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AN IMPROVEMENT OF FIXED POINT ALGORITHMS BY USING A GOOD TRIANGULATION

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We consider measures for triangulations of R^n. A new measure is introduced based on the ratio of the length of the sides and the content of the subsimplices of the triangulation. In a subclass of triangulations, which is appropriate for computing fixed points using simplicial subdivisions, the optimal one according to this measure is calculated and some of its properties are given. It is proved that for the average directional density this triangulation is optimal (within the subclass) as n goes to infinity. Furthermore, we compare the measures of the optimal triangulation with those of other triangulations. We also propose a new triangulation of the affine hull of the unit simplex. Finally, we report some computational experience that confirms the theoretical results.

Key words: Triangulation, Average Directional Density, Fixed Point, Equilateral Triangles.

1. Introduction

For computing fixed points of a convex upper semi-continuous point-to-set mapping, there are a rather great number of algorithms available using triangulations and complementary pivoting techniques (see Scarf [13, 14], Eaves [1], Eaves and Saigal [2], Merrill [10], Kuhn and MacKinnon [5], Van der Laan and Talman [6, 7, 8]). The efficiency of these algorithms depends clearly on the particular triangulation used (see Saigal [11] and Todd [15, 16]). Until now, three theoretical measures of the efficiency of a certain triangulation have been developed. The first one counts the number of simplices into which the triangulation divides the unit cube. However, many triangulations have the same number, whereas computational results show that they in fact differ in efficiency (see Todd [16] and Saigal [11]). For $n = 3$, 4 and 5 Mara [9] discovered a triangulation with fewer simplices then the well-known triangulations. In the case $n = 3$, it yields 5 simplices, whereas the K, H and J_1 triangulations yield 6 simplices. The generalization to arbitrary n is not known.

A second measure has been given by Saigal et al. [12]. They introduced the concept of the diameter of a triangulation. The calculation by Saigal of the diameter of K and H was confirmed by his computational experience. Also this measure can only be used for triangulations that divide the unit cube.

The last measure is the average directional density (a.d.d.) introduced by Todd

[15]. It is based on counting the number of simplices met by a straight line with direction d, by averaging this number over all vectors d of unit length. Unfortunately, it is not known how to calculate the best triangulation according to this criterion (Todd [16]).

In this paper we introduce a new measure for comparing triangulations. In an appropriate subclass M of triangulations of R^n we will compute the optimal one according to this criterion. Moreover, it is shown that the a.d.d. of this triangulation is lower than the a.d.d. of the K and H triangulations. We will prove that in the limit our triangulation is optimal within M according to the a.d.d. criterion. The new measure provides information about the local geometry of a triangulation, and is invariant with respect to homogeneous scaling in R^n. The a.d.d. measure, though, is global in character and depends on scale.

Section 2 contains some notation and definitions. In Section 3 the subclass M of triangulations is specified and the new measure is presented. In Section 4 the optimal triangulation within M is calculated and some of its properties are given. Section 5 compares the latter with the H and K triangulations by computing for each both their a.d.d. and their measure introduced in this paper. Section 6 presents a new triangulation of the affine hull of the unit simplex having, when it is orthogonally projected on R^n, the same measure as the optimal one within M. Section 7 shows that the theoretical superiority of the new triangulation is confirmed by computational experience.

2. Notation and definitions

Let $e(1), \dots, e(n)$ be the unit vectors in the n-dimensional real space R^n. Define I_n as the set of integers $\{1, \dots, n\}$ and let $\gamma = (\gamma_1, \dots, \gamma_n)$ be a permutation of the elements of I_n. A k-dimensional simplex or k-simplex σ_k in R^n $(k \leq n)$ is the convex hull of $k+1$ independent points $y^0, \dots, y^k$, the vertices of $\sigma_k = \sigma(y^0, \dots, y^k)$. A j-dimensional simplex τ_j is called a j-face of the k-simplex σ_k $(j \leq k)$ if the vertices of τ_j are a subset of the vertices of σ_k. Let $\tau_1(\sigma_k)$ be the set of 1-faces of σ_k. The diameter of a simplex σ_k is defined by $\max_{\tau \in \tau_1(\sigma_k)} \|\tau\|$ where $\|\tau\|$ denotes the length of τ using the Euclidean norm.

