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SUMMARY

An improved finite difference procedure for determining the natural frequencies and mode shapes of tapered cantilever beams undergoing uncoupled vibrations is presented. Boundary conditions are derived in the form of simple recursive relations involving the second order central differences. Any approximation error resulting from this process is discussed. Results obtained by using the conventional first order central differences and the present second order central differences are compared, and it is observed that the present second order scheme is more efficient than the conventional approach. An important advantage offered by the present approach is that the results converge to exact values rapidly, and thus the extrapolation of the results is not necessary. Consequently, the basic handicap with the classical finite difference method of solution that requires the Richardson's extrapolation procedure is eliminated. Furthermore, for the cases considered herein, the present approach produces consistent lower bound solutions.

INTRODUCTION

Several methods of solution of beam vibration problems, which can be broadly classified as either belonging to the continuum model approach or to the discrete model approach, have been published. Principles of the minimum potential energy, or complimentary energy, the Reissner mixed method and the Dean and Plass principle have been used in the continuum model approach. It has been established that the potential and complimentary energy principles give an upper bound to the solutions while the Reissner method and Dean and Plass method may produce upper bound solutions depending on the type of formulation and choice of shape functions. However, bounds for these mixed methods have not been established theoretically. Each of these methods has its inherent advantages (refs. 1 to 3). In the discrete model approach, the Holzer, Stodola, polynomial frequency equation, Myklestad, and finite element methods are all well known. Solution of the equations of motion in the continuum model approach is possible under certain conditions by the application of the Galerkin process, the collocation method, and the finite difference method. Among these techniques, the Galerkin process is known to be equivalent to the Rayleigh-Ritz methods and generates upper bound solutions identical to those obtained by the Ritz method, provided that similar shape functions are used in both techniques. It has been observed that the accuracy of the results obtained by using the collocation method depend on the choice of the collocation

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points and their location (ref. 4). The finite difference method has attracted considerable attention and several contributions exist which make use of the first-order forward, backward, or central difference schemes or their combinations (refs. 5 to 12). In almost all the works, it has been observed that the finite difference method is subject to relatively slow convergence with mesh refinement, although Richardson's extrapolation procedure (ref. 13), when applied to two or three successive iterations with different mesh sizes, can produce results which may be close to the exact solutions. However, such an extrapolation procedure requires that the convergence of the results be monotonic, and the extrapolated result may not necessarily give a bound.

Relatively few works exist which deal with the application of higher order finite differences. Greenwood (ref. 14) used first order and second order finite difference schemes, which produce truncation errors of the order $O(h^2)$ and $O(h^4)$, respectively, for the analysis of uniform beams in flexure and for another case having uniform breadth but varying depth with fixed-free end conditions. The fourth order differential equation for the uncoupled vibration was transformed into four first order equations, the slopes and shearing forces were evaluated at half integer stations while the deflections and bending moments were evaluated at integer stations. Central difference approximation of order $O(h^4)$ with staggered stations was used. One-sided approximations were used to satisfy the boundary conditions wherein all the stations encountered were inside the beam and fictitious stations outside the beam were not required. As an alternative approach, another complicated, but symmetric, method of representing the boundary conditions was also illustrated. It was concluded that one-sided approximation gave better results while the symmetry assumptions gave results of moderate accuracy. It is interesting to note from Greenwood's results that, for the tapered beam case, the $O(h^2)$ approximation yields better accuracy than the higher order $O(h^4)$ approximation. It was stated in reference 14 that the lumping of nonuniform mass caused the largest loss of accuracy in the $O(h^4)$ approximation.

Gawain and Ball (ref. 15) presented finite difference formulae having consistent errors of $O(h^2)$ by representing the function with a truncated power series at the boundary in such a manner that the boundary conditions are satisfied. A consistent error of $O(h^2)$ was obtained for the error estimate. However, higher order central difference expressions having truncation errors of order $O(h^4)$ have not been reported.

In what follows, a second order central difference approach is presented which eliminates most of the problems discussed above. Clamped free beams are analyzed for axial, torsional, and flexural vibrations. Boundary conditions are enforced which eliminate the fictitious stations outside the beam. These conditions are obtained from the simple and logical extensions of the first order theory. Any approximation error encountered in this process is discussed. It will be shown that the present theory produces accurate results with rapid convergence and, consequently, the extrapolation procedures needed in the classical first order central difference theory can be successfully obviated with the present improved formulations.

Professor A. W. Leissa provided facilities at Ohio State University from March to April 1983, during which time the uniform beam cases were solved. The results of the uniform beam cases are reproduced in this report for completeness.

SYMBOLS

A	area at any section
b_o	breadth of beam
C	torsional rigidity
$\frac{d(\ )}{dz}$	derivative with respect to z
$\frac{d^2(\ )}{d\eta^2}$	second derivative with respect to η
E	Young's modulus
G	modulus of rigidity
h	length of each elemental beam segment
I	second moment of area about flexible plane
I_p	polar moment of inertia about centroid
L	length of beam
n	number of beam segments
p	natural radian frequency
t	time
t_o	depth of beam
w	dynamic displacement in longitudinal direction
y	dynamic displacement in flexible direction
z	coordinate measured along longitudinal direction of beam
β	breadth taper parameter
γ	pretwist over length of beam
δ	depth taper parameter
η	axial fractional length
θ	dynamic torsional displacement
ρ	mass density

Subscripts:

i	arbitrary station
$-1, -2, n+1, n+2, n+3$	fictitious stations outside beam domain

Superscripts:

$'$	differentiation with respect to z or η
$\cdot$	differentiation with respect to time

ANALYSIS

Axial Vibrations of Tapered Cantilever Beams

The governing differential equation for free axial vibrations of a cantilever beam is

$$\frac{d}{dz}\left(EA\frac{dw}{dz}\right) + \rho p^2 w = 0 \tag{1}$$

For a beam with breadth taper β and depth taper δ, equation (1) reduces to

$$aw'' + bw' + cp^2 w = 0 \tag{2}$$

where

$$\left.\begin{aligned} a &= (1 - \beta\eta)(1 - \delta\eta)E/L^2 \\ b &= -E[\beta(1 - \delta\eta) + \delta(1 - \beta\eta)]/L^2 \\ c &= \rho(1 - \beta\eta)(1 - \delta\eta) \\ \eta &= z/L,\ 0 \leq \eta \leq 1 \end{aligned}\right\} \tag{3}$$

The boundary conditions for the cantilever beam reduce to

$$w(0) = w'(1) = 0 \tag{4}$$

<u>Solution by first order central differences.</u> - Substituting the central differences for the derivatives of w given in the appendix in equation (2) and simplifying, one can write the following equation for any arbitrary station i of the beam as follows:

$$A_i w_{i-1} + B_i w_i + C_i w_{i+1} = D_i p^2 w_i \qquad i = 0, 1, \ldots, n \tag{5}$$

In the preceding equation

$$\left.\begin{aligned} &A_i = a_i - b_i h/2;\ B_i = -2a_i;\ C_i = a_i + b_i h/2;\ D_i = -c_i h^2; \\ &a_i = (1 - \beta ih)(1 - \delta ih)E/L^2;\ b_i = -E[\beta(1 - \delta ih) + \delta(1 - \beta ih)]/L^2; \\ &c_i = \rho(1 - \beta ih)(1 - \delta ih) \end{aligned}\right\} \tag{6}$$

The boundary conditions in terms of the first order central differences are

$$w_o = 0;\ w_n' = 0 \quad \text{or} \quad w_{n+1} = w_{n-1} \tag{7}$$

where n represents the station located at the free end of the cantilever. Equation (5) together with the boundary conditions given by equation (7) leads to the frequency equation in the form of n-equations for $i = 1, 2, \ldots, n$:

$$\begin{bmatrix} B_1 & C_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ A_2 & B_2 & C_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & A_3 & B_3 & C_3 & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & A_{n-1} & B_{n-1} & C_{n-1} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & A_n+C_n & B_n \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ w_{n-1} \\ w_n \end{Bmatrix} = p^2 \begin{Bmatrix} w_1 \cdot D_1 \\ w_2 \cdot D_2 \\ w_3 \cdot D_3 \\ \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \\ w_{n-1} \cdot D_{n-1} \\ w_n \cdot D_n \end{Bmatrix} \tag{8}$$

Solution by second order central differences. - The boundary conditions for the fixed-free case of a beam in axial vibration given by equation (4) in terms of second order central differences (see appendix) are

$$w_o = 0;\ w_n' = \frac{1}{12h}\left(w_{n-2} - 8w_{n-1} + 8w_{n+1} - w_{n+2}\right) = 0 \tag{9}$$

Assumption of the symmetry condition (ref. 14) at the free end for the functions (w_{n-1}, w_{n+1}) and (w_{n-2}, w_{n+2}) leads to

$$w_{n-1} = w_{n+1};\ w_{n-2} = w_{n+2} \tag{10}$$

and the fictitious stations w_{n+1} and w_{n+2} can thus be eliminated. As can be seen from equation (7), the physically consistent condition, $w_{n-1} = w_{n+1}$, as given by the first order theory is extended to cover the fictitious station w_{n+2} also. Another important observation can be made by inspecting equation (5). If this equation is evaluated for the station $i = 0$, the built in end, one obtains the condition

$$A_o w_{-1} + B_o w_o + C_o w_1 = p^2 D_o w_o \tag{11}$$

which implies that

$$w_{-1} = -\frac{C_o}{A_o} w_1 = -\left[\frac{1 - \frac{(\beta + \delta)h}{2}}{1 + \frac{(\beta + \delta)h}{2}}\right] w_1 \tag{12}$$

yielding $w_{-1} = w_1$ for a uniform beam. Substituting the second order finite difference expressions for the derivatives given in the appendix in equation (5) and simplifying, one obtains the following equation for any arbitrary station i:

$$A_i w_{i-2} + B_i w_{i-1} + C_i w_i + D_i w_{i+1} + E_i w_{i+2} = F_i p^2 w_i \tag{13}$$

In the previous equation

$$\left.\begin{aligned}
A_i &= (b_i h - a_i)/12h^2;\; B_i = (16a_i - 8b_i h)/12h^2; \\
C_i &= -30a_i/12h^2;\; D_i = (16a_i + 12b_i h)/12h^2; \\
E_i &= -(a_i + b_i h)/12h^2;\; F_i = -\rho(1 - \delta ih)(1 - \beta ih); \\
a_i &= E(1 - \beta ih)(1 - \delta ih)/L^2;\; b_i = -E[\beta(1 - \delta ih) + \delta(1 - \beta ih)]/L^2
\end{aligned}\right\} \tag{14}$$

If equation (13) is evaluated at $\eta = 0$, it can be shown that

$$\left.\begin{aligned}
w_{-1} &= -(D_o/B_o)w_1 = -\left\{[1 - h(\beta + \delta)/2]/[1 + h(\beta + \delta)/2]\right\} w_1 \\
w_{-2} &= -(E_o/A_o)w_2 = -\left\{[1 - h(\beta + \delta)]/[1 + h(\beta + \delta)]\right\} w_2
\end{aligned}\right\} \tag{15}$$

and it can be seen that the first of equations (15) is identical to the corresponding expression given by the first order central differences.

