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AN IMPROVED FUZZY MCDM MODEL BASED ON IDEAL AND ANTI-IDEAL CONCEPTS

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Abstract Liang presented a fuzzy multiple criteria decision making (MCDM) method based on the concepts of ideal and anti-ideal points. Despite its merits, Liang method has the following limitations: (i) the objective criteria are converted into dimensionless indices and the subjective criteria are not converted, which may prevent compatibility for these criteria, (ii) the formulas for converting objective criteria are not reliable, and (iii) an unreliable ranking method, *i.e.* maximizing set and minimizing set, is applied to rank the fuzzy numbers. This paper applies the Hsu and Chen method and suggests a fuzzy number ranking method to propose an improved fuzzy MCDM model based on ideal and anti-ideal concepts to overcome the shortcomings of the Liang method. Numerical examples demonstrate the effectiveness and feasibility of the proposed ranking method and the improved model, respectively.

1. Introduction

Liang [17] proposed a fuzzy multiple criteria decision making (MCDM) method based on the concepts of ideal and anti-ideal points. His method can be regarded as an application of fuzzy set theory [23] to TOPSIS (technique for order performance by similarity to ideal solution).

The TOPSIS technique was initiated by Hwang and Yoon [13] and has become one of the most commonly used multiple criteria decision-making (MCDM) methods [4, 22]. This technique is based on the concept that an alternative to be evaluated by n attributes can be represented as a point in n-dimensional space. Geometrical relationships among m points can be constructed. The ideal alternative has the best level for all attributes considered, while the anti (or negative)-ideal as the one with all the worst attribute values. Solutions from TOPSIS are defined as the points which are simultaneously farthest from the anti-ideal point and closest to the ideal point. To consider the uncertainty associated with the mapping of human perception to a number, the application of fuzzy numbers to TOPSIS was suggested by Negi [19] and Chen *et al.* [4]. However, what they presented are just prototype models.

In 1999, Liang [17] proposed a fuzzy TOPSIS model to solve the ill-defined MCDM problems. Despite the merits, the Liang method does contain several shortcomings however. First, the objective criteria are converted into dimensionless indices and the subjective criteria are not converted, which may not ensure the compatibility of these criteria. For example, if the linguistic terms represented by fuzzy numbers used to assess the suitability of alternatives under subjective criteria are not defined in the range $[0,1]$, but in $[1,10]$, the converted objective criteria and the aggregated ratings of subjective criteria are incommensurable. Second, the formulas for converting objective criteria are not reliable. For example, assume that the evaluation of three alternatives under a benefit criterion ($\$ \times 10^6$)

are $A_1 = (0.2, 0.35, 0.45, 0.65)$, $A_2 = (0.3, 0.45, 0.55, 0.75)$, and $A_3 = (0.32, 0.5, 0.73, 0.925)$. By Liang's direct relationship formula [17], the conversion of A_3 is $(0.14, 0.29, 0.56, 1.13)$. This does not fall between $[0, 1]$ and results in incompatibility between the converted A_3 and fuzzy numbers ranged in $[0, 1]$. This same problem also exists in Liang's inverse relationship formula [17]. For example, suppose the evaluation of three alternatives under a cost criterion ($\$ \times 10^6$) are $A_1 = (8, 11, 18, 20)$, $A_2 = (7, 9, 17, 19)$, and $A_3 = (6, 8, 16, 18)$. The Liang formula conversion of A_3 is $(0.13, 0.19, 0.71, 1.05)$, which again is not ranged in $[0, 1]$. Third, Liang applied an unreliable ranking method (*i.e.* Chen's maximizing set and minimizing set [3]) to rank fuzzy numbers (*i.e.* weighted suitability elements) to help complete his model. Although the Chen method [3] was verified illogical by Liou and Wang [18] in 1992, the Liou and Wang method also contains shortcomings as it inconsistently ranks fuzzy numbers and their opposites (see Section 3.2). To overcome the above shortcomings, this paper applies the Hsu and Chen method [12] and suggests a ranking method to propose an improved fuzzy MCDM model based on ideal and anti-ideal concepts. The proposed model provides an accurate means of applying fuzzy set theory to TOPSIS.

A review of many of the numerous fuzzy number ranking methods can be found in Bortolan *et al.* [1], Chen *et al.* [4] and [5, 6, 7, 8, 9, 11, 14, 16, 20, 21]. This paper proposes ranking fuzzy numbers by the average of the relative regions. The left relative region is defined as the area bounded by the left membership function of the fuzzy number and the axis at $x_{\min}-$ a relative minimal value on the x-axis. Similarly, the right relative region is defined as the area bounded by the right membership function of the fuzzy number and the axis at $x_{\min}$. The arithmetic average of the left and right relative regions is referred to herein as the average of the relative regions, which is used to rank fuzzy numbers. This ranking method can consistently rank all fuzzy numbers and their opposites. Formulas for ranking triangular and trapezoidal fuzzy numbers are also presented for easily performing applications. Comparative examples from Liou and Wang [18] demonstrate the advantages of the proposed ranking method. This ranking method is also utilized to determine the ideal/anti-ideal solutions and to measure the distance between the weighted converted elements and the ideal/anti-ideal solutions in order to complete the proposed fuzzy MCDM model. The numerical example in Liang [17] demonstrates the feasibility of the proposed model.

The rest of this paper is organized as follows. Section 2 introduces fuzzy numbers. Section 3 presents ranking fuzzy numbers by the average of the relative regions. Comparative examples are also included in this section to illustrate the advantages of the proposed ranking method. Meanwhile, Section 4 proposes an improved fuzzy MCDM model based on the ideal and anti-ideal concepts and the numerical example in Section 5 demonstrates the feasibility of the proposed model. Conclusions are finally made in Section 6.

