An Improved Modified Extended tanh-Function Method

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In this paper we further improve the modified extended tanh-function method to obtain new exact solutions for nonlinear partial differential equations. Numerical applications of the proposed method are verified by solving the improved Boussinesq equation and the system of variant Boussinesq equations. The new exact solutions for these equations include Jacobi elliptic doubly periodic type, Weierstrass elliptic doubly periodic type, triangular type and solitary wave solutions.

Key words: Nonlinear Evolution Equation; Modified Extended tanh-Function Method; Travelling Wave.

1. Introduction

The rapid development of science and technology and the need to simulate complex physical phenomena demand an increasing need to solve quite a number of complicated high-dimensional and nonlinear partial differential equations (PDEs) with variable coefficients. The use of the integrable system technique to determine explicit exact solutions for these kinds of equations has recently drawn the attention of many scientists and engineers. The investigation of these explicit exact solutions, for instance soliton, periodic and quasi-periodic solutions of soliton equations, plays an important role in the study of nonlinear physical phenomena. The wave phenomena observed in fluid dynamics, plasma, and elastic media are often modelled by the bell-shaped sech soliton solutions and the kink-shaped tanh soliton solutions. Exact solutions of these kinds of nonlinear equations are crucial in the stability analysis and efficiency comparison with available numerical solvers of the problems.

During the last decades, effective methodologies such as inverse scattering method [1 – 3], Darboux transformation [4 – 8], Hirota bilinear method [9 - 11] and tanh-function method [12 – 19] have been proposed for the determination of solitons. Among these proposed methods the tanh-function method provides an effective and direct algebraic method for solving nonlinear equations. Based on the localized nature of soliton solutions, the tanh-function method overcomes the complex integration process to obtain explicit solutions to various types of nonlinear equations. Attention has been concentrated on the various extensions and applications of the tanh-function method. For instance, Parkes and Duffy automated to some degree the method by using *Mathematica* for the solitary wave solutions to nonlinear evolution equations [15] and Korteweg – de Vries (KdV) – Burgers equations [16], respectively. The soliton solutions of these nonlinear equations are usually expressed as polynomials of tanh-functions. During the solution process, the original equation will be transformed into a nonlinear system of algebraic equations. Based on an extension of the tanh-function method, Fan [17, 18] developed a new algebraic method with symbolic computation for obtaining a series of travelling wave solutions in a unified way.

Recently Elwakil et al. [19, 20] proposed a modified extended tanh-function method to obtain new exact solutions for some nonlinear evolution equations. Fan and Hon [21] proposed a generalized tanh method to obtain more general explicit solutions to nonlinear soliton equations. In this paper, we further improve the modified extended tanh-function method and successfully obtain more new exact solutions for the nonlinear improved Boussinesq equation and the system of variant Boussinesq equations.

The outline of the paper is as follows: The main idea to improve the modified extended tanh-function method will be given in Section 2. In Section 3, we verify the numerical applications of the proposed method by solving the improved Boussinesq equation and the system of variant Boussinesq equations. Numerical results indicated that more new exact solutions can be obtained. A conclusion is given in Section 4.

2. Improved Modified Extended tanh-Function Method

For simplicity, we consider the following nonlinear PDE with only two independent variables:

$$H(u,u_t,u_x,u_{xx},\cdots)=0. \tag{1}$$

Using the wave transformation $u(x,t)=u(\zeta)$, $\zeta=x+ct$, we can reduce (1) to the following ordinary differential equation (ODE):

$$H(u,u',u'',\cdots)=0. \tag{2}$$

The idea of the modified extended tanh-function method [19] is to express the solution u for (1) in the form of

$$u(\zeta)=\sum_{i=0}^{n}a_i\omega^i+\sum_{i=1}^{n}b_i\omega^{-i}, \tag{3}$$

where ω satisfies the well-known Riccati equation

$$\omega'=b+\omega^2, \tag{4}$$

with $\omega=\omega(\zeta)$, $\omega'=\mathrm{d}\omega/\mathrm{d}\zeta$ and $a_i, b_i, b\neq 0$ and $c\neq 0$ are parameters to be determined. The value of n can be found by balancing the highest-order linear term with the nonlinear terms of (2). Substituting (3) and (4) into the ODE (2) and setting all the coefficients of ω^i to zero, a system of algebraic equations can be obtained for determining the unknown parameters a_i, b_i, b and c. It is well-known that the Riccati equation (4) has the general solutions $\omega=-\sqrt{-b}\tanh(\sqrt{-\mathrm{b}}\zeta)$ and $\omega=-\sqrt{-b}\coth(\sqrt{-\mathrm{b}}\zeta)$ for $b<0$, $\omega=-\frac{1}{\zeta}$ for $b=0$, $\omega=\sqrt{b}\tan(\sqrt{\mathrm{b}}\zeta)$ and $\omega=-\sqrt{b}\cot(\sqrt{\mathrm{b}}\zeta)$ for $b>0$. These solutions are called ‘fundamental solutions’ throughout this paper. From these fundamental solutions w and the determined values of the parameters a_i, b_i, b, and c, five kinds of travelling wave solutions for (1) can be obtained [19].

Based on the idea of Fan and Hon [21], we further improve the method by extending the Riccati equation (4) to the following general ODE:

$$\omega'=\varepsilon\sqrt{\sum_{j=0}^{r}c_j\omega^j}, \tag{5}$$

where $\varepsilon=\pm 1$. Balancing the highest-order linear term with the nonlinear terms in (2), we obtain a relationship between the positive integers r and n. In fact, if $r=4$, $c_1=c_3=0$, $c_0=b^2$, $c_2=2b$, $c_4=1$, the ODE (5) degenerates to the Riccati equation (4), and in this case the proposed method is the modified extended tanh-function method. In the following we show that the general ODE (5) gives various kinds of fundamental solutions which include the known fundamental solutions for (4). From these new fundamental solutions, more new exact solutions for (1) can be obtained. We remark here that the solutions for (1) depend on the exact solvability of (5) which will become more difficult if the value of r is too large. In this paper we consider the case $r=4$ in (5) so that (5) can be reduced to

$$\omega'=\varepsilon\sqrt{c_0+c_1\omega+c_2\omega^2+c_3\omega^3+c_4\omega^4}. \tag{6}$$

From the different possible values of c_0, c_1, c_2, c_3 and c_4, we obtain from (6) the various kinds of fundamental solutions as follows:

Case 1. $c_0=c_1=c_3=0$.

A bell-shaped solitary wave solution, a triangular type solution and a rational solution are obtained:

$$\omega=\sqrt{-\frac{c_2}{c_4}}\mathrm{sech}(\sqrt{\mathrm{c}_2}\zeta), \quad \mathrm{c}_2>0, \mathrm{c}_4<0, \tag{7}$$

$$\omega=\sqrt{-\frac{c_2}{c_4}}\sec(\sqrt{-\mathrm{c}_2}\zeta), \quad \mathrm{c}_2<0, \mathrm{c}_4>0, \tag{8}$$

$$\omega=-\frac{\varepsilon}{\sqrt{c_4}\zeta}, \quad c_2=0, c_4>0. \tag{9}$$

Case 2. $c_1=c_3=0$.