A triangulation G of a k-dimensional subset F of R^n is a collection of k-simplices that satisfies the following conditions:

(1) F is the union of the simplices in G.

(2) The intersection of two k-simplices is empty or is a common face.

The mesh of a triangulation G is defined by mesh $G = \sup(\text{diam}\, \sigma \mid \sigma \in G)$.

The K triangulation of R^n is the collection of simplices $\sigma(y^0, \dots, y^n)$, such that all components of y^0 are integer and $y^i = y^{i-1} + e(\gamma_i)$ for some permutation γ of I_n. Note that the K triangulation has mesh $\sqrt{n}$. Let A be a non-singular $n \times n$ matrix. The triangulation AK of R^n is the collection of simplices $\sigma(y^0, \dots, y^n)$ such that all components of $A^{-1}y^0$ are integer and $y^i = y^{i-1} + a(\gamma_i)$ with $a(j)$ the

jth column of A. When A has 1's on the diagonal, -1's on the upper diagonal, and zeroes elsewhere, we have the H triangulation.

In his papers [15, 16] Todd proved that the a.d.d. of the K triangulation denoted by $N(K)$ is equal to $\{n + n(n-1)/\sqrt{2}\}g_n$ where

$$g_n = 2\Gamma(\tfrac{1}{2}n)/(n-1)\sqrt{\pi}\Gamma(\tfrac{1}{2}(n-1)).$$

The a.d.d. of a triangulation AK is (cf. [16])

$$N(AK) = \Big(\sum_i \|B^i\| + \sum_{i<j} \|B^i - B^j\|\Big)g_n,$$

where B^i is the ith row of A^{-1}.

Since the a.d.d. of a triangulation depends on its mesh, we will compare throughout this paper only triangulations with mesh $\sqrt{n}$.

A triangulation G is called congruent if all its simplices are mutually congruent. Note that the K triangulation is congruent. The simplex of a congruent triangulation AK with y^0 equal to the zero vector and with $y^i = y^{i-1} + a(i)$, $i = 1, \ldots, n$ will be called the standard simplex of AK and is denoted by s_n or $s(y^0, \ldots, y^n)$.

3. The subclass M of triangulations and the SC measure

Throughout the paper we will restrict ourself to the subclass M of (α, β) triangulations of R^n. A triangulation AK is called an (α, β) triangulation if the matrix A is of the following form

$$A = \begin{bmatrix} \alpha & \beta & \cdots & & \beta \\ \beta & \alpha & \ddots & & \vdots \\ \vdots & & & & \beta \\ \beta & & \cdots & \beta & \alpha \end{bmatrix}$$

with the restriction that β is non-positive and that the sum of all columns is strictly positive i.e. that $\alpha + (n-1)\beta > 0$. An element of this subclass will be denoted by $A(\alpha, \beta)$. It is easy to see that an (α, β) triangulation is congruent. It seems to be adequate to use an element of this subclass in algorithms for computing fixed points. Adding the ith column of $A(\alpha, \beta)$ to a point y of R^n means that only the ith component of y is increased, whereas all other components are affected by the same amount. Thus going from y^0 to y^n all components have equally been increased by $\alpha + (n-1)\beta$. Note that the K triangulation is $A(1, 0)$ and thus is an element of M.

To compare the a.d.d. of the triangulations in this subclass we have to normalize the columns of the matrix A. To achieve triangulations with mesh $\sqrt{n}$ it is only required to compute the maximum over the lengths of the one-faces of the standard simplex. Unfortunately it is not easy to find an optimal triangulation according the a.d.d. criterion, not even in this subclass, since $\max_{\tau \in \tau_1(s_n)} \|\tau\|$ is a nondifferentiable function of α and β. However, we can give a lower bound of the a.d.d. for the subclass of (α, β)-triangulations.

