An improvement of sufficient condition for *k*-leaf-connected graphs*

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Abstract For integer $k \ge 2$, a graph *G* is called *k*-leaf-connected if $|V(G)| \ge k + 1$ and given any subset $S \subseteq V(G)$ with |S| = k, *G* always has a spanning tree *T* such that *S* is precisely the set of leaves of *T*. Thus a graph is 2-leaf-connected if and only if it is Hamilton-connected. In this paper, we present a best possible condition based upon the size to guarantee a graph to be *k*-leaf-connected, which not only improves the results of Gurgel and Wakabayashi [On *k*-leaf-connected graphs, J. Combin. Theory Ser. B 41 (1986) 1-16] and Ao, Liu, Yuan and Li [Improved sufficient conditions for *k*-leafconnected graphs, Discrete Appl. Math. 314 (2022) 17-30], but also extends the result of Xu, Zhai and Wang [An improvement of spectral conditions for Hamilton-connected graphs, Linear Multilinear Algebra, 2021]. Our key approach is showing that an (n+k-1)closed non-*k*-leaf-connected graph must contain a large clique if its size is large enough. As applications, sufficient conditions for a graph to be *k*-leaf-connected in terms of the (signless Laplacian) spectral radius of *G* or its complement are also presented.

Keywords: *k*-leaf-connected, Hamilton-connected, spectral radius, signless Laplacian, closure, complement

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1 Introduction

In this paper, we consider simple, undirected and connected graphs. Let G be a graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and edge set E(G). The order and size of G are denoted by |V(G)| = n and |E(G)| = e(G), respectively. For any vertex $u \in V(G)$, we denote by $d_G(u)$ the degree of vertex u in G and by $(d_1, d_2, ..., d_n)$ the degree sequence

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of *G* with $d_1 \leq d_2 \leq \cdots \leq d_n$. Let G_1 and G_2 be two vertex-disjoint graphs. We denote by $G_1 + G_2$ the disjoint union of G_1 and G_2 . The join $G_1 \vee G_2$ is the graph obtained from $G_1 + G_2$ by adding all possible edges between $V(G_1)$ and $V(G_2)$. We denote by δ , \overline{G} , $\omega(G)$ the minimum degree, the complement and the clique number of *G*, respectively. For undefined terms and notions one can refer to [3] and [4].

Let A(G) be the adjacency matrix and D(G) be the diagonal degree matrix of G. Let Q(G) = D(G) + A(G) be the signless Laplacian matrix of G. The largest eigenvalues of A(G) and Q(G), denoted by $\rho(G)$ and q(G), are called the spectral radius and the signless Laplacian spectral radius of G, respectively.

The concept of closure of a graph was used implicitly by Ore [13], and formally introduced by Bondy and Chvatal [2]. Fix an integer $l \ge 0$, the *l*-closure of a graph *G* is the graph obtained from *G* by successively joining pairs of nonadjacent vertices whose degree sum is at least *l* until no such pair exists. Denote by $C_l(G)$ the *l*-closure of *G*. Then we have

$$d_{C_l(G)}(u) + d_{C_l(G)}(v) \le l - 1$$

for every pair of nonadjacent vertices u and v of $C_l(G)$.

For integer $k \ge 2$, a graph *G* is called *k*-leaf-connected if $|V(G)| \ge k + 1$ and given any subset $S \subseteq V(G)$ with |S| = k, *G* always has a spanning tree *T* such that *S* is precisely the set of leaves of *T*. Thus a graph is 2-leaf-connected if and only if it is Hamilton-connected. Hence *k*-leaf-connectedness of a graph is a natural generalization of Hamilton-connectedness. Gurgel and Wakabayashi [9] proved that if *G* is a *k*-leaf-connected graph of order *n*, where $2 \le k \le n - 2$, then *G* is (k + 1)-connected. Hence $\delta \ge k + 1$ is a trivial necessary condition for a graph to be *k*-leaf-connected.

Determining whether a given graph is *k*-leaf-connected is NP-complete. Gurgel and Wakabayashi [9] initially proved the following sufficient condition in terms of e(G) to guarantee a graph G to be *k*-leaf-connected.

Theorem 1.1 (Gurgel and Wakabayashi [9]). *Let G* be a connected graph of order *n* with minimum degree $\delta \ge k + 1$, where $2 \le k \le n - 4$. If

$$e(G) \ge \binom{n-1}{2} + k + 1,$$

then G is k-leaf-connected.

Ao, Liu, Yuan and Li [1] presented the following sufficient condition for a graph to be *k*-leaf-connected and improved the result of Theorem 1.1.

Theorem 1.2 (Ao, Liu, Yuan and Li [1]). *Let G* be a connected graph of order *n* and minimum degree $\delta \ge k + 1$, where $2 \le k \le n - 4$. If

$$e(G) \ge \binom{n-2}{2} + 2k + 2,$$

then G is k-leaf-connected unless $G \in \{K_3 \lor (K_{n-5} + 2K_1), K_4 \lor (K_2 + 3K_1), K_6 \lor 6K_1, K_5 \lor 5K_1, K_4 \lor (K_{1,4} + K_1), K_3 \lor K_{2,5}, K_4 \lor 4K_1, K_3 \lor (K_{1,3} + K_1), K_2 \lor K_{2,4}\}.$

As a special case of *k*-leaf-connectedness, there are many sufficient conditions to assure a graph to be 2-leaf-connected (see for example [14, 16–18]). By introducing the minimum degree δ as a new parameter, Chen and Zhang [5] presented a sufficient condition for a graph with $\delta \ge t \ge 2$ to be Hamilton-connected: $e(G) \ge {\binom{n-t+1}{2}} - \frac{t^2-3t-2}{2}$. Zhou and Wang [19] proved a better condition for a graph to be Hamilton-connected: $e(G) \ge {\binom{n-t+1}{2}} + t^2 + t$. Recently, Xu, Zhai and Wang [15] improved the results of [5] and [19]. Define $L_n^t = K_2 \lor (K_{n-t-1} + K_{t-1})$ $(2 \le t \le \frac{n}{2}), N_n^t = K_t \lor (K_{n-2t+1} + (t-1)K_1)$ $(2 \le t \le \frac{n}{2}),$ and $M_n^t = K_{t+1} \lor (K_{n-2t-1} + tK_1)$ $(2 \le t \le \frac{n-1}{2})$.