Making use of equations (10) and (15) and evaluating equation (13) at each station $i = 1, 2, \ldots, n$ results in the following frequency equation:

$$
\begin{bmatrix}
\frac{C_1-D_oA_1}{B_o} & D_1 & E_1 & & & & & & & \\
B_2 & C_2 & D_2 & E_2 & & & & & & \\
A_3 & B_3 & C_3 & D_3 & E_3 & & & \mathbf{0} & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & & & & & \\
 & & & & \cdots & A_{n-2} & B_{n-2} & C_{n-2} & D_{n-2} & E_{n-2} \\
 & \mathbf{0} & & & \cdots & & A_{n-1} & B_{n-1} & C_{n-1}+E_{n-1} & D_{n-1} \\
 & & & & \cdots & & & A_n+E_n & B_n+D_n & C_n
\end{bmatrix}
\begin{Bmatrix}
w_1 \\ w_2 \\ w_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ w_{n-2} \\ w_{n-1} \\ w_n
\end{Bmatrix}
$$

$$
= p^2\left\{(F_1w_1)(F_2w_2)\ \ldots\ (F_nw_n)\right\}^T \tag{16}
$$

Torsional Vibrations

Torsional vibrations of tapered cantilever beams: second order finite differences. - The governing equation for free torsional vibrations of a beam is given by

$$
\frac{d}{dz}\left(C\,\frac{d\theta}{dz}\right) + \rho I_p p^2 \theta = 0 \tag{17}
$$

The boundary conditions for a fixed-free beam are

$$
\theta(0) = 0;\ \theta'(L) = 0 \tag{18}
$$

For a tapered cantilever beam of rectangular cross section with width taper β and depth taper δ, torsional rigidity C, and polar moment of inertia I_p take the form

$$C = \frac{Gb_o t_o^3}{3}\left\{(1 - \beta z)(1 - \delta z)^3 - \frac{192}{\pi^5}\frac{t_o}{b_o}\sum_{N=1,3,5}\left[\frac{1}{N^5}\tanh\frac{N\pi}{2}\frac{b_o(1 - \beta z)}{t_o(1 - \delta z)}\right]\right\}$$

$$I_p = \frac{b_o t_o}{12}(1 - \beta z)(1 - \delta z)\left[b_o^2(1 - \beta z)^2 + t_o^2(1 - \delta z)^2\right] \qquad (19)$$

Substituting equations (19) in equation (17), performing the necessary differentiation and writing in nondimensional form, one obtains the following equation in terms of second order finite differences for any station i:

$$A_i\theta_{i-2} + B_i\theta_{i-1} + C_i\theta_i + D_i\theta_{i+1} + E_i\theta_{i+2} = F_i p^2\theta_i \qquad (20)$$

and

$$A_i = S1(S2\ h - S3);\ B_i = S1(16\ S3 - 8h\ S2);\ C_i = -30\ S1\ S3;$$

$$D_i = S1(16\ S3 + 8h\ S2);\ E_i = -S1(S3 + h\ S2);\ F_i = \rho I_p$$

$$S1 = 1/12h^2$$

$$S2 = \frac{Gb_o t_o^3}{3L^2}\left\{\left[-3\delta(1 - \beta ih)(1 - \delta ih)^2 - \beta(1 - \delta ih)^3\right]\right.$$

$$+ \frac{768}{\pi^5 b_o} t_o\delta(1 - \delta ih)^3 \sum_{N=1,3,5}\left[\frac{1}{N^5}\left(\frac{e^{S4} - e^{-S4}}{e^{S4} + e^{-S4}}\right)\right]$$

$$\left. - \frac{192}{\pi^5 b_o}(1 - \delta ih)^4 \sum_{N=1,3,5}\left[\frac{2\pi b_o(\delta - \beta)}{t_o(1 - \delta ih)^2 N^4}(e^{S4} + e^{-S4})\right]\right\} \qquad (21)$$

$$S3 = \frac{Gb_o t_o^3}{3L^2}\left\{(1 - \beta ih)(1 - \delta ih)^3 - \frac{192}{\pi^5 b_o}(1 - \delta ih)^4 \sum_{N=1,3,5}\left[\frac{1}{N^5}\frac{(e^{S4} - e^{-S4})}{(e^{S4} + e^{-S4})}\right]\right\}$$

$$S4 = \frac{N\pi b_o(1 - \beta ih)}{2t_o(1 - \delta ih)}$$

By proceeding on the same lines as presented for the case of axial vibration and noting that the boundary conditions in terms of the finite differences can easily be written analogously to those given in equations (10) and (15), one can develop the frequency equation which will be identical to the one presented in equation (16), with A_i, B_i, . . ., F_i defined by equation (21).

Torsional vibrations of pretwisted-tapered cantilever beams. - When a beam is pretwisted, an increase in the torsional rigidity takes place due to the inclination of the blade fibers in addition to the fiber bending effects. Carnegie (ref. 16) derived a correction factor for fiber bending and pretwist effects for thin rectangular cross section blades. So far, no rigorous mathematical solutions are available for the fiber bending effects of tapered blades. Among the recent contributions on the torsional vibrations of pretwisted blades, at least two works require careful consideration. Duggan and Slyper (ref. 10) observed that the boundary conditions adopted by Carnegie (ref. 16) do not yield accurate solutions for low aspect ratio blades. They drew attention to the boundary condition discussion by Barr (ref. 16) and used a modified set of boundary conditions to obtain the torsional frequencies for low aspect ratio blades. Kaza and Kielb (private communication) reported on the case of rotating pretwisted cantilever blades, allowing for the effects of warping rigidity. The torsional equation of motion was derived, and the boundary conditions were established through a variational formulation (Hamilton's principle). They studied the effect of structural warping and of inertial warping for blades having wide ranges of aspect ratios and rotational speeds. Results were generated by using the Galerkin method. Closed form exact solutions were obtained for the nonrotating case. These equations were used by Subrahmanyam and Kaza (unpublished data) with a first order finite difference method of solution for uniform pretwisted blade cases of small and large aspect ratios; close agreement between the two approaches was shown.

If warping is included, the torsional equation of motion will be of fourth order with variable coefficients, and incorporation of the effects of taper will be more involved. On the other hand, for thin rectangular blades of large thickness and aspect ratios, Carnegie's formulation, which leads to a second order equation of motion and to an appropriate set of boundary conditions, will be adequate, and the equations developed in the preceding section for untwisted beams in torsion can be used with only slight modifications.

If the additional torsional rigidity due to pretwist is incorporated, the net torsional rigidity (neglecting structural warping effects) can be written as

$$C_t = C + C_s \tag{22}$$

where C is given by equation (19) and the increase in torsional rigidity due to pretwist, over and above that of St. Venant, is

$$\left.\begin{aligned} C_s &= \frac{Et_o b_o^5}{180}(1 - \beta z)^5(1 - \delta z)(\xi')^2 \\ \xi &= \gamma z/L = \gamma\eta \end{aligned}\right\} \quad (23)$$

The frequency equation for the case of a pretwisted tapered blade will be obtained by replacing S2 and S3 by $\overline{S2}$ and $\overline{S3}$, which are defined as

$$\left.\begin{aligned} \overline{S2} &= S2 - \frac{Et_o b_o^5 \gamma^2}{180L^2}\left[\delta(1 - \beta z)^5 + 5\beta(1 - \delta z)(1 - \beta z)^4\right] \\ \overline{S3} &= S3 + \frac{Et_o b_o^5}{180L^4}\gamma^2(1 - \beta z)^5(1 - \delta z) \end{aligned}\right\} \quad (24)$$

Flexural Vibrations of Tapered Cantilever Beams

The differential equation for flexural vibration of beams neglecting shear deflection and rotary inertia effects is of the form

$$\frac{d^2}{dz^2}\left(EI\frac{d^2y}{dz^2}\right) - A\rho p^2 y = 0 \quad (25)$$

which can be written in the following nondimensional form

$$\frac{d^2}{d\eta^2}\left(EI\frac{d^2y}{d\eta^2}\right) - \rho A L^4 p^2 y = 0 \quad (26)$$

The boundary conditions for a fixed-free beam can be reduced to

$$\left.\begin{aligned} y = 0 \quad \text{and} \quad \frac{dy}{d\eta} = 0 \quad \text{at} \quad \eta = 0 \\ \frac{d^2y}{d\eta^2} = 0 \quad \text{and} \quad \frac{d^3y}{d\eta^3} = 0 \quad \text{at} \quad \eta = 1 \end{aligned}\right\} \quad (27)$$

Equation (26) is rewritten as follows for a tapered beam of rectangular cross section:

$$a \frac{d^4y}{d\eta^4} + b \frac{d^3y}{d\eta^3} + c \frac{d^2y}{d\eta^2} = dp^2y \tag{28}$$

where

$$\left.\begin{aligned} a &= (1 - \beta\eta)(1 - \delta\eta)^3 \\ b &= -2\beta(1 - \delta\eta)^3 - 6\delta(1 - \beta\eta)(1 - \delta\eta)^2 \\ c &= 6\beta\delta(1 - \delta\eta)^2 + 6\delta^2(1 - \beta\eta)(1 - \delta\eta) \\ d &= 12\rho L^4(1 - \beta\eta)(1 - \delta\eta)/Et_0^2 \end{aligned}\right\} \tag{29}$$

Solution by first order central differences. - The boundary conditions represented by equation (27) when written in terms of first order central differences give

$$\left.\begin{aligned} y_0 = 0; \; y_{-1} = y_1; \; y_{n+1} &= 2y_n - y_{n-1} \\ y_{n+2} &= y_{n-2} - 4y_{n-1} + 4y_n \end{aligned}\right\} \tag{30}$$

By using the finite difference expressions for the derivatives from the appendix, incorporating the relations given by equation (30) and following on the lines described earlier, the frequency equation for the flexural vibrations can be developed easily and is identical to that given by Carnegie and Thomas (ref. 11).