2. Fuzzy numbers

The concept of fuzzy number can be defined as follows [10]:

Definition 1. A real fuzzy number A is described as any fuzzy subset of the real line R with membership function f_A which processes the following properties:

(a) f_A is a continuous mapping from R to the closed interval $[0, 1]$;
(b) $f_A(x) = 0$, for all $x \in (-\infty, a]$;
(c) f_A is strictly increasing on $[a, b]$;
(d) $f_A(x) = 1$, for all $x \in [b, c]$;
(e) f_A is strictly decreasing on $[c, d]$;

(f) $f_A(x) = 0$, for all $x \in (d, \infty]$,

where a, b, c and d are real numbers. We may let $a = -\infty$, or $a = b$, or $b = c$, or $c = d$, or $d = +\infty$. Unless elsewhere specified, it is assumed that A is convex, normal and bounded, *i.e.* $-\infty < a$, $d < \infty$. For convenience, the fuzzy number in Definition 1 can be denoted by $A = [a, b, c, d; 1]$. The opposite of A can be given by $-A = [-d, -c, -b, -a; 1]$ [15].

The membership function f_A of A can be expressed as

$$f_A(x) = \begin{cases} f_A^{\mathrm{L}}(x), & a \leq x \leq b, \\ 1, & b \leq x \leq c, \\ f_A^{\mathrm{R}}(x), & c \leq x \leq d, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where $f_A^{\mathrm{L}} : [a, b] \to [0, 1]$ is the left membership function and $f_A^{\mathrm{R}} : [c, d] \to [0, 1]$ is the right membership function of the fuzzy number A. Since the left membership function, *i.e.* $f_A^{\mathrm{L}}(x)$, is continuous on $[a, b]$, $f_A^{\mathrm{L}}(x)$ is integrable on $[a, b]$. That is, $\int_a^b f_A^{\mathrm{L}}(x)dx$ exists. Similarly $\int_c^d f_A^{\mathrm{R}}(x)dx$ exists [18].

Definition 2. The α-cut of fuzzy number A can be defined as [15]

$$A^\alpha = \{x \mid f_A(x) \geq \alpha\}, \qquad \text{where } x \in R, \alpha \in [0, 1].$$

A^α is a non-empty bounded closed interval contained in R and it can be denoted by $A^\alpha = [A_l^\alpha, A_u^\alpha]$, where A_l^α and A_u^α are the lower and upper bounds of the closed interval respectively.

Given any two fuzzy numbers, $A = [a_1, b_1, c_1, d_1]$ and $B = [a_2, b_2, c_2, d_2]$ as in (1), the α-cut of A and B are $A^\alpha = [A_l^\alpha, A_u^\alpha]$ and $B^\alpha = [B_l^\alpha, B_u^\alpha]$ respectively. According to [15], some main operations of A and B can be expressed as follows:

$$(A(+)B)^\alpha = [A_l^\alpha + B_l^\alpha, A_u^\alpha + B_u^\alpha], \tag{2}$$

$$(A(-)B)^\alpha = [A_l^\alpha - B_u^\alpha, A_u^\alpha - B_l^\alpha], \tag{3}$$

$$(A(\times)B)^\alpha = [A_l^\alpha \cdot B_l^\alpha, A_u^\alpha \cdot B_u^\alpha], \quad \text{if } a_1 \geq 0, a_2 \geq 0, \tag{4}$$

$$(A(\div)B)^\alpha = \left[\frac{A_l^\alpha}{B_u^\alpha}, \frac{A_u^\alpha}{B_l^\alpha}\right], \quad \text{if } a_1 \geq 0, a_2 > 0, \tag{5}$$

$$(A(\times)r)^\alpha = [A_l^\alpha \cdot r, A_u^\alpha \cdot r], \quad \text{if } r \geq 0. \tag{6}$$

3. Ranking Fuzzy Numbers by the Average of the Relative Regions

Suppose n fuzzy numbers, A_i, $i = 1 \sim n$, as in Definition 1, each with membership function $f_{A_i}(x)$, $A_i = [a_i, b_i, c_i, d_i]$, $i = 1 \sim n$, must be compared to decide their ranking order. The left relative region, *i.e.* $S_{\mathrm{L}}(A_i)$, and the right relative region, *i.e.* $S_{\mathrm{R}}(A_i)$, are defined as

$$S_{\mathrm{L}}(A_i) = (b_i - x_{\min}) \times 1 - \int_{a_i}^{b_i} f_{A_i}^{\mathrm{L}}(x)dx, \tag{7}$$

$$S_{\mathrm{R}}(A_i) = (c_i - x_{\min}) \times 1 + \int_{c_i}^{d_i} f_{A_i}^{\mathrm{R}}(x)dx, \tag{8}$$

where $x_{\min} = \inf P$, $P = \bigcup_{i=1}^{n} P_i$, $P_i = \{x \mid f_{A_i}(x) > 0\}$, both $S_{\mathrm{L}}(A_i)$ and $S_{\mathrm{R}}(A_i) \geq 0$.

The $S_{\mathrm{L}}(A_i)$ stretches from the left membership function of the fuzzy number A_i to the axis at $x_{\min}$. The $x_{\min}$ denotes a relative minimal value on the x-axis. The $S_{\mathrm{R}}(A_i)$

stretches from the right membership function of the fuzzy number A_i to the axis at $x_{\min}$. The meanings of $S_{\rm L}(A_i)$ and $S_{\rm R}(A_i)$ are expressed in Figires 1 and 2, respectively. Clearly, the fuzzy number A_i becomes larger if $S_{\rm L}(A_i)$ and/or $S_{\rm R}(A_i)$ are larger. Thus, both $S_{\rm L}(A_i)$ and $S_{\rm R}(A_i)$ must be considered when ranking fuzzy numbers.