Lemma 3.1. *Let AK be an (α, β)-triangulation with mesh $AK \leq \sqrt{n}$. Then*

$$N(AK) \geq \{\tfrac{1}{2}(n-1) + (n-1)^2/\sqrt{8}\}g_n.$$

Proof. Let y^i, $i = 0, \ldots, n$ be the vertices of s_n.

Suppose $\alpha > 2$. If $0 \geq \beta > -1/(n-1)$, then $\alpha + (n-1)\beta > 1$, which implies that

$$\|y^n\| = \{\alpha + (n-1)\beta\}\sqrt{n} > \sqrt{n},$$

contradicting the fact that $\|y^k\| \leq \sqrt{n}$ for all k. Suppose now $\beta \leq -1/(n-1)$. Then there exists an integer k, $1 \leq k \leq n-1$, such that $k\beta \leq -1$ and $(k-1)\beta > -1$. Hence $\alpha + (k-1)\beta > 1$ and $|k\beta| \geq 1$. Thus

$$\|y^k\| = [k\{\alpha + (k-1)\beta\}^2 + (n-k)(k\beta)^2]^{1/2} > \sqrt{n},$$

again contradicting the fact that the mesh is less than or equal to $\sqrt{n}$.

This together implies $\alpha \leq 2$. It is easily seen that the diagonal elements of $B(\alpha, \beta)$, the inverse of $A(\alpha, \beta)$, are

$$\{\alpha + (n-2)\beta\}/(\alpha - \beta)\{\alpha + (n-1)\beta\}$$

and that its off-diagonal elements are

$$-\beta/(\alpha - \beta)\{\alpha + (n-1)\beta\}.$$

Note that $\alpha - \beta > 0$. Hence, $\|B^i\| \geq 1/(\alpha - \beta)$ for all i and $\|B^i - B^j\| = \sqrt{2}/(\alpha - \beta)$ for all $i < j$. So,

$$N(AK) \geq g_n\{n + n(n-1)/\sqrt{2}\}/(\alpha - \beta)$$

for all admissible α, β. The restriction $\alpha + (n-1)\beta > 0$ implies $\alpha - \beta < n\alpha/(n-1) \leq 2n/(n-1)$. The lemma follows now immediately.

Observe that for n goes to infinity the lower bound of the a.d.d. converges to $g_n n^2/\sqrt{8}$.

Next we introduce the *SC* measure for the efficiency of a triangulation. Within the class of (α, β) triangulations this measure is a differentiable function of (α, β), and it is rather easy to find the optimal triangulation by analytical computations.

Clearly for computing fixed points an efficient triangulation of R^2 is that which divides the plane into equilateral triangles. This triangulation is congruent and, moreover, all one-faces have the same length. However, in higher dimensions there does not exist a congruent triangulation with simplices having one-faces with equal length.

The triangulation of R^2 with equilaterial triangles has the property that it maximizes the area of the triangle, given the total length of its sides. With some modifications the SC measure is based on this property and is given by

$$SC(AK) = \max_{\sigma_n \in AK} \left\{ \sum_{\tau \in \tau_1(\sigma_n)} \|\tau\|^2 \right\}^{1/2} / \{|\det A|/n!\}^{1/n}.$$

Note that $|\det A|/n!$ is the content of any simplex of AK, and that $\sum_{\tau \in \tau_1(\sigma_n)} \|\tau\|^2$ is the total squared length of the one-faces of the simplex σ_n. By raising the numerator to the power $\frac{1}{2}$ and the denominator to the power $1/n$ the measure becomes homogeneous of degree zero in (α, β). It is obvious that for $n = 2$ the equilateral triangulation minimizes the measure SC.