Theorem 1.3 (Xu, Zhai and Wang [15]). *Let G* be a connected graph of order $n \ge 6t + 3$ with $\delta \ge t \ge 2$. If

$$e(G) \ge \binom{n-t}{2} + t^2 + 2,$$

then G is Hamilton-connected unless $C_{n+1}(G) \in \{L_n^t, N_n^t, M_n^t\}$.

Inspired by the ideas from the conjecture by Erdős and Hajnal [6] and the result on Hamilton-connected graphs by Xu, Zhai and Wang [15], we first show that an (n + k - 1)-closed non-k-leaf-connected graph G must contain a large clique if its number of edges is large enough. Using the key approach and typical spectral techniques, we present a best possible condition based upon the size to guarantee a graph to be k-leaf-connected as follows. Our main result not only improves the result of Theorem 1.2, but also extends the result on Hamilton-connected graphs in Theorem 1.3.

Theorem 1.4. Let G be a connected graph of order $n \ge k + 17$ and minimum degree $\delta \ge k + 1$, where $k \ge 2$. If

$$e(G) \ge \binom{n-3}{2} + 3k + 5,$$

then G is k-leaf-connected unless $C_{n+k-1}(G) \in \{K_k \lor (K_{n-k-2} + K_2), K_3 \lor (K_{n-5} + 2K_1), K_4 \lor (K_{n-7} + 3K_1)\}.$

2 Preliminaries

We will present in this section some important results that will be used in our subsequent arguments. Gurgel and Wakabayashi [9] proved a sufficient condition in terms of the degree sequence for a graph to be k-leaf-connected.

Lemma 2.1 (Gurgel and Wakabayashi [9]). Let k and n be such that $2 \le k \le n-3$. Let G be a graph with degree sequence $d_1 \le d_2 \le \cdots \le d_n$. Suppose that there is no integer i with $k \le i \le \frac{n+k-2}{2}$ such that $d_{i-k+1} \le i$ and $d_{n-i} \le n-i+k-2$. Then G is k-leaf-connected.

Lemma 2.2 (Gurgel and Wakabayashi [9]). Let *G* be a graph and *k* be an integer with $2 \le k \le n-1$. Then *G* is *k*-leaf-connected if and only if the (n + k - 1)-closure $C_{n+k-1}(G)$ of *G* is *k*-leaf-connected.

An important upper bound on the spectral radius $\rho(G)$ is as follows.

Lemma 2.3 (Hong, Shu and Fang [11], Nikiforov [12]). *Let G be a graph with minimum degree* δ *. Then*

$$\rho(G) \le \frac{\delta - 1}{2} + \sqrt{2e(G) - \delta n + \frac{(\delta + 1)^2}{4}}.$$

The following observation is very useful when we use the above upper bound on $\rho(G)$.

Proposition 2.1 (Hong, Shu and Fang [11], Nikiforov [12]). For graph G with $2e(G) \le n(n-1)$, the function

$$f(x) = \frac{x-1}{2} + \sqrt{2e(G) - nx + \frac{(x+1)^2}{4}}$$

is decreasing with respect to x for $0 \le x \le n - 1$.

Feng and Yu [7] proved an upper bound on q(G), which has been widely used in the literature.

Lemma 2.4 (Feng and Yu [7]). Let G be a connected graph on n vertices and e(G) edges. Then

$$q(G) \le \frac{2e(G)}{n-1} + n - 2.$$

Let *M* be the following $n \times n$ matrix

$$M = \begin{pmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,m} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m,1} & M_{m,2} & \cdots & M_{m,m} \end{pmatrix},$$

whose rows and columns are partitioned into subsets X_1, X_2, \ldots, X_m of $\{1, 2, \ldots, n\}$. The quotient matrix R(M) of the matrix M (with respect to the given partition) is the $m \times m$ matrix whose entries are the average row sums of the blocks $M_{i,j}$ of M. The above partition is called equitable if each block $M_{i,j}$ of M has constant row (and column) sum.

Lemma 2.5 (Brouwer and Haemers [4], Godsil and Royle [8], Haemers [10]). Let M be a real symmetric matrix and let R(M) be its equitable quotient matrix. Then the eigenvalues of the quotient matrix R(M) are eigenvalues of M. Furthermore, if M is nonnegative and irreducible, then the spectral radius of the quotient matrix R(M) equals to the spectral radius of M.

3 Proof of Theorem 1.4

Before presenting our main result, we first show that an (n + k - 1)-closed non-k-leafconnected graph G must contain a large clique if its number of edges is large enough. We denote by $\omega(G)$ the clique number of G. Let (d_1, d_2, \ldots, d_n) be the degree sequence of G, where $d_1 \le d_2 \le \cdots \le d_n$. **Lemma 3.1.** Let G be an (n+k-1)-closed non-k-leaf-connected graph of order $n \ge k+17$ with $\delta \ge k+1$ and $k \ge 2$. If

$$e(G) \ge \binom{n-3}{2} + 3k + 5,$$

then $\omega(G) = n - 2$ unless $G \cong K_4 \vee (K_{n-7} + 3K_1)$.