Solution by second order central differences. - Substituting the second order finite difference equivalents for the derivatives from the appendix in equation (28), one obtains the following equation for any arbitrary station i:

$$\begin{aligned} &A_iy_{i-3} + B_iy_{i-2} + C_iy_{i-1} + D_iy_i + E_iy_{i+1} \\ &\qquad + F_iy_{i+2} + G_iy_{i+3} = H_ip^2y_i \qquad i = 1, 2, \ldots, n \end{aligned} \tag{31}$$

where

$$A_i = \frac{b_i}{8h^3} - \frac{a_i}{6h^4}\ ;\ B_i = \frac{2a_i}{h^4} - \frac{b_i}{h^3} - \frac{c_i}{12h^2}\ ;\ C_i = \frac{13b_i}{8h^3} - \frac{39a_i}{6h^4} + \frac{16c_i}{12h^2}\ ;$$

$$D_i = \frac{56a_i}{6h^4} - \frac{30c_i}{12h^2}\ ;\ E_i = \frac{4c_i}{3h^2} - \frac{39a_i}{6h^4} - \frac{13b_i}{8h^3}\ ;\ F_i = \frac{2a_i}{h^4} + \frac{b_i}{h^3} - \frac{c_i}{12h^2}\ ;$$

$$G_i = -\frac{a_i}{6h^4} + \frac{b_i}{8h^3}\ ;\ H_i = 12\rho L^4(1 - \beta ih)(1 - \delta ih)/Et_0^2 \tag{32}$$

$$a_i = (1 - \beta ih)(1 - \delta ih)^3$$

$$b_i = -2\beta(1 - \delta ih)^3 - 6\delta(1 - \beta ih)(1 - \delta ih)^2$$

$$c_i = 6\beta\delta(1 - \delta ih)^2 + 6\delta^2(1 - \beta ih)(1 - \delta ih)$$

In the case of first order central differences, the fictitious stations y_{-1}, y_{n+1}, and y_{n+2} can directly be eliminated by using the finite difference equivalents of the boundary conditions given by equations (30).

In the present case of second order central differences, additional fictitious stations y_{-2} and y_{n+3} must be eliminated in addition to those mentioned earlier. This is accomplished by again using the symmetry conditions

$$y_{-1} = y_1;\ y_{-2} = y_2 \tag{33}$$

Elimination of the fictitious stations y_{n+1}, y_{n+2}, and y_{n+3} is accomplished by following the conditions obtained from the first order theory in the following manner:

$$y_n'' = 0 \quad \text{leads to} \quad y_{n+1} = -y_{n-1} + 2y_n \tag{34}$$

$$y_n''' = 0 \quad \text{leads to} \quad y_{n+2} = y_{n-2} - 4y_{n-1} + 4y_n \tag{35}$$

These equations can be written in the following alternative forms:

$$(y_{n+1} - y_{n-1}) = 2(y_n - y_{n-1}) \tag{36}$$

$$(y_{n+2} - y_{n-2}) = 4(y_n - y_{n-1}) \tag{37}$$

and by recursion, one can eliminate y_{n+3} from

$$(y_{n+3} - y_{n-3}) = 6(y_n - y_{n-1}) \tag{38}$$

If y_{n+1}, y_{n+2}, and y_{n+3} from equations (36) to (38) are introduced into the second order central difference equivalent of y_n''' given in equation (A8), one can show that $y_n''' \equiv 0$. By using equations (36) and (37) in the finite difference equivalent of y_n'' and setting the result to zero, one obtains the following relation:

$$y_{n-2} - 2y_{n-1} + y_n = 0 \tag{39}$$

Equation (39) states that the deflection at the $(n - 1)^{th}$ station is the average of the deflections at the preceding and succeeding stations. Thus, the deflection curve near the tip of the cantilever beam assumes a straight line form. Since the bending moment at the free end is zero, the condition of constant slope near the tip is justified and, thus, the boundary conditions represented by equations (36) to (38) should give accurate results for n suitably large.

The eigenvalue problem that results by using the second order central differences in equation (31) together with equations (36) to (38) is as follows:

$$[R]\{y\} = p^2 \frac{\rho A L^4}{EI} [S]\{y\} \tag{40}$$

where

$$
[R] = \begin{bmatrix}
B_1+D_1 & A_1+E_1 & F_1 & G_1 & 0 & 0 & 0 & 0 & 0 & & & & & & & \\
A_2+C_2 & D_2 & E_2 & F_2 & G_2 & 0 & 0 & 0 & 0 & & & & 0 & & & \\
B_3 & C_3 & D_3 & E_3 & F_3 & G_3 & 0 & 0 & 0 & & & & & & & \\
A_4 & B_4 & C_4 & D_4 & E_4 & F_4 & G_4 & 0 & 0 & & & & & & & \\
0 & A_5 & B_5 & C_5 & D_5 & E_5 & F_5 & G_5 & 0 & & & & & & & \\
 & & & & & & & \text{Nonzero elements} & & & & & & & & \\
 & & & & & & & & 0 & A_{n-3} & B_{n-3} & C_{n-3} & D_{n-3} & E_{n-3} & F_{n-3} & G_{n-3} \\
 & & 0 & & & & & & 0 & 0 & A_{n-2} & B_{n-2} & C_{n-2} & D_{n-2} & E_{n-2}-G_{n-2} & F_{n-2}+2G_{n-2} \\
 & & & & & & & & 0 & 0 & 0 & A_{n-1} & B_{n-1} & C_{n-1}+G_{n-1} & D_{n-1}-F_{n-1}-4G_{n-1} & E_{n-1}+2F_{n-1}+4G_{n-1} \\
 & & & & & & & & 0 & 0 & 0 & 0 & A_n+G_n & B_n+F_n & C_n-E_n-4F_n-6G_n & D_n+2E_n+4F_n+6G_n
\end{bmatrix}
\tag{41a}
$$

$$[S] = \begin{bmatrix} H_1 & & & & & \\ & H_2 & & & \mathbf{0} & \\ & & \cdot & & & \\ & & & \cdot & & \\ & \mathbf{0} & & & \cdot & \\ & & & & & H_n \end{bmatrix} \tag{41b}$$

and

$$\{y\} = \left\{y_1 y_2 \cdot \cdot \cdot y_n\right\}^T \tag{41c}$$

RESULTS AND DISCUSSION

The eigenvalue problems given by equations (8), (16), and (40) for the representative cases were solved on an IBM 370 computer by using the IMSL routine EIGZF. The theoretical results obtained thus with the first and second order central difference theories are compared with the results available in the literature and these are presented in what follows.

Axial Vibrations

The following numerical data were used to determine the axial vibratory modes of cantilever beams having a length of 0.1524 m (6 in.), various taper parameters, a thickness ratio $t_0/b_0 = 0.20$, and an aspect ratio $L/b_0 = 2.4$. Table I shows the relative convergence rates of the first and second order central difference theories for the case of a uniform beam and one tapered beam with β (or δ) = 0.6 and δ (or β) = 0.8. It can be seen from this table that with $n = 5$, the second order central difference theory produces the fundamental frequency of the uniform beam to within 0.005 percent error and the second mode frequency to within 0.4 percent error. A solution with at least $n = 15$ is required in the case of the first order central differences to attain this accuracy. The convergence of higher modes is, however, not so rapid in the second order theory with $n = 5$. Comparing the second order central difference solution with $n = 10$ and the first order solution with $n = 25$, one can observe that the maximum error encountered in the fourth mode frequency is of the order of 0.7 percent using second order theory with $n = 10$, while the corresponding error with $n = 25$ and using first order theory is about 0.8 percent. From these results, it can be seen that the convergence is very rapid in the case of second order central differences and that the lowest five axial mode frequencies can be obtained to within 0.027 percent error with a beam divided into 30 segments ($n = 30$). The first order theory produces results with errors of the order of 1.0 percent, and Richardson's extrapolation of first order theory with $n = 10, 20, 30$ produces accurate results.

Similar trends of convergence are observed for the case of tapered beams also, as can be seen from table I. It may be noted here that the percentage error magnitudes are relatively higher than the corresponding values obtained for the uniform beam. Accurate results up to the fifth mode can be obtained for all the taper parameters studied here by using a second order theory with $n = 30$ for all practical purposes. The frequency ratios presented in table II can be considered as close lower bound solutions. A further comparison of the first two axial mode frequencies with those presented in reference 17 indicates a close agreement.

Torsional Vibrations

In order to study the torsional vibrational characteristics, use is made of the same numerical data employed in the case of axial vibrations. As is well known, the torsional equation of motion can be obtained by replacing E in the equation for axial motion by (C/I_p), and consequently, the convergence patterns for the torsional vibrational frequencies for the case of uniform beam are identical to the corresponding values presented in table I for the case of axial motion. Since the convergence trends have already been established by the axial vibratory motion study, the torsional frequency parameter ratios (defined as the ratio of the square of the natural torsional frequency of any mode to the square of the fundamental torsional frequency) were obtained by using the second order central difference theory with $n = 30$. The ratios for the lowest five torsional modes are presented in figures 1 to 5. Comparison of the present theoretical frequencies with results obtained from the Holzer method (ref. 18) and the Reissner method (ref. 20) indicates close agreement.

Table III shows a comparison of the fundamental torsional characteristic function of a uniform beam obtained from the second order central difference theory with $n = 30$ and the exact characteristic function. It can be seen that the theoretical results agree with the exact solution (ref. 19) up to five significant digits. Further, the higher mode characteristic functions are also in extremely close agreement with the corresponding exact values.

The numerical data given in references 18 and 21 were used to determine the torsional frequencies of pretwisted cantilever beams. These frequencies were modified by applying a correction factor given by Carnegie (ref. 16) to account for the fiber bending effects, and these results are presented in figure 6. Second order central difference theory was used with $n = 15$. It can be seen that the present results agree closely with the experimental results of Carnegie (ref. 16) and with the theoretical results from the Reissner method (ref. 20). Figure 7 shows a comparison of the present natural frequencies of pretwisted tapered blading with experimental results (ref. 22) and the theoretical results obtained by using the Galerkin technique. A correction factor, applicable to uniform beams (ref. 16), was applied to the torsional frequencies obtained here by using second order central difference theory with $n = 15$. Close agreement between the various methods can be seen here also.

Flexural Vibrations

The following numerical data were used to study the flexural vibrations of cantilever beams having a length of 0.254 m (10 in.) and various breadth and thickness taper ratios: thickness ratio $t_0/b_0 = 1.0$; aspect ratio $L/b_0 = 40$. The shape factor for this uniform beam $\sqrt{I/AL^2}$ is 0.00721688 so that the higher order effects like shear deformation and rotary inertia can safely be ignored. The breadth and thickness taper parameters were varied from -0.75 to 0.75 in steps of 0.25, and various combinations of these taper parameters were studied.

A convergence study has been made for the case of a uniform beam with the beam divided into an odd number of segments ($n = 5$, 11, 15, 17, 23, 25, 29) and an even number of segments ($n = 6$, 10, 12, 18, 20, 24, 30). Both first and second order central difference theories were used, and the frequency ratios are shown in table IV. These values are also shown in graphical form in figure 8; it can be seen from this figure that the convergence is monotonic from below for both the first and second order central differences and that the convergence is continuous for even and odd values of n.