Since for a congruent triangulation the maximum is achieved by every n-simplex σ_n, we can restrict ourself to the standard simplex of a triangulation. So, for the class of congruent triangulations

$$SC(AK) = \left\{ \sum_{\tau \in \tau_1(s_n)} \|\tau\|^2 \right\}^{1/2} / \{|\det A|/n!\}^{1/n}.$$

Because of the homogeneity of degree zero in (α, β), we will set β equal to -1 in the following lemma.

Lemma 3.2. *For an $(\alpha, -1)$ triangulation, we have*

$$SC(\alpha, -1) = r_n(n^2 - n - 2\alpha n + 2\alpha + 2\alpha^2)^{1/2} / \{(\alpha - n + 1)(\alpha + 1)^{n-1}\}^{1/n},$$

where $r_n = (n!)^{1/n}\{n(n+1)(n+2)/12\}^{1/2}$.

Proof. The components of the $(k+1)$th vertex of the standard simplex are given by $y_i^k = \alpha - k + 1$, $i \le k$ and $y_i^k = -k$, $i > k$.

So,

$$\|y^k - y^j\| = \|y^{k-j} - y^0\| = \|y^{k-j}\|, \quad 0 \le j < k \le n.$$

Hence, the number of one-faces with length $\|y^k\|$ is $n - k + 1$, $k = 1, \ldots, n$. This means that the square of the numerator is equal to

$$\begin{aligned} \sum \|\tau\|^2 &= \sum_{k=1}^{n} (n - k + 1)\{k(\alpha - k + 1)^2 + (n - k)k^2\} \\ &= n(n+1)(n+2)(n^2 - n - 2\alpha n + 2\alpha + 2\alpha^2)/12. \end{aligned}$$

Moreover, it is easy to see that Det $A(\alpha, -1) = (\alpha - n + 1)(\alpha + 1)^{n-1}$. From these results the lemma follows immediately.

4. The optimal triangulation

In this section we compute the optimal triangulation within M according to the SC measure. Again we set β equal to -1 because of the homogeneity.

Theorem 4.1. $SC(n+\sqrt{n+1}, -1) = \min_{\alpha>n-1} SC(\alpha, -1)$.

Proof. Denote $(n^2-n-2\alpha n+2\alpha+2\alpha^2)/\{(\alpha-n+1)(\alpha+1)^{n-1}\}^{2/n}$ by $C(\alpha, n)$, then

$$\begin{aligned}\frac{\mathrm{d}SC(\alpha,-1)}{\mathrm{d}\alpha} &= \tfrac{1}{2}C(\alpha,n)^{-1/2}r_n\frac{\mathrm{d}C(\alpha,n)}{\mathrm{d}\alpha}\\ &= D(\alpha,n)[(2\alpha+1-n)(\alpha-n+1)(\alpha+1)\\ &\quad -(n^2-n-2\alpha n+2\alpha^2+2\alpha)\{\alpha+1+(\alpha-n+1)(n-1)\}/n]\end{aligned}$$

where $D(\alpha, n) = C(\alpha, n)^{-1/2}r_n(\alpha+1)^{-(3n-2)/n}(\alpha-n+1)^{-(n+2)/n}$.

Certainly $D(\alpha, n)$ is positive for $\alpha > n-1$, all n. Setting the expression between the main brackets equal to zero, we obtain after simple calculations that the only feasible solution is $\alpha = n+\sqrt{n+1}$. It is left to the reader to verify that this solution indeed minimizes the function $SC(\alpha, -1)$ for $\alpha > n-1$.

Next we give some properties of the simplices of the optimal triangulation.

Lemma 4.2. *The barycenter of any simplex of the optimal $(\alpha, -1)$ triangulation has equal distance to every vertex of the simplex. This distance is equal to* $n^{1/2}(n+2)^{1/2}(1+\sqrt{n+1})/2\sqrt{3}$.

