An improved lower bound on the number of limit cycles bifurcating from a quintic Hamiltonian planar vector field under quintic perturbation

Tomas Johnson
Department of Mathematics
Uppsala University
Box 480, 751 06 Uppsala, Sweden
johnson@math.uu.se

Warwick Tucker
Department of Mathematics
University of Bergen
Johannes Brunsgate 12, 5008 Bergen, Norway
warwick.tucker@math.uib.no

April 23, 2009

Abstract

The limit cycle bifurcations of a $\mathbf{Z}_2$ equivariant quintic planar Hamiltonian vector field under $\mathbf{Z}_2$ equivariant quintic perturbation is studied. We prove that the given system can have at least 27 limit cycles. This is an improved lower bound on the possible number of limit cycles that can bifurcate from a quintic planar Hamiltonian system under quintic perturbation.

Keywords. Limit cycles, bifurcation theory, planar Hamiltonian systems, interval analysis.
2000 *Mathematics Subject Classification.* Primary: 37G15, 37M20; Secondary: 34C07, 65G20.

1 Introduction

Determining the number and location of (isolated) limit cycles for planar polynomial ordinary differential equations was posed as a grand challenge in Hilbert's seminal address to the International Congress of Mathematicians in 1900. Of the 23 problems presented by Hilbert, this (the 16th) turned out to be one of the most persistent: despite more than a century of intense research, not even the quadratic case has been resolved. For an overview of the progress that has been made to solve this problem we refer to [12]. Partial results for the quadratic case, and a general introduction to the bifurcation theory of planar polynomial vector fields can be found in [22]. What is known, is that any given polynomial vector field can have only a finite number of limit cycles; this is proved in [7, 11].

A restricted version of Hilbert's 16th problem introduced by Arnol'd, see e.g. [2], known as the *weak* or *tangential* Hilbert's 16th problem, asks for the number of limit cycles that can bifurcate from a perturbation of a Hamiltonian system, see e.g. [5]. The weak Hilbert's 16th problem has been solved for the quadratic case, see [4].

In order to find Hamiltonian systems such that their perturbations have a maximum number of zeros, it is common to study symmetric Hamiltonians with a maximal number of centres, see [3, 15, 27, 28]. The specific perturbations are often constructed using the so-called *detection function* method, see [16]. In [15] a quintic perturbation of a $\mathbf{Z}_6$ equivariant system, with 24 limit cycles is constructed. As far as we know, this is the largest previously known lower bound on the number of limit cycles that can bifurcate through quintic perturbation of a quintic Hamiltonian vector field.

The aim of the present paper is to study the $\mathbf{Z}_2$ equivariant system from [28], and prove that at least 27 limit cycles can bifurcate from it. We locate a suitable perturbation by conducting a similar study as in [14]; this is described in detail in Section 4.2. We stress that our approach is completely rigorous, seeing that all numerics is done in interval arithmetic with directed rounding.

2 Abelian integrals

A classical method to prove the existence of limit cycles bifurcating from a continuous family of level curves of a Hamiltonian, $\gamma_h \subset H^{-1}(h)$, depending continuously on h, is to study Abelian integrals, or, more generally, the Melnikov function, see e.g. [5, 8]. The closed level-curves of a polynomial Hamiltonian are called *ovals*. We denote the interior of an oval D_h, i.e. $\partial D_h = \gamma_h$. Given a Hamiltonian system and a perturbation,

$$\left\{ \begin{array}{rcl} \dot{x} &=& -H_y(x,y) + \epsilon f(x,y) \\ \dot{y} &=& H_x(x,y) + \epsilon g(x,y), \end{array} \right. \tag{1}$$

the Abelian integral, in general multi-valued, is defined as

$$I(h) = \int_{\gamma_h} f(x,y)\,dy - g(x,y)\,dx. \tag{2}$$

We denote the integrand ω, and call it the 1-form associated with the perturbation. In this paper all perturbations are polynomial.

The most important property of Abelian integrals is described by the Poincaré-Pontryagin theorem.