The convergence rates of the two methods can be compared from table IV or figure 8. As has been observed in the earlier cases of axial or torsional vibrations, the lowest two flexural mode frequencies given by the second order theory with $n = 5$ are better than those given by the first order theory with $n = 10$. With $n = 30$, the second order central difference theory produces the lowest five flexural frequencies to within 0.2 percent error while the first order theory shows errors of the order of 2.6 percent. Using the second order theory with $n = 30$, the frequency parameter ratios (defined as the ratio of the square of the natural frequency of any mode of a tapered beam to the square of the fundamental flexural frequency of a uniform beam of comparable dimensions at the root) are calculated and presented in figures 9 to 19 for the lowest five flexural modes. Graphs are drawn showing the effects of breadth (or depth) taper for a given depth (or breadth) taper. Comparisons are made with the theoretical results, obtained earlier by using the Reissner method (ref. 23) and the Galerkin process (ref. 24), and also with experimental results (refs. 11 and 25). Close agreement of the results obtained by the various approaches is observed. The characteristic functions obtained by using the first and second order theories are presented in table V for the case of a uniform beam where further comparisons are made with the exact characteristic functions (ref. 19). It has been observed that the second order central difference theory gives characteristic functions close to the exact ones. Figures 20 to 24 show the mode shapes of tapered cantilever beams obtained by means of the second order theory. These mode shapes agree very closely with the experimental mode shapes obtained by Carnegie and Dawson (ref. 25).

CONCLUDING REMARKS

The second order finite difference method has been successfully applied to determine the uncoupled dynamic characteristics of cantilevered beams having variable mass and elasticity properties. Simple recursive relations have been used to eliminate the fictitious stations outside the beam domain by making logical extensions from the first order theory. The present approach is shown to produce accurate natural frequencies and mode shapes. The present improved finite difference method has the following specific advantages compared to the classical approach of using the first order central differences:

1. For the same mesh size (step length h), the second order finite difference method produces natural frequencies with greater accuracy than the first order theory.

2. The convergence of the lower mode frequencies is very rapid in the case of second order central differences compared to that of the first order theory.

3. Because of the rapid convergence shown by the present approach, accurate natural frequencies and mode shapes can be obtained directly by using a suitable number of segments without any necessity of the extrapolations that are customary with the first order central difference theory.

4. Finally, there are few methods which produce close lower bounds; the present technique may be invaluable in obtaining close lower bound solutions without requiring extrapolations. It may be noted that even though the finite difference method produces lower bound solutions in general, an extrapolated result obtained by using the Richardson method does not necessarily give a bound. Thus, the present improved approach eliminates most of the shortcomings associated with the conventional approaches.

The method developed in this report has the potential for extension to complex blade vibration problems involving coupling between in-plane and out-of-plane bending and torsional motions. Further extension to plate theory may prove beneficial since rapid convergence in the two-dimensional case may reduce the computational space and time considerably.

APPENDIX - FINITE-DIFFERENCE EQUATIONS FOR DERIVATIVES

First Order Central Differences:

$$\varphi'_i = \frac{1}{2h}(-\varphi_{i-1} + \varphi_{i+1}) \tag{A1}$$

$$\varphi''_i = \frac{1}{h^2}(\varphi_{i-1} - 2\varphi_i + \varphi_{i+1}) \tag{A2}$$

$$\varphi'''_i = \frac{1}{2h^3}(-\varphi_{i-2} + 2\varphi_{i-1} - 2\varphi_{i+1} + \varphi_{i+2}) \tag{A3}$$

$$\varphi^{iv}_i = \frac{1}{h^4}(\varphi_{i-2} - 4\varphi_{i-1} + 6\varphi_i - 4\varphi_{i+1} + \varphi_{i+2}) \tag{A4}$$

Second Order Central Differences:

$$\varphi'_i = (\varphi_{i-2} - 8\varphi_{i-1} + 8\varphi_{i+1} - \varphi_{i+2})/12h \tag{A5}$$

$$\varphi''_i = (-\varphi_{i-2} + 16\varphi_{i-1} - 30\varphi_i + 16\varphi_{i+1} - \varphi_{i+2})/12h^2 \tag{A6}$$

$$\varphi'''_i = (\varphi_{i-3} - 8\varphi_{i-2} + 13\varphi_{i-1} - 13\varphi_{i+1} + 8\varphi_{i+2} - \varphi_{i+3})/8h^3 \tag{A7}$$

$$\varphi^{iv}_i = (-\varphi_{i-3} + 12\varphi_{i-2} - 39\varphi_{i-1} + 56\varphi_i - 39\varphi_{i+1} + 12\varphi_{i+2} - \varphi_{i+3})/6h^4 \tag{A8}$$

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TABLE I. - RELATIVE CONVERGENCE RATES OF FREQUENCY RATIOS* OF AXIAL VIBRATION FOR TYPICAL TAPER PARAMETERS

n	First mode		Second mode		Third mode		Fourth mode		Fifth mode	
	First order	Second order	First order	Second order	First order	Second order	First order	Second order	First order	Second order
Breadth taper ß = 0; depth taper δ = 0										
5	0.99589	0.99994644	2.89019	2.98783	4.50158	4.86226	5.67232	6.37886	6.28782	7.23831
10	.99897	.99999702	2.97232	2.99920	4.87248	4.98999	6.65266	6.94877	8.26903	8.83121
15	.99954	.99999941	2.98768	2.99984	4.93408	4.99796	6.84432	6.98929	8.67058	8.96348
20	.99974	.99999987	2.99307	2.99995	4.96794	4.99935	6.91218	6.99654	8.81380	8.98805
25	.99984	.99999996	2.99556	2.99998	4.97946	4.99973	6.94372	6.99857	8.88056	8.99503
30	.99989	1.00000000	2.99692	2.99999	4.95873	4.99987	6.96089	6.99931	8.91696	8.99758
Exact solution (ref. 19)	1.0		3.0		5.0		7.0		9.0	
Breadth taper ß = 0.6, depth taper δ = 0.8 or breadth taper ß = 0.8, depth taper δ = 0.6										
5	1.47717	1.49710	3.06602	3.20418	4.50158	4.88551	5.57925	6.26438	6.19245	7.10365
10	1.50305	1.50851	3.23515	3.27744	5.01479	5.15540	6.72215	7.04261	8.28857	8.86965
15	1.50785	1.51035	3.26716	3.28681	5.11520	5.18261	6.95531	7.11627	8.74112	9.05151
20	1.50955	1.51097	3.27840	3.28966	5.15069	5.18970	7.03833	7.13296	8.90366	9.09013
25	1.51032	1.51124	3.28362	3.29089	5.16718	5.19249	7.07701	7.13885	8.97962	9.10270
30	1.51074	1.51139	3.28645	3.29154	5.17615	5.19387	7.09808	7.14153	8.02107	9.10801

$$\text{*Frequency ratio} = \frac{\text{Natural frequency of tapered beam in any mode}}{\text{Natural frequency of uniform beam in fundamental mode}}.$$

TABLE II. - COMPARISON OF FREQUENCY RATIOS* FOR TAPERED BLADING IN AXIAL VIBRATIONS:

FINITE DIFFERENCE SOLUTIONS; n = 30

Breadth taper, β	Depth taper, δ	First mode		Second mode		Third mode		Fourth mode		Fifth mode	
		First order	Second order	First order	Second order	First order	Second order	First order	Second order	First order	Second order
0	0	0.99989	1.00000	2.99692	2.99999	4.98573	4.99987	6.96089	6.99931	8.91696	8.99758
0	.2	1.04543	1.04557	3.01268	3.01502	4.99504	5.00916	6.96733	7.00518	8.92176	9.00002
0	.4	1.01460	1.10481	3.03616	3.03950	5.00911	5.02368	6.97713	7.01614	8.92908	9.01044
0	.6	1.18625	1.18653	3.07709	3.08070	5.03462	5.04963	6.99510	7.03472	8.94257	9.02467
0	.8	1.31006	1.31047	3.16917	3.17333	5.09932	5.11536	7.04296	7.08405	8.97936	9.06332
.2	.2	1.09189	1.09208	3.02991	3.03322	5.00528	5.01980	6.97444	7.01340	8.92707	9.00835
.2	.4	1.15215	1.15240	3.05523	3.05872	5.02055	5.03537	6.98509	7.02445	8.93508	9.01680
.2	.6	1.23511	1.23545	3.09858	3.10236	5.04772	5.06303	7.00427	7.04430	8.94943	9.03204
.2	.8	1.36046	1.36093	3.19405	3.19843	5.11506	5.13147	7.05413	7.09573	8.89777	9.07238
.4	.4	1.21366	1.21397	3.08289	3.08659	5.03739	5.05256	6.99687	7.03700	8.94384	9.02620
.4	.6	1.29802	1.29843	3.12938	3.13340	5.06679	5.08251	7.01768	7.05827	8.95949	9.04280
.4	.8	1.42465	1.42521	3.22938	3.23404	5.13781	5.15474	7.07036	7.11269	9.00003	9.08554
.6	.6	1.38368	1.38418	3.18019	3.18459	5.09947	5.11584	7.04094	7.08243	8.97702	9.06146
.6	.8	1.51074	1.51139	3.28645	3.29154	5.17615	5.19387	7.09808	7.14153	9.02107	9.10801
.8	.8	1.63529	1.63605	3.40099	3.40678	5.26289	5.28214	7.16392	7.20971	9.07223	9.16222

$$\text{*Frequency ratio} = \frac{\text{Natural frequency of tapered beam in any mode}}{\text{Natural frequency of uniform beam in fundamental mode}} .$$

TABLE III. - COMPARISON OF CHARACTERISTIC FUNCTIONS OF AXIAL OR TORSIONAL VIBRATIONS OF UNIFORM CANTILEVER BEAM: SECOND ORDER CENTRAL DIFFERENCE THEORY; n = 30

Axial fractional length, $\eta = z/L$	First mode		Second mode; theoretical	Third mode; theoretical	Fourth mode; theoretical	Fifth mode; theoretical
	Exact solution (ref. 19)	Theoretical				
0.0	0.00000	0.00000	-0.00000	0.00000	0.00000	0.00000
.1	.15643	.15643	-.45399	-.70711	-.89101	-.98769
.2	.30902	.30902	-.80902	-1.00000	-.80902	-.30902
.3	.45399	.45399	-.98769	-.70711	.15643	.89101
.4	.58779	.58779	-.95106	0.45157×10^{-13}	.95106	.58779
.5	.70711	.70711	-.70711	.70711	.70711	-.70711
.6	.80902	.80902	-.30902	1.00000	-.30902	-.80902
.7	.89101	.89101	.15643	.70711	-.98769	.45399
.8	.95106	.95106	.58779	$-.47591\times10^{-14}$	-.58779	.95106
.9	.98769	.98769	.89101	-.70711	.45399	-.15643
1.0	1.00000	1.00000	1.00000	-1.00000	1.00000	-1.00000

