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An Improved Bound for *k*-Sets in Three Dimensions*

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Abstract. We prove that the maximum number of k-sets in a set S of n points in $\mathbb{R}^3$ is $O(nk^{3/2})$. This improves substantially the previous best known upper bound of $O(nk^{5/3})$ (see [7] and [1]).

1. Introduction

Let S be a set of n points in $\mathbb{R}^d$. A *k-set* of S is a subset $S' \subset S$ such that $S' = S \cap H$ for some halfspace H and $|S'| = k$. The problem of determining tight asymptotic bounds on the maximum number of k-sets is one of the most intriguing open problems in combinatorial geometry. Due to its importance in analyzing geometric algorithms [5], [9], the problem has caught the attention of computational geometers as well [3], [7], [8], [14], [16]. A close to optimal solution for the problem remains elusive even in the plane. The best asymptotic upper and lower bounds in the plane are $O(nk^{1/3})$ (see [6]) and $n \cdot 2^{\Omega(\sqrt{\log k})}$ (see [15]), respectively. In this paper we obtain the following result:

Theorem 1.1. *The number of k-sets in a set of n points in $\mathbb{R}^3$ is $O(nk^{3/2})$.*

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This result improves the previous best known asymptotic upper bound of $O(nk^{5/3})$ (see [7] and [1]). The best known asymptotic lower bound for the number of k-sets in three dimensions is $nk \cdot 2^{\Omega(\sqrt{\log k})}$ (see [15]).

2. An Overview of our Technique

(a) The paper by Agarwal et al. [1] presents a general technique, based on random sampling, for transforming an upper bound on the number of k-sets that is independent of k to a bound that does depend on k. Our main thrust thus is to establish the upper bound $O(n^{5/2})$ for the number of k-sets. This, combined with the technique of [1], will imply Theorem 1.1.

(b) We assume that the set S is in general position, meaning that no four points in S lie in a common plane. Applying a small perturbation to the points of any set S yields a set of points in general position and the number of k-sets does not decrease.

(c) We consider the set T of halving triangles spanned by S: A triangle $\Delta = abc$, with vertices $a, b, c \in S$, is a *halving triangle* if the plane containing Δ has the same number of points of S on either side. (Note that n has to be odd for halving triangles to exist, and we indeed assume, without loss of generality, that n is odd.) We show that $|T| = O(n^{5/2})$. This implies that the number of k-triangles, for any k, is also bounded by $O(n^{5/2})$, where a *k-triangle* is a triangle Δ spanned by three points in S with exactly k points of S on one side of the plane containing Δ. Indeed, choose a direction d not contained in the plane of any k-triangle and add $|n - 3 - 2k|$ extra points to S far enough in the direction d or $-d$. Each k-triangle in S turns into a halving triangle in one of the two resulting configurations. It is well known [2] that the $O(n^{5/2})$ bound on the number of k-triangles for any k carries over to the same bound on the number of k-sets.

(d) All the previous approaches are based on (the three-dimensional extension of) Lovász' lemma [4]: Any line crosses (the relative interiors of) at most $O(n^2)$ halving triangles. The preceding techniques aimed to derive a general lower bound for the number of such crossings. Specifically, they showed that for any collection of t triangles spanned by the points of S there exists a line that crosses many triangles, where the best lower bound for this number of crossings is $\Omega(t^3/n^6)$ [7]. Combining this lower bound with the upper bound provided by Lovász' lemma, one obtains an upper bound of $O(n^{8/3})$ for the number of k-sets.

(e) In contrast, our technique focuses on the specific set T of halving triangles, and exploits the structure of this set. The main property of this set, which is also used in deriving Lovász' lemma, is the *antipodality* property, which we re-establish rigorously in Lemma 3.4 below. Informally, it asserts that the halving triangles with a common edge pq alternate sides, as we rotate a plane containing pq. See Fig. 1 for an illustration of this property. This is the only property of the set T that is needed in the proof.