Proof. The ith component of the barycenter b of the standard simplex of any $(\alpha, -1)$ triangulation is equal to $\{(n+1)\alpha - i\alpha - \frac{1}{2}n(n-1) - i + 1\}/(n+1)$. Clearly, the distance between the barycenter and the jth vertex (of the standard simplex) is

$$\left[\sum_{i=1}^{j}\{b_i-\alpha+(j-1)\}^2+\sum_{i=j+1}^{n}(b_i+j)^2\right]^{1/2}, \quad j=0,\dots,n.$$

After simple calculations it can be seen that for the optimal triangulation this distance is independent of j and is equal to

$$\left(\sum_{i=1}^{n} b_i^2\right)^{1/2} = n^{1/2}(n+2)^{1/2}(n+2+2\sqrt{n+1})^{1/2}/2\sqrt{3},$$

which completes the proof, since the triangulation is congruent.

The lemma means that the simplices are as "round" as possible within M.

Lemma 4.3. *The standard simplex of the optimal $(\alpha, -1)$ triangulation has the following properties*:

(a) $\|y^k\| = \|y^{n-k+1}\|, \quad k=1,\dots,n;$

(b) $$\max_{\tau\in\tau_1(s_n)}\|\tau\| = \begin{cases}\|y^{(n+1)/2}\| = \frac{1}{2}(n+1)(1+\sqrt{n+1}) & \text{if } n \text{ is odd},\\ \|y^{n/2}\| = \|y^{n/2+1}\| = \frac{1}{2}n^{1/2}(n+2)^{1/2}(1+\sqrt{n+1}) & \text{if } n \text{ is even}.\end{cases}$$

Proof. (a) Clearly $y_i^k = n + \sqrt{n+1} - (k-1)$ for $i \le k$ and $y_i^k = -k$ for $i > k$, $k = 1, \dots, n$. Consequently,

$$\|y^k\|^2 - \|y^{n-k+1}\|^2 = k(n-k+1+\sqrt{n+1})^2 + (n-k)k^2 - (n-k+1)(k+\sqrt{n+1})^2 - (k-1)(n-k+1)^2 = 0, \quad k = 1, \dots, n.$$

(b) Since $\|y^k - y^j\| = \|y^{k-j}\|$, for $0 \le j < k \le n$,

$$\max \|\tau\| = \max_k \|y^k\|.$$

Hence,

$$\max \|\tau\| = \max_k \{k(n + \sqrt{n+1} - k + 1)^2 + (n-k)k^2\}^{1/2}.$$

It is easy to see that this maximum is attained for $k = \frac{1}{2}(n+1)$ if n is odd and for $k = \frac{1}{2}n$ and $k = \frac{1}{2}(n+2)$ if n is even.

Part (a) of the lemma means that the distance between y^n and y^0 is equal to that of y^i and y^{i-1}, $i = 1, \dots, n$. Thus using the fixed point algorithm of Van der Laan and Talman [7] the length of the extra direction they use is equal to the length of all other directions. Moreover, the angles between any two rays (see Todd [17]) are equal.

5. The comparison of various triangulations

To compare the optimal (α, β) triangulation with the well-known K and H triangulations we will compute both their average directional density and their SC measure.

The a.d.d. of the $(n + \sqrt{n+1}, -1)$ triangulation is equal to $g_n n\{\frac{1}{2}(n+1)\}^{1/2}/(1 + \sqrt{n+1})$. Using Lemma 4.3 for normalizing the mesh, we easily obtain

$$N(A^*K) = \begin{cases} g_n\{n(n+1)/8\}^{1/2}\{n(n+2)\}^{1/2} & \text{if } n \text{ is even,} \\ g_n\{n(n+1)/8\}^{1/2}(n+1) & \text{if } n \text{ is odd,} \end{cases}$$

where A^*K is the normalized $(n + \sqrt{n+1}, -1)$ triangulation with mesh $\sqrt{n}$. Clearly, $N(A^*K)$ converges to the lower bound of Lemma 3.1 as $n \to \infty$.

Table 1 gives $N(G)/g_n$ for various values of n and G equal to H, K and A^*K. The values of $N(K)$ and $N(H)$ were obtained from Table 1 of Todd [16].