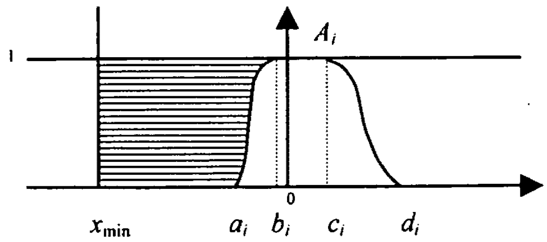

Figure 1: The left relative surface $S_{\rm L}(A_i)$ of A_i

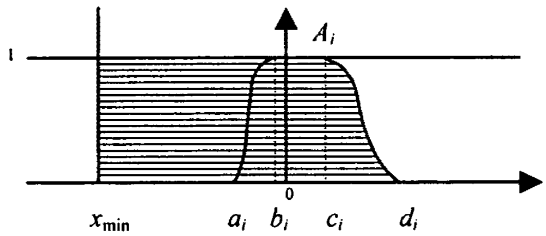

Figure 2: The right relative surface $S_{\rm R}(A_i)$ of A_i

The arithmetic average of the left and right relative regions for A_i can then be defined as

$$S(A_i) = \frac{1}{2}(S_{\rm L}(A_i) + S_{\rm R}(A_i)) \tag{9}$$

where $S_{\rm L}(A_i)$ and $S_{\rm R}(A_i)$ are the left and right relative regions of A_i.

Herein, $S(A_i)$ is used to rank fuzzy numbers. The larger the $S(A_i)$, the larger the fuzzy number A_i. Therefore, for any two fuzzy numbers A_i and A_j, if $S(A_i) > S(A_j)$, then $A_i > A_j$. If $S(A_i) = S(A_j)$, then $A_i = A_j$. Finally, if $S(A_i) < S(A_j)$, then $A_i < A_j$. If A_i is a non-normal fuzzy number, f_{A_i} can always be normalized by dividing the maximal value of f_{A_i} before ranking, that is, $f_{\bar{A}_i}(x) = f_{A_i}(x)/\max_x f_{A_i}(x)$,where $\bar{A}_i$ denotes the normalized fuzzy number of A_i and $f_{\bar{A}_i}(x)$ denotes the membership function of $\bar{A}_i$. Thus, the relative region of $\bar{A}_i$ represents the relative region of A_i when being ranked, *i.e.* $S_{\rm L}(A) = S_{\rm L}(\bar{A})$ and $S_{\rm R}(A) = S_{\rm R}(\bar{A})$.

Property 1. For any two fuzzy numbers $A_1 = [a_1, b_1, c_1, d_1]$ and $A_2 = [a_2, b_2, c_2, d_2]$ as in Definition 1. If $S(A_1) > S(A_2)$, then $S(-A_1) < S(-A_2)$; if $S(A_1) = S(A_2)$, then $S(-A_1) = S(-A_2)$; and if $S(A_1) < S(A_2)$, then $S(-A_1) > S(-A_2)$.

Proof. $x_{\max} = \sup P$, $x_{\min} = \inf P$, $P = \bigcup_{i=1}^{2} P_i$, $P_i = \{x \mid f_{A_i}(x) > 0\}$. Let $x_{\max} = x_{\min} + \delta, \delta \in R^+$.

$$-A_1 = [-d_1, -c_1, -b_1, -a_1], \quad -A_2 = [-d_2, -c_2, -b_2, -a_2].$$

Let $x'_{\min} = \inf P', P' = \bigcup_{i=1}^{2} P'_i, P'_i = \{x \mid f_{-A_i}(x) > 0\}$. Obviously, $x'_{\min} = -x_{\max}$. By (7)~(9),

$$S(-A_1) \;=\; \frac{1}{2}\{S_{\rm L}(-A_1) + S_{\rm R}(-A_1)\}$$

$$
\begin{aligned}
&= \frac{1}{2}\left\{\left[(-c_1 - x'_{\min}) - \int_{-d_1}^{-c_1} f^{\mathrm{L}}_{-A_1}(x)dx\right] + \left[(-b_1 - x'_{\min}) + \int_{-b_1}^{-a_1} f^{\mathrm{R}}_{-A_1}(x)dx\right]\right\} \\
&= \frac{1}{2}\left\{\left[(-c_1 + x_{\min} + \delta) - \int_{c_1}^{d_1} f^{\mathrm{R}}_{A_1}(x)dx\right] + \left[(-b_1 + x_{\min} + \delta) + \int_{a_1}^{b_1} f^{\mathrm{L}}_{A_1}(x)dx\right]\right\} \\
&= \frac{-1}{2}\left\{\left[(b_1 - x_{\min}) - \int_{a_1}^{b_1} f^{\mathrm{L}}_{A_1}(x)dx\right] + \left[(c_1 - x_{\min}) + \int_{c_1}^{d_1} f^{\mathrm{R}}_{A_1}(x)dx\right] - 2\delta\right\} \\
&= -S(A_1) + \delta.
\end{aligned}
$$

Similarly, $S(-A_2) = -S(A_2) + \delta$.
Thus,

$$S(A_1) > S(A_2) \Rightarrow -S(A_1) < -S(A_2) \Rightarrow -S(A_1)+\delta < -S(A_2)+\delta \Rightarrow S(-A_1) < S(-A_2).$$

Similarly, if $S(A_1) = S(A_2)$, then $S(-A_1) = S(-A_2)$, and if $S(A_1) < S(A_2)$, then $S(-A_1) > S(-A_2)$. □

Property 2. The ranking of any two given fuzzy numbers, $A_1 = [a_1, b_1, c_1, d_1]$ and $A_2 = [a_2, b_2, c_2, d_2]$ as in Definition 1, will not be altered when $x_{\min}$ is changed.