Proof. Note that $\delta \ge k + 1$. First we claim that $\omega(G) \le n - 2$. Otherwise, suppose that $\omega(G) \ge n-1$, then *G* contains an (n-1)-clique, and hence for any two vertices $u, v \in V(G)$, we always have $d_G(u) + d_G(v) \ge n + k - 1$. If there exists two vertices $uv \notin E(G)$, then $d_G(u) + d_G(v) \le n + k - 2$ since *G* is an (n + k - 1)-closed graph, a contradiction. Hence any two vertices of *G* are adjacent. That is, $G \cong K_n$, and obviously *G* is *k*-leaf-connected, a contradiction.

Let $(d_1, d_2, ..., d_n)$ be the degree sequence of *G* with $d_1 \le d_2 \le \cdots \le d_n$. Note that *G* is not *k*-leaf-connected. By Lemma 2.1, there exists an integer *i* with $k \le i \le \frac{n+k-2}{2}$ such that $d_{i-k+1} \le i$ and $d_{n-i} \le n-i+k-2$. Then we have

$$\begin{split} e(G) &= \frac{1}{2} \sum_{j=1}^{n} d_j \\ &= \frac{1}{2} \left(\sum_{j=1}^{i-k+1} d_j + \sum_{j=i-k+2}^{n-i} d_j + \sum_{j=n-i+1}^{n} d_j \right) \\ &\leq \frac{1}{2} \left[(i-k+1)i + (n-2i+k-1)(n-i+k-2) + i(n-1) \right] \\ &= \binom{n-3}{2} + 3k + 5 + \frac{f_1(i)}{2}, \end{split}$$

where

$$f_1(i) = 3i^2 - (2n + 4k - 5)i + (2k + 4)n + k^2 - 9k - 20$$

By the assumption $e(G) \ge {\binom{n-3}{2}} + 3k + 5$, then we have $f_1(i) \ge 0$. Note that $k + 1 \le \delta \le d_{i-k+1} \le i \le \frac{n+k-2}{2}$. We shall divide the proof into the following three cases. **Case 1.** $k + 3 \le i \le \frac{n+k-2}{2}$.

Since $f_1''(i) = 6 > 0$, then $f_1(i)$ is a concave function on *i*. For $n \ge k + 17$, we have

$$f_1(k+3) = -2n + 2k + 22 < 0,$$

$$n+k-2 = n^2 + 11 + 12 + 11k$$

and
$$f_1(\frac{n+k-2}{2}) = -\frac{n^2}{4} + \frac{k+11}{2}n - \frac{k^2}{4} - \frac{11k}{2} - 22 < 0.$$

This implies that $f_1(i) < 0$, a contradiction.

Case 2. i = k + 2.

Then the corresponding degree sequence of G is

$$\underbrace{d_1 \le d_2 \le d_3 \le k+2}_{V_1}, \ \underbrace{d_4 \le d_5 \le \dots \le d_{n-k-2} \le n-4}_{V_2}, \ \underbrace{d_{n-k-1} \le d_{n-k} \le \dots \le d_n \le n-1}_{V_3}$$

According to the above degree sequence, we divide V(G) into three parts: V_1 , V_2 and V_3 .

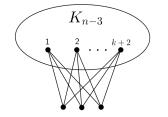


Fig. 1: Graph $K_{k+2} \vee (K_{n-k-5} + 3K_1)$.

Claim 1. There is no vertex of degree less than $\frac{n+k-1}{2}$ in V_2 .

Proof. Suppose that there exists a vertex of degree less than $\frac{n+k-1}{2}$ in V_2 . Then

$$\begin{split} e(G) &= \frac{1}{2} \sum_{j=1}^{n} d_{j} \\ &< \frac{1}{2} [3(k+2) + (n-k-6)(n-4) + (k+2)(n-1) + \frac{n+k-1}{2}] \\ &= \binom{n-3}{2} + 3k + 5 - \frac{n-k-11}{4} \\ &\leq \binom{n-3}{2} + 3k + 5 - \frac{3}{2} \\ &< e(G), \end{split}$$

a contradiction, since $n \ge k + 17$.

By Claim 1, it follows that $d_G(u) + d_G(v) \ge n + k - 1$ for any two different vertices $u, v \in V_2 \cup V_3$. Note that *G* is (n + k - 1)-closed. Then $V_2 \cup V_3$ is a clique of *G*, and hence

$$\omega(G) \ge |V_2 \cup V_3| \ge (n-k-5) + (k+2) = n-3.$$

Recall that $\omega(G) \le n - 2$. Then we have

$$n-3 \le \omega(G) \le n-2.$$

If $\omega(G) = n - 2$, then $d_3 \ge n - 3$. Note that $d_3 \le k + 2$. Then $n \le k + 5$, which contradicts $n \ge k + 17$. Thus, we have $\omega(G) = n - 3$. Let $C = V_2 \cup V_3$. Note that |C| = n - 3. Then C is a maximum clique of G, and $V(G) = V_1 \cup C$. Notice that $k + 1 \le \delta \le d_G(v) \le k + 2$ for each $v \in V_1$. Let $V_1 = \{v_1, v_2, v_3\}$ and $V_1^* = \{v_i \in V_1 \mid d_G(v_i) = k + 2\}$.