TABLE IV. - CONVERGENCE PATTERN OF FLEXURAL FREQUENCIES OF UNIFORM CANTILEVER BEAM USING FIRST AND SECOND ORDER CENTRAL DIFFERENCE THEORIES: NONDIMENSIONAL FREQUENCIES $p\sqrt{\rho AL^4/EI}$

n	First mode		Second mode		Third mode		Fourth mode		Fifth mode	
	First order	Second order	First order	Second order	First order	Second order	First order	Second order	First order	Second order
5	3.4021	3.5062	18.870	21.204	45.334	54.421	72.469	91.02	92.603	118.85
6	3.4359	3.5104	19.107	21.553	49.370	57.320	83.386	101.85	114.54	144.60
10	3.4866	3.5148	21.134	21.934	56.603	60.774	104.52	116.59	160.58	185.93
11	3.4916	3.5152	21.284	21.960	57.423	61.014	107.06	117.71	166.45	189.46
12	3.4955	3.5153	21.340	21.977	58.063	61.179	109.07	118.48	171.13	191.94
15	3.5029	3.5158	21.623	22.006	59.318	61.443	113.07	119.72	180.62	195.99
17	3.5057	3.5158	21.713	22.015	59.827	61.526	114.72	120.12	184.59	197.30
18	3.5069	3.5158	21.750	22.018	60.024	61.555	115.35	120.25	186.14	197.74
20	3.5086	3.5159	21.801	22.023	60.334	61.595	116.37	120.44	188.62	198.36
23	3.5104	3.5159	21.857	22.027	60.660	61.632	117.44	120.61	191.25	198.92
24	3.5108	3.5159	21.872	22.028	60.744	61.640	117.72	120.65	191.93	199.04
25	3.5113	3.5159	21.884	22.029	60.817	61.647	117.96	120.68	192.53	199.15
29	3.5125	3.5160	21.923	22.031	61.041	61.666	118.70	120.76	194.37	199.42
30	3.5127	3.5160	21.930	22.031	61.083	61.669	118.85	120.78	194.72	199.47
Exact solution (ref. 19)	3.5160		22.0345		61.69073		120.9010		199.8604	
Percent error based on n = 30	-0.094	0.000	-0.474	-0.016	-0.996	-0.046	-1.696	-0.100	-2.572	-0.195

TABLE V. - COMPARISON OF CHARACTERISTIC FUNCTIONS OF FLEXURAL

VIBRATION OF UNIFORM CANTILEVER BEAM: n = 30

Axial fractional length, $\eta = z/L$	Exact solution (ref. 19)	First order central differences	Second order central differences
First mode			
0.0	0.000000	0.0	0.0
.1	.016775	.016848	.016783
.2	.063870	.063992	.063887
.3	.136485	.136629	.136508
.4	.229885	.230038	.229919
.5	.339525	.339671	.339565
.6	.461135	.461267	.461181
.7	.590875	.590985	.590922
.8	.725480	.725557	.725517
.9	.86240	.862443	.862424
1.0	1.000000	1.000000	1.000000
Second mode			
0.0	0.0	0.0	0.0
.1	.092630	.094109	.092979
.2	.301056	.303290	.301818
.3	.526135	.528641	.527350
.4	.683470	.685965	.685029
.5	.713665	.715997	.715351
.6	.589475	.591552	.591031
.7	.317050	.318772	.318252
.8	-.070035	-.068810	-.069318
.9	-.523750	-.523154	-.523497
1.0	-1.000000	-1.000000	-1.000000
Third mode			
0.0	0.0	0.0	0.0
.1	.228070	.234307	.230020
.2	.604505	.612659	.608300
.3	.756240	.763252	.760600
.4	.525925	.530052	.528838
.5	.019685	.020298	.019714
.6	-.473765	-.476490	-.476608
.7	-.657425	-.662509	-.661663
.8	-.394875	-.400383	-.398434
.9	.228510	.225100	.226988
1.0	1.000000	1.000000	1.000000
Fourth mode			
0.0	0.0	0.0	0.0
.1	.385010	.400286	.390900
.2	.753790	.768408	.762570
.3	.433870	.437896	.438329
.4	-.315560	-.322786	-.319340
.5	-.707120	-.719090	-.715030
.6	-.326495	-.334367	-.330318
.7	.397390	.399734	.401725
.8	.643040	.654228	.651489
.9	.052035	.062183	.057070
1.0	1.000000	1.000000	1.000000
Fifth mode			
0.0	0.0	0.0	0.0
.1	.537245	.565373	.550025
.2	.659625	.673812	.671266
.3	-.211285	-.224220	-.216402
.4	-.696550	-.715362	-.709451
.5	.000850	.000992	.000859
.6	.700255	.719567	.713189
.7	.225730	.240056	.230950
.8	-.600450	-.610899	-.611704
.9	-.294005	-.314576	-.305421
1.0	1.000000	1.000000	1.000000

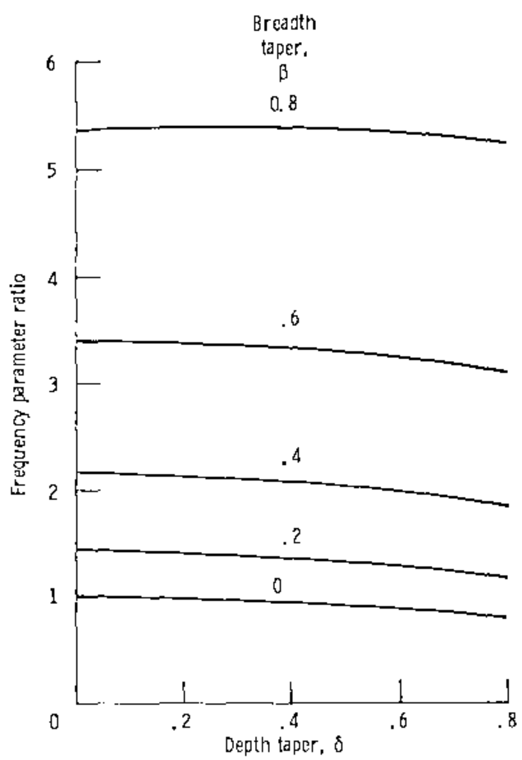

Figure 1. - Torsional frequencies of tapered canti-levered beams. First mode.

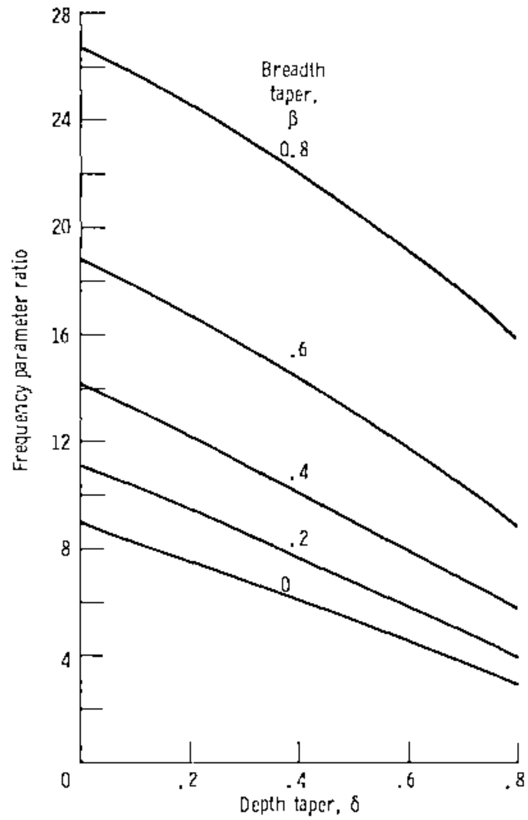

Figure 2. - Torsional frequencies of tapered canti-levered beams. Second mode.

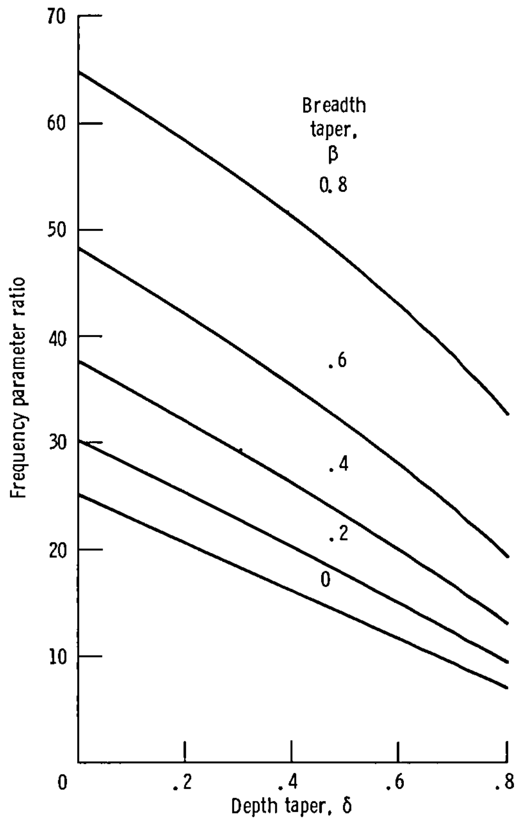

Figure 3. - Torsional frequencies of tapered cantilevered beams. Third mode.

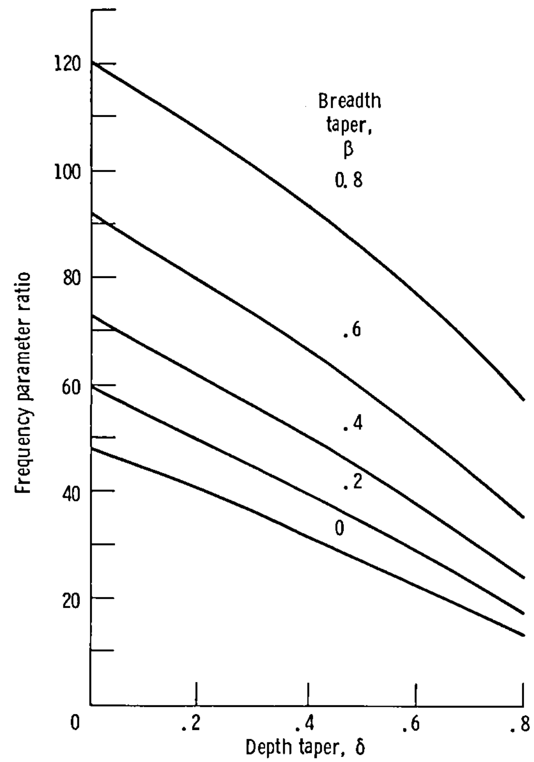

Figure 4. - Torsional frequencies of tapered cantilevered beams. Fourth mode.