Proof.

$$x_{\min} = \inf P, \quad P = \bigcup_{i=1}^{2} P_i, \quad P_i = \{x \mid f_{A_i}(x) > 0\}.$$

Suppose fuzzy number A_3 is added to compare with A_1 and A_2.

$$x'_{\min} = \inf P', \quad P' = \bigcup_{i=1}^{3} P'_i, \quad P'_i = \{x \mid f_{A_i}(x) > 0\}.$$

Let $x'_{\min} = x_{\min} - \delta, \delta \in R^+$. By(7)$\sim$ (9),

$$
\begin{aligned}
S'(A_1) &= \frac{1}{2}\left\{\left[(b_1 - x'_{\min}) - \int_{a_1}^{b_1} f^{\mathrm{L}}_{A_1}(x)dx\right] + \left[(c_1 - x'_{\min}) + \int_{c_1}^{d_1} f^{\mathrm{R}}_{A_1}(x)dx\right]\right\} \\
&= \frac{1}{2}\left\{\left[(b_1 - x_{\min} + \delta) - \int_{a_1}^{b_1} f^{\mathrm{L}}_{A_1}(x)dx\right] + \left[(c_1 - x_{\min} + \delta) + \int_{c_1}^{d_1} f^{\mathrm{R}}_{A_1}(x)dx\right]\right\} \\
&= S(A_1) + \delta.
\end{aligned}
$$

Similarly, $S'(A_2) = S(A_2) + \delta$. Thus, if $S(A_1) > S(A_2)$, then $S'(A_1) > S'(A_2)$; if $S(A_1) = S(A_2)$, then $S'(A_1) = S'(A_2)$; and if $S(A_1) < S(A_2)$, then $S'(A_1) < S'(A_2)$. □

3.1. The average of the relative regions for trapezoidal and triangular fuzzy numbers

Definition 3. The fuzzy number B is a trapezoidal fuzzy number if its membership function f_B is given by [15]

$$f_B(x) = \begin{cases} (x-a)/(b-a), & a \le x \le b, \\ 1, & b \le x \le c, \\ (x-d)/(c-d), & c \le x \le d, \\ 0, & \text{otherwise}, \end{cases}$$

where a, b, c and d are real numbers. For convenience, B can be denoted by (a, b, c, d).

Suppose $B = (a, b, c, d)$ is a trapezoidal fuzzy number as in Definition 3. By (7)$\sim$(9), the average of the relative regions for B can be obtained as:

$$S(B) = \frac{1}{2}\{S_{\mathrm{L}}(B) + S_{\mathrm{R}}(B)\} = \frac{1}{4}(a + b + c + d) - x_{\min}. \tag{10}$$

Definition 4. The fuzzy number A is a triangular fuzzy number if its membership function f_A is given by [15]

$$f_A(x) = \begin{cases} (x-a)/(b-a), & a \le x \le b, \\ (x-c)/(b-c), & b \le x \le c, \\ 0, & \text{otherwise,} \end{cases}$$

where a, b and c are real numbers. For convenience A can be denoted by (a, b, c).

Suppose $A = (a, b, c)$ is a triangular fuzzy number as in Definition 4. By (7)~(9), the average of the relative regions for A can be obtained as:

$$S(B) = \frac{1}{2}\{S_{\mathrm{L}}(B) + S_{\mathrm{R}}(B)\} = \frac{1}{4}(a + 2b + d) - x_{\min}. \tag{11}$$

Notably, when $x_{\min} = 0$, formula (11) is the same as the one in Kaufmann and Gupta [14].

3.2. Comparative examples

Consider the two triangular fuzzy numbers, $A_1 = (3, 5, 7; 1), A_2 = (3, 5, 7; 0.8)$, and the three trapezoidal fuzzy numbers, $B_1 = (5, 7, 9, 10; 1), B_2 = (6, 7, 9, 10; 0.6), B_3 = (7, 8, 9, 10; 0.4)$ as in Figure 3 from Liou and Wang [18].

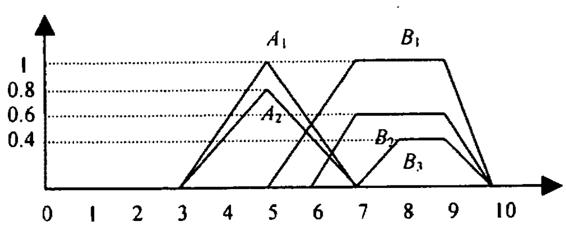

Figure 3: Triangular fuzzy numbers $A_1 = (3, 5, 7; 1)$ and $A_2 = (3, 5, 7; 0.8)$, and trapezoidal fuzzy numbers, $B_1 = (5, 7, 9, 10; 1), B_2 = (6, 7, 9, 10; 0.6)$ and $B_3 = (7, 8, 9, 10; 0.4)$

By Liou and Wang method, $I_{\mathrm{T}}^{\alpha}(A_1) = I_{\mathrm{T}}^{\alpha}(A_2) = 4 + 2\alpha$, obtaining $A_1 = A_2$ for $\alpha \in [0, 1]$. Moreover, $I_{\mathrm{T}}^{\alpha}(B_1) = 6 + 3.5\alpha$, $I_{\mathrm{T}}^{\alpha}(B_2) = 6.5 + 3\alpha$, and $I_{\mathrm{T}}^{\alpha}(B_3) = 7.5 + 2\alpha$, obtaining $B_1 < B_2 < B_3$ for $0 \le \alpha < 1$ and $B_1 = B_2 = B_3$ for $\alpha = 1$. From this result, we logically infer $-A_1 = -A_2$ for $\alpha \in [0, 1]$, $-B_1 > -B_2 > -B_3$ for $0 \le \alpha < 1$ and $-B_1 = -B_2 = -B_3$ for $\alpha = 1$. However, by their method, $I_{\mathrm{T}}^{\alpha}(-A_1) = I_{\mathrm{T}}^{\alpha}(-A_2) = -6 + 2\alpha$, $I_{\mathrm{T}}^{\alpha}(-B_1) = -9.5 + 3.5\alpha$, $I_{\mathrm{T}}^{\alpha}(-B_2) = -9.5 + 3\alpha$, and $I_{\mathrm{T}}^{\alpha}(-B_3) = -9.5 + 2\alpha$. We obtain $-A_1 = -A_2$ for $\alpha \in [0, 1]$, $-B_1 > -B_2 > -B_3$ for $0 < \alpha \le 1$ and $-B_1 = -B_2 = -B_3$ for $\alpha = 0$. Obviously, the Liou and Wang method inconsistently ranks fuzzy numbers and their opposites. By the proposed method, $x_{\min} = 3, S(A_1) = S(A_2) = 2, M(B_1) = 4.75, M(B_2) = 5$, and $M(B_3) = 5.5$, producing the ranking order $A_1 = A_2 < B_1 < B_2 < B_3$. In addition, $S(-A_1) = S(-A_2) = 6.25, M(-B_1) = 2.25, M(-B_2) = 2$, and $M(-B_3) = 1.5$, implying that $-A_1 = -A_2 > -B_1 > -B_2 > -B_3$. Therefore, the proposed method can overcome the shortcomings of the Liou and Wang method.

