# An improvement of sufficient condition for $k$-leaf-connected graphs* 

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#### Abstract

For integer $k \geq 2$, a graph $G$ is called $k$-leaf-connected if $|V(G)| \geq k+1$ and given any subset $S \subseteq V(G)$ with $|S|=k, G$ always has a spanning tree $T$ such that $S$ is precisely the set of leaves of $T$. Thus a graph is 2-leaf-connected if and only if it is Hamilton-connected. In this paper, we present a best possible condition based upon the size to guarantee a graph to be $k$-leaf-connected, which not only improves the results of Gurgel and Wakabayashi [On $k$-leaf-connected graphs, J. Combin. Theory Ser. B 41 (1986) 1-16] and Ao, Liu, Yuan and Li [Improved sufficient conditions for $k$-leafconnected graphs, Discrete Appl. Math. 314 (2022) 17-30], but also extends the result of Xu, Zhai and Wang [An improvement of spectral conditions for Hamilton-connected graphs, Linear Multilinear Algebra, 2021]. Our key approach is showing that an ( $n+k-1$ )closed non- $k$-leaf-connected graph must contain a large clique if its size is large enough. As applications, sufficient conditions for a graph to be $k$-leaf-connected in terms of the (signless Laplacian) spectral radius of $G$ or its complement are also presented.


Keywords: $k$-leaf-connected, Hamilton-connected, spectral radius, signless Laplacian, closure, complement

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## 1 Introduction

In this paper, we consider simple, undirected and connected graphs. Let $G$ be a graph with vertex set $V(G)=\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$ and edge set $E(G)$. The order and size of $G$ are denoted by $|V(G)|=n$ and $|E(G)|=e(G)$, respectively. For any vertex $u \in V(G)$, we denote by $d_{G}(u)$ the degree of vertex $u$ in $G$ and by $\left(d_{1}, d_{2}, \ldots, d_{n}\right)$ the degree sequence

[^0]of $G$ with $d_{1} \leq d_{2} \leq \cdots \leq d_{n}$. Let $G_{1}$ and $G_{2}$ be two vertex-disjoint graphs. We denote by $G_{1}+G_{2}$ the disjoint union of $G_{1}$ and $G_{2}$. The join $G_{1} \vee G_{2}$ is the graph obtained from $G_{1}+G_{2}$ by adding all possible edges between $V\left(G_{1}\right)$ and $V\left(G_{2}\right)$. We denote by $\delta, \bar{G}$, $\omega(G)$ the minimum degree, the complement and the clique number of $G$, respectively. For undefined terms and notions one can refer to [3] and [4].

Let $A(G)$ be the adjacency matrix and $D(G)$ be the diagonal degree matrix of $G$. Let $Q(G)=D(G)+A(G)$ be the signless Laplacian matrix of $G$. The largest eigenvalues of $A(G)$ and $Q(G)$, denoted by $\rho(G)$ and $q(G)$, are called the spectral radius and the signless Laplacian spectral radius of $G$, respectively.

The concept of closure of a graph was used implicitly by Ore [13], and formally introduced by Bondy and Chvatal [2]. Fix an integer $l \geq 0$, the $l$-closure of a graph $G$ is the graph obtained from $G$ by successively joining pairs of nonadjacent vertices whose degree sum is at least $l$ until no such pair exists. Denote by $C_{l}(G)$ the $l$-closure of $G$. Then we have

$$
d_{C_{l}(G)}(u)+d_{C_{l}(G)}(v) \leq l-1
$$

for every pair of nonadjacent vertices $u$ and $v$ of $C_{l}(G)$.
For integer $k \geq 2$, a graph $G$ is called $k$-leaf-connected if $|V(G)| \geq k+1$ and given any subset $S \subseteq V(G)$ with $|S|=k, G$ always has a spanning tree $T$ such that $S$ is precisely the set of leaves of $T$. Thus a graph is 2-leaf-connected if and only if it is Hamiltonconnected. Hence $k$-leaf-connectedness of a graph is a natural generalization of Hamiltonconnectedness. Gurgel and Wakabayashi [9] proved that if $G$ is a $k$-leaf-connected graph of order $n$, where $2 \leq k \leq n-2$, then $G$ is $(k+1)$-connected. Hence $\delta \geq k+1$ is a trivial necessary condition for a graph to be $k$-leaf-connected.

Determining whether a given graph is $k$-leaf-connected is NP-complete. Gurgel and Wakabayashi [9] initially proved the following sufficient condition in terms of $e(G)$ to guarantee a graph $G$ to be $k$-leaf-connected.

Theorem 1.1 (Gurgel and Wakabayashi [9]). Let $G$ be a connected graph of order $n$ with minimum degree $\delta \geq k+1$, where $2 \leq k \leq n-4$. If

$$
e(G) \geq\binom{ n-1}{2}+k+1
$$

then $G$ is $k$-leaf-connected.
Ao, Liu, Yuan and Li [1] presented the following sufficient condition for a graph to be $k$-leaf-connected and improved the result of Theorem 1.1.

Theorem 1.2 (Ao, Liu, Yuan and Li [1]). Let $G$ be a connected graph of order $n$ and minimum degree $\delta \geq k+1$, where $2 \leq k \leq n-4$. If

$$
e(G) \geq\binom{ n-2}{2}+2 k+2
$$

then $G$ is $k$-leaf-connected unless $G \in\left\{K_{3} \vee\left(K_{n-5}+2 K_{1}\right), K_{4} \vee\left(K_{2}+3 K_{1}\right), K_{6} \vee 6 K_{1}, K_{5} \vee\right.$ $\left.5 K_{1}, K_{4} \vee\left(K_{1,4}+K_{1}\right), K_{3} \vee K_{2,5}, K_{4} \vee 4 K_{1}, K_{3} \vee\left(K_{1,3}+K_{1}\right), K_{2} \vee K_{2,4}\right\}$.

As a special case of $k$-leaf-connectedness, there are many sufficient conditions to assure a graph to be 2-leaf-connected (see for example [14, 16-18]). By introducing the minimum degree $\delta$ as a new parameter, Chen and Zhang [5] presented a sufficient condition for a graph with $\delta \geq t \geq 2$ to be Hamilton-connected: $e(G) \geq\binom{ n-t+1}{2}-\frac{t^{2}-3 t-2}{2}$. Zhou and Wang [19] proved a better condition for a graph to be Hamilton-connected: $e(G) \geq\binom{ n-t}{2}+t^{2}+t$. Recently, Xu, Zhai and Wang [15] improved the results of [5] and [19]. Define $L_{n}^{t}=K_{2} \vee\left(K_{n-t-1}+K_{t-1}\right)\left(2 \leq t \leq \frac{n}{2}\right), N_{n}^{t}=K_{t} \vee\left(K_{n-2 t+1}+(t-1) K_{1}\right)\left(2 \leq t \leq \frac{n}{2}\right)$, and $M_{n}^{t}=K_{t+1} \vee\left(K_{