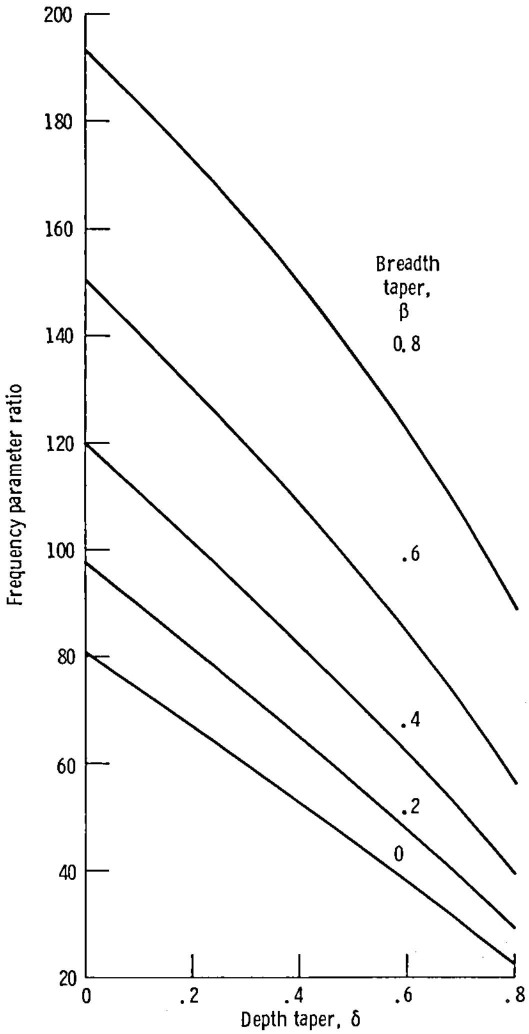

Figure 5. - Torsional frequencies of tapered cantilevered beams. Fifth mode.

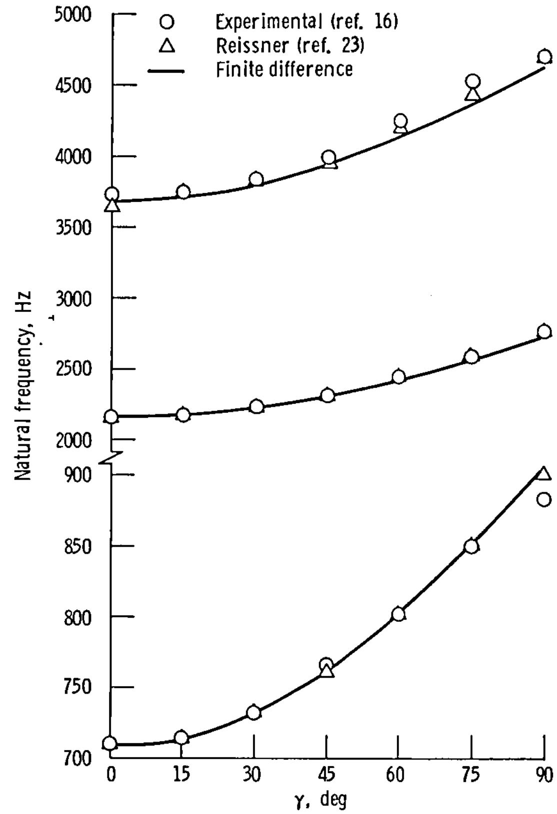

Figure 6. - Effect of pretwist on the lowest three torsional modes.

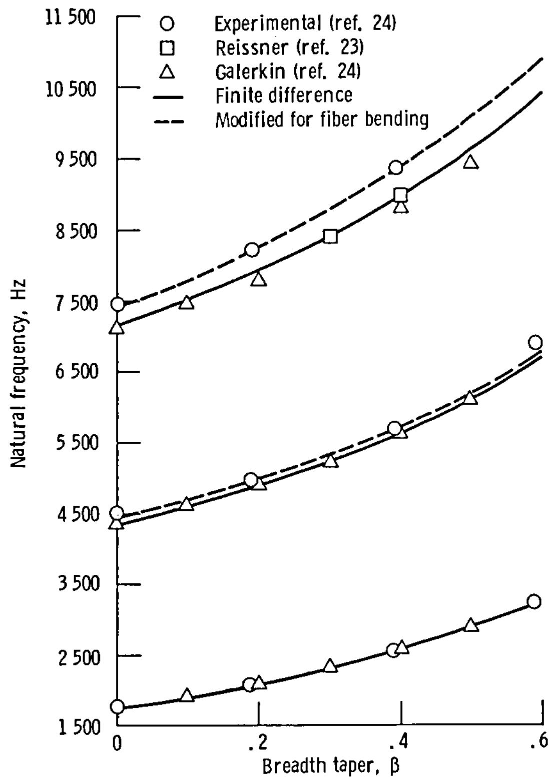

Figure 7. - Effect of width taper parameter (β) for depth taper parameter δ = 0.436464 and pretwist γ = 45° on three lowest torsional frequencies.

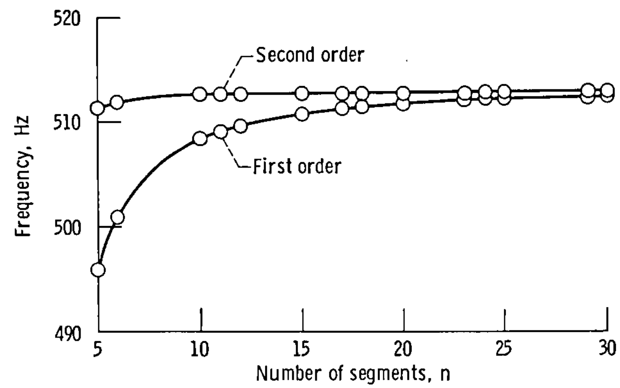

Figure 8. - Relative convergence pattern of fundamental flexural frequency.

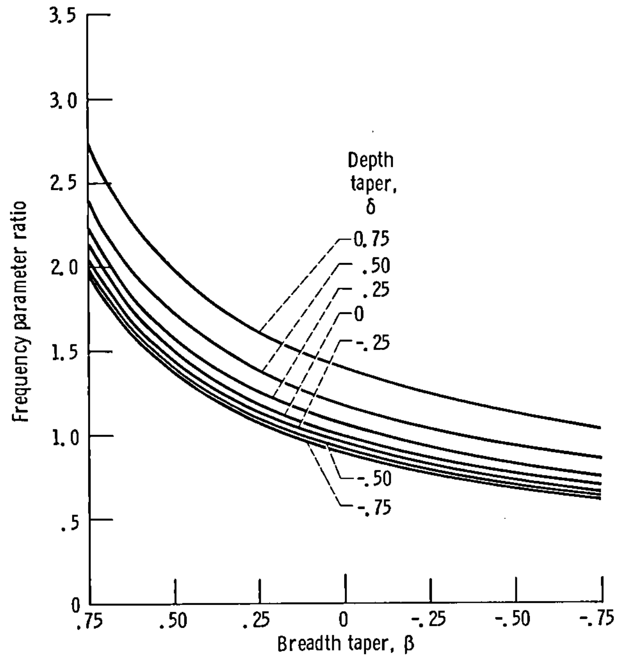

Figure 9. - Frequency parameter ratios of tapered cantilevered beams vibrating in flexure. First mode - effect of breadth taper.

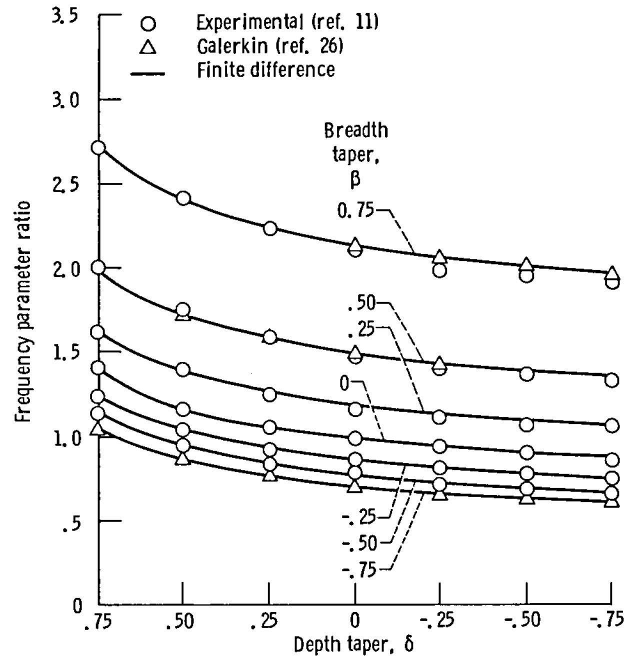

Figure 10. - Frequency parameter ratios of tapered cantilevered beams vibrating in flexure. First mode - effect of depth taper.

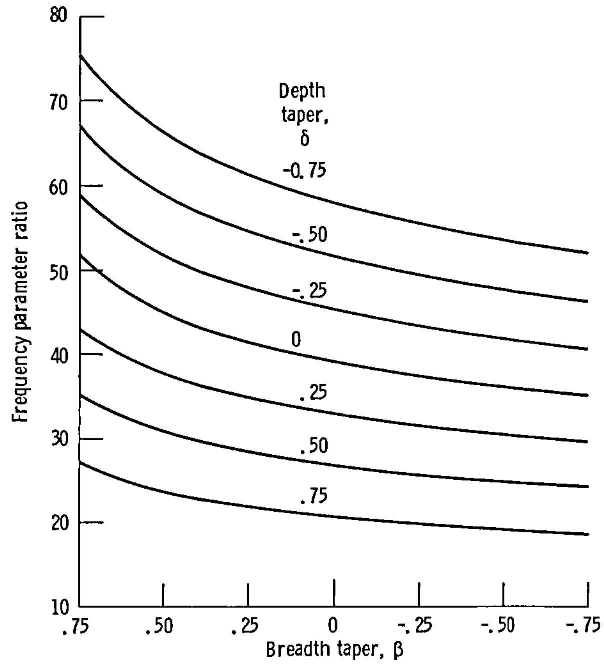

Figure 11. - Frequency parameter ratios of tapered cantilevered beams vibrating in flexure. Second mode - effect of breadth taper.

