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AN IMPROVED UPPER BOUND FOR GLOBAL DIMENSION OF SEMIGROUP ALGEBRAS

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ABSTRACT. An upper bound for the global dimension of the semigroup algebra of a finite regular monoid in terms of an ideal series for the monoid is determined by the partially ordered set of $\mathcal{J}$-classes of the monoid. In particular, if the monoid is combinatorial, the global dimension of the algebra is bounded by the sum of the global dimension of the coefficient ring and twice the length of the longest chain of $\mathcal{J}$-classes in the monoid.

In [3] we considered a finite regular monoid S and a composition series of ideals $S=I_1\supset I_2\supset\cdots\supset I_n$. For a commutative ring with identity k we then defined $\sigma(I_j)=0$, 1, or 2, respectively, depending on whether the algebra kI_j/kI_{j+1} has a two-sided identity, has no two-sided identity but has a right or left identity, or has neither right nor left identity. Then we set $\mu(S)=\sigma(I_1)+\sigma(I_2)+\cdots+\sigma(I_n)$ and found an upper bound for l.gl.dim kS which involved $\mu(S)$. In particular, if S is combinatorial (i.e., each subgroup is trivial), or if kG is semisimple for each subgroup G of S, then l.gl.dim $kS\leqq\mu(S)+$l.gl.dim k. The purpose of this paper is to show that an estimate $\tau(S)$ can be computed as above using an ideal series for S which is in general shorter than a composition series. For semigroup terminology we follow [1] or [2].

We form an ideal series $S=I'_1\supset I'_2\supset\cdots\supset I'_m$ as follows: Let I'_m be the unique minimal ideal of S, and for $r=m-1,\cdots,1$ let I'_r be such that I'_r/I'_{r+1} is the union of all 0-minimal ideals in S/I'_{r+1}. In this ideal series m is the length of the longest chain in the partially ordered set of $\mathcal{J}$-classes of S. In particular, $m=n$ if and only if S is a chain of $\mathcal{J}$-classes; otherwise $m<n$.

Define $\sigma(I'_j)$ in terms of identities of kI'_j/kI'_{j+1} as was done for the ideals in the composition series above. Let $\tau(S)=\sigma(I'_1)+\cdots+\sigma(I'_m)$. Then

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we have

THEOREM. *Let S be a finite regular monoid and k be a commutative ring with identity. If S is combinatorial or if kG is semisimple for each subgroup G of S, then* $\text{l.gl.dim } kS \leqq \tau(S) + \text{l.gl.dim } k$.

The proof is by induction and the following lemma.

LEMMA. *Let S be a regular monoid with zero, and let I be the union of the 0-minimal ideals of S. Then $I = J_1^0 \cup \cdots \cup J_p^0$ where each J_i is a regular $\mathcal{J}$-class of S with maximal subgroup G_i. Let $T = S/I$, k be a commutative ring with identity, and k_0S be the contracted semigroup algebra of S. If for any left k_0S-module M we let $d_{k_0S}(M)$ be the projective dimension of M, then*

$$d_{k_0S}(M) \leqq \max\{\max\{\text{l.gl.dim } kG_i : i = 1, \cdots, p\}, \text{l.gl.dim } k_0T + \sigma(I)\}.$$

PROOF. Observe $k_0T = k_0S/k_0I$. As in [3] we see that $d_{k_0S}(k_0T) = 0$, 1 resp., depending on whether k_0I has or lacks a right identity.

Letting $M^* = M/(k_0I)M$, we see that M^* is a k_0T-module and that $d_{k_0S}(M^*) \leqq d_{k_0T}(M^*) + d_{k_0S}(k_0T) \leqq \text{l.gl.dim } k_0T + d_{k_0S}(k_0T)$ by a standard result.

We observe that $k_0I = k_0J_1 \oplus k_0J_2 \oplus \cdots \oplus k_0J_p$, so that $(k_0I)M = (k_0J_1)M + \cdots + (k_0J_p)M$. Now if e_i is the identity of G_i, it is known that $J_i^0 = Se_iS$. We may thus define M^{**} by the exact sequence

$$0 \to M^{**} \to \bigoplus_{i=1}^{p} (k_0Se_i \otimes_{kG_i} e_iM) \to (k_0I)M \to 0,$$

where the right hand map is given by $(x_1 \otimes m_1, \cdots, x_p \otimes m_p) \mapsto x_1m_1 + \cdots + x_pm_p$. Now if $(u_1, \cdots, u_p) \in M^{**}$ and $x \in J_1$, then $x(u_1, \cdots, u_p) = (xu_1, 0, \cdots, 0)$ since J_1 annihilates J_i, $i = 2, \cdots, p$. Now the argument of [3] shows that $x_1u_1 = 0$ also; hence we show that M^{**} is a k_0T-module and that if k_0I has a left identity, $M^{**} = 0$.

Thus we have $d_{k_0S}(M^{**}) \leqq \text{l.gl.dim } k_0T + d_{k_0S}(k_0T)$. By [3],

$$d_{k_0S}(k_0Se_i \otimes_{kG_i} e_iM) = d_{kG_i}(e_iM).$$

By standard results we have

$$d_{k_0S}(M) \leqq \max\{d_{k_0S}(k_0IM), d_{k_0S}(M^*)\}$$

and

$$d_{k_0S}(k_0IM) \leqq \max\{d_{k_0S}(M^{**}), \max\{d_{k_0S}(k_0Se_i \otimes_{kG_i} e_iM) : i = 1, \cdots, p\}\}.$$

These inequalities establish the lemma.

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