Theorem 2.1 (Poincaré-Pontryagin)**.** *Let P be the return map defined on some section transversal to the ovals of H, parametrised by the values h of H, where h is taken from some bounded interval (a,b). Let $d(h) = P(h) - h$ be the displacement function. Then, $d(h) = \epsilon(I(h) + \epsilon\phi(h,\epsilon))$,* as $\epsilon \to 0$, *where $\phi(h,\epsilon)$ is analytic and uniformly bounded on a compact neighbourhood of $\epsilon = 0,\ h \in (a,b)$.*

Proof. see e.g. [5]. □

As a consequence of the above theorem, one can prove that a simple zero of $I(h)$ corresponds to a unique limit cycle bifurcating from the Hamiltonian system as $\epsilon \to 0$. In fact, to prove existence of a limit cycle, it suffices to have a zero of odd order.

2.1 Computer-aided proofs

Seeing that our proof relies upon a great deal of numerical computations, we have been very careful in validating the computational results. A numerical algorithm is said to be *auto-validating* if it produces a mathematically correct result, incorporating not only the discretisation errors of the numerical method, but also the computer's internal representation of the floating point numbers and its rounding procedures. The basic object in any such algorithm is an interval, whose endpoints are computer-representable floating points. All mathematical operations are performed in interval arithmetic with directed rounding to ensure the correctness of the result. For a thorough introduction to this topic we refer to [1, 18, 19, 20, 21].

2.2 Computer-aided computation of Abelian integrals

We use the method developed in [13] to enclose the values of all Abelian integrals $I(h)$ appearing in our proof. This enables us to rigorously sample their values, i.e., for some discrete values of h, we can determine intervals such that $I(h) \in [I^-(h), I^+(h)]$. If we can find two ovals γ_{h_1}, and γ_{h_2}, such that all elements of $[I^-(h_1), I^+(h_1)]$ have the opposite sign as those of $[I^-(h_2), I^+(h_2)]$ then, by the intermediate value theorem, there exists $h^* \in (h_1, h_2)$, such that $I(h^*) = 0$, and a neighbourhood of γ_{h^*} that is either attracting or repelling for the perturbed vector field.

Since P_ϵ, the return map of the perturbed vector field, is analytic and non-constant, it has isolated fixed points. Thus, a zero of I implies the existence of (at least) one limit cycle bifurcating from γ_{h^*}.

In order to construct a perturbation such that the associated Abelian integral has a given number of zeros, the perturbation has to be chosen in a careful manner. The heuristic approach we have used to construct such a perturbation is described in Section 4.1.

3 The Hamiltonian

We study the Hamiltonian, described in [28] :

$$H(x,y) = \frac{x^2}{2} - \frac{9x^4}{8} + \frac{x^6}{3} + \frac{y^2}{2} - \frac{73y^4}{144} + \frac{2y^6}{27} \qquad (3)$$

corresponding to the differential system,

$$\begin{cases} \dot{x} &= -y\left(1 - \frac{16y^2}{9}\right)\left(1 - \frac{y^2}{4}\right) \\ \dot{y} &= x\left(1 - 4x^2\right)\left(1 - \frac{x^2}{2}\right) \end{cases} . \qquad (4)$$

The system has 25 equilibrium points and 19 periodic annuli, appearing in 9 classes, see Figure 1. We label the classes of periodic annuli $\Gamma_1 - \Gamma_9$, in the order of increasing h, see Figure 2 and Table 1.