(f) Our technique only considers interaction between pairs of triangles of T *with a common vertex*. Specifically, we consider *crossings* between such pairs of triangles, where two triangles pab and pcd cross each other if their relative interiors have a nonempty intersection (in this case p is the only common vertex of these triangles).

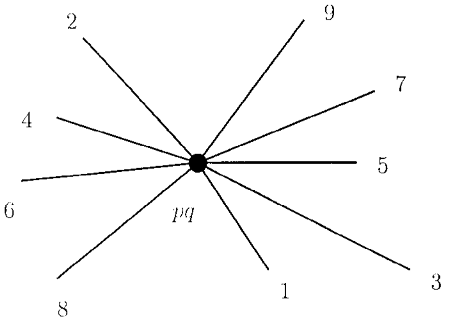

Fig. 1. The antipodality property of halving triangles: the common edge pq is shown head-on, as a point; as we rotate a plane containing pq we encounter the other endpoints of these triangles in the order shown.

(g) Our proof proceeds by deriving both an upper bound and a lower bound on the number of triangle crossings (of the above special type) in T. The upper bound is $O(n^4)$ and it is an easy consequence of Lovász' lemma in 3-space. The lower bound is $\Omega(t^2/n)$, and is proven using arguments that extend those that were used in [6] for the analysis of k-sets in the plane. These upper and lower bounds immediately yield the desired bound on the number of k-sets.

3. Proof of the Theorem

Let n be odd, let S be a set of n points in $\mathbb{R}^3$ in general position, and let T be the set of all halving triangles of S. Put $t = |T|$.

Definition 3.1. We say that two triangles $\Delta_1, \Delta_2 \in T$ *cross* if Δ_1 and Δ_2 share exactly one vertex, say p, and the edge opposite to p in one of the triangles crosses the other triangle (this is equivalent to the definition given in Section 2). Let X denote the number of crossing pairs of triangles in T.

The following extension of the two-dimensional Lovász' lemma [10] has been derived in [4] and used in [3] and [4]. We say that a line *crosses* a triangle if it intersects the triangle but not any of its edges. One can prove this lemma using the Antipodality Lemma below by translating a line from infinity to the given location, and by observing how the number of triangles crossed by the line changes as it moves—this number changes only when the line crosses a segment connecting two points and then it changes by ± 1.

Lemma 3.2 [4], [11]. *Any line crosses fewer than* $n^2/4$ *halving triangles.*

As a consequence we obtain:

Lemma 3.3. *The number* X *of crossing pairs of halving triangles for a set* S *as above is less than* $3n^4/8$.

Proof. Fix an edge $e = pq$ with endpoints in S. This edge crosses fewer than $n^2/4$ triangles. For each triangle $\Delta = abc$ that it crosses, e can contribute at most three crossings to X, namely, a crossing between abc and apq, between abc and bpq, and between abc and cpq. Since there are only $\binom{n}{2}$ edges, we have in total fewer than $3n^4/8$ crossings. □

The following well-known lemma, which is the basis for the three-dimensional version of Lovász' lemma (see, e.g., [3] and [4]), is crucial for our analysis. We include a proof for the sake of completeness.

Lemma 3.4 (Antipodality Lemma). *Let $p, q \in S$ and let T_{pq} denote the subset of all triangles in T incident to both p and q. Rotate a halfplane h, bounded by the line ℓ passing through p and q, about ℓ; h meets the triangles in T_{pq} in a cyclic order. Let Δ and Δ' be two consecutive elements of T_{pq} in this cyclic order, let W be the wedge swept by h as it rotates from Δ to Δ', and let W' denote the antipodal wedge, emanating from ℓ and bounded by the same pair of planes. Then there is a unique "antipodal" triangle $\Delta'' \in T_{pq}$ contained in W'.*

Proof. Consider the halfplane h rotating about pq. If during the rotation h contains a halving triangle pqr and the next such triangle is pqr', then as h leaves r the plane containing h has one more point of S on its side containing r than on the opposite side. Just before reaching r' the plane containing h has one more point on its side containing r' than on the opposite side containing r. Since the difference between the number of points of S contained in the two sides changes by one each time the plane reaches or leaves a point of S, there must be a position in between when the difference is zero. At that point the plane containing h contains a halving triangle from T_{pq}, but since Δ and Δ' are consecutive, this halving triangle is not contained in h but in the opposite halfplane and therefore in W'.