Furthermore, consider the triangular fuzzy number, $A = (1, 2, 5; 1)$, and the general fuzzy number, $B = [1, 2, 2, 4; 1]$, as in Figure 4 from Liou and Wang [18]. The membership function of B is defined as

$$f_B(x) = \begin{cases} [1 - (x-2)^2]^{1/2}, & 1 \le x \le 2, \\ [1 - \frac{1}{4}(x-2)^2]^{1/2}, & 2 \le x \le 4, \\ 0, & \text{otherwise,} \end{cases}$$

where $f_B^{\mathrm{L}}(x) = [1-(x-2)^2]^{1/2}$ and $f_B^{\mathrm{R}}(x) = [1-\frac{1}{4}(x-2)^2]^{1/2}$.

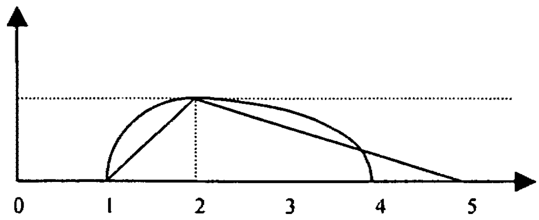

Figure 4: The triangular fuzzy numbers $A = (1,2,5;1)$ and the general fuzzy number $B = [1,2,2,4;1]$

By Liou and Wang method, the total integral value of A is $I_T^\alpha(A) = 1.5 + 2\alpha$, and the total integral value of B is $I_T^\alpha(B) = 1.2 + 2.4\alpha$. For an optimistic decision maker, with $\alpha = 1$: $A < B$; for a moderate decision maker, with $\alpha = 0.5$: $A > B$; and for a pessimistic decision maker, with $\alpha = 0$: $A > B$. From this result, we logically infer $-A > -B$ for $\alpha = 1$, $-A < -B$ for $\alpha = 0.5$, and $-A < -B$ for $\alpha = 0$. However, by their method, $I_T^\alpha(-A) = -3.5+2\alpha, I_T^\alpha(-B) = -3.6+2.4\alpha$, which implies $-A < -B$ for $\alpha = 1$, $-A < -B$ for $\alpha = 0.5$, and $-A > -B$ for $\alpha = 0$. Again, the Liou and Wang method inconsistently ranks fuzzy numbers and their opposites. By our method, $x_{\min} = 1, S(A) = 1.5$ and $S(B) = 1.39$, obtaining the ranking order $A > B$. And by Property 1, we obtain $-A < -B$. Therefore, the method proposed herein overcomes the shortcomings of the Liou and Wang method by simpler results.

The proposed ranking method is also utilized to establish an improved fuzzy MCDM model based on the concepts of the ideal and anti-ideal points as shown in the next section.

4. An Improved Fuzzy MCDM Model Based on Ideal and Anti-ideal Concepts

This section presents an improved fuzzy MCDM model based on the ideal and anti-ideal concepts, where the importance weights of all criteria and the ratings of alternatives under subjective criteria are assessed in linguistic terms [24] represented by trapezoidal fuzzy numbers. Rigorously, extensive experiment is needed to determine the linguistic terms and their corresponding fuzzy numbers to solve the fuzzy MCDM problems. Herein, for convenience, the linguistic terms and fuzzy numbers used in Liang [17] will be applied to the numerical example in Section 5. Assume that a committee of n decision-makers (D_j, $j = 1 \sim n$) is responsible for assessing m alternatives (A_i, $i = 1 \sim m$) under each of k criteria (C_t, $t = 1 \sim k$) as well as assessing the importance weights of the criteria, $k, m, n \in N$.

4.1. Aggregate the importance weights of criteria

Many methods are available to pool the decision-makers' opinions, for example, mean, median, max, min and mixed operators [2]. Each of operators has its limitations. Criteria for selecting appropriate aggregation operator can be found in Zimmermann [25]. Since the average operation is the most commonly used aggregation method, in here, the mean operator is used to pool the decision-makers' opinions [17]. Let $W_{tj} = (a_{tj}, b_{tj}, c_{tj}, d_{tj})$, $t = 1 \sim k, j = 1 \sim n$, be the linguistic weight assigned to criterion C_t by decision-maker D_j. The aggregated linguistic weight, $W_t = (a_t, b_t, c_t, d_t)$, $t = 1 \sim k$, for criterion t from n decision-makers' opinions can be calculated by:

$$W_t = (1/n) \otimes (W_{t1} \oplus W_{t2} \oplus \cdots \oplus W_{tn}), \tag{12}$$

where $a_t = \sum_{j=1}^{n} a_{tj}/n,\ b_t = \sum_{j=1}^{n} b_{tj}/n,\ c_t = \sum_{j=1}^{n} c_{tj}/n$ and $d_t = \sum_{j=1}^{n} d_{tj}/n$.