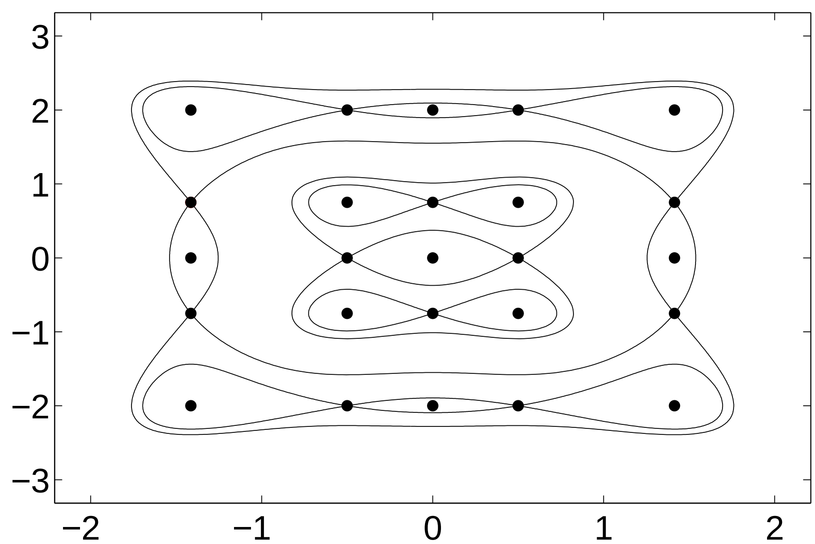

Figure 1: Phase portrait of the unperturbed Hamiltonian system.

We are interested in limit cycles bifurcating from the periodic solutions of (4), corresponding to integral curves of (3).

We follow [28], and study the following $\mathbf{Z}_2$ equivariant perturbation of the Hamiltonian system (4),

$$\begin{array}{rcl} p(x,y) & := & \frac{\alpha_{00}}{2} + \frac{\alpha_{20}}{4}x^2 + \frac{\alpha_{02}}{4}y^2 + \frac{\alpha_{40}}{6}x^4 + \frac{\alpha_{22}}{6}x^2y^2 + \frac{\alpha_{04}}{6}y^4 \\ f(x,y) & := & xp(x,y) \\ g(x,y) & := & yp(x,y) \end{array} \qquad (5)$$

Thus, the Abelian integral (2) reads,

$$\begin{array}{rcl} I(h) & = & \int_{\gamma_h} f\,dy - g\,dx = \int_{\gamma_h} xp\,dy - yp\,dx = \int_{D_h} \left(2p + x\frac{\partial p}{\partial x} + y\frac{\partial p}{\partial y}\right)\,dx \wedge dy \\ & = & \int_{D_h} \left(\alpha_{00} + \alpha_{20}x^2 + \alpha_{02}y^2 + \alpha_{40}x^4 + \alpha_{22}x^2y^2 + \alpha_{04}y^4\right)\,dx \wedge dy. \end{array}$$

4 Results

Theorem 4.1. *Consider the Hamiltonian vector field (4), perturbed as in (5). Then one can choose α_{ij}, such that, as $\epsilon \to 0$, at least 27 limit cycles appear in the configuration,*