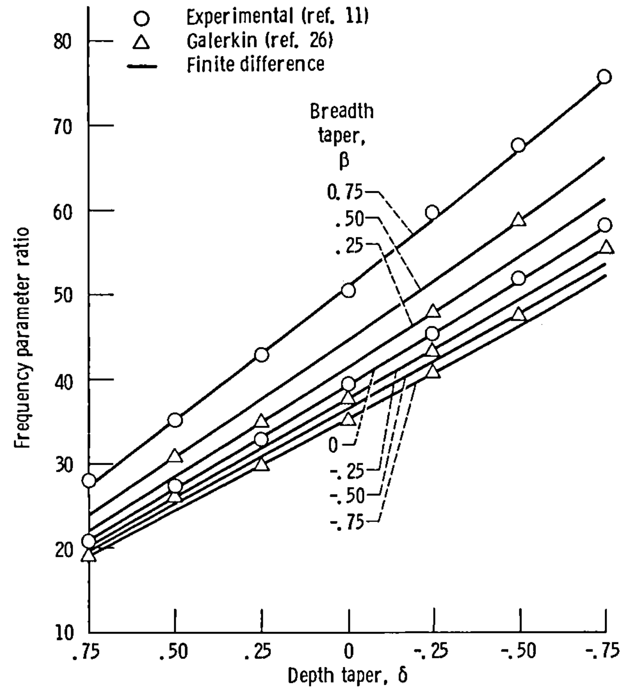

Figure 12. - Frequency parameter ratios of tapered cantilevered beams vibrating in flexure. Second mode - effect of depth taper.

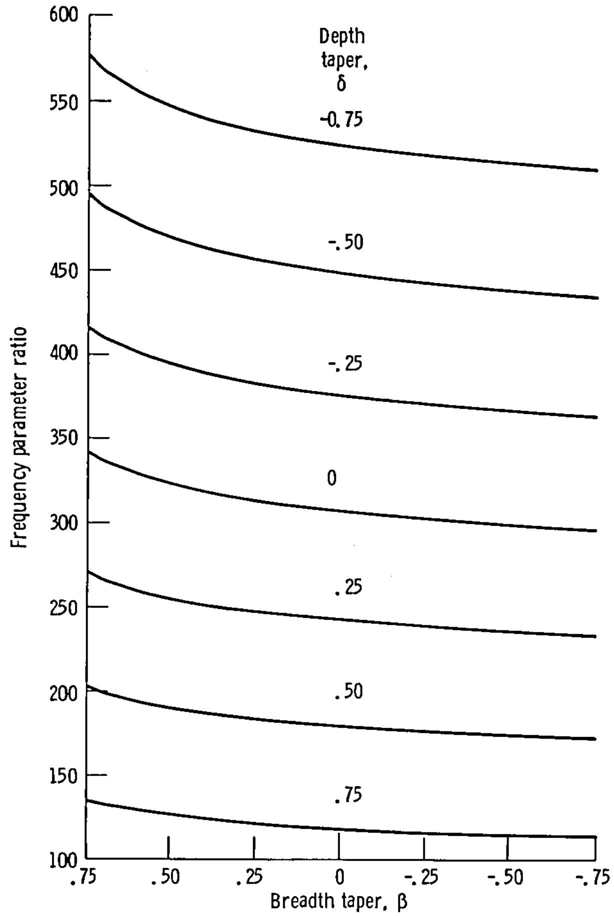

Figure 13. - Frequency parameter ratios of tapered cantilevered beams vibrating in flexure. Third mode - effect of breadth taper.

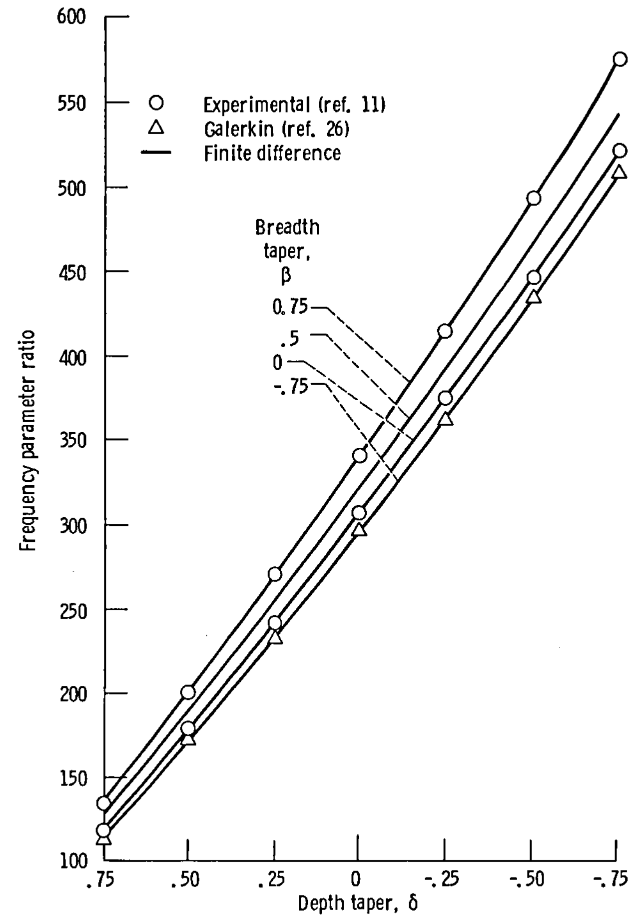

Figure 14. - Frequency parameter ratios of tapered cantilevered beams vibrating in flexure. Third mode - effect of depth taper. (Breadth taper, β = 0.75, 0.5, 0, and -0.75.)

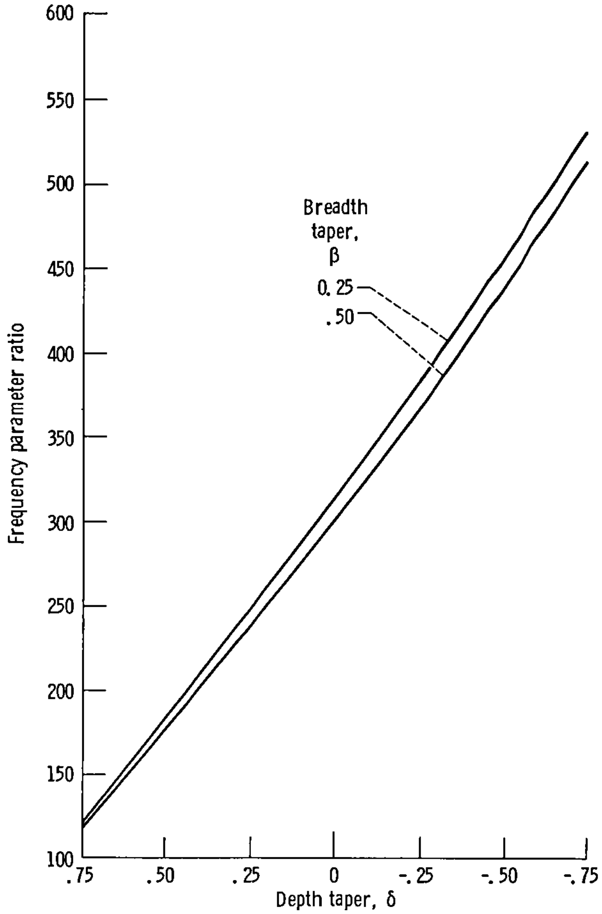

Figure 15. - Frequency parameter ratios of tapered cantilevered beams vibrating in flexure. Third mode - effect of depth taper. (Breadth taper, β = 0.25 and -0.50.)

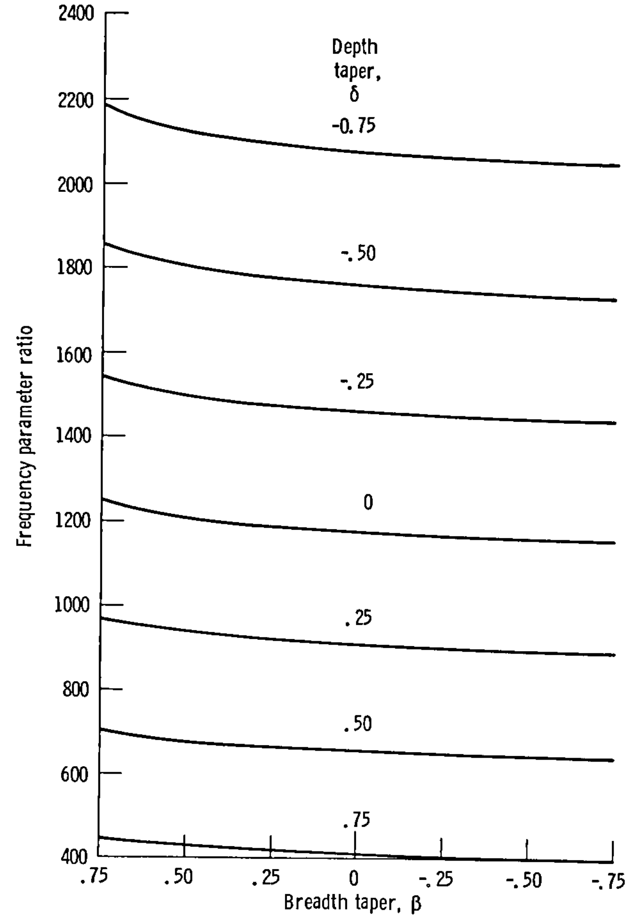

Figure 16. - Frequency parameter ratios of tapered cantilevered beams vibrating in flexure. Fourth mode - effect of breadth taper.

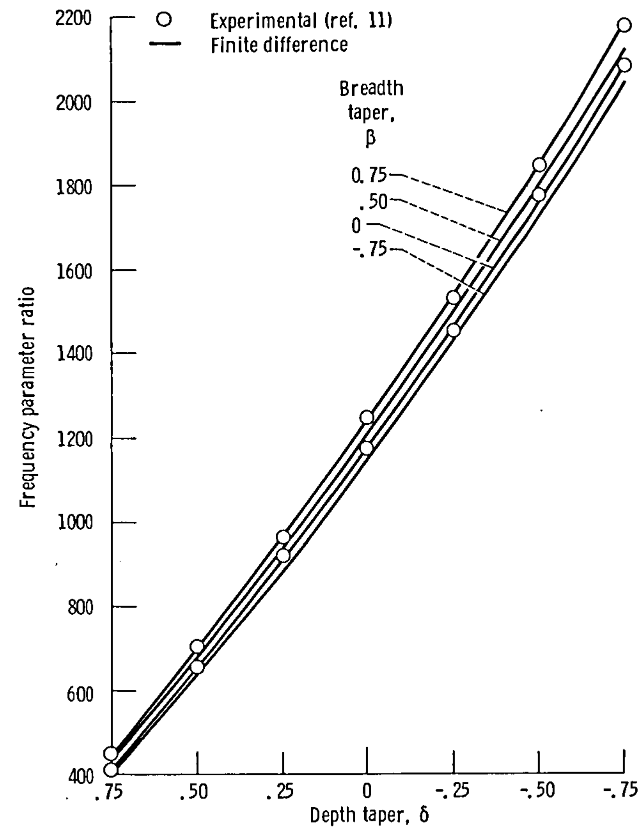

Figure 17. - Frequency parameter ratios of tapered cantilevered beams vibrating in flexure. Fourth mode - effect of depth taper.