Claim 2. $|V_1^*| \ge 2$.

Proof. Suppose, to the contrary, that $|V_1^*| \le 1$. Note that $k + 1 \le d_G(v_i) \le k + 2$ for any $v_i \in V_1$. Then

$$e(G) \le e(C) + \sum_{i=1}^{3} d_G(v_i) \le \binom{n-3}{2} + 2(k+1) + (k+2) = \binom{n-3}{2} + 3k + 4 < e(G),$$

a contradiction.

Define $C^* = \{v \in C \mid N_G(v) \cap V_1 \neq \emptyset\}.$

Claim 3. $|C^*| = k + 2$.

Proof. By the definition of C^* , we know that $d_G(v) \ge n-3$ for each $v \in C^*$. Then $d_G(v) + d_G(v_i) \ge (n-3) + (k+2) = n+k-1$ for any $v \in C^*$ and $v_i \in V_1^*$. Note that G is (n+k-1)-closed. It follows that each vertex of C^* is adjacent to each vertex of V_1^* . Combining Claim 2, we have $d_G(v) \ge d_C(v) + |V_1^*| \ge (n-4) + 2 = n-2$ for each $v \in C^*$. Therefore, $d_G(v) + d_G(v_i) \ge (n-2) + (k+1) = n+k-1$ for any $v \in C^*$ and $v_i \in V_1$. Then each vertex of V_1 is adjacent to each vertex of C^* , which implies that $|C^*| \le d_G(v_i) \le k+2$, where $v_i \in V_1$.

On the other hand, let $e(V_1, C)$ denote the number of edges between V_1 and C. Notice that $e(V_1, C) = e(V_1, C^*) = |V_1||C^*| = 3|C^*|$ and $e(V_1) = \frac{1}{2}(\sum_{v_i \in V_1} d_G(v_i) - 3|C^*|) \le \frac{3(k+2-|C^*|)}{2}$. Then

$$e(G) = e(C) + e(V_1, C^*) + e(V_1) \le \binom{n-3}{2} + \frac{3(k+2+|C^*|)}{2}$$

Combining the assumption $e(G) \ge \binom{n-3}{2} + 3k + 5$, we have $|C^*| \ge k + 2$. Therefore, $|C^*| = k + 2$.

Recall that $d_G(v_i) \le k+2$ for each $v_i \in V_1$. According to Claim 3, V_1 is an independent set. This implies that $G \cong K_{k+2} \lor (K_{n-k-5} + 3K_1)$ (see Fig. 1). Define

$$L = V(K_{k+2}), M = V(K_{n-k-5}) \text{ and } N = V(3K_1).$$

Notice that the vertices of *N* are only adjacent to those of *L*. When $k \ge 3$, for any $S \subseteq V(G)$ with |S| = k, we always find a spanning tree *T* (see Fig. 2) such that *S* is precisely the set of leaves (labeled by red vertices) of *T*. Hence $K_{k+2} \lor (K_{n-k-5} + 3K_1)$ is *k*-leaf-connected, which contradicts the assumption. However, $K_4 \lor (K_{n-7} + 3K_1)$ is not 2-leaf-connected. Therefore, $G \cong K_4 \lor (K_{n-7} + 3K_1)$.

Case 3. i = k + 1.

Then the degree sequence of G is given by

$$\underbrace{d_1 = d_2 = k+1}_{V_1}, \ \underbrace{d_3 \le d_4 \le \dots \le d_{n-k-1} \le n-3}_{V_2}, \ \underbrace{d_{n-k} \le d_{n-k+1} \le \dots \le d_n \le n-1}_{V_3}.$$

Claim 4. There are at most three vertices of degree less than $\frac{n+k-1}{2}$ in V_2 .

Proof. Assume that there exist four vertices of degree less than $\frac{n+k-1}{2}$ in V_2 . Then we have

$$\begin{split} e(G) &= \frac{1}{2} \sum_{j=1}^{n} d_{j} \\ &< \frac{1}{2} [2(k+1) + (n-k-7)(n-3) + (k+1)(n-1) + 4 \cdot \frac{n+k-1}{2}] \\ &= \binom{n-3}{2} + 3k + 4, \\ &< e(G), \end{split}$$

a contradiction.

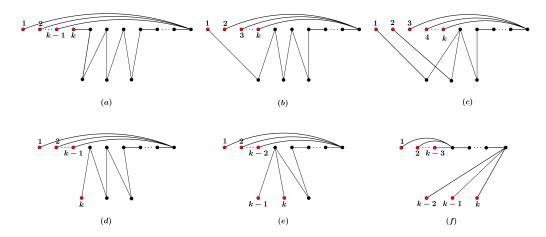


Fig. 2: (*a*). *k* vertices are chosen from *M*; (*b*). One of *k* vertices belongs to *L*, and the rest belong to *M*; (*c*). At least two vertices come from *L*, and the rest come from *M*; (*d*). One vertex is from *N*, and the remaining vertices come from $L \cup M$; (*e*). Two vertices belong to *N*, and the remaining vertices come from $L \cup M$. (*f*). Three vertices belong to *N*, and the remaining vertices come from $L \cup M$.

Let $V_2^* = \{v \in V_2 \mid d_G(v) \ge \frac{n+k-1}{2}\}$. By Claim 4, we have $|V_2^*| \ge |V_2| - 3 = n-k-6 > 0$. It is clear that $d_G(u) + d_G(v) \ge n+k-1$ for any $u, v \in V_2^* \cup V_3$. Note that *G* is an (n+k-1)-closed graph. This implies that $V_2^* \cup V_3$ is a clique of *G*, and hence $\omega(G) \ge |V_2^* \cup V_3| \ge (n-k-6) + (k+1) = n-5$. Note that $\omega(G) \le n-2$. Then we have

$$n-5 \le \omega(G) \le n-2.$$

Define $C = V_2^* \cup V_3$.