$$(\Gamma_1^2)^4(\Gamma_2)^2(\Gamma_3)^2(\Gamma_6)(\Gamma_8)^2(\Gamma_9^3)^4,$$

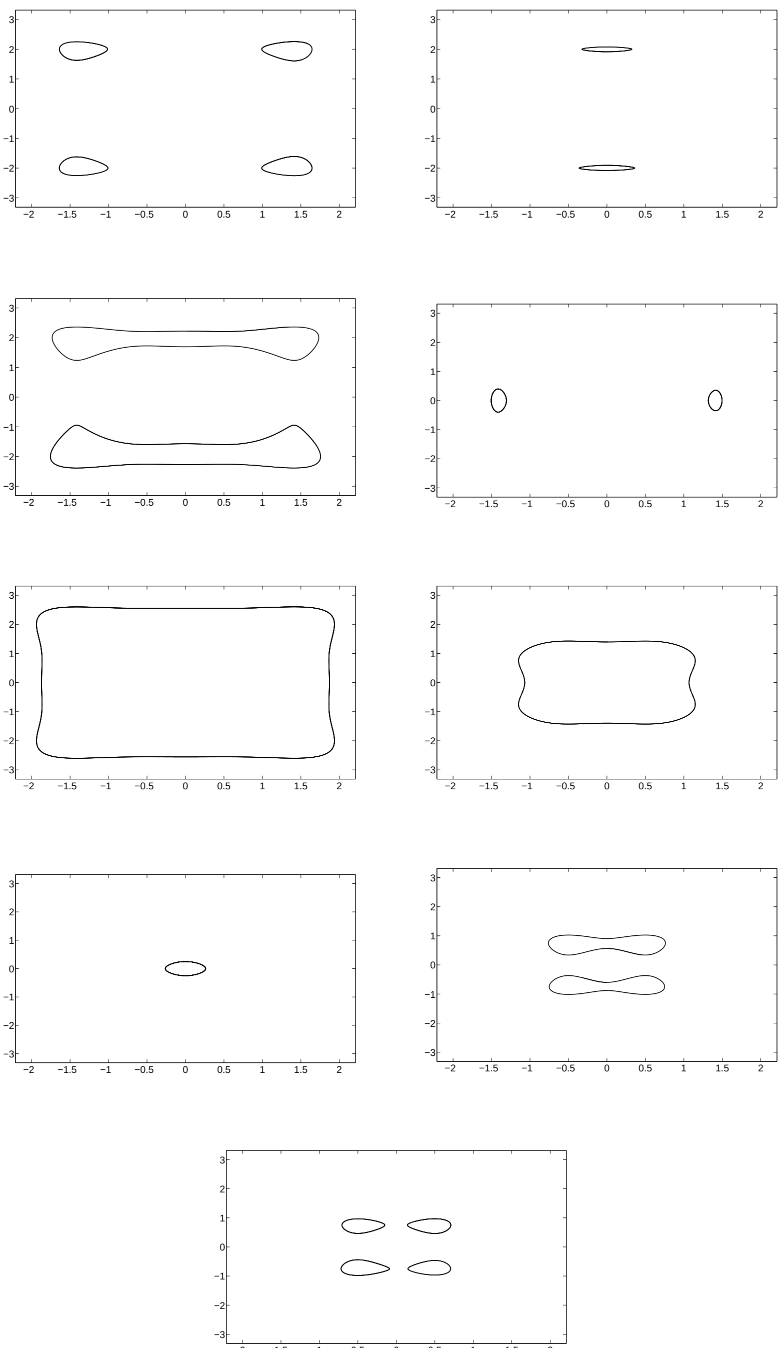

Figure 2: The periodic annuli, $\Gamma_1 - \Gamma_9$.

Periodic annulus	h_{min}	h_{max}	Expand/Contract
1	-2.2037	-1.3105	`expand`
2	-1.3704	-1.3105	`expand`
3	-1.3105	-0.6993	`expand`
4	-0.8333	-0.6993	`expand`
5	-0.6693	∞	`expand`
6	-0.6693	0.0599	`contract`
7	0.0000	0.0599	`expand`
8	0.0599	0.1340	`contract`
9	0.1340	0.1939	`contract`

Table 1: The domains of the periodic annuli. The labels `contract` and `expand` refer to the behaviour of the ovals in an annulus as h increases.

see Figure 3.

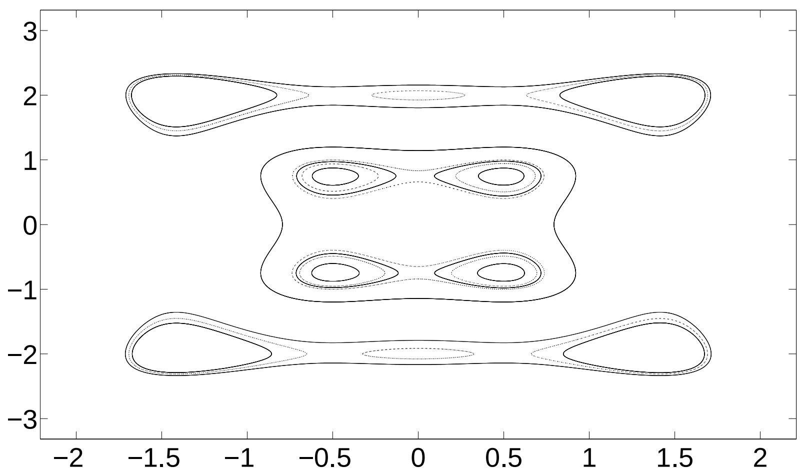

Figure 3: The ovals, from which the limit cycles bifurcate.