4.2. Convert the aggregated linguistic ratings of alternatives under subjective criteria

Let $X_{itj} = (l_{itj}, m_{itj}, n_{itj}, o_{itj}), i = 1 \sim m, t = 1 \sim h, j = 1 \sim n$, denote the linguistic rating assigned to alternative A_i by decision-maker D_j for subjective criterion C_t. The mean operator is also used to pool the decision-makers' opinions [17]. The aggregated linguistic rating, $X_{it} = (l_{it}, m_{it}, n_{it}, o_{it}),\ i = 1 \sim m, t = 1 \sim h$, of alternative A_i under subjective criterion C_t from n decision-makers' opinions can be calculated by:

$$X_{it} = (1/n) \otimes (X_{it1} \oplus X_{it2} \oplus \cdots \oplus X_{itn}), \tag{13}$$

where $l_{it} = \sum_{j=1}^{n} l_{itj}/n,\ m_{it} = \sum_{j=1}^{n} m_{itj}/n,\ n_{it} = \sum_{j=1}^{n} n_{itj}/n$ and $o_{it} = \sum_{j=1}^{n} o_{itj}/n$.

In Section 4.3, the ranges of the converted objective criteria belong to $[0,1]$. If the linguistic ratings are not defined in $[0,1]$, e.g. $[1,10]$, the aggregated linguistic ratings are incommensurable with the converted objective criteria. Thus, to ensure compatibility between the converted objective criteria and the aggregated linguistic ratings, the aggregated linguistic ratings must also be converted. Herein, the conversion is performed by applying the Hsu and Chen method [12], which preserves the property that the ranges of the converted trapezoidal fuzzy numbers belong to $[0,1]$. By our concept, the linguistic ratings can be defined anywhere in R^+. The converted aggregated linguistic ratings, $M_{it} = (p_{it}, q_{it}, r_{it}, s_{it}),\ i = 1 \sim m, t = 1 \sim h$, can be calculated by:

$$M_{it} = (l_{it}/o_t^*, m_{it}/o_t^*, n_{it}/o_t^*, o_{it}/o_t^*), \tag{14}$$

where $o_t^* = \max_i o_{it}, p_{it} = l_{it}/o_t^*, q_{it} = m_{it}/o_t^*, r_{it} = n_{it}/o_t^*, s_{it} = o_{it}/o_t^*$.

4.3. Convert the objective criteria

The objective criteria (fuzzy or non-fuzzy) can be classified to two categories: cost and benefit. Objective criteria have incommensurable units. To ensure compatibility, they need to be converted into a compatible scale (into dimensionless indices). Herein, the conversion is performed by applying the Hsu and Chen method [12] since it preserves the property that the ranges of converted trapezoidal fuzzy numbers belong to $[0,1]$. If $T_{it} = (g_{it}, u_{it}, v_{it}, w_{it}),\ i = 1 \sim m, t = h+1 \sim k$, represents the fuzzy (or non-fuzzy) total cost/benefit assigned to alternative A_i versus objective criterion C_t, then the converted objective criteria, $M_{it} = (p_{it}, q_{it}, r_{it}, s_{it}),\ i = 1 \sim m, t = h+1 \sim k$, can be calculated by:

(1) For benefit criteria

$$M_{it} = (g_{it}/w_t^*, u_{it}/w_t^*, v_{it}/w_t^*, w_{it}/w_t^*), \tag{15}$$

where $w_t^* = \max_i w_{it},\ p_{it} = g_{it}/w_t^*,\ q_{it} = u_{it}/w_t^*,\ r_{it} = v_{it}/w_t^*,\ s_{it} = w_{it}/w_t^*,\ i = 1 \sim m, t = h+1 \sim k$.

(2) For cost criteria

$$M_{it} = (g_t^-/w_{it}, g_t^-/v_{it}, g_t^-/u_{it}, g_t^-/g_{it}), \tag{16}$$

where $g_t^- = \min_i w_{it},\ p_{it} = g_t^-/w_{it},\ q_{it} = g_t^-/v_{it},\ r_{it} = g_t^-/u_{it},\ s_{it} = g_t^-/g_{it},\ i = 1 \sim m, t = h+1 \sim k$.

4.4. Construct the weighted converted fuzzy decision matrix

Let $D_{it}, i = 1 \sim m, t = 1 \sim k$, be an element of the weighted converted fuzzy decision matrix D, *i.e.* $D = [D_{it}]_{m\times k}$. The weighted converted fuzzy rating, D_{it}, of each alternative versus each criterion can be calculated by:

$$D_{it} = W_t \otimes M_{it}, \tag{17}$$

where $W_t = (a_t, b_t, c_t, d_t), M_{it} = (p_{it}, q_{it}, r_{it}, s_{it})$ and $i = 1 \sim m, t = 1 \sim k$.

Herein, D_{it} does not yield a trapezoidal shape. By formula (4), the membership function for D_{it} can be developed as follows:

$$\begin{aligned} D_{it}^{\alpha} &= (W_t(\times)M_{it})^{\alpha} \\ &= (W_{tl}^{\alpha} \cdot M_{itl}^{\alpha}, W_{tu}^{\alpha} \cdot M_{itu}^{\alpha}) \\ &= ([\alpha(b_t - a_t) + a_t] \cdot [\alpha(q_{it} - p_{it}) + p_{it}], [\alpha(c_t - d_t) + d_t] \cdot [\alpha(r_{it} - s_{it}) + s_{it}]) \\ &= ((b_t - a_t)(q_{it} - p_{it})\alpha^2 + [a_t(q_{it} - p_{it}) + p_{it}(b_t - a_t)]\alpha + a_t p_{it}, \\ &\qquad (c_t - d_t)(r_{it} - s_{it})\alpha^2 + [d_t(r_{it} - s_{it}) + s_{it}(c_t - d_t)]\alpha + d_t s_{it}) \end{aligned}$$