The uniqueness of this antipodal triangle is a consequence of the existence proof: if there were two or more antipodal triangles in T_{pq} for Δ and Δ', then one could choose two consecutive ones and this pair of two consecutive elements of T_{pq} would have no antipodal triangle. □

Remarks. (a) Note that $|T_{pq}|$ must be odd to satisfy the assertion of the lemma, unless T_{pq} is empty. It is easy to show that T_{pq} is not empty for any edge pq. If T_{pq} has a single element the assertion of the lemma holds automatically. In all other cases the lemma implies that any halfspace with p and q on its boundary contains at least one element of T_{pq}.

(b) We say that a collection T of triangles that is spanned by S is *antipodal* if it satisfies the property in Lemma 3.4. Inspecting the foregoing proof, it is easily verified that it also applies to any antipodal collection T. Hence any such collection can have at most $O(n^{5/2})$ triangles. As a matter of fact, this also holds for *weakly antipodal* collections T, meaning that, for each edge pq, the antipodality property holds for all but a constant number of consecutive pairs of triangles in T_{pq}.

We fix a coordinate frame and assume that no horizontal plane (i.e., one parallel to the xy plane) contains more than one point of S. We further assume that the plane of no triangle in T is parallel to the y-axis. This can be achieved by a suitable rotation.

Fix a point $p \in S$, and let T_p denote the set of triangles in T that are incident to p. Let h_p be the horizontal plane passing through p. Let π_p be any horizontal plane above p. Clip each triangle in T_p to the halfspace above h_p, and project each (nonempty) clipped triangle centrally from p onto π_p. The resulting set of projected triangles has the following structure. Each point $u \in S$ that lies above h_p is mapped to a point $u^* \in \pi_p$. Each triangle puv in T_p for which both u and v lie above h_p is mapped to the segment u^*v^*, and each triangle puv in T_p for which u lies above h_p but v lies below h_p is mapped to a ray emanating from u^*. Triangles puv in T_p for which both of u and v lie below h_p are excluded from the analysis. Let G_p denote this geometric graph drawn on π_p (strictly speaking, G_p is not a geometric graph in the sense of [12], because of the infinite rays that it contains), and let S_p^* be its set of vertices, the projected images of points of S above h_p. We refer to both the bounded edges and the rays as *edges* of G_p.

Notice that a crossing pair of edges in G_p corresponds to a crossing pair of triangles in T_p. We do not necessarily get all crossing pairs of triangles in T_p this way, nevertheless Lemma 3.3 bounds the total number of edge crossings in the graphs G_p.

Let e_p and r_p be the number of (bounded or unbounded) edges and the number of rays in G_p, respectively. In the following lemma we find the average of these numbers.

Lemma 3.5.

(a) $\sum_{p \in S} e_p = 2t$;
(b) $\sum_{p \in S} r_p = t$.

Proof. Consider any triangle Δ in T and let the vertices of Δ in ascending order of their z-coordinates be p, q, and r. The triangle Δ contributes a bounded edge to G_p since q and r are both above h_p. Δ contributes a ray to G_q since r is above h_q but p is below it. Finally, Δ does not contribute to G_r since both p and q are below h_r. Each triangle in T contributes two to the sum in (a) and one to the sum in (b), thus proving the lemma. □

We next observe that G_p has the following antipodality property, which is immediate from the antipodality property of Lemma 3.4.