Table 1
The a.d.d. of the H, K and A^*K triangulations for various n (mesh equal to $\sqrt{n}$)

G \ n	1	2	3	4	5	9	15	20	30	50	100	n
H	1	3.4	7.6	13.7	22.1	82.0	269.4	534	1420	4942	27316	$\sim 4n^{5/2}/15$
K	1	3.4	7.2	12.5	19.1	59.9	163.5	289	645	1782	7100	$\sim n^2/\sqrt{2}$
A^*K	1	2.4	4.9	7.7	11.6	33.5	87.6	152	334	910	3553	$\sim n^2/\sqrt{8}$

Next we compute the SC measure for the three triangulations. Since the H triangulation is non-congruent we have to compute

$$\max_{\sigma_n \in H} \left\{ \sum_{\tau \in \tau_1(\sigma_n)} \|\tau\|^2 \right\}^{1/2} .$$

It can easily be shown that this maximum is achieved for the simplices with vertices $y^i = y^{i-1} + h(\gamma_i)$ where $h(j)$ is the vector with jth component equal to 1, $(j-1)$th component equal to -1 and zeroes elsewhere, and where γ is the permutation

$$\gamma = \begin{cases} (2, 4, \dots, n, 1, 3, \dots, n-1) & \text{if } n \text{ is even,} \\ (1, 3, \dots, n, 2, 4, \dots, n-1) & \text{if } n \text{ is odd.} \end{cases}$$

The maximum is then equal to $\frac{1}{2}(n^3 + 2n^2 - n + 2)^{1/2}$ if n is odd and $\frac{1}{2}(n^3 + 2n^2)^{1/2}$ if n is even. Since for the H triangulation $|\text{Det } A| = 1$, its measure is

$$SC(H) = \begin{cases} r_n\{3n/(n+1)\}^{1/2} & \text{if } n \text{ is even,} \\ r_n\{3(n^3 + 2n^2 - n + 2)/n(n+1)(n+2)\}^{1/2} & \text{if } n \text{ is odd,} \end{cases}$$

both converging to $r_n\sqrt{3}$ as $n \to \infty$.

The SC measure of the K-triangulation is $r_n\sqrt{2}$, for all n, which follows from Lemma 3.2 for $\alpha \to \infty$.

From the same lemma we get that the SC measure of the $(n + \sqrt{n+1}, -1)$ triangulation is $r_n(n+1)^{1/2n}$, which converges to r_n.

Table 2 gives SC/r_n of the H, K and A^*K triangulation for various values of n.

Table 2
The SC measure of the H, K and A^*K triangulations for various n

G \ n	1	2	3	4	5	9	15	20	30	50	100	n
H	1.41	1.41	1.48	1.55	1.57	1.64	1.67	1.69	1.70	1.71	1.72	~1.73
K	1.41					1.41						1.41
A^*K	1.41	1.32	1.26	1.22	1.20	1.14	1.10	1.08	1.06	1.04	1.02	~1

6. Triangulations of the affine hull of the unit simplex

In this section a new triangulation of the affine hull T^n of the n-dimensional set $S^n = \{x \in R^n_+ \mid \sum_{i=1}^{n+1} x_i = 1\}$ is proposed and compared with the standard triangulation QK of S^n (see Kuhn [4] and Scarf [14]), where Q is defined as by Todd [16]. When applying a fixed point algorithm on S^n, finding label i according to the standard labelling rule means that there is an incentive to decrease the ith component to find other labels. Instead of increasing only one component with that amount as done for the standard triangulation, it seems more natural to increase all other components with the same amount which must be equal to a fraction n^{-1} of the amount with which the ith component is decreased. Therefore we propose to triangulate the affine hull of S^n according to the $(n+1)\times n$ triangulation matrix T defined by

$$T = \begin{bmatrix} -n & 1 & \cdots & 1 \\ 1 & -n & \cdots & \vdots \\ & & \ddots & 1 \\ \vdots & & & -n \\ 1 & & \cdots & 1 \end{bmatrix}$$

