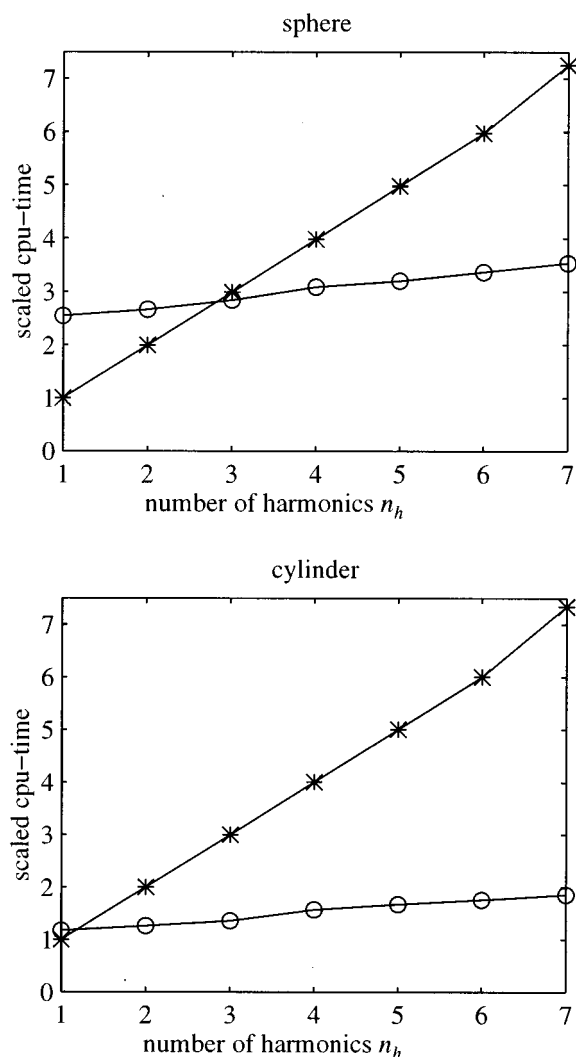


FIG. 3. Scaled computation time for spherical and cylindrical radiation for Gauss (*) and FFT (O) based quadrature.

cylinder were discretized using quadratic elements. The normal velocity boundary condition for both problems was defined as

$$u_r(\mathbf{y}) = \frac{1}{-i\rho\omega} p'(\mathbf{y}) = \sum_{n=1}^{n_h} \cos(n\theta),$$

where the number of harmonics n_h ranged from 1 to 7 to assess the usability of the methods for constructions with more complicated boundary conditions in circumferential direction. The acoustic pressure response was computed with both the FFT based and Gauss based integration methods. The differences between the results obtained with the FFT method and the Gauss method were negligible, indicating that the circumferential integrals were computed with the same accuracy. The efficiency of the new method can be assessed by making a comparison between cpu time needed for the radiation computation for both problems as a function of the number of harmonics n_h in the boundary conditions as depicted in Fig. 3.

The cpu times were scaled by dividing the actual computation time by the time needed to compute the response of the first harmonic using the Gaussian quadrature method. For

both methods and for both problems, the cpu time needed to compute the acoustic response increased linearly with each extra harmonic in the boundary conditions. However, the cpu time for the Gaussian quadrature method increased more per extra harmonic than for the FFT method. This can be explained by the most important difference between the methods: when the Gauss method is used, the circumferential integrals are computed for each Fourier mode separately, but when the FFT method is used, these integrals are computed for all modes simultaneously. In other words, the characteristic total cpu time for the radiation computation with the Gauss method is the computation time for one harmonic multiplied by the total number of harmonics in the boundary conditions, while for the FFT method, the computation time for evaluating *one* harmonic is typical of the total cpu time for all harmonics. For one or two harmonics in the boundary conditions for the sphere radiation, and for one harmonic in the cylinder boundary conditions, the calculations with Gaussian quadrature were faster than those with the FFT method (Fig. 3). However, for *all* other problem configurations that were tested, using the FFT method was clearly advantageous. In conclusion, for more than one harmonic in the boundary conditions, the FFT method is generally more efficient than the Gaussian quadrature method, while obtaining the same accuracy.

IV. CONCLUSIONS

A new method was presented to handle the integrals over the angle of revolution arising in the Fourier boundary element method (FBEM) for the calculation of the acoustic response of axisymmetric bodies with arbitrary boundary conditions. Previously proposed methods to handle these integrals required a separate (expensive) integral evaluation for each Fourier term (circumferential harmonic) that is present in the boundary conditions. With the integration method proposed here, these integrals are computed simultaneously for all harmonics. The method efficiently uses fast Fourier transform (FFT) algorithms to obtain numerical values for these integrals.

Simulations were performed comparing the efficiency of the new FFT integration method with that of a method based on Gaussian quadrature. They showed that the FFT method is generally more efficient, especially when more than one circumferential harmonic is present in the boundary conditions.

It is expected that the utility of the new integration method presented here is not limited to acoustic problems only. The method should be usable whenever the integrand of the circumferential integral is periodic and (fast) Fourier transformable.

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