Lemma 3.6. *Let $u^* \in S_p^*$ and sort the edges of G_p incident to u^* in the angular order around u^*. For any two consecutive elements e_1 and e_2 of this cyclic order there is a unique "antipodal" edge e_3 in G_p incident to u^*, namely, one that extends from u^* into the wedge that is antipodal to the wedge formed between e_1 and e_2.*

Proof. The edges e in G_p incident to u^* are in 1–1 correspondence with the triangles in T that are incident to both p and u. (Here u^* is the projected image of the point $u \in S$.) Our lemma follows from Lemma 3.4 since the cyclic ordering of these edges coincides with the cyclic ordering of the triangles around the line pu and antipodality for edges corresponds to antipodality of triangles. □

We use the antipodality established above to decompose the edges of each G_p into a collection of *x-monotone convex chains*, in a manner similar to that in [6]. We include a description of this construction so as to make our paper self-contained and to handle properly the presence of infinite rays in our graphs.

Notice that our assumption on the coordinate system implies that no edge of G_p is parallel to the y-axis, and thus we can distinguish between left and right endpoints of edges. For defining the chains we describe how to continue a chain to the right past an edge e of G_p. We extend e to the right past its right endpoint q^* and turn the extended segment about q^* counterclockwise (looking from above) until we encounter the first edge e' in G_p incident to q^* and extending from it to the right. The chain containing e continues through e'. If e is a ray having no right endpoint or if there is no such e' as required, then e is the rightmost edge in its chain. A chain is extended to the left in a fully symmetric manner, replacing "right" by "left" and "counterclockwise" by "clockwise." The proof of Lemma 3.7 below implies that these right-extension and left-extension rules are consistent with each other. See Fig. 2 for an illustration of the decomposition of G_p into chains.

Lemma 3.7.

(a) *Each edge of G_p appears in a unique chain.*
(b) *A single chain terminates at any given vertex of S_p^* (either on its right side or on its left side).*
(c) *The number of chains c_p is at least $r_p/2$.*

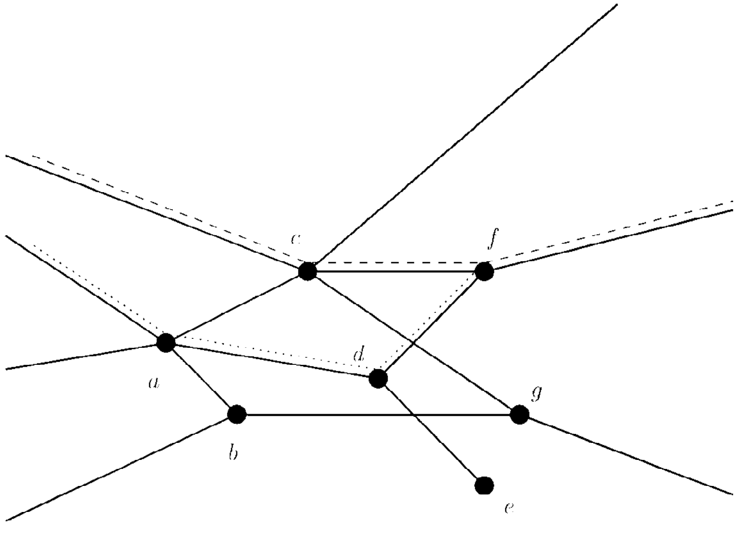

Fig. 2. An illustration of the graph G_p. One convex chain is drawn as dashed and one as dotted. The remaining chains are: $(-\infty, a, c, +\infty)$, $(c, g, +\infty)$, (d, e), (a, b, g), $(-\infty, b)$. Here $-\infty/+\infty$ means that the chain starts/stops on a ray. The $(-\infty, a, c, +\infty)$ chain contains the lower ray ending at a.

