# Technical Notes and Correspondence

## An Improved Stability Criterion for the Damped Mathieu Equation

*Abstract*—An improved stability criterion is derived for the damped Mathieu equation using periodic Lyapunov functions.

Considering the damped Mathieu equation

$$x + 0.5\delta\dot{x} + (\omega^2 + \epsilon\cos 2t)x = 0 \quad (1)$$

(with 6 > 0) Michael [1] has shown that for large *w* and small 6, stability is assured if $|\epsilon| < 6$. In this correspondence, an improved stability criterion is obtained using periodic Lyapunov functions.

Putting $x = x_1$ and $\dot{x} = x_2$, (1) can be written as

$$\dot{x}_1 = x_2 \quad (2)$$

$$\dot{x}_2 = -0.5622 - (\omega^2 + \epsilon\cos 2t)\, x_1.$$

Consider the Lyapunov function

$$V_1 = (\omega^2 + 0.5k\delta + k\epsilon\sin 2t)x_1^2 + 2kx_1x_2 + x_2^2 \quad (3)$$

where $k$ is to be chosen. Differentiating (3) with respect to time and using (2),

$$\dot{V}_1 = -2\omega^2 k x_1^2 - 2\epsilon(\cos 2t - k\sin 2t)x_1x_2 - (6 - 2k)x_2^2.$$

For $\dot{V}_1$ to be negative definite, we require

$$0 < 2k < \delta \quad (4)$$

and

$$\epsilon^2 < \frac{2\omega^2 k(\delta - 2k)}{(1 + k^2)} \quad (5)$$

Now choose the value of k as

$$k = \frac{1/62 + 4 - 2}{6} \quad (6)$$

which maximizes the right-hand side of (5) and also satisfies (4). It can be easily proved that, for the above value of $k$, the function $V_1$ is positive definite. Substitution of the value of $k$ given in (6) into (5) yields the stability criterion as

$$|\epsilon| < [\sqrt{\delta^2 + 4} - 2]^{1/2}\omega. \quad (7)$$

Now consider another Lyapunov function given by

$$V_2 = (\omega^2 + \epsilon\cos 2t)x_1^2 + (0.5\delta x_1 + x_2)^2. \quad (8)$$

Differentiating (8) and using (2),

$$\dot{V}_2 = -[\delta\omega^2 + \delta\epsilon\cos 2t + 2\epsilon\sin 2t]x_1^2.$$

Here we see that $\dot{V}_2$ can only be negative semidefinite since $\dot{V}_2 \equiv 0$

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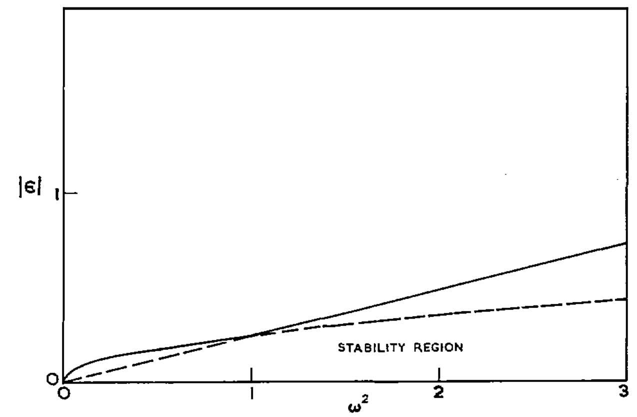


Fig. 1. Stability region for $\delta = 0.5$.

when $x_1 \equiv 0$. It is still possible to prove asymptotic stability in the large (MIL) of the system (2) by using the theorem of Krasovskii and Barbasin [2] for periodic Lyapunov functions if we can prove that $\dot{V}_2 \equiv 0$ only when $x_1 \equiv x_2 \equiv 0$. For the system (2), it can be easily seen that when $x_1 \equiv 0$, $x_2$ is also identically zero. Hence it is sufficient if $\dot{V}_2$ is negative semidefinite which requires that

$$|\epsilon| < \frac{\delta\omega^2}{\sqrt{\delta^2 + 4}}. \tag{9}$$

Combining (7) and (9), the stability criterion can be written as

$$|\epsilon| < \max\left[\{\sqrt{\delta^2 + 4} - 2\}^{1/2}\omega, \frac{\delta\omega^2}{\sqrt{\delta^2 + 4}}\right]. \tag{10}$$

The stability region using (10) for $\delta = 0.5$ is shown in Fig. 1. It may be noted that the stability region obtained by Michael [1] is contained in the stability region obtained here.


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## References


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[2] T. A. Burton, "Some Liapunov theorems," *J. SIAM Contr.*, vol. 4, 1966, pp. 460–465.