We use $Z(n+1,m)$ to denote the maximum number of limit cycles that can bifurcate from a Hamiltonian vector field of degree n, under a perturbation of order m. Obviously, $Z(n) := Z(n+1,n) \leq H(n)$, where $H(n)$ denotes the maximum number of limit cycles that a nth degree planar polynomial system can have. Some known results are $Z(2) = 2$ [4], $Z(3) \geq 12$ [25], $Z(4) \geq 15$ [29], $Z(5) \geq 24$ [3], $Z(6) \geq 35$ [23], $Z(7) \geq 49$ [17], $Z(9) \geq 80$ [26], and $Z(11) \geq 121$ [24].

Corollary 4.2. $Z(5) \geq 27$.

4.1 Strategy

In this section we apply the methods developed in [13] to the Hamiltonian system described above. The first part of our approach is to integrate monomial forms at some points, $h_1, \ldots, h_N$, and then to specify the coefficients of

$$d\omega = \left(\alpha_{00} + \alpha_{20}x^2 + \alpha_{02}y^2 + \alpha_{40}x^4 + \alpha_{22}x^2y^2 + \alpha_{04}y^4\right)\, dx \wedge dy, \qquad (6)$$

such that the Abelian integrals vanish:

$$I(h_\ell) = \int_{\gamma_{h_\ell}} \omega = 0, \quad \ell = 1, \ldots, N. \qquad (7)$$

Therefore, let

$$I_{ij}(h) = \int_{D_h} x^i y^j \, dx \wedge dy, \qquad (8)$$

where $\partial D_h = \gamma_h$. Then we have the following linear decomposition

$$I(h) = \alpha_{00}I_{00}(h)+\alpha_{20}I_{20}(h)+\alpha_{02}I_{02}(h)+\alpha_{40}I_{40}(h)+\alpha_{22}I_{22}(h)+\alpha_{04}I_{04}(h). \quad (9)$$

Given some candidate coefficients of the form ω, we calculate the $I_{ij}(h)$ at intermediate ovals, $\tilde{h}_1 < h_1 < \tilde{h}_2 < \cdots < h_N < \tilde{h}_{N+1}$. If the linear combination (9) of the $I_{ij}(\tilde{h})$ has validated sign changes between the sample points we are done: it has been proved that the corresponding perturbation yields bifurcations with at least the given number of limit cycles as $\epsilon \to 0$.

We recall that, in general, the Abelian integral is multi-valued, and the abovementioned computations are done for each continuous family of ovals separately. In the equivariant case at hand, $I(h)$ is identically the same on each of the different annuli within one annulus-class. Thus, one can trivially split the set of ovals corresponding to $H = h$ into natural subsets. This is crucial for the success of our approach, since for each limit cycle we find there will be one [three] additional cycles that can be found be rotating the first cycle by an angle π [$\frac{\pi}{2}$] for the annuli $\Gamma_2, \Gamma_3, \Gamma_4$, and Γ_8 [Γ_1, and Γ_9].

4.2 Generating candidate coefficients

Using the tools developed in [13], the computation of verified sign changes of the Abelian integral is automatic, once we have a set of proper coefficients. To choose such candidate coefficients, however, is non-trivial. The reason is that the regions in the parameter space yielding a large number of zeros is, typically, small. We sample each of the monomials I_{ij}^k, where $k \in \{1, ..., 9\}$ denotes in which annulus the integral is computed, at 100 uniformly distributed points in the respective domains. Generically, it is expected that the space of Abelian integrals on one branch should be Chebyshev, see e.g. [10], i.e. the number of zeros of a function in the space is one less than the dimension of the space.

The first step is to choose some ovals where we force the Abelian integral to be zero, by solving the corresponding linear system for the coefficients of ω. Note, this method can automatically give any configuration of limit cycles generated by 5 zeros. We see from Figure 1 that it is desirable to maximise the number of zeros on Γ_1 and Γ_9, since such limit cycles have multiplicity four. If this procedure is done arbitrarily, only up to 20 limit cycles would be guaranteed.