Proof. For (a) it suffices to show that no two different edges of G_p with a common right endpoint can have the same right neighbor in their respective chains. Consider a vertex $q^* \in S_p^*$ and let the edges in G_p extending from q^* to the left in counterclockwise angular order be $e_1, \ldots, e_k$. Using Lemma 3.6 we find a unique edge f_i incident to q^* in the wedge antipodal to $e_i e_{i+1}$ for each of the values $i = 1, \ldots, k-1$. Note that since the wedges are pairwise openly disjoint, the edges f_i are distinct, and extend from q^* to the right. Our construction guarantees that the chain containing e_i continues through f_i for $i = 1, \ldots, k-1$ and the chain through e_k does not continue through any of the edges f_i.

For (b), notice that if there are no edges incident to q^* other than the edges e_i and f_i, then the chain containing e_k terminates at q^* (and this is the only chain terminating there). If, however, there are more edges of G_p incident to q^*, then (again by Lemma 3.6) there are exactly two more edges, both extending from q^* to the right, and the chain containing e_k extends through one of them, while the other edge represents a chain that terminates (on its left) at q^*.

We remark here that the above arguments also prove that the dual definition (of continuing chains to the left) results in the same set of convex chains.

For (c) notice that each chain contains at most two rays. □

The following lemma implies that for typical values of e_p and r_p (which are both $\Theta(t/n)$) and for $t \geq 100n^2$, a positive fraction of all pairs of edges in G_p are crossing. This is a substantial improvement over the $\Omega(|T_p|^3/n^2)$ bound on the crossing number of the graph obtained by projecting T_p centrally from p to a sphere around p (see, e.g., Theorem 14.12 of [13]). (Notice that G_p is the central projection from p onto π_p of the portion of this spherical graph that lies in the upper hemisphere.) Using this weaker bound instead (and comparing it with the upper bound of Lemma 3.3), would yield a simple proof of the known result [7] that a set of n points in 3-space has $O(n^{8/3})$ k-sets, for any fixed k.

Lemma 3.8. *The number of edge-crossings in G_p is at least $r_p^2/8 - 3e_p n$.*

Proof. We call a pair of chains C_1, C_2 *crossing* if there exist edges $e_1 \in C_1$, $e_2 \in C_2$ that cross each other (in their relative interiors). That is, pairs of chains "crossing" at a vertex do not count. In view of Lemma 3.7(a), it suffices to obtain a lower bound for the number of pairs of chains that cross each other. Instead, we derive an upper bound for the number of noncrossing pairs of chains. Let C_1, C_2 be a noncrossing pair of chains. Then either (a) C_1 and C_2 are disjoint, or (b) C_1 and C_2 meet at a vertex. We assume that both C_1 and C_2 start and end on rays of G_p. The total number of pairs of chains that violate this assumption is at most $c_p n$, as follows from Lemma 3.7(b).

Suppose that C_1 and C_2 are disjoint, in which case one of the chains, say C_1, lies fully above C_2 (in the y-direction). Take any edge e_2 of C_2, and let ℓ_1 be the line tangent to C_1 and parallel to e_2. (The line ℓ_1 exists because C_1 lies above C_2 and C_2 lies above the line containing e_2.) Let p_1 be a vertex of C_1 incident to ℓ_1; see Fig. 3. The pair (p_1, e_2) determines the pair (C_1, C_2). Indeed, the edge e_2 identifies the chain C_2 uniquely, by Lemma 3.7(a). The pair (e_2, p_1) determine the tangent line ℓ_1, and the construction of