i.e. the so-called T triangulation of T^n with gridsize m $(m>0)$ is the collection of simplices $\sigma(y^0, \dots, y^n)$ such that for an a priori chosen gridpoint z in T^n, $y^0 = z + m^{-1}\sum_{i=1}^{n} \alpha_i t(i)$ for (unique) integers α_i, $i = 1, \dots, n$ and $y^j = y^{j-1} + m^{-1}t(\gamma_j)$, $j = 1, \dots, n$ for some permutation $\gamma = (\gamma_1, \dots, \gamma_n)$ of the elements of I_n.

We will prove now that the SC measure of the T triangulation of T^n is equal to $SC(A^*K)$. Of course, in the definition of the measure of a triangulation of T^n the term $|\det A|/n!$ must be changed into the content of the simplex σ_n.

Lemma 6.1. *For the T triangulation of T^n holds*

$$SC(T) = r_n(n+1)^{1/2n}.$$

Proof. Note that the SC measure of a triangulation of T^n is again homogeneous and that the T triangulation is congruent. To compute the content of a simplex of this triangulation, T^n is mapped orthogonally into R^n by the $n\times(n+1)$ matrix F defined by Todd [16].

Then the content of a simplex of T^n is equal to the content of the FTK triangulation on R^n. Since FT is an upper triangular matrix with ith diagonal element equal to $(n+1)\{(n-i+1)/(n-i+2)\}^{1/2}$, $|\det FT| = (n+1)^{n-1/2}$. Clearly the numerator of $SC(T)$ is equal to

$$\sum_{k=1}^{n} (n+1-k)\{k(n+1-k)^2 + (n+1-k)k^2\} = \{n(n+1)^3(n+2)/12\}^{1/2}.$$

Hence $SC(T) = r_n(n+1)^{1/2n}$.

Obviously the T triangulation has the same nice properties as the $(n+\sqrt{n+1}, -1)$ triangulation of R^n. note that the simplices of the T triangulation meeting S^n do not all lie in S^n. Therefore, in a fixed point algorithm on S^n it is necessary to extend the function (or mapping) to T^n. Furthermore the F^TA^*K triangulation of T^n (see Todd [16]) has the same SC measure as the T triangulation. However, the F^TA^*K triangulation is less appropriate for computing fixed points on T^n.

Finally we calculate the SC measure of the standard triangulation QK to compare it with $SC(T)$ for various n.

Lemma 6.2. *For the standard triangulation of T^n holds*

$$SC(QK) = \begin{cases} r_n\sqrt{3}(n+1)^{-1/2n} & \text{if } n \text{ is even,} \\ r_n\sqrt{3}\{(n^3+3n^2-n+5)/(n^3+3n^2+2n)\}^{1/2}(n+1)^{-1/2n} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Following the proof of Theorem 6.1 the content of any simplex is equal to $|\det FQ|/n! = (n+1)^{1/2}/n!$. However, the triangulation is not congruent. It can be rather easily seen that the maximal sum of the squared length of the one-faces is attained for any simplex $\sigma(y^1, \pi)$ with

$$\pi = \begin{cases} (2,4,6,\dots,n,1,3,5,\dots,n-1) & \text{if } n \text{ is even,} \\ (2k-1,\ 2k+1,\ 2k-3,\ 2k+3,\dots,1,\ n,\ 2k,\ 2k-2,\ 2k+2,\dots,2,\ n-1) & \text{if } n = 4k-1,\ k=1,2,\dots, \\ (2k-1,\ 2k+1,\ 2k-3,\ 2k+3,\dots,n,\ 1,\ 2k,\ 2k-2,\ 2k+2,\dots,n-1,\ 2) & \text{if } n = 4k-3,\ k=1,2,\dots. \end{cases}$$

For these simplices the sum of the squared length of the one-faces is equal to $\frac{1}{4}(n^3+3n^2+2n)$ if n is even and equal to $\frac{1}{4}(n^3+3n^2-n+5)$ if n is odd.