To find a better set of candidate coefficients we note that the system has the following properties: (i) the domains of $I^1(h)$ and $I^2(h)$ overlap, see Table 1; (ii) the annuli Γ_1^2 and Γ_2 are surrounded by the annulus Γ_3, and the annuli Γ_9^2 are surrounded by the annulus Γ_8, see Figure 1. Property (i) indicates that it should be possible, as in [14], to force $I^1(h)$ and $I^2(h)$ to oscillate together, see Figure 4. Property (ii) indicates that it should be possible, as in the figure eight case in [13], to get two extra cycles surrounding the annuli Γ_1^2 and Γ_2, and Γ_9^2, respectively. By taking property (i), and (ii) into account, we can locate a suitable candidate form ω.

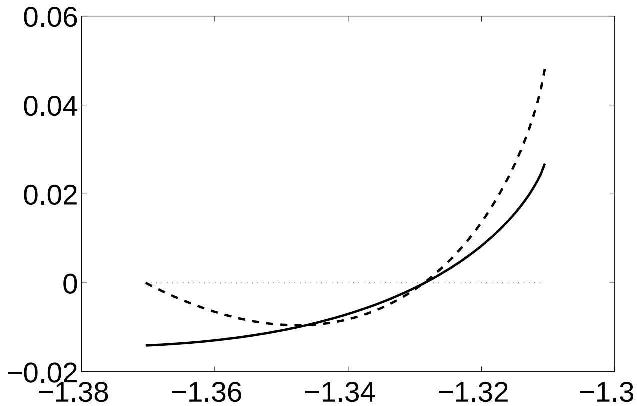

Figure 4: The Abelian integrals on ovals 1 (solid) and 2 (dashed), oscillating together.

4.3 Computational results

Using the method described above to generate candidate coefficients for ω, we get the result listed in Table 2.

α_{00}	2.176832745375219
α_{20}	0.203687169951339
α_{02}	-4.663680776344302
α_{40}	-8.410822908376025
α_{22}	4.313536179874701
α_{04}	1.000000000000000

Table 2: The generated coefficients of the perturbation (5).

The next step is to validate that the generated coefficients yield the expected behaviour. Therefore, we enclose the value of the corresponding Abelian integrals at intermediate ovals. As is shown in Table 3, the generated coefficients correspond to a perturbation for which the claimed number of limit cycles bifurcate from the given Hamiltonian. The graphs of the Abelian integrals for

$\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_6, \Gamma_8$, and Γ_9, from which it bifurcates limit cycles, are shown in Figure 5.

Periodic annulus	h	$I(h)$
1	-1.840000	$[6.88,7.11]\times10^{-2}$
1	-1.380000	$-[1.55,1.19]\times10^{-2}$
1	-1.310500	$[2.43,2.83]\times10^{-2}$
2	-1.345600	$-[1.18,1.12]\times10^{-2}$
2	-1.310500	$[4.16,4.31]\times10^{-2}$
3	-1.267300	$[2.17,2.27]\times10^{-1}$
3	-0.700000	$-[7.88,7.85]\times10^{0}$
6	-0.600000	$-[1.66,1.65]\times10^{1}$
6	0.058000	$[1.33,1.34]\times10^{0}$
8	0.060000	$[1.86,1.87]\times10^{-1}$
8	0.131500	$-[1.38,1.04]\times10^{-3}$
9	0.134100	$-[3.22,3.06]\times10^{-4}$
9	0.141300	$[9.56,11.0]\times10^{-5}$
9	0.163700	$-[3.36,2.36]\times10^{-5}$
9	0.186100	$[2.04,2.53]\times10^{-5}$

Table 3: The computed enclosures of the Abelian integrals.

All computations were performed on a AMD Opteron 848 2.2GHz, 64bit processor with 16Gb of RAM. The program was compiled with `gcc`, version 3.4.6. The software for interval arithmetic was provided by the `CXS-C` package, version 2.1.1, see [6, 9]. The total run-time of the validated program [13], to calculate the 15 Abelian integrals necessary for the proof, was 50 minutes.