There are two equations to solve, that is,

$$(b_t - a_t)(q_{it} - p_{it})\alpha^2 + [a_t(q_{it} - p_{it}) + p_{it}(b_t - a_t)]\alpha + a_t p_{it} - x = 0, \tag{18}$$

$$(c_t - d_t)(r_{it} - s_{it})\alpha^2 + [d_t(r_{it} - s_{it}) + s_{it}(c_t - d_t)]\alpha + d_t s_{it} - x = 0. \tag{19}$$

Let $F_{it1} = (b_t - a_t)(q_{it} - p_{it}), G_{it1} = a_t(q_{it} - p_{it}) + p_{it}(b_t - a_t), F_{it2} = (c_t - d_t)(r_{it} - s_{it}), H_{it1} = G_{it1}/2F_{it1}, G_{it2} = d_t(r_{it} - s_{it}) + s_{it}(c_t - d_t), H_{it2} = -G_{it2}/2F_{it2}, U_{it} = a_t p_{it}, V_{it} = b_t q_{it}, Y_{it} = c_t r_{it}, Z_{it} = d_t s_{it}$. Equations (18) and (19) can be presented as

$$F_{it1}\alpha^2 + G_{it1}\alpha + U_{it} - x = 0, \tag{20}$$

$$F_{it2}\alpha^2 + G_{it2}\alpha + Z_{it} - x = 0. \tag{21}$$

Only two roots in $[0, 1]$ are retained,

$$\text{for (20),} \quad \alpha = -H_{it1} + [H_{it1}^2 + (x - U_{it})/F_{it1}]^{1/2},$$
$$\text{for (21),} \quad \alpha = H_{it2} - [H_{it2}^2 + (x - Z_{it})/F_{it2}]^{1/2}.$$

Finally, $\forall x \in R^+$,

$$f_{D_{it}}(x) = \begin{cases} -H_{it1} + [H_{it1}^2 + (x - U_{it})/F_{it1}]^{1/2}, & U_{it} \le x \le V_{it}, \\ 1, & V_{it} \le x \le Y_{it}, \\ H_{it2} - [H_{it2}^2 + (x - Z_{it})/F_{it2}]^{1/2}, & Y_{it} \le x \le Z_{it}, \\ 0, & \text{otherwise.} \end{cases} \tag{22}$$

For convenience, D_{it} can be expressed as [17]:

$$D_{it} = (U_{it}, V_{it}, Y_{it}, Z_{it}; H_{it1}, F_{it1}; H_{it2}, F_{it2}), \ i = 1 \sim m, t = 1 \sim k.$$

4.5. Determine the ideal (I^+) and anti-ideal (I^-) solutions

By the proposed ranking method, *i.e.* formulas (7)~(9), the ranking values of D_{it}, $i = 1 \sim m$ and fixed t, of the alternatives can be easily calculated to determine the ideal (I^+) and anti-ideal (I^-) solutions as:

$$I^+ = (I_1^+, I_2^+, \cdots, I_k^+), \tag{23}$$

$$I^- = (I_1^-, I_2^-, \cdots, I_k^-), \tag{24}$$

where $I_t^+ = \max_i\{D_{it}\}$ and $I_t^- = \min_i\{D_{it}\}$, $t = 1 \sim k$.

4.6. Calculate the distance between the different alternatives versus I^+ and I^-

Let d_i^+ and d_i^- denote the distance of the alternative A_i versus I^+ and I^-, respectively. Define

$$d_i^+ = \sum_{t=1}^{k}\left(1-\frac{S(D_{it})}{S(I_t^+)}\right), \quad i=1\sim m, \tag{25}$$

$$d_i^- = \sum_{t=1}^{k}\left(1-\frac{S(I_t^-)}{S(D_{it})}\right), \quad S(I_t^-)>0, \quad i=1\sim m, \tag{26}$$

where $S(D_{it}), S(I_t^+)$ and $S(I_t^-)$ denote the ranking values of D_{it}, I_t^+ and I_t^-, respectively.

4.7. Calculate the closeness coefficient

The closeness coefficient, c_i^*, $i=1\sim m$, can be calculated by:

$$c_i^* = \frac{d_i^-}{d_i^+ + d_i^-}, \; i=1\sim m. \tag{27}$$

Since alternative A_i is closer to the ideal point (I_t^+) and farther from the anti-ideal point (I_t^-) as c_i^* approaches 1, the ranking order of all alternatives can be determined and the optimum choice can be selected according to the closeness coefficient.

5. Example

The numerical example presented in Liang [17] is applied to illustrate the feasibility of the improved model. Assume that a high technology company must choose a site to build a new plant. Three alternative sites A_1, A_2 and A_3 remain after preliminary screening. A committee of four decision-makers, D_1, D_2, D_3 and D_4, is formed to determine the most appropriate site. Three subjective criteria: climate (C_1), labour force quality (C_2), transportation availability (C_3), and one objective criterion, investment cost (C_4), are considered.

The decision-makers employ a linguistic weighting set $W=\{VL, L, M, H, VH\}$, where VL =very low, L =low, M =medium, H =high, and VH =very high, to evaluate the importance of the criteria. And $VL=(0,0,0,0.3)$, $L=(0,0.3,0.3,0.5)$, $M=(0.2.0.5,0.5,0.8)$, $H=(0.5,0.7,0.7,1)$, $VH=(0.7,1,1,1)$. The decision-makers utilize the linguistic rating set $S=\{VP, B.VP\&P, P, B.P\&F, F, B.F\&G, G, B.G\&VG, VG\}$, where VP =very poor, $B.VP\&P$ = between very poor and poor, P =poor, $B.P\&F$ =between poor and fair, F =fair, $B.F\&G$ =between fair and good, G =good $B.G\&VG$ =between good and very good, and VG =very good, to assess the suitability of alternatives under each of the subjective criteria. And $VP=(0,0,0,0.2)$, $B.VP\&P=(0,0,0.2,0.4)$, $P=(0,0.2,0.2,0.4)$, $B.P\&F=(0,0.2,0.5,0.7)$, $F=(0.3,0.5,0.5,0.7)$, $B.F\&G=(0.3,0.5,0.8,1)$, $G=(0.6,0.8,0.8,1)$, $B.G\&VG=(0.6,0.8,1,1)$, and $VG=(0.8,1,1,1)$. The investment cost ($\$ \times 10^6$) are $A_1=(33,35,45,47)$, $A_2=(22,25,30,32)$ and $A_3=(22,26,26,28)$.