Claim 5. C is a maximum clique of G.

Proof. By the definition of V_2^* , we know that $d_G(u) < \frac{n+k-1}{2} \le n-9 < n-5$ for any $u \in V_1 \cup (V_2 \setminus V_2^*)$, since $n \ge k + 17$. Hence there exists at least one vertex $v \in C$ such that $uv \notin E(G)$ for any $u \in V_1 \cup (V_2 \setminus V_2^*)$, and thus $u \notin C$. This implies that *C* is a maximum clique of *G*.

Next let $\omega(G) = \omega$ for short.

Claim 6. $d_G(u) \le n + k - \omega - 1$ for each $u \in V_2 \setminus V_2^*$.

Proof. Suppose, to the contrary, that $d_G(u) \ge n + k - \omega$ for each $u \in V_2 \setminus V_2^*$. Then $d_G(u) + d_G(v) \ge (n + k - \omega) + (\omega - 1) = n + k - 1$ for $u \in V_2 \setminus V_2^*$ and $v \in C$. Note that *G* is an (n + k - 1)-closed graph. Then *u* is adjacent to every vertex of *C*, and hence $C \cup \{u\}$ is a larger clique, which contradicts Claim 5.

Notice that $|V_2 \setminus V_2^*| = n - |V_1| - |V_2^* \cup V_3| = n - \omega - 2$. Hence by Claim 6, we obtain

$$\sum_{u \in V_2 \setminus V_2^*} d_G(u) \le (n - \omega - 2)(n + k - \omega - 1).$$

Then we have

$$\begin{split} e(G) &\leq \sum_{u \in V_1} d_G(u) + \sum_{u \in V_2 \setminus V_2^*} d_G(u) + e(V_2^* \cup V_3) \\ &\leq 2(k+1) + (n-\omega-2)(n+k-\omega-1) + \binom{\omega}{2} \\ &= \frac{3}{2}\omega^2 - (2n+k-\frac{5}{2})\omega + n^2 + kn - 3n + 4 \\ &\triangleq f_2(\omega). \end{split}$$

Note that $f_2(\omega)$ is a concave function on ω . If $n - 5 \le \omega(G) \le n - 3$, then

$$e(G) \le \max\{f_2(n-5), f_2(n-3)\} = \binom{n-3}{2} + 3k + 4 < e(G).$$

a contradiction. Therefore, $\omega(G) = n - 2$. This completes the proof.

Remark 3.1. The sufficient condition in terms of edge in Lemma 3.1 is best possible. Let $G \cong K_3 \lor (K_{n-6} + K_2 + K_1)$. Note that $C_{n+1}(G) = G$. Then G is not 2-leaf-connected and $e(G) = \binom{n-3}{2} + 10$. However, $\omega(G) = n - 3$.

Using the above technical Lemma 3.1, we will present the proof of Theorem 1.4.

Proof of Theorem 1.4. Suppose, to the contrary, that *G* is not *k*-leaf-connected, where $n \ge k + 17, \delta \ge k + 1$ and $k \ge 2$. Let $H = C_{n+k-1}(G)$. By Lemma 2.2, *H* is not *k*-leaf-connected. Note that $G \subseteq H$. By the assumption $e(G) \ge {\binom{n-3}{2}} + 3k + 5$, then $e(H) \ge {\binom{n-3}{2}} + 3k + 5$. By Lemma 3.1, either $\omega(H) = n - 2$ or $H \cong K_4 \lor (K_{n-7} + 3K_1)$.

Assume that $\omega(H) = n - 2$. Next we will characterize the structure of *H*. Let *C* be an (n-2)-clique of *H* and *F* be a subgraph of *H* induced by $V(H)\setminus C$, and let $V(F) = \{v_1, v_2\}$.

Claim 7. $d_H(v_i) = k + 1$ for each $v_i \in V(F)$.

Proof. Suppose there exists a vertex $v_i \in V(F)$ with $d_H(v_i) \ge k + 2$. Then $d_H(v_i) + d_H(v) \ge (k+2) + (n-3) = n + k - 1$ for any $v \in C$. Recall that $H = C_{n+k-1}(G)$. Then v_i is adjacent to vertex v. Note that v is an arbitrary vertex of C. Hence v_i is adjacent to all vertices of C. This implies that $\omega(H) \ge n - 1$, a contradiction.

Claim 8. $N_H(v_1) \cap C = N_H(v_2) \cap C$.

Proof. Without loss of generality, assume that a vertex v of C is adjacent to v_1 of F, then $d_H(v) \ge n-2$. Therefore, $d_H(v) + d_H(v_2) \ge (n-2) + (k+1) = n+k-1$. Note that $H = C_{n+k-1}(G)$. Then v is also adjacent to vertex v_2 . Hence $N_H(v_1) \cap C = N_H(v_2) \cap C$. \Box

Let $|N_H(v_i) \cap C| = t$. Note that |V(F)| = 2. By Claim 7, we know that $d_H(v_i) = k + 1$. Then $t \ge k$. On the other hand, $t \le d_H(v_i) = k + 1$. Hence $k \le t \le k + 1$. Next, we will discuss the following two cases.

Case 1. t = k.