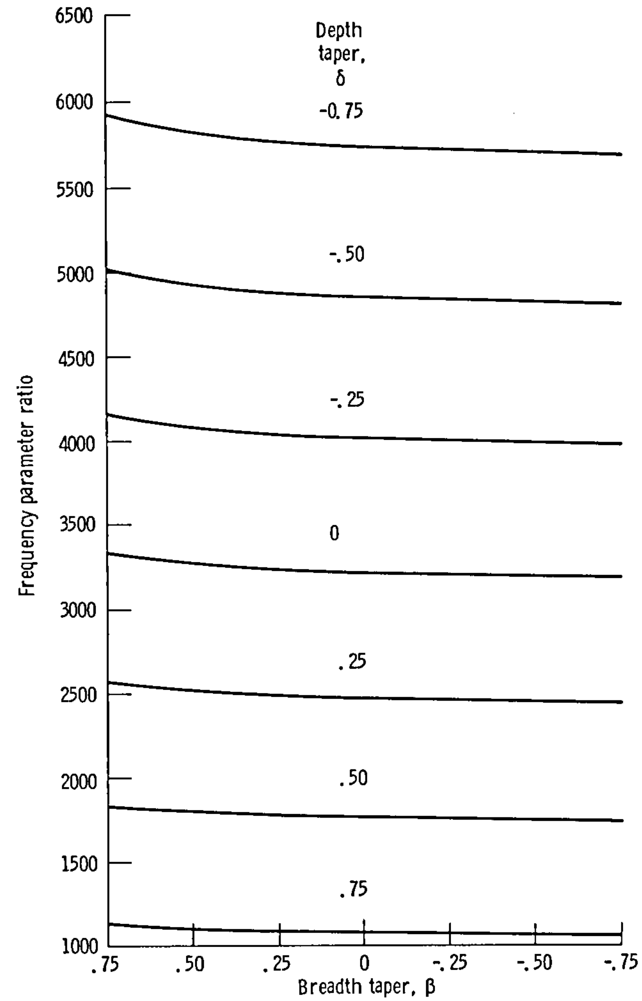

Figure 18. - Frequency parameter ratios of tapered cantilevered beams vibrating in flexure. Fifth mode - effect of breadth taper.

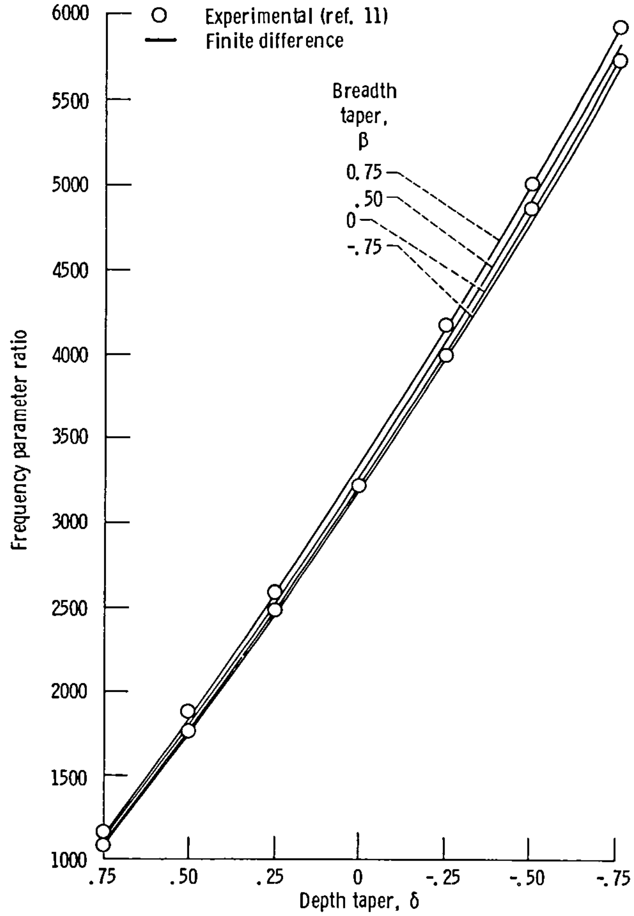

Figure 19. - Frequency parameter ratios of tapered cantilevered beams vibrating in flexure. Fifth mode - effect of depth taper.

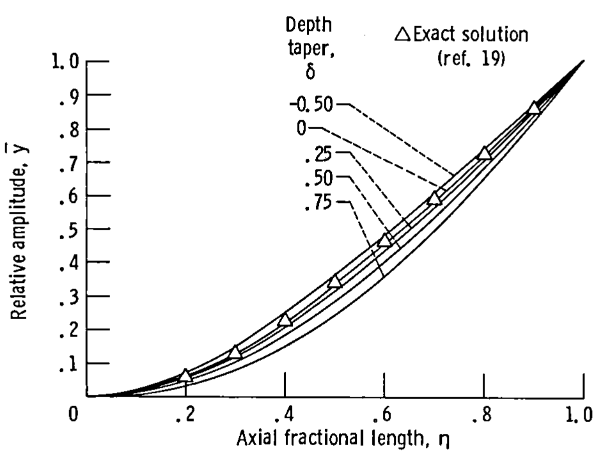

Figure 20. - Mode shapes of tapered cantilevered beams vibrating in flexure. First mode. Breadth taper β = 0.

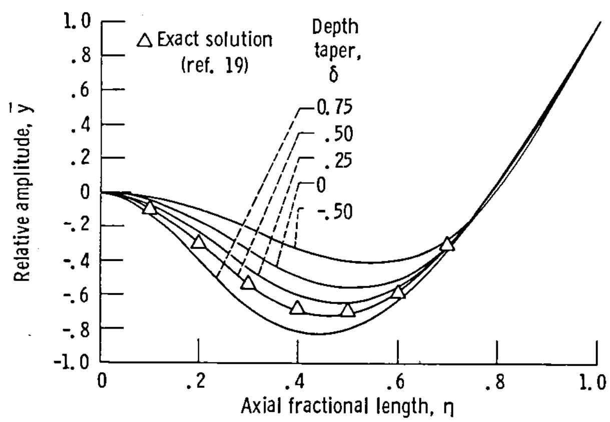

Figure 21. - Mode shapes of tapered cantilevered beams vibrating in flexure. Second mode. Breadth taper $\beta = 0$.

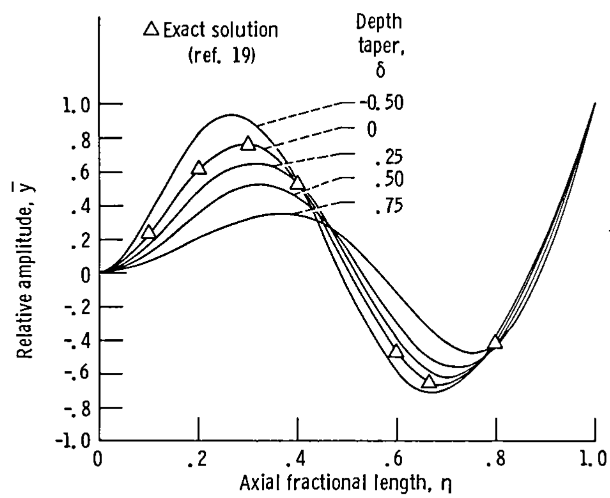

Figure 22. - Mode shapes of tapered cantilevered beams vibrating in flexure. Third mode. Breadth taper $\beta = 0$.

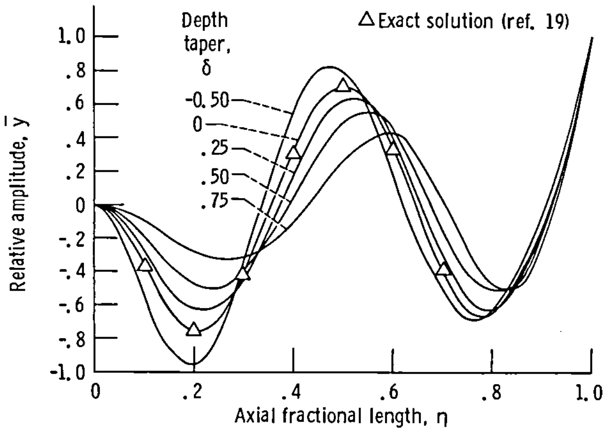

Figure 23. - Mode shapes of tapered cantilevered beams vibrating in flexure. Fourth mode. Breadth taper $\beta = 0$.

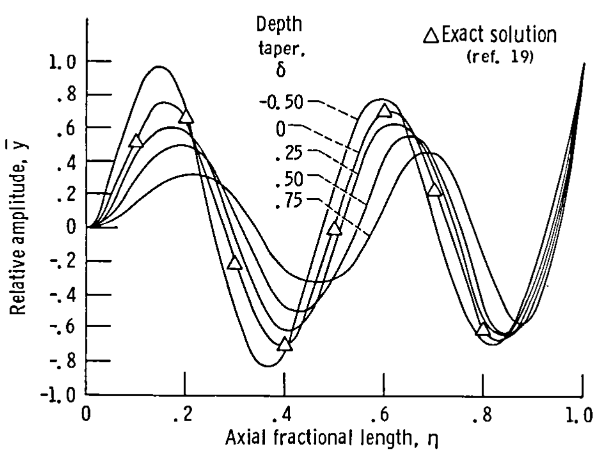

Figure 24. - Mode shapes of tapered cantilevered beams vibrating in flexure. Fifth mode. Breadth taper $\beta = 0$.

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15. Supplementary Notes

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16. Abstract

An improved finite difference procedure for determining the natural frequencies and mode shapes of tapered cantilever beams undergoing uncoupled vibrations is presented. Boundary conditions are derived in the form of simple recursive relations involving the second order central differences. Results obtained by using the conventional first order central differences and the present second order central differences are compared, and it is observed that the present second order scheme is more efficient than the conventional approach. An important advantage offered by the present approach is that the results converge to exact values rapidly, and thus the extrapolation of the results is not necessary. Consequently, the basic handicap with the classical finite difference method of solution that requires the Richardson's extrapolation procedure is eliminated. Furthermore, for the cases considered herein, the present approach produces consistent lower bound solutions.

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