5 Conclusions

We have applied the method developed in [13] to study an equivariant quintic Hamiltonian vector field under quintic perturbation, previously studied in [28]. Our approach differs from the one utilised in [28] in two ways. First, we vary all parameters together, whereas in [28] the detection function method, developed in [16], is used, which means that the parameters are determined in two steps; in the first step all parameters except one are determined, and in the final search only one parameter is varied. Our approach to variation of the coefficients yields a perturbation with a larger number of limit cycles than was previously established for the quintic case. The new bound $Z(5) \geq 27$, improves the result $Z(5) \geq 24$ in [3]. Second, after determining candidate coefficients, we validate that they have the desired properties. The second step makes our result mathematically rigorous.

References

[1] G. Alefeld, and J. Herzberger, Introduction to Interval Computations, Academic Press, New York, 1983.

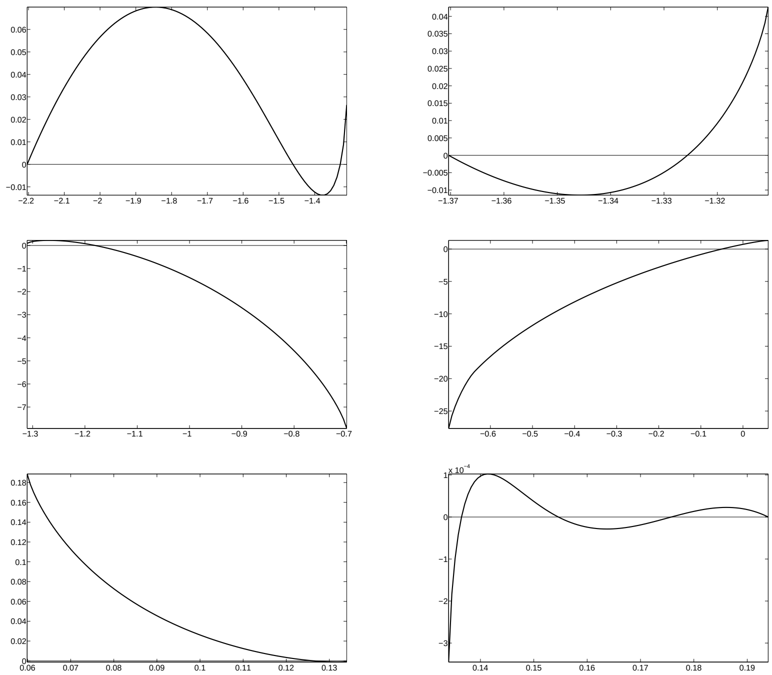

Figure 5: The graphs of the Abelian integrals.

[2] V.I. Arnol'd, Ten problems. Theory of singularities and its applications, 1–8, Adv. Soviet Math., 1, Amer. Math. Soc., Providence, RI, 1990.

[3] H.S.Y. Chan, K. W. Chung, J. Li, Bifurcations of limit cycles in Z_Q-equivariant planar vector fields of degree 5. Foundations of computational mathematics (Hong Kong, 2000), 61–83, World Sci. Publ., River Edge, NJ, 2002.

[4] F. Chen, C. Li, J. Llibre, Z. Zhang, A unified proof on the weak Hilbert 16th problem for $n = 2$. J. Differential Equations 221 (2006), no. 2, 309–342.

[5] C. Christopher, C. Li, Limit cycles of differential equations. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2007.

[6] CXSC – C++ eXtension for Scientific Computation, version 2.0. Available from http://www.math.uni-wuppertal.de/org/WRST/xsc/cxsc.html

[7] Écalle, Jean Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac. (French) [Introduction to analyzable functions and constructive proof of the Dulac conjecture] Actualités Mathématiques. [Current Mathematical Topics] Hermann, Paris, 1992.

[8] J. Guckenheimer, P. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Applied Mathematical Sciences, 42. Springer-Verlag, New York, 1983.

[9] R. Hammer, M. Hocks, U. Kulisch, and D. Ratz, C++ Toolbox for Verified Computing, Springer-Verlag, New York, 1995.