Then $H \cong K_k \vee (K_{n-k-2} + K_2)$. Note that $G - V(K_k)$ is not connected. Then G has no spanning tree such that $V(K_k)$ is precisely the set of leaves, and this implies that G

is not *k*-leaf-connected. Note that $e(H) = \binom{n-2}{2} + 2k + 1 > \binom{n-3}{2} + 3k + 5$. Hence $H \cong K_k \vee (K_{n-k-2} + K_2)$.

Case 2. t = k + 1.

Then $H \cong K_{k+1} \lor (K_{n-k-3} + 2K_1)$. By Theorem 1.5 in [1], we know that $K_{k+1} \lor (K_{n-k-3} + 2K_1)$ is k-leaf-connected for $k \ge 3$, a contradiction. However, $K_3 \lor (K_{n-5} + 2K_1)$ is not 2-leaf-connected. Notice that $e(H) = \binom{n-2}{2} + 6 > \binom{n-3}{2} + 11$. Therefore, $H \cong K_3 \lor (K_{n-5} + 2K_1)$.

By the above proof, we have $H = C_{n+k-1}(G) \in \{K_k \lor (K_{n-k-2} + K_2), K_3 \lor (K_{n-5} + 2K_1), K_4 \lor (K_{n-7} + 3K_1)\}$, as desired.

4 Applications

As applications, we will provide sufficient spectral conditions to guarantee a graph to be k-leaf-connected. The following lemmas are used in the sequel.

Lemma 4.1. Let $H \cong K_k \lor (K_{n-k-2} + K_2)$. (i) If $n \ge 2k + 8$, then $\rho(H) > \frac{k}{2} + \sqrt{n^2 - (k+8)n + \frac{k^2}{4} + 7k + 23}$. (ii) If $n \ge 3k + 10$, then $q(H) > 2n - 8 + \frac{6k+16}{n-1}$. (iii) If $n \ge 3k + 9$, then $\rho(\overline{H}) < \sqrt{\frac{(n-k)(3n-3k-11)}{n}}$.

Proof. (i) Note that K_{n-2} is a proper subgraph of *H*. Then for $n \ge 2k + 8$, we have

$$\rho(H) > \rho(K_{n-2}) = n - 3 > \frac{k}{2} + \sqrt{n^2 - (k+8)n + \frac{k^2}{4} + 7k + 23}.$$

(ii) For $n \ge 3k + 10$, by a direct calculation, we obtain that

$$q(H) > q(K_{n-2}) = 2n - 6 > 2n - 8 + \frac{6k + 16}{n - 1}.$$

(iii) Obviously, $\overline{H} \cong kK_1 \cup [(n-k-2)K_1 \vee 2K_1]$. For $n \ge 3k+9$, we have

$$\rho(\overline{H}) = \rho(K_{2,n-k-2}) = \sqrt{2(n-k-2)} < \sqrt{\frac{(n-k)(3n-3k-11)}{n}}$$

as desired.

Lemma 4.2. Let $H \cong K_3 \vee (K_{n-5} + 2K_1)$. (*i*) If $n \ge 9$, then $\rho(H) > 1 + \sqrt{n^2 - 10n + 38}$. (*ii*) If $n \ge 10$, then $q(H) > 2n - 8 + \frac{28}{n-1}$. (*iii*) If $n \ge 17$, then $\rho(\overline{H}) < \sqrt{\frac{(n-2)(3n-17)}{n}}$.

Proof. (i) Let R(A) be an equitable quotient matrix of the adjacency matrix A(H) with respect to the partition $(V(K_3), V(K_{n-5}), V(2K_1))$. In the proof of Theorem 4.2 [1], we known that the characteristic polynomial of R(A) is $P_{R(A)}(x) = x^3 - (n-4)x^2 - (n+3)x + 6n - 36$, and $P_{R(A)}(x)$ is a monotonically increasing function on $\left[\frac{n-4+\sqrt{n^2-5n+25}}{3}, +\infty\right)$. Note that $\rho(H) = \lambda_1(R(A)) > \frac{n-4+\sqrt{n^2-5n+25}}{3}$ and

$$1 + \sqrt{n^2 - 10n + 38} > \frac{n - 4 + \sqrt{n^2 - 5n + 25}}{3}$$

By Maple, $P_{R(A)}(1 + \sqrt{n^2 - 10n + 38}) < 0 = P_{R(A)}(\rho(H))$ for $n \ge 9$. This implies that $\rho(H) > 1 + \sqrt{n^2 - 10n + 38}$.

(ii) Let R(Q) be an equitable quotient matrix of the signless Laplacian matrix Q(H) with respect to the partition $(V(K_3), V(K_{n-5}), V(2K_1))$. In the proof of Theorem 4.7 [1], the characteristic polynomial of R(Q) is $P_{R(Q)}(x) = x^3 - (3n-5)x^2 + (2n^2 - n - 24)x - 6n^2 + 42n - 72$, and $P_{R(Q)}(x)$ is a monotonically increasing function on $\left[\frac{3n-5+\sqrt{3n^2-27n+97}}{3}, +\infty\right)$. Note that $q(H) > \frac{3n-5+\sqrt{3n^2-27n+97}}{3}$ and

$$2n - 8 + \frac{28}{n - 1} > \frac{3n - 5 + \sqrt{3n^2 - 27n + 97}}{3}$$

By a simple calculation, we have $P_{R(Q)}(2n-8+\frac{28}{n-1}) < 0 = P_{R(Q)}(q(H))$ for $n \ge 10$. Hence, $q(H) > 2n-8+\frac{28}{n-1}$.