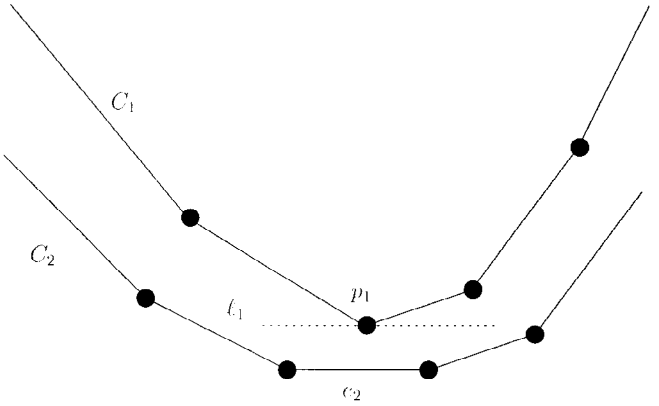

Fig. 3. A pair of noncrossing chains.

the chains is easily seen to imply that p_1 and ℓ_1 uniquely identify C_1. Hence, the number of disjoint pairs of chains is at most $e_p n$.

Suppose next that C_1 and C_2 meet at a vertex. Let $e_1 \in C_1$ and $e_2 \in C_2$ be edges of the chains with a common right endpoint. Clearly, e_1 and e_2 determine C_1 and C_2. Here e_1 is one of the e_p edges of G_p and e_2 is one of the at most n edges in G_p incident to the right endpoint of e_1. (Here we use the fact that the maximum degree of G_p is bounded by n, since at most n triangles in T are incident to a fixed pair of points of S.) Hence, the number of pairs of chains having a common vertex is at most $e_p n$.

We thus have at least $\binom{c_p}{2} - c_p n - 2e_p n$ crossing pairs of edges in G_p which is, by Lemma 3.7(c), at least the claimed number $r_p^2/8 - 3e_p n$. □

We finish the proof by comparing the upper bound in Lemma 3.3 and the lower bound in Lemma 3.8 for the number X of crossing pairs of triangles in T with a common vertex. We have

$$3n^4/8 \geq X \geq \sum_{p \in S} (r_p^2/8 - 3e_p n) \geq t^2/(8n) - 6tn,$$

where the last inequality follows from Lemma 3.5. We thus have $t^2 \leq 3n^5 + 48tn^2$, which implies that $t = O(n^{5/2})$.

This, and the observations in paragraphs (a) and (c) of Section 2, complete the proof of Theorem 1.1.

4. Open Problems

(a) Our analysis is based on the upper bound $O(n^4)$ on the number of crossings derived in Lemma 3.3. However, this bound seems to be weak, because, for an edge ab connecting two points a, b of S, we want to count the number of halving triangles pcd that it crosses, with the additional constraint that pab is also a halving triangle. In our derivation we do not exploit this constraint at all, so the first open problem is whether this bound can be improved, taking into account this constraint.

(b) We conjecture that the following holds: given a set S of n points in 3-space in general position and an arbitrary set T of t triangles spanned by S, there exists a line that crosses $\Omega(t^2/n^3)$ triangles of T. This bound is significantly larger than the bound $\Omega(t^3/n^6)$ of [7] and it would yield a trivial proof of Theorem 1.1 (using Lovász' lemma). We are not aware of any construction that contradicts this conjectured bound. This bound is best possible, for $t = \Omega(n^2)$, which can be shown by a simple construction.

(c) An even stronger conjecture is the following: given a set S of n points in the plane in general position and an arbitrary set T of t triangles spanned by S, there exists a point that lies in $\Omega(t^2/n^3)$ triangles of T. The best known lower bound, due to [3], is $\Omega(t^3/(n^6 \log^5 n))$. Again, the conjectured bound is best possible for $t = \Omega(n^2)$. (Note that if (c) is true, then the following strengthening of (b) also holds: given S and T as in (b), then for any direction u there exists a line parallel to u that crosses $\Omega(t^2/n^3)$ triangles of T.)

(d) Finally, can the technique used in this paper be extended to higher dimensions? A main difficulty in such an extension is that, already in four dimensions, the fact that two halving simplices (even with some common vertices) cross each other does not necessarily imply that an edge of one of them crosses the relative interior of the other. This precludes an immediate extension of Lemma 3.3 to four dimensions.

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