[10] Horozov, Emil; Iliev, Iliya D. Linear estimate for the number of zeros of Abelian integrals with cubic Hamiltonians. Nonlinearity 11 (1998), no. 6, 1521–1537.

[11] Yu. S. Il'yashenko, Finiteness theorems for limit cycles. Translated from the Russian by H. H. McFaden. Translations of Mathematical Monographs, 94. American Mathematical Society, Providence, RI, 1991.

[12] Yu. S. Il'yashenko, Centennial history of Hilbert's 16th problem. Bull. Amer. Math. Soc. (N.S.) 39 (2002), no. 3, 301–354

[13] T. Johnson, W. Tucker, On a computer-aided approach to the computation of Abelian integrals, submitted.

[14] T. Johnson, W. Tucker, A rigorous study of possible configurations of limit cycles bifurcating from a hyper-elliptic Hamiltonian of degree five, to appear in Dynamical Systems - An international journal, doi:/10.1080/14689360802641206.

[15] J. Li, H.S.Y. Chan, K. W. Chung, Bifurcations of limit cycles in a Z_6-equivariant planar vector field of degree 5. Sci. China Ser. A 45 (2002), no. 7, 817–826.

[16] J.B. Li, G.M. Huang, Bifurcations of limit cycles forming compound eyes in the cubic system. Chinese Ann. Math. Ser. B 8 (1987), no. 4, 391–403.

[17] J. Li, M. Zhang, Bifurcations of limit cycles in a Z_8-equivariant planar vector field of degree 7. J. Dynam. Differential Equations 16 (2004), no. 4, 1123–1139.

[18] R.E. Moore, Interval Analysis, Prentice-Hall, Englewood Cliffs, New Jersey, 1966.

[19] R.E. Moore, Methods and Applications of Interval Analysis, SIAM Studies in Applied Mathematics, Philadelphia, 1979.

[20] A. Neumaier, Interval Methods for Systems of Equations. Encyclopedia of Mathematics and its Applications 37, Cambridge Univ. Press, Cambridge, 1990

[21] M.S. Petković, L.D. Petković, Complex interval arithmetic and its applications. Mathematical Research, 105. Wiley-VCH Verlag Berlin GmbH, Berlin, 1998.

[22] R. Roussarie, Bifurcation of planar vector fields and Hilbert's sixteenth problem. Progress in Mathematics, 164. Birkhäuser Verlag, Basel, 1998.

[23] S. Wang, P. Yu, Bifurcation of limit cycles in a quintic Hamiltonian system under a sixth-order perturbation. Chaos Solitons Fractals 26 (2005), no. 5, 1317–1335.

[24] S. Wang, P. Yu, Existence of 121 limit cycles in a perturbed planar polynomial Hamiltonian vector field of degree 11. Chaos Solitons Fractals 30 (2006), no. 3, 606–621.

[25] P. Yu, M. Han, Twelve limit cycles in a cubic case of the 16th Hilbert problem. Internat. J. Bifur. Chaos Appl. Sci. Engrg. 15 (2005), no. 7, 2191–2205.

[26] S. Wang, P. Yu, J. Li, Bifurcation of limit cycles in Z_{10}-equivariant vector fields of degree 9. Internat. J. Bifur. Chaos Appl. Sci. Engrg. 16 (2006), no. 8, 2309–2324.

[27] H. Zhou, W. Xu, S. Li, Y. Zhang, On the number of limit cycles of a cubic polynomials Hamiltonian system under quintic perturbation. (English summary) Appl. Math. Comput. 190 (2007), no. 1, 490–499.

[28] H. Zhou, W. Xu, X. Zhao, S. Li, Detection function method and its application to a class of quintic Hamiltonian systems with quintic perturbations. Appl. Math. Comput. 191 (2007), no. 2, 490–503.

[29] T. Zhang, M. Han, H. Zang, M. Hong, Xinzhu Bifurcations of limit cycles for a cubic Hamiltonian system under quartic perturbations. Chaos Solitons Fractals 22 (2004), no. 5, 1127–1138.