(iii) We have $\overline{H} \cong 3K_1 \cup [(n-5)K_1 \vee K_2]$. Let RC(A) be an equitable quotient matrix of the adjacency matrix $A(\overline{H})$ with respect to the partition $(V(3K_1), V((n-5)K_1), V(K_2))$. One can see that

$$RC(A) = \left(\begin{array}{rrr} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & n-5 & 1 \end{array}\right)$$

Then the characteristic polynomial of RC(A) is given by $P_{RC(A)}(x) = x(x^2 - x - 2n + 10)$. By a direct calculation, $\rho(\overline{H}) = \frac{1+\sqrt{8n-39}}{2} < \sqrt{\frac{(n-2)(3n-17)}{n}}$ for $n \ge 17$.

Lemma 4.3. Let $H \cong K_4 \vee (K_{n-7} + 3K_1)$. (*i*) If $n \ge 9$, then $\rho(H) < 1 + \sqrt{n^2 - 10n + 38}$. (*ii*) If $n \ge 9$, then $q(H) < 2n - 8 + \frac{28}{n-1}$. (*iii*) If $n \ge 7$, then $\rho(\overline{H}) > \sqrt{\frac{(n-2)(3n-17)}{n}}$.

Proof. (i) Let R(A) be an equitable quotient matrix of the adjacency matrix A(H) with respect to the partition $(V(K_4), V(K_{n-7}), V(3K_1))$. One can see that

$$R(A) = \left(\begin{array}{rrrr} 3 & n-7 & 3 \\ 4 & n-8 & 0 \\ 4 & 0 & 0 \end{array}\right).$$

Then the characteristic polynomial of R(A) is given by $P_{R(A)}(x) = x^3 - (n-5)x^2 - (n+8)x + 12n - 96$. By Lemma 2.5, we know that $\rho(H) = \lambda_1(R(A))$ is the largest root of the equation $P_{R(A)}(x) = 0$. Let $P'_{R(A)}(x) = 3x^2 - 2(n-5)x - n - 8 = 0$. We can solve this equation to obtain that

$$x_1 = \frac{n-5-\sqrt{n^2-7n+49}}{3}$$
 and $x_2 = \frac{n-5+\sqrt{n^2-7n+49}}{3}$

Then $P_{R(A)}(x)$ is a monotonically increasing function on $[x_2, +\infty)$. Note that $\rho(H) = \lambda_1(R(A)) > x_2$ and $1 + \sqrt{n^2 - 10n + 38} > x_2$. By Maple, $P_{R(A)}(1 + \sqrt{n^2 - 10n + 38}) > 0 = P_{R(A)}(\rho(H))$ for $n \ge 9$. This implies that $\rho(H) < 1 + \sqrt{n^2 - 10n + 38}$.

(ii) Let R(Q) be an equitable quotient matrix of the signless Laplacian matrix Q(H) with respect to the partition $(V(K_4), V(K_{n-7}), V(3K_1))$. Then

$$R(Q) = \begin{pmatrix} n+2 & n-7 & 3\\ 4 & 2n-12 & 0\\ 4 & 0 & 4 \end{pmatrix}.$$

Then the characteristic polynomial of R(Q) is given by $P_{R(Q)}(x) = x^3 - 3(n-2)x^2 + (2n^2 - 48)x - 8n^2 + 72n - 160$. By Lemma 2.5, we have $q(H) = \lambda_1(R(Q))$ is the largest root of the equation $P_{R(Q)}(x) = 0$. Let $P'_{R(Q)}(x) = 3x^2 - 6(n-2)x + 2n^2 - 48 = 0$. The two roots x_1 and x_2 of this equation are as follows:

$$x_1 = \frac{3n - 6 - \sqrt{3n^2 - 36n + 180}}{3}$$
 and $x_2 = \frac{3n - 6 + \sqrt{3n^2 - 36n + 180}}{3}$

Then $P_{R(Q)}(x)$ is a monotonically increasing function on $[x_2, +\infty)$. Note that $q(H) > x_2$ and $2n-8+\frac{28}{n-1} > x_2$. By a simple calculation, we have $P_{R(Q)}(2n-8+\frac{28}{n-1}) > 0 = P_{R(Q)}(q(H))$ for $n \ge 9$. Hence $q(H) < 2n-8+\frac{28}{n-1}$.

(iii) It is easy to see that $\overline{H} \cong 4K_1 \cup [(n-7)K_1 \vee K_3]$. Let RC(A) be an equitable quotient matrix of the adjacency matrix $A(\overline{H})$ with respect to the partition $(V(4K_1), V((n-7)K_1), V(K_3))$. One can see that

$$RC(A) = \left(\begin{array}{rrr} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & n-7 & 2 \end{array}\right).$$

Then the characteristic polynomial of RC(A) is given by $P_{RC(A)}(x) = x(x^2 - 2x - 3n + 21)$. By a direct calculation, we have $\rho(\overline{H}) = 1 + \sqrt{3n - 20} > \sqrt{\frac{(n-2)(3n-17)}{n}}$ for $n \ge 7$. \Box

Ao, Liu, Yuan and Li [1] presented sufficient conditions to guarantee a graph to be *k*-leaf-connected in terms of the (signless Laplacian) spectral radius of *G* or its complement.

Theorem 4.1 (Ao, Liu, Yuan and Li [1]). Let *G* be a connected graph of order *n* and minimum degree $\delta \ge k + 1$, where $2 \le k \le n - 4$. Then

(i) If $\rho(G) \geq \frac{k}{2} + \sqrt{n^2 - (k+6)n + \frac{k^2}{4} + 5k + 11}}$, then G is k-leaf-connected unless $G \in \{K_3 \vee 3K_1, K_4 \vee 4K_1\}$. (ii) If $q(G) \geq 2n - 6 + \frac{4k+6}{n-1}$, then G is k-leaf-connected unless $G \cong K_4 \vee 4K_1$. (iii) If $\rho(\overline{G}) \leq \sqrt{\frac{(n-k)(2n-2k-5)}{n}}$, then G is k-leaf-connected.