Consequently,

$$\begin{aligned} SC(QK) &= \{\tfrac{1}{4}n(n+1)(n+2)\}^{1/2}/\{(n+1)^{1/2}/n!\}^{1/n} \\ &= r_n\sqrt{3}(n+1)^{-1/2n} \qquad \text{if } n \text{ is even} \end{aligned}$$

and

$$\begin{aligned} SC(QK) &= \{\tfrac{1}{4}(n^3+3n^2-n+5)\}^{1/2}/\{(n+1)^{1/2}/n!\}^{1/n} \\ &= r_n\sqrt{3}[n^3+3n^2-n+5)/\{n(n+1)(n+2)\}]^{1/2}(n+1)^{-1/2n} \qquad \text{if } n \text{ is odd.} \end{aligned}$$

From the Lemmas 6.1 and 6.2 it follows that for large n $SC(T)$ is of the order r_n and $SC(QK)$ is of the order $r_n\sqrt{3}$ which corresponds to the SC measures of respectively the A^*K and the H triangulations of R^n. Note however, that for $n=2$ $SC(T) = SC(QK)$ which is clear since both triangulations yield equilateral triangles. But we may expect even for $n=2$ that the T triangulation will perform better than the standard triangulation because the first one is more natural for use in a fixed point algorithm.

Finally, in Table 3 $SC(G)/r_n$ is given for various values of n for both the QK and the T triangulation.

7. Computational experience

We apply the algorithm of Van der Laan and Talman [7], for the K triangulation as well as for the triangulation proposed in this paper, to a ten-

Table 3
The SC measure of the QK and T triangulations for various n

G \ n	1	2	3	4	10	20	50	100	n
QK	1.41	1.32	1.33	1.42	1.50	1.60	1.67	1.70	~1.73
T	1.41	1.32	1.26	1.22	1.14	1.08	1.04	1.02	~1

dimensional variant of the problem considered by Kellogg et al. [3]. In Table 4 the number of iterations is given for various starting points. For both triangulations the mesh was $(2\sqrt{10})^{-1}$ for all applications. The accuracy, $\max_i |f_i(x^*) - x_i^*|$, where x^* is the approximate fixed point, was of the order of 10^{-4}.

As will be noted, the version with the new triangulation does considerably better than the normal application with the K triangulation. By Table 4 the ratio between the number of iterations is 1.306 which agrees very well with the theoretical ratio 1.254 (for $n = 10$). Note that the K triangulation is very appropriate for the first two starting points, because the path from the starting point to the approximation is in the direction e respectively $-e$, where $e = (1, 1, \ldots, 1)$. However, for some other starting points the algorithm needs many iterations for the K triangulation because the path to the fixed point is along an inappropriate direction. Since the new triangulation has as "round" cells as possible, the number of iterations does not vary so much and the average will be lower.

Table 4
The number of iterations for various starting points

Starting point \ Triangulation	K	A^*K
0 0 0 0 0 0 0 0 0 0	62	99
1 1 1 1 1 1 1 1 1 1	76	83
1 0 1 0 1 0 1 0 1 0	161	132
1 1 1 0 0 0 0 0 0 0	255	153
0 0 0 0 0 0 0 0 0 1	280	148
1 0 0 1 0 0 1 1 1 1	170	144
0 1 1 0 0 1 1 0 1 0	212	153
1 0 0 1 0 1 1 0 1 1	196	153
1 1 1 0 1 1 1 0 1 1	119	117
1 0 0 1 0 0 1 0 0 1	229	155
0 1 0 0 1 0 1 1 0 1	270	171
1 0 1 1 1 1 0 1 1 1	119	115
0 0 1 1 0 0 1 1 0 0	165	127
0 0 0 1 1 1 1 1 1 1	149	136
Total number of iterations	2463	1886
Average number of iterations	176	135