Liang [17] assumed that the aggregated linguistic weights, *i.e.* W_t, $t=1\sim 4$, for the four criteria from the three decision-makers' opinions are:

$$W_1=(0.400,0.675,0.675,0.900), \quad W_2=(0.525,0.800,0.800,0.950),$$
$$W_3=(0.600,0.840,0.840,1.000), \quad W_4=(0.650,0.900,0.900,0.925).$$

Liang [17] also presumed that the aggregated linguistic ratings of different alternative A_i under various subjective criteria C_t from the three decision-makers' opinions are:

$$X_{11}=(0.230,0.425,0.580,0.780), \quad X_{12}=(0.200,0.350,0.450,0.650),$$

$$X_{13} = (0.525, 0.730, 0.825, 1.000), \quad X_{21} = (0.550, 0.700, 0.775, 0.930),$$
$$X_{22} = (0.300, 0.450, 0.550, 0.750), \quad X_{23} = (0.400, 0.550, 0.725, 0.850),$$
$$X_{31} = (0.260, 0.430, 0.675, 0.850), \quad X_{32} = (0.320, 0.500, 0.730, 0.925),$$
$$X_{33} = (0.350, 0.500, 0.650, 0.850).$$

By formula (14), these aggregated linguistic ratings can be converted into:

$$M_{11} = (0.247, 0.457, 0.624, 0.839), \quad M_{12} = (0.216, 0.378.0.486, 0.703),$$
$$M_{13} = (0.525, 0.730, 0.825, 1.000), \quad M_{21} = (0.591, 0.753, 0.833, 1.000),$$
$$M_{22} = (0.324, 0.486, 0.595, 0.811), \quad M_{23} = (0.400, 0.550, 0.725, 0.850),$$
$$M_{31} = (0.280, 0.462, 0.726, 0.914), \quad M_{32} = (0.346, 0.541, 0.789, 1.000),$$
$$M_{33} = (0.350, 0.500, 0.650, 0.850).$$

By applying formula (16), the investment costs (objective criterion) can be converted into:

$$M_{14} = (0.468, 0.489, 0.629, 0.667), \; M_{24} = (0.688, 0.733, 0.880, 1.000),$$
$$M_{34} = (0.786, 0.846, 0.846, 1.000).$$

By formulas (17)$\sim$ (22), the elements of the weighted converted fuzzy decision matrix, $D = [D_{it}]_{3\times 4}$, can be obtained as:

$$D_{11} = (0.099, 0.308, 0.421, 0.755; 1.315, 0.058; 3.951, 0.048),$$
$$D_{21} = (0.236, 0.508, 0.562, 0.900; 2.551, 0.045; 4.994, 0.038),$$
$$D_{31} = (0.112, 0.312, 0.490, 0.823; 1.497, 0.050; 4.431, 0.042),$$
$$D_{12} = (0.113, 0.302, 0.389, 0.668; 1.621, 0.045; 4.786, 0.033),$$
$$D_{22} = (0.170, 0.389, 0.476, 0.770; 1.955, 0.045; 5.044, 0.032),$$
$$D_{32} = (0.182, 0.433, 0.631, 0.950; 1.842, 0.054; 5.536, 0.032),$$
$$D_{13} = (0.315, 0.613, 0.693, 1.000; 2.530, 0.049; 5.982, 0.028),$$
$$D_{23} = (0.240, 0.462, 0.609, 0.850; 2.583, 0.036; 6.525, 0.020),$$
$$D_{33} = (0.210, 0.420, 0.546, 0.850; 2.417, 0.036; 5.250, 0.032),$$
$$D_{14} = (0.304, 0.440, 0.566, 0.617; 12.443, 0.005; 27.276, 0.001),$$
$$D_{24} = (0.447, 0.660, 0.792, 0.925; 8.944, 0.011; 22.667, 0.003),$$
$$D_{34} = (0.511, 0.761, 0.761, 0.925; 7.850, 0.015; 21.747, 0.004).$$

Formulas (7)$\sim$ (9) propose that the ideal (I^+) and anti-ideal (I^-) solutions can be obtained via:

$$I^+ = (D_{21}, D_{32}, D_{13}, D_{34}), \quad I^- = (D_{11}, D_{12}, D_{33}, D_{14}).$$

Formulas (25)$\sim$ (26) reveal that the distance between alternative A_i versus I^+ and I^- can be obtained via:

$$d_1^+ = 1.368, \; d_1^+ = 0.557, \; d_1^+ = 0.6, \; d_1^- = 0.337, \; d_1^- = 1.276, \text{ and } d_1^- = 1.137.$$

Formula (27) verifies the closeness coefficient for each alternative i, $i = 1, 2, 3$, can be obtained via:

$$c_1^* = 0.498, \; c_2^* = 0.696 \text{ and } c_3^* = 0.655.$$

The optimal selection is alternative A_2 since the ranking order for the three alternatives is A_2, A_3 and A_1: a result that coincides with Liang [17].

6. Conclusions

Liang [17] proposed a fuzzy MCDM method based on the concepts of ideal and anti-ideal points, which can be regarded as an application of fuzzy set theory to TOPSIS. Several shortcomings in Liang method have been verified in this paper. By applying the Hsu and Chen method [12] and a proposed ranking method, an improved MCDM model based on ideal and anti-ideal concepts was proposed to solve the shortcomings of the Liang method. Numerical examples have demonstrated the effectiveness and feasibility of the proposed ranking method and the improved model, respectively. The proposed model provides an accurate means of applying fuzzy set theory to TOPSIS.

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