In this paper, we improve the above result as follows.

Theorem 4.2. Let G be a connected graph of order $n \ge k + 17$ and minimum degree $\delta \ge k + 1$, where $k \ge 2$. If one of the following holds, (i) $\rho(G) \ge \frac{k}{2} + \sqrt{n^2 - (k+8)n + \frac{k^2}{4} + 7k + 23}$, (ii) $q(G) \ge 2n - 8 + \frac{6k+16}{n-1}$, (iii) $\rho(\overline{G}) \le \sqrt{\frac{(n-k)(3n-3k-11)}{n}}$, then G is k-leaf-connected unless $C_{n+k-1}(G) \in \{K_k \lor (K_{n-k-2} + K_2), K_3 \lor (K_{n-5} + 2K_1)\}$. **Proof.** Suppose, to the contrary, that *G* is not *k*-leaf-connected.

(i) By Lemma 2.3 and Proposition 2.1, we have

$$\rho(G) \le \frac{\delta - 1}{2} + \sqrt{2e(G) - \delta n + \frac{(\delta + 1)^2}{4}} \le \frac{k}{2} + \sqrt{2e(G) - (k + 1)n + \frac{k^2}{4} + k + 1}.$$

Since $\rho(G) \ge \frac{k}{2} + \sqrt{n^2 - (k+8)n + \frac{k^2}{4} + 7k + 23}}$, we have $e(G) \ge \binom{n-3}{2} + 3k + 5$. Let $H = C_{n+k-1}(G)$. By Theorem 1.4, we have $H \in \{K_k \lor (K_{n-k-2} + K_2), K_3 \lor (K_{n-5} + 2K_1), K_4 \lor (K_{n-7} + 3K_1)\}$. Assume that $H \cong K_4 \lor (K_{n-7} + 3K_1)$. According to (i) of Lemma 4.3, $\rho(G) \le \rho(H) < 1 + \sqrt{n^2 - 10n + 38}$, a contradiction. For $H \in \{K_k \lor (K_{n-k-2} + K_2), K_3 \lor (K_{n-5} + 2K_1)\}$ and $n \ge k + 17$, by (i) of Lemmas 4.1 and 4.2, we can not compare completely $\rho(G)$ with $\frac{k}{2} + \sqrt{n^2 - (k+8)n + \frac{k^2}{4} + 7k + 23}$. For the brevity of discussion, we have $C_{n+k-1}(G) = H \in \{K_k \lor (K_{n-k-2} + K_2), K_3 \lor (K_{n-5} + 2K_1)\}$.

(ii) By Lemma 2.4, we have $q(G) \leq \frac{2e(G)}{n-1} + n - 2$. Note that $q(G) \geq 2n - 8 + \frac{6k+16}{n-1}$. Then $e(G) \geq {n-3 \choose 2} + 3k + 5$. Let $H = C_{n+k-1}(G)$. By Theorem 1.4, we have $H \in \{K_k \lor (K_{n-k-2} + K_2), K_3 \lor (K_{n-5} + 2K_1), K_4 \lor (K_{n-7} + 3K_1)\}$. Suppose that $H \cong K_4 \lor (K_{n-7} + 3K_1)$. By (ii) of Lemma 4.3, $q(G) \leq q(H) < 2n - 8 + \frac{28}{n-1}$, a contradiction. Therefore, $C_{n+k-1}(G) = H \in \{K_k \lor (K_{n-k-2} + K_2), K_3 \lor (K_{n-5} + 2K_1)\}$.

(iii) Let $H = C_{n+k-1}(G)$. Similar to the proof of Theorem 4.4 in [1], we can obtain that

$$\rho(\overline{H}) \geq \sqrt{\frac{(n-k)e(\overline{H})}{n}}$$

Note that $\overline{H} \subseteq \overline{G}$. Then we have

$$\rho(\overline{H}) \le \rho(\overline{G}) \le \sqrt{\frac{(n-k)(3n-3k-11)}{n}},$$

and therefore,

$$\sqrt{\frac{(n-k)e(\overline{H})}{n}} \le \rho(\overline{H}) \le \rho(\overline{G}) \le \sqrt{\frac{(n-k)(3n-3k-11)}{n}}.$$

It is easy to check that $e(\overline{H}) \leq 3n - 3k - 11$ and

$$e(H) = \binom{n}{2} - e(\overline{H}) \ge \binom{n-3}{2} + 3k + 5.$$

Applying Theorem 1.4 on *H*, we have $C_{n+k-1}(H) = H \in \{K_k \lor (K_{n-k-2} + K_2), K_3 \lor (K_{n-5} + 2K_1), K_4 \lor (K_{n-7} + 3K_1)\}$. Assume that $H \cong K_4 \lor (K_{n-7} + 3K_1)$. By (iii) of Lemma 4.3, $\rho(\overline{G}) \ge \rho(\overline{H}) > \sqrt{\frac{(n-2)(3n-17)}{n}}$, a contradiction. Hence $C_{n+k-1}(G) = H \in \{K_k \lor (K_{n-k-2} + K_2), K_3 \lor (K_{n-5} + 2K_1)\}$. This completes the proof of Theorem 4.2.

Declaration of competing interest

The authors declare that they have no conflict of interest.

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