A kink-shaped solitary wave solution, a triangular type solution and three Jacobi elliptic doubly periodic-type solutions are obtained:

$$\omega=\varepsilon\sqrt{-\frac{c_2}{2c_4}}\tanh\Big(\sqrt{-\frac{\mathrm{c}_2}{2}}\zeta\Big), \quad c_2<0,\ c_4>0,\ c_0=\frac{c_2^2}{4c_4}, \tag{10}$$

$$\omega=\varepsilon\sqrt{\frac{c_2}{2c_4}}\tan\Big(\sqrt{\frac{\mathrm{c}_2}{2}}\zeta\Big), \quad c_2>0,\ c_4>0,\ c_0=\frac{c_2^2}{4c_4}, \tag{11}$$

$$\omega=\sqrt{\frac{-c_2m^2}{c_4(2m^2-1)}}\mathrm{cn}(\sqrt{\frac{c_2}{2m^2-1}}\zeta),\quad c_2>0,\ c_4<0,\ c_0=\frac{c_2^2m^2(1-m^2)}{c_4(2m^2-1)^2},\tag{12}$$

$$\omega=\sqrt{\frac{-m^2}{c_4(2-m^2)}}\mathrm{dn}(\sqrt{\frac{c_2}{2-m^2}}\zeta),\quad c_2>0,\ c_4<0,\ c_0=\frac{c_2^2(1-m^2)}{c_4(2-m^2)^2},\tag{13}$$

$$\omega=\varepsilon\sqrt{\frac{-c_2m^2}{c_4(m^2+1)}}\mathrm{sn}(\sqrt{-\frac{c_2}{m^2+1}}\zeta),\quad c_2<0,\ c_4>0,\ c_0=\frac{c_2^2m^2}{c_4(m^2+1)^2},\tag{14}$$

where m denotes a modulus.

Case 3. $c_0=c_1=c_4=0$.

A bell-shaped solitary wave solution, a triangular type solution and a rational type solution are obtained:

$$\omega=-\frac{c_2}{c_3}\mathrm{sech}^2(\frac{\sqrt{c_2}}{2}\zeta),\quad c_2>0,\tag{15}$$

$$\omega=-\frac{c_2}{c_3}\sec^2(\frac{\sqrt{-c_2}}{2}\zeta),\quad c_2<0,\tag{16}$$

$$\omega=\frac{4}{c_3\zeta^2},\quad c_2=0.\tag{17}$$

Case 4. $c_2=c_4=0,\ c_0\neq 0,\ c_1\neq 0, c_3>0$.

A Weierstrass elliptic doubly periodic type solution is obtained:

$$\omega=\wp(\frac{\sqrt{c_3}}{2}\zeta,g_2,g_3),\tag{18}$$

where $g_2=-4c_1/c_3$ and $g_3=-4c_0/c_3$ are called invariants of the Weierstrass elliptic function.

Case 5. $c_0=c_1=c_2=0$.

A rational type solution and an exponential type solution are obtained:

$$\omega=\frac{4c_3}{c_3^2\zeta^2-4c_4},\quad c_4\neq 0,\tag{19}$$

$$\omega=\frac{c_3}{2c_4}\exp(\frac{\varepsilon c_3}{2\sqrt{-c_4}}\zeta),\quad c_4<0.\tag{20}$$

Case 6. $c_3=c_4=0$.

An exponential type solution, two triangular type solutions and two hyperbolic type solutions are obtained:

$$\omega=-\frac{c_1}{2c_2}+\exp(\varepsilon\sqrt{c_2}\zeta),\ c_2>0,\ c_0=\frac{c_1^2}{4c_2},\tag{21}$$

$$\omega=-\frac{c_1}{2c_2}+\frac{\varepsilon c_1}{2c_2}\sin(\sqrt{-c_2}\zeta),\ c_0=0,\ c_2<0,\tag{22}$$

$$\omega=-\frac{c_1}{2c_2}+\frac{\varepsilon c_1}{2c_2}\sinh(2\sqrt{c_2}\zeta),\ c_0=0,\ c_2>0\tag{23}$$

$$\omega=\varepsilon\sqrt{-\frac{c_0}{c_2}}\sin(\sqrt{-c_2}\zeta),\ c_1=0,\ c_0>0,\ c_2<0,\tag{24}$$

$$\omega=\varepsilon\sqrt{\frac{c_0}{c_2}}\sinh(\sqrt{-c_2}\zeta),\ c_1=0,\ c_0>0,\ c_2>0.\tag{25}$$

Case 7. $c_0=c_1=0, c_4>0$.

A triangular type solution and two solitary wave solutions are obtained:

$$\omega=-\frac{c_2\sec^2(\frac{1}{2}\sqrt{-c_2}\zeta)}{2\varepsilon\sqrt{-c_2c_4}\tan(\frac{1}{2}\sqrt{-c_2}\zeta)+c_3},\quad c_2<0,\tag{26}$$

$$\omega=\frac{c_2\mathrm{sech}^2(\frac{1}{2}\sqrt{c_2}\zeta)}{2\varepsilon\sqrt{c_2c_4}\tanh(\frac{1}{2}\sqrt{c_2}\zeta)-c_3},\quad c_2>0,\ c_3\neq 2\varepsilon\sqrt{c_2c_4},\tag{27}$$

$$\omega=\frac{1}{2}\varepsilon\sqrt{\frac{c_2}{c_4}}(1+\tanh(\frac{1}{2}\sqrt{c_2}\zeta)),\quad c_2>0,\ c_3=2\varepsilon\sqrt{c_2c_4}.\tag{28}$$

Remark. The Jacobi elliptic functions are doubly periodical and possess properties of triangular functions as:

$$\mathrm{sn}^2\zeta+\mathrm{cn}^2\zeta=1,\quad \mathrm{dn}^2\zeta=1-\mathrm{m}^2\mathrm{sn}^2\zeta,$$
$$(\mathrm{sn}\zeta)'=\mathrm{cn}\zeta\mathrm{dn}\zeta,\quad (\mathrm{cn}\zeta)'=-\mathrm{sn}\zeta\mathrm{dn}\zeta,$$
$$(\mathrm{dn}\zeta)'=-\mathrm{m}^2\mathrm{sn}\zeta\mathrm{cn}\zeta.$$

When $m\to 1$, the Jacobi functions degenerate to the hyperbolic functions, i. e.,

$$\mathrm{sn}\zeta\to\tanh\zeta,\quad \mathrm{cn}\zeta\to\mathrm{sech}\zeta,\quad \mathrm{dn}\zeta\to\mathrm{sech}\zeta.$$

When $m \to 0$, the Jacobi functions degenerate to the triangular functions, i. e.

$$\mathrm{sn}\zeta \to \sin\zeta, \quad \mathrm{cn}\zeta \to \cos\zeta, \quad \mathrm{dn}\zeta \to 1.$$

A more detailed notation for the Weierstrass and Jacobi elliptic functions can be found in [22, 23]. When $m \to 1$, the Jacobi doubly periodic solutions (12) and (13) degenerate to the solitary wave solutions (7), and the solution (14) degenerates to the solitary wave solutions (10).

3. Numerical Verifications

Example 1. Consider the improved Boussinesq equation [24]

$$u_{tt} - u_{xx} - uu_{xx} - (u_x)^2 - u_{xxtt} = 0. \tag{29}$$

Using the wave transformations $u = u(\zeta)$, $\zeta = x + ct$, we can reduce (29) to the following ODE:

$$c^2u'' - u'' - uu'' - (u')^2 - c^2u^{(4)} = 0. \tag{30}$$

Using the method proposed in the last section, we expand the solution of (29) as (3), where ω satisfies (5). By balancing the highest-order linear term with the nonlinear terms in (30) we obtain

$$2n - 2 + 2 \times \frac{r}{2} = n - 4 + 4 \times \frac{r}{2}, \tag{31}$$

which gives the relation $n = r - 2$. Since $r = 4$, we have $n = 2$. Substituting $n = 2$ into (3) we then have

$$u = a_0 + a_1\omega + a_2\omega^2 + b_1\omega^{-1} + b_2\omega^{-2}, \tag{32}$$

where ω satisfies (6). Substituting (32) and (6) into (30) and setting the coefficients of all powers like ω^i and $\omega^i\sqrt{\sum_{j=0}^4 c_j\omega^j}$ to zero yields a system of algebraic equations for the unknown parameters a_i $(i = 0,1,2)$, b_i $(i = 1,2)$, c_i $(i = 0,\cdots,4)$, and c. With the aid of *Mathematica*, we can classify the solutions of the system according to the solutions for (29) as follows:

Case 1. $c_0 = c_1 = c_3 = 0$. We have

$$a_1 = b_1 = b_2 = 0, \quad a_0 = c^2 - 1 - 4c^2c_2,$$
$$a_2 = -12c^2c_4.$$

From equations (7)–(9) and (32) we obtain a bell-shaped solitary wave solution, a triangular type solution and a rational type solution:

$$u_{11} = c^2 - 1 - 4c^2c_2 + 12c^2c_2\mathrm{sech}^2(\sqrt{\mathrm{c}_2}(\mathrm{x}+\mathrm{ct})),$$
$$c_2 > 0, c_4 < 0,$$
$$u_{12} = c^2 - 1 - 4c^2c_2 + 12c^2c_2\sec^2(\sqrt{-\mathrm{c}_2}(\mathrm{x}+\mathrm{ct})),$$
$$c_2 < 0,\ c_4 > 0,$$
$$u_{13} = c^2 - 1 - \frac{12c^2}{(x+ct)^2}, \quad c_2 = 0, c_4 > 0.$$

Case 2. $c_1 = c_3 = 0$. We have

(i) $a_1 = b_1 = b_2 = 0$, $a_0 = c^2 - 1 - 4c^2c_2$, $a_2 = -12c^2c_4$;

(ii) $a_1 = a_2 = b_1 = 0$, $a_0 = c^2 - 1 - 4c^2c_2$, $b_2 = -12c^2c_0$;

(iii) $a_1 = b_1 = 0$, $a_0 = c^2 - 1 - 4c^2c_2$, $a_2 = -12c^2c_4$, $b_2 = -12c^2c_0$.

If c_0, c_2, c_4 satisfy $c_0 = c_2^2/4c_4$, from (10), (11) and (32), we have three solitary wave solutions and three triangular type solutions:

$$u_{21} = c^2 - 1 - 4c^2c_2 + 6c^2c_2\tanh^2(\sqrt{-\frac{c_2}{2}}(x+ct)), \quad c_2 < 0,\ c_4 > 0,$$

$$u_{22} = c^2 - 1 - 4c^2c_2 + 6c^2c_2\coth^2(\sqrt{-\frac{c_2}{2}}(x+ct)), \quad c_2 < 0,\ c_4 > 0,$$

$$u_{23} = c^2 - 1 - 4c^2c_2 + 6c^2c_2\tanh^2(\sqrt{-\frac{c_2}{2}}(x+ct)) + 6c^2c_2\coth^2(\sqrt{-\frac{c_2}{2}}(x+ct)), \quad c_2 < 0, c_4 > 0,$$

$$u_{24} = c^2 - 1 - 4c^2c_2 - 6c^2c_2\tan^2(\sqrt{\frac{c_2}{2}}(x+ct)), \quad c_2 > 0,\ c_4 < 0,$$

$$u_{25} = c^2 - 1 - 4c^2c_2 - 6c^2c_2\cot^2(\sqrt{\frac{c_2}{2}}(x+ct)), \quad c_2 > 0,\ c_4 < 0,$$

$$u_{26} = c^2 - 1 - 4c^2c_2 - 6c^2c_2\tan^2\left(\sqrt{\frac{c_2}{2}}(x+ct)\right) - 6c^2c_2\cot^2\left(\sqrt{\frac{c_2}{2}}(x+ct)\right), \quad c_2 < 0,\ c_4 > 0.$$

If c_0, c_2, c_4 satisfy $c_2 > 0$, $c_4 < 0$, $c_0 = \frac{c_2^2 m^2(1-m^2)}{c_4(2m^2-1)^2}$, from (12) and (32) we have three Jacobi elliptic doubly periodic type solutions:

$$u_{27} = c^2 - 1 - 4c^2c_2 + \frac{12c^2c_2m^2}{2m^2-1}\mathrm{cn}^2\left(\sqrt{\frac{c_2}{2m^2-1}}(x+ct)\right),$$

$$u_{28} = c^2 - 1 - 4c^2c_2 + \frac{12c^2c_2(1-m^2)}{2m^2-1}\mathrm{nc}^2\left(\sqrt{\frac{c_2}{2m^2-1}}(x+ct)\right),$$

$$u_{29} = c^2 - 1 - 4c^2c_2 + \frac{12c^2c_2m^2}{2m^2-1}\mathrm{cn}^2\left(\sqrt{\frac{c_2}{2m^2-1}}(x+ct)\right) + \frac{12c^2c_2(1-m^2)}{2m^2-1}\mathrm{nc}^2\left(\sqrt{\frac{c_2}{2m^2-1}}(x+ct)\right).$$

If c_0, c_2, c_4 satisfy $c_2 > 0$, $c_4 < 0$, $c_0 = \frac{c_2^2(1-m^2)}{c_4(2-m^2)^2}$, from (13) and (32) we obtain three Jacobi elliptic doubly periodic type solutions:

$$u_{210} = c^2 - 1 - 4c^2c_2 + \frac{12c^2m^2}{2-m^2}\mathrm{dn}^2\left(\sqrt{\frac{c_2}{2-m^2}}(x+ct)\right),$$

$$u_{211} = c^2 - 1 - 4c^2c_2 + \frac{12c^2c_2^2(1-m^2)}{(2-m^2)m^2}\mathrm{nd}^2\left(\sqrt{\frac{c_2}{2-m^2}}(x+ct)\right),$$

$$u_{212} = c^2 - 1 - 4c^2c_2 + \frac{12c^2m^2}{2-m^2}\mathrm{dn}^2\left(\sqrt{\frac{c_2}{2-m^2}}(x+ct)\right) + \frac{12c^2c_2^2(1-m^2)}{(2-m^2)m^2}\mathrm{nd}^2\left(\sqrt{\frac{c_2}{2-m^2}}(x+ct)\right).$$

If c_0, c_2, c_4 satisfy $c_2 < 0$, $c_4 > 0$, $c_0 = \dfrac{c_2^2m^2}{c_4(m^2+1)^2}$, from (14) and (32) we obtain three Jacobi elliptic doubly periodic type solutions:

$$u_{213} = c^2 - 1 - 4c^2c_2 + \frac{12c^2c_2m^2}{m^2+1}\mathrm{sn}^2\left(\sqrt{-\frac{c_2}{m^2+1}}(x+ct)\right),$$

$$u_{214} = c^2 - 1 - 4c^2c_2 + \frac{12c^2c_2}{m^2+1}\mathrm{ns}^2\left(\sqrt{-\frac{c_2}{m^2+1}}(x+ct)\right),$$

$$u_{215} = c^2 - 1 - 4c^2c_2 + \frac{12c^2c_2m^2}{m^2+1}\mathrm{sn}^2\left(\sqrt{-\frac{c_2}{m^2+1}}(x+ct)\right) + \frac{12c^2c_2}{m^2+1}\mathrm{ns}^2\left(\sqrt{-\frac{c_2}{m^2+1}}(x+ct)\right).$$

Case 3. $c_0 = c_1 = c_4 = 0$. We have

$$a_2 = b_1 = b_2 = 0,\ a_0 = c^2 - 1 - c^2c_2,\ a_1 = -3c^2c_3.$$

From (15) – (17) and (32), we obtain a bell-shaped solitary wave solution, a triangular type solution and a rational type solution:

$$u_{31} = c^2 - 1 - c^2c_2 + 3c^2c_2\,\mathrm{sech}^2\left(\frac{\sqrt{c_2}}{2}(x+ct)\right),\ c_2 > 0,$$

$$u_{32} = c^2 - 1 - c^2c_2 + 3c^2c_2\sec^2\left(\frac{\sqrt{c_2}}{2}(x+ct)\right),\ c_2 < 0,$$

$$u_{33} = c^2 - 1 - \frac{12c^2}{(x+ct)^2}, \quad c_2 = 0.$$

Case 4. $c_2 = c_4 = 0$. We have

(i) $a_2 = b_1 = b_2 = 0,\ a_0 = c^2 - 1,\ a_1 = -3c^2c_3;$

(ii) $a_1 = a_2 = 0,\ a_0 = c^2 - 1 + \dfrac{c_3}{2c_1},$

$b_1 = -6c^2c_1,\ b_2 = -12c^2c_0,\ 8c_0^2c_3 + c_1^3 = 0.$

From (18) and (32) we have two Weierstrass elliptic doubly periodic type solutions:

$$u_{41} = c^2 - 1 - 3c^2 c_3 \wp(\frac{\sqrt{c_3}}{2}, g_2, g_3),$$

$$u_{42} = c^2 - 1 + \frac{c_3}{2c_1} - 6c^2 c_1 (\wp(\frac{\sqrt{c_3}}{2}, g_2, g_3))^{-1} - 12c^2 c_0 (\wp(\frac{\sqrt{c_3}}{2}, g_2, g_3))^{-2},$$

where $g_2 = -4c_1/c_3$ and $g_3 = -4c_0/c_3$ are called invariants of the Weierstrass elliptic function.

Case 5. $c_3 = c_4 = 0$. We have

(i) $a_1 = a_2 = 0,\ a_0 = c^2 - 1 - c^2 c_2,$
$b_1 = -6c^2 c_1,\ b_2 = -12c^2 c_0,\ c_0 = \frac{c_1^2}{4c_2};$

(ii) $a_1 = a_2 = b_2 = 0,\ a_0 = c^2 - 1 - c^2 c_2,$
$b_1 = -3c^2 c_1,\ c_0 = 0;$

(iii) $a_1 = a_2 = b_1 = 0,\ a_0 = c^2 - 1 - 4c^2 c_2,$
$b_2 = -12c^2 c_0,\ c_1 = 0.$

From (21) – (25) and (32) we have an exponential type solution, two solitary wave solutions and two triangular type solutions:

$$u_{51} = c^2 - 1 - c^2 c_2 - 6c^2 c_1 (-\frac{c_1}{2c_2} + \exp(\varepsilon\sqrt{c_2}(x+ct)))^{-1} - 12c^2 c_0 (-\frac{c_1}{2c_2} + \exp(\varepsilon\sqrt{c_2}(x+ct)))^{-2}, \quad c_2 > 0,$$

$$u_{52} = c^2 - 1 - c^2 c_2 + 6c^2 c_2 (1 - \varepsilon \sinh(2\sqrt{c_2}(x+ct)))^{-1}, \quad c_2 > 0,$$
$$u_{53} = c^2 - 1 - 4c^2 c_2 - 12c^2 c_2 \operatorname{csch}^2(\sqrt{c_2}(x+ct)), \quad c_0 > 0,\ c_2 > 0,$$
$$u_{54} = c^2 - 1 - c^2 c_2 + 6c^2 c_2 (1 - \varepsilon \sin(\sqrt{-c_2}(x+ct)))^{-1}, \quad c_2 < 0,$$
$$u_{55} = c^2 - 1 - 4c^2 c_2 + 12c^2 c_2 \csc^2(\sqrt{-c_2}(x+ct)), \quad c_0 > 0, c_2 < 0.$$

Case 6. $c_0 = c_1 = 0$. We have

$$b_1 = b_2 = 0, \quad a_0 = c^2 - 1 - c^2 c_2, \quad a_1 = -6c^2 c_3, \quad a_2 = -12c^2 c_4, \quad c_3 = 2\varepsilon\sqrt{c_2 c_4}.$$

From (28) and (32) we have a kink-shaped solitary wave solution:

$$u_{61} = c^2 - 1 + 2c^2 c_2 - 3c^2 c_2 (2 + \tanh(\frac{1}{2}\sqrt{c_2}(x+ct)))^2,$$
$$c_2 > 0,\ c_4 > 0.$$

Remark. From the transformation $c_2 \to -c_2/2$, u_{11}, u_{53}, u_{12}, and u_{55} can be transformed to u_{21}, u_{22}, u_{24}, and u_{25}, respectively. From the transformation $c_2 \to -2c_2$, u_{31}, and u_{32} can be transformed to u_{21} and u_{24}, respectively. It can be seen that u_{13} and u_{33} are actually the same solution. As $m \to 1$, the Jacobi elliptic doubly periodic type solutions u_{27} - u_{215} can be reduced to u_{21} - u_{23} through proper transformation. The solutions u_{21} - u_{26} and u_{13} have been found by Elwakil et al. [20]. Theoretically, the solutions u_{11} - u_{13}, u_{21}, u_{22}, u_{24}, u_{25}, u_{27}, u_{210}, u_{213}, u_{31} - u_{33}, u_{41}, and u_{61} can also be deduced from Fan and Hon [21]. To the knowledge of the authors, the solutions u_{28}, u_{29}, u_{211}, u_{212}, u_{214}, u_{215}, u_{42}, u_{51}, u_{52} and u_{54} are new exact solutions for the improved Boussinesq equation (29).

The plots of some of the solutions are given in Figs. 1 – 5 to illustrate their properties.

Example 2. Consider the system

$$\begin{aligned} &u_t + v_x + uu_x + pu_{xxt} = 0, \\ &v_t + (uv)_x + qu_{xxx} = 0, \end{aligned} \tag{33}$$

which is called the system of variant Boussinesq equations [25]. Using the wave transformations $u = u(\zeta)$, $\zeta = x + ct$, we can reduce (33) to the following system of ODEs:

$$\begin{aligned} &cu' + v' + uu' + cpu''' = 0, \\ &cv' + (uv)' + qu''' = 0. \end{aligned} \tag{34}$$

By expanding the solutions of (33), u and v are given

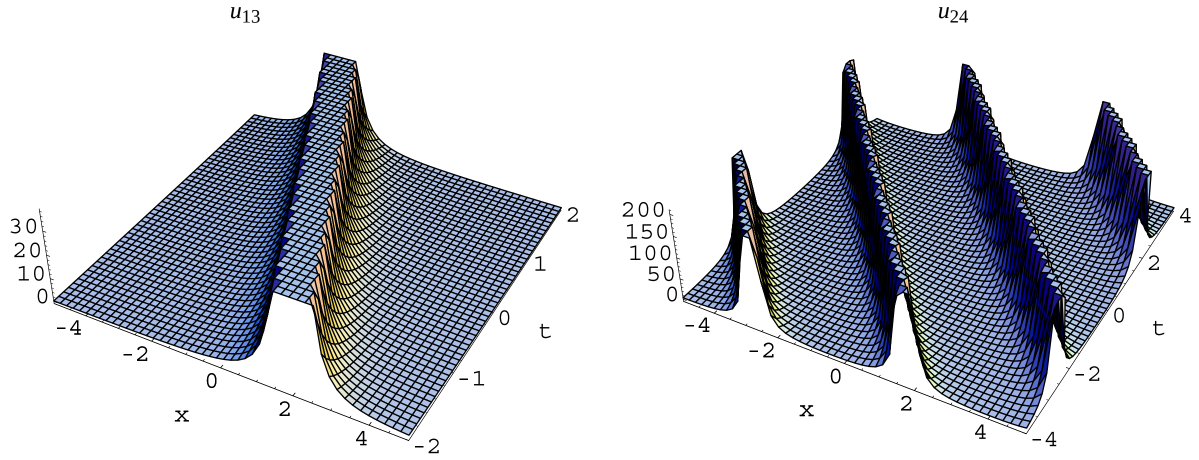

Fig. 1. u_{13} with $c = 1$ and u_{24} with $c = 1$, $c_2 = 1$.

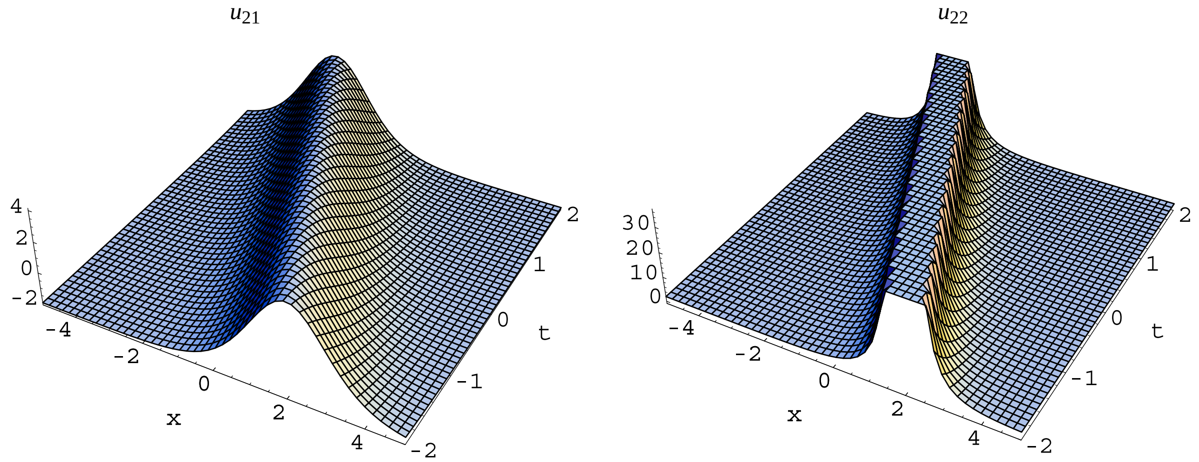

Fig. 2. u_{21} and u_{22} with $c = 1$, $c_2 = -1$.

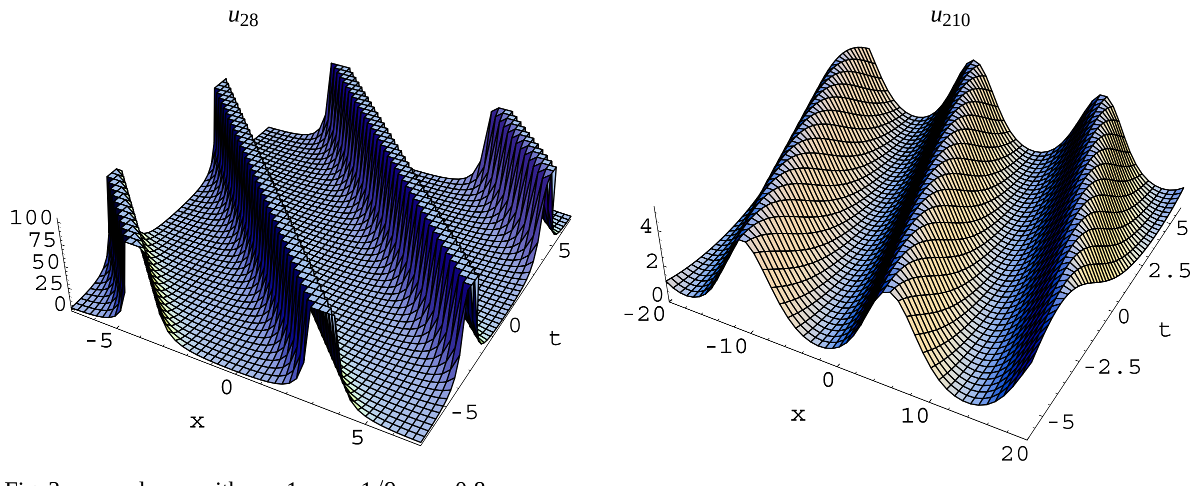

Fig. 3. u_{28} and u_{210} with $c = 1$, $c_2 = 1/9$, $m = 0.8$.

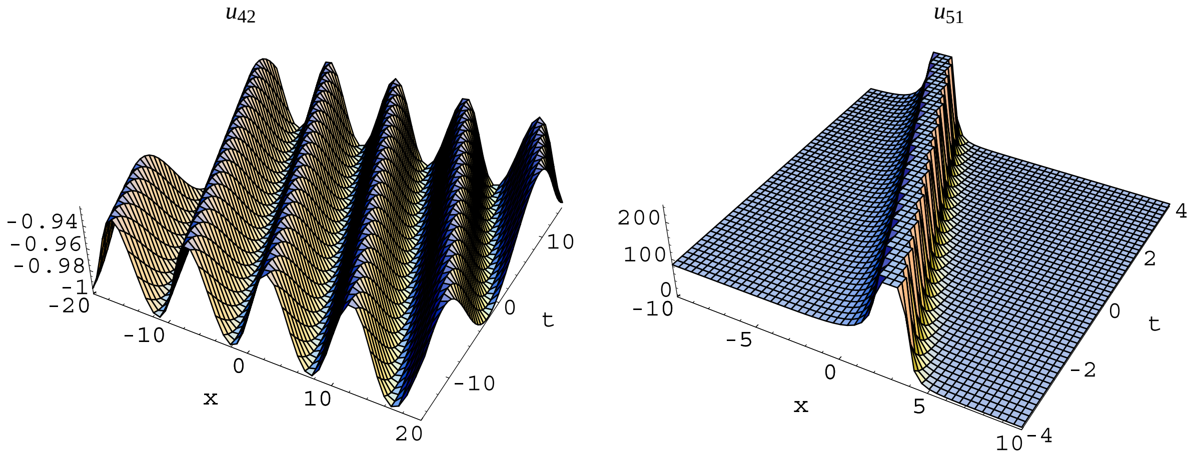

Fig. 4. u_{42} with $c = 0.5$, $c_0 = 0.1$, $c_1 = -0.2$, $c_3 = 0.1$ and u_{51} with $c = 1$, $c_0 = 2$, $c_1 = 1$, $c_2 = 1$, $\varepsilon = 1$.

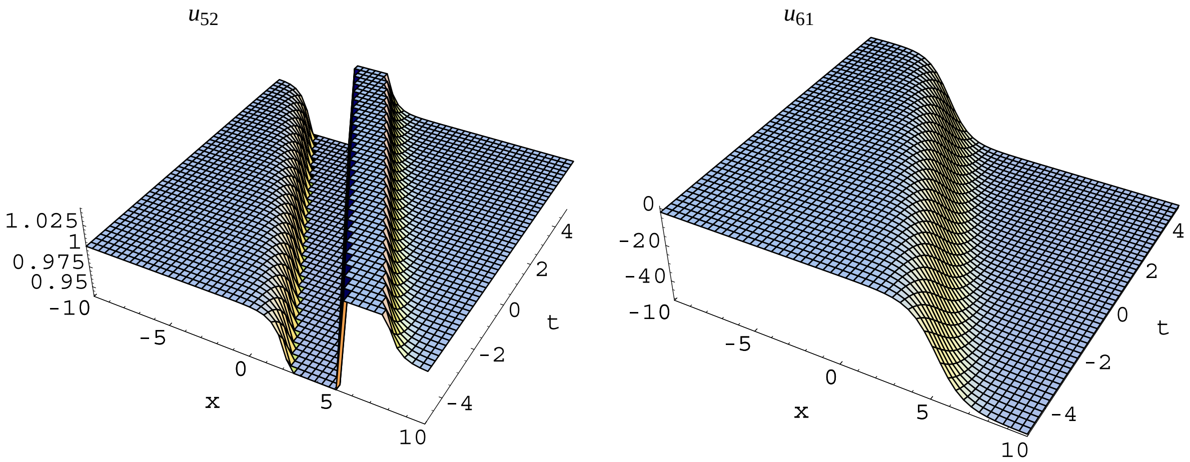

Fig. 5. u_{52} with $c = 1$, $c_2 = 1$, $\varepsilon = 1$ and u_{61} with $c = 1$, $c_2 = 1$.

as

$$u = \sum_{i=0}^{n} a_{1i}\omega^i + \sum_{i=1}^{n} b_{1i}\omega^{-i},$$
$$v = \sum_{i=0}^{m} a_{2i}\omega^i + \sum_{i=1}^{m} b_{2i}\omega^{-i}, \tag{35}$$

where ω satisfies (5). Balancing the highest-order linear term with the nonlinear terms in (34) gives

$$2n - 1 + \frac{r}{2} = n - 3 + 3 \times \frac{r}{2},$$
$$m + n - 1 + \frac{r}{2} = n + 3 \times \frac{r}{2}. \tag{36}$$

Setting $r = 4$, we get $n = 2$ and $m = 2$ from (36). Substituting $n = 2$ and $m = 2$ into (35), we have

$$u = a_{10} + a_{11}\omega + a_{12}\omega^2 + b_{11}\omega^{-1} + b_{12}\omega^{-2},$$
$$v = a_{20} + a_{21}\omega + a_{22}\omega^2 + b_{21}\omega^{-1} + b_{22}\omega^{-2}. \tag{37}$$

Substituting (37) and (6) into (34) and setting the coefficients of all powers like ω^i and $\omega^i\sqrt{\sum_{j=0}^{4} c_j\omega^j}$ to zero yields a system of algebraic equations for the determination of the unknown parameters $a_{ij}\,(i = 0,1,2;\, j = 1,2)$, $b_{ij}\,(i,j = 1,2)$, $c_i\,(i = 0,\cdots,4)$, and c, respectively. Again with the aid of *Mathematica*, the following solutions of the system for (33) are obtained:

Case 1. $c_0 = c_1 = c_3 = 0$. We have

$$a_{11} = a_{21} = b_{11} = b_{12} = b_{21} = b_{22} = 0,$$

$$a_{12} = -12cpc_4, \quad a_{22} = -6qc_4, \quad a_{10} = \frac{-2c^2p - q - 8c^2p^2c_2}{2cp}, \quad b_{20} = \frac{q^2 - 8c^2p^2qc_2}{4c^2p^2}.$$

From (7), (8), (9), and (37), we obtain a bell-shaped solitary wave solution, a triangular type solution and a rational type solution:

$$u_{11} = \frac{-2c^2p - q - 8c^2p^2c_2}{2cp} + 12cpc_2\mathrm{sech}^2(\sqrt{\mathrm{c}_2}(\mathrm{x}+\mathrm{ct})),$$

$$v_{11} = \frac{q^2 - 8c^2p^2qc_2}{4c^2p^2} + 6qc_2\mathrm{sech}^2(\sqrt{\mathrm{c}_2}(\mathrm{x}+\mathrm{ct})), \quad \mathrm{c}_2 > 0,\ \mathrm{c}_4 < 0,$$

$$u_{12} = \frac{-2c^2p - q - 8c^2p^2c_2}{2cp} + 12cpc_2\sec^2(\sqrt{-\mathrm{c}_2}(\mathrm{x}+\mathrm{ct})),$$

$$v_{12} = \frac{q^2 - 8c^2p^2qc_2}{4c^2p^2} + 6qc_2\sec^2(\sqrt{-\mathrm{c}_2}(\mathrm{x}+\mathrm{ct})), \quad \mathrm{c}_2 < 0,\ \mathrm{c}_4 > 0,$$

$$u_{13} = \frac{-2c^2p - q}{2cp} - \frac{12cp}{(x+ct)^2}, \quad v_{13} = \frac{q^2}{4c^2p^2} - \frac{6q}{(x+ct)^2}, \quad c_2 = 0,\ c_4 > 0.$$

Case 2. $c_1 = c_3 = 0$. We have

(i) $a_{11} = a_{21} = b_{11} = b_{12} = b_{21} = b_{22} = 0, \quad a_{12} = -12cpc_4, \quad a_{22} = -6qc_4,$
$a_{10} = \frac{-2c^2p - q - 8c^2p^2c_2}{2cp}, \quad b_{20} = \frac{q^2 - 8c^2p^2qc_2}{4c^2p^2};$

(ii) $a_{11} = a_{12} = a_{21} = a_{22} = b_{11} = b_{21} = 0, \quad b_{12} = -12cpc_0, \quad b_{22} = -6qc_0,$
$a_{10} = \frac{-2c^2p - q - 8c^2p^2c_2}{2cp}, \quad b_{20} = \frac{q^2 - 8c^2p^2qc_2}{4c^2p^2};$

(iii) $a_{11} = a_{21} = b_{11} = b_{21} = 0, \quad a_{12} = -12cpc_4, \quad a_{22} = -6qc_4, \quad b_{12} = -12cpc_0,$
$b_{22} = -6qc_0, \quad a_{10} = \frac{-2c^2p - q - 8c^2p^2c_2}{2cp}, \quad b_{20} = \frac{q^2 - 8c^2p^2qc_2}{4c^2p^2}.$

If c_0, c_2, c_4 satisfy $c_0 = c_2^2/4c_4$, from (10), (11), and (37) we obtain three solitary wave solutions and three triangular type solutions:

$$u_{21} = \frac{-2c^2p - q - 8c^2p^2c_2}{2cp} + 6cpc_2\tanh^2(\sqrt{-\frac{\mathrm{c}_2}{2}}(\mathrm{x}+\mathrm{ct})),$$

$$v_{21} = \frac{q^2 - 8c^2p^2qc_2}{4c^2p^2} + 3qc_2\tanh^2(\sqrt{-\frac{\mathrm{c}_2}{2}}(\mathrm{x}+\mathrm{ct})), \quad \mathrm{c}_2 < 0,\ \mathrm{c}_4 > 0,$$

$$u_{22} = \frac{-2c^2p - q - 8c^2p^2c_2}{2cp} + 6cpc_2\coth^2(\sqrt{-\frac{\mathrm{c}_2}{2}}(\mathrm{x}+\mathrm{ct})),$$

$$v_{22} = \frac{q^2 - 8c^2p^2qc_2}{4c^2p^2} + 3qc_2\coth^2(\sqrt{-\frac{\mathrm{c}_2}{2}}(\mathrm{x}+\mathrm{ct})), \quad \mathrm{c}_2 < 0,\ \mathrm{c}_4 > 0,$$

$$u_{23} = \frac{-2c^2p - q - 8c^2p^2c_2}{2cp} + 6cpc_2\tanh^2(\sqrt{-\frac{\mathrm{c}_2}{2}}(\mathrm{x}+\mathrm{ct})) + 6\mathrm{cpc}_2\coth^2(\sqrt{-\frac{\mathrm{c}_2}{2}}(\mathrm{x}+\mathrm{ct})),$$

$$v_{23}=\frac{q^2-8c^2p^2qc_2}{4c^2p^2}+3qc_2\tanh^2(\sqrt{-\frac{c_2}{2}}(x+ct))+3qc_2\coth^2(\sqrt{-\frac{c_2}{2}}(x+ct)),\quad c_2<0, c_4>0,$$

$$u_{24}=\frac{-2c^2p-q-8c^2p^2c_2}{2cp}-6cpc_2\tan^2(\sqrt{\frac{c_2}{2}}(x+ct)),$$

$$v_{24}=\frac{q^2-8c^2p^2qc_2}{4c^2p^2}-3qc_2\tan^2(\sqrt{\frac{c_2}{2}}(x+ct)),\quad c_2>0,\ c_4<0,$$

$$u_{25}=\frac{-2c^2p-q-8c^2p^2c_2}{2cp}-6cpc_2\cot^2(\sqrt{\frac{c_2}{2}}(x+ct)),$$

$$v_{25}=\frac{q^2-8c^2p^2qc_2}{4c^2p^2}-3qc_2\cot^2(\sqrt{\frac{c_2}{2}}(x+ct)),\quad c_2>0,\ c_4<0,$$

$$u_{26}=\frac{-2c^2p-q-8c^2p^2c_2}{2cp}-6cpc_2\tan^2(\sqrt{\frac{c_2}{2}}(x+ct))-6cpc_2\cot^2(\sqrt{\frac{c_2}{2}}(x+ct)),$$

$$v_{26}=\frac{q^2-8c^2p^2qc_2}{4c^2p^2}-3qc_2(\tan^2(\sqrt{\frac{c_2}{2}}(x+ct))-3qc_2\cot^2(\sqrt{\frac{c_2}{2}}(x+ct))),\quad c_2>0,\ c_4<0.$$

If c_0, c_2, c_4 satisfy $c_2<0$, $c_4>0$, $c_0=\frac{c_2^2m^2}{c_4(m^2+1)^2}$, from (14) and (37) we obtain three Jacobi elliptic doubly periodic type solutions:

$$u_{27}=\frac{-2c^2p-q-8c^2p^2c_2}{2cp}+\frac{12cpc_2m^2}{m^2+1}\mathrm{sn}^2(\sqrt{-\frac{c_2}{m^2+1}}(x+ct)),$$

$$v_{27}=\frac{q^2-8c^2p^2qc_2}{4c^2p^2}+\frac{6qc_2m^2}{m^2+1}\mathrm{sn}^2(\sqrt{-\frac{c_2}{m^2+1}}(x+ct)),$$

$$u_{28}=\frac{-2c^2p-q-8c^2p^2c_2}{2cp}+\frac{12cpc_2}{m^2+1}\mathrm{ns}^2(\sqrt{-\frac{c_2}{m^2+1}}(x+ct)),$$

$$v_{28}=\frac{q^2-8c^2p^2qc_2}{4c^2p^2}+\frac{6qc_2}{m^2+1}\mathrm{ns}^2(\sqrt{-\frac{c_2}{m^2+1}}(x+ct)),$$

$$u_{29}=\frac{-2c^2p-q-8c^2p^2c_2}{2cp}+\frac{12cpc_2m^2}{m^2+1}\mathrm{sn}^2(\sqrt{-\frac{c_2}{m^2+1}}(x+ct))+\frac{12cpc_2m^2}{m^2+1}\mathrm{ns}^2(\sqrt{-\frac{c_2}{m^2+1}}(x+ct)),$$

$$v_{29}=\frac{q^2-8c^2p^2qc_2}{4c^2p^2}+\frac{6qc_2m^2}{m^2+1}\mathrm{sn}^2(\sqrt{-\frac{c_2}{m^2+1}}(x+ct))+\frac{6qc_2m^2}{m^2+1}\mathrm{ns}^2(\sqrt{-\frac{c_2}{m^2+1}}(x+ct)).$$

Case 3. $c_0=c_1=c_4=0$. We have

$$a_{12}=a_{22}=b_{11}=b_{12}=b_{21}=b_{22}=0,\quad a_{11}=-3cpc_3,\ a_{22}=-\frac{3}{2}qc_3,$$

$$a_{10}=\frac{-2c^2p-q-2c^2p^2c_2}{2cp},\quad b_{20}=\frac{q^2-2c^2p^2qc_2}{4c^2p^2}.$$

From (15) – (17) and (37) we obtain a bell-shaped solitary wave solution, a triangular type solution and a rational type solution:

$$u_{31}=\frac{-2c^2p-q-2c^2p^2c_2}{2cp}+3cpc_2\mathrm{sech}^2(\sqrt{\frac{c_2}{2}}(x+ct)),$$

$$v_{31} = \frac{q^2 - 2c^2p^2qc_2}{4c^2p^2} + \frac{3}{2}qc_2\mathrm{sech}^2(\sqrt{\frac{\mathrm{c}_2}{2}}(\mathrm{x}+\mathrm{ct})), \quad \mathrm{c}_2 > 0,$$

$$u_{32} = \frac{-2c^2p - q - 2c^2p^2c_2}{2cp} + 3cpc_2\sec^2(\sqrt{\frac{\mathrm{c}_2}{2}}(\mathrm{x}+\mathrm{ct})),$$

$$v_{32} = \frac{q^2 - 2c^2p^2qc_2}{4c^2p^2} + \frac{3}{2}qc_2\sec^2(\sqrt{\frac{\mathrm{c}_2}{2}}(\mathrm{x}+\mathrm{ct})), \quad \mathrm{c}_2 < 0,$$

$$u_{33} = \frac{-2c^2p - q}{2cp} - \frac{12cp}{(x+ct)^2}, \quad v_{33} = \frac{q^2}{4c^2p^2} - \frac{6q}{(x+ct)^2}, \quad c_2 = 0.$$

Case 4. $c_2 = c_4 = 0$. We have

(i) $a_{12} = a_{22} = b_{11} = b_{12} = b_{21} = b_{22} = 0, \quad a_{11} = -3cpc_3, \quad a_{22} = -\frac{3}{2}qc_3,$
$a_{10} = \frac{-2c^2p - q}{2cp}, \quad b_{20} = \frac{q^2}{4c^2p^2};$

(ii) $a_{12} = a_{12} = a_{21} = a_{22}, \quad b_{11} = -6cpc_1, \quad b_{12} = -12cpc_0, \quad b_{21} = -3qc_1, \quad b_{22} = -6qc_0,$
$a_{10} = \frac{-4c^2pc_0 - 2qc_0 - 3c^2p^2c_1^2}{4cpc_0}, \quad b_{20} = \frac{2q^2c_0 + 3c^2p^2qc_1^2}{8c^2p^2c_0}, \quad 8c_0^2c_3 + c_1^3 = 0.$

From (18) and (37) we obtain two Weierstrass elliptic doubly periodic type solutions:

$$u_{41} = \frac{-2c^2p - q}{2cp} - 3cpc_3\wp(\frac{\sqrt{c_3}}{2}, g_2, g_3), \quad v_{41} = \frac{q^2}{4c^2p^2} - \frac{3}{2}qc_3\wp(\frac{\sqrt{c_3}}{2}, g_2, g_3),$$

$$u_{42} = \frac{-4c^2pc_0 - 2qc_0 - 3c^2p^2c_1^2}{4cpc_0} - 6cpc_1(\wp(\frac{\sqrt{c_3}}{2}, g_2, g_3))^{-1} - 12cpc_0(\wp(\frac{\sqrt{c_3}}{2}, g_2, g_3))^{-2},$$

$$v_{42} = \frac{2q^2c_0 + 3c^2p^2qc_1^2}{8c^2p^2c_0} - 3qc_1(\wp(\frac{\sqrt{c_3}}{2}, g_2, g_3))^{-1} - 6qc_0(\wp(\frac{\sqrt{c_3}}{2}, g_2, g_3))^{-2},$$

where $g_2 = -4c_1/c_3$ and $g_3 = -4c_0/c_3$ are called invariants of the Weierstrass elliptic function.

Case 5. $c_3 = c_4 = 0$. We have

(i) $a_{12} = a_{12} = a_{21} = a_{22} = 0, \quad b_{11} = -6cpc_1, \quad b_{12} = -12cpc_0, \quad b_{21} = -3qc_1, \quad b_{22} = -6qc_0,$
$a_{10} = \frac{-2c^2p - q - 2c^2p^2c_2}{2cp}, \quad b_{20} = \frac{q^2 - 2c^2p^2qc_2}{4c^2p^2}, \quad c_0 = \frac{c_1^2}{4c_2};$

(ii) $a_{12} = a_{12} = a_{21} = a_{22} = b_{12} = b_{22} = 0, \quad b_{11} = -3cpc_1, \quad b_{21} = -\frac{3}{2}qc_1,$
$a_{10} = \frac{-2c^2p - q - 2c^2p^2c_2}{2cp}, \quad b_{20} = \frac{q^2 - 2c^2p^2qc_2}{4c^2p^2}, \quad c_0 = 0;$

(iii) $a_{12} = a_{12} = a_{21} = a_{22} = b_{11} = b_{21} = 0, \quad b_{12} = -12cpc_0, \quad b_{22} = -6qc_0,$
$a_{10} = \frac{-2c^2p - q - 8c^2p^2c_2}{2cp}, \quad b_{20} = \frac{q^2 - 8c^2p^2qc_2}{4c^2p^2}, \quad c_1 = 0.$

From (21) – (25) and (37) we obtain an exponential type solution, two solitary wave solutions and two triangular type solutions:

$$u_{51} = \frac{-2c^2p - q - 2c^2p^2c_2}{2cp} - 6cpc_1(-\frac{c_1}{2c_2} + \exp(\varepsilon\sqrt{\mathrm{c}_2}(\mathrm{x}+\mathrm{ct})))^{-1} - 12\mathrm{cpc}_0(-\frac{\mathrm{c}_1}{2\mathrm{c}_2} + \exp(\varepsilon\sqrt{\mathrm{c}_2}(\mathrm{x}+\mathrm{ct})))^{-2},$$

$$v_{51} = \frac{q^2 - 2c^2p^2qc_2}{4c^2p^2} - 3qc_1(-\frac{c_1}{2c_2} + \exp(\varepsilon\sqrt{c_2}(x+ct)))^{-1} - 6qc_0(-\frac{c_1}{2c_2} + \exp(\varepsilon\sqrt{c_2}(x+ct)))^{-2}, c_2 > 0,$$

$$u_{52} = \frac{-2c^2p - q - 2c^2p^2c_2}{2cp} + \frac{6cpc_2}{1 - \varepsilon\sinh(2\sqrt{c_2}(x+ct))},$$

$$v_{52} = \frac{q^2 - 2c^2p^2qc_2}{4c^2p^2} + \frac{3qc_2}{1 - \varepsilon\sinh(2\sqrt{c_2}(x+ct))}, \quad c_2 > 0,$$

$$u_{53} = \frac{-2c^2p - q - 8c^2p^2c_2}{2cp} - 12cpc_2\operatorname{csch}^2(\sqrt{c_2}(x+ct)),$$

$$v_{53} = \frac{q^2 - 8c^2p^2qc_2}{4c^2p^2} - 6qc_2\operatorname{csch}^2(\sqrt{c_2}(x+ct)), \quad c_0 > 0,\ c_2 > 0,$$

$$u_{54} = \frac{-2c^2p - q - 2c^2p^2c_2}{2cp} + \frac{6cpc_2}{1 - \varepsilon\sin(\sqrt{-c_2}(x+ct))},$$

$$v_{54} = \frac{q^2 - 2c^2p^2qc_2}{4c^2p^2} + \frac{3qc_2}{1 - \varepsilon\sin(\sqrt{-c_2}(x+ct))}, \quad c_2 < 0,$$

$$u_{55} = \frac{-2c^2p - q - 8c^2p^2c_2}{2cp} + 12cpc_2\csc^2(\sqrt{-c_2}(x+ct)),$$

$$v_{55} = \frac{q^2 - 8c^2p^2qc_2}{4c^2p^2} + 6qc_2\csc^2(\sqrt{-c_2}(x+ct)), \quad c_0 > 0,\ c_2 < 0.$$

Case 6. $c_0 = c_1 = 0$. We have

$$b_{12} = b_{12} = b_{21} = b_{22} = 0, \quad a_{11} = -6cpc_3, \quad a_{12} = -12cpc_4, \quad a_{21} = -3qc_3,$$
$$b_{22} = -6qc_4, \quad a_{10} = \frac{-2c^2p - q - 2c^2p^2c_2}{2cp}, \quad b_{20} = \frac{q^2 - 2c^2p^2qc_2}{4c^2p^2}, \quad c_3 = 2\varepsilon\sqrt{c_2c_4}.$$

From (28) and (37) we obtain a kink-shaped solitary wave solution:

$$u_{61} = \frac{-2c^2p - q + 4c^2p^2c_2}{2cp} - 3cpc_2(2 + \tanh(\frac{1}{2}\sqrt{c_2}(x+ct)))^2,$$

$$v_{61} = \frac{q^2 + 4c^2p^2qc_2}{4c^2p^2} - \frac{3}{2}qc_2(2 + \tanh(\frac{1}{2}\sqrt{c_2}(x+ct)))^2, \quad c_2 > 0.$$

Remark. From the transformation $c_2 \to -c_2/2$, (u_{11}, v_{11}), (u_{53}, v_{53}), (u_{12}, v_{12}) and (u_{55}, v_{55}) can be transformed to (u_{21}, v_{21}), (u_{22}, v_{22}), (u_{24}, v_{24}) and (u_{25}, v_{25}), respectively. From the transformation $c_2 \to -2c_2$, (u_{31}, v_{31}) and (u_{32}, v_{32}) can be transformed to (u_{21}, v_{21}) and (u_{24}, v_{24}), respectively. It can be seen that the solutions (u_{13}, v_{13}) and (u_{33}, v_{33}) are the same rational type solutions. Since the Jacobi elliptic doubly periodic type solutions obtained from (12) and (13) are similar to the solutions obtained from (14), we omit these solutions in the above example. As $m \to 1$, the Jacobi elliptic doubly periodic type solutions (u_{27}, v_{27}) - (u_{29}, v_{29}) can be reduced to (u_{21}, v_{21}) - (u_{23}, v_{23}). The solutions (u_{21}, v_{21}) - (u_{26}, v_{26}) and (u_{13}, v_{13}) have been found by Elwakil et al. [19]. To the knowledge of the authors, the solutions (u_{27}, v_{27}) - (u_{29}, v_{29}), (u_{41}, v_{41}), (u_{42}, v_{42}), (u_{51}, v_{51}), (u_{52}, v_{52}), (u_{54}, v_{54}) and (u_{61}, v_{61}) are new exact solutions for (33). Because the types of solutions obtained for (33) are similar to the types of solutions for (29), the properties of these solutions are similar as well. Thus, we do not give the plots of these solutions for (33).

4. Conclusion

In this paper we further improved the applicability of the modified extended tanh-function method. The validity of our method was verified by solving the improved Boussinesq equation and the system of variant Boussinesq equations. Numerical results indicated that more new exact solutions can be found. The proposed method can also be applied to solve many nonlinear partial differential equations which will be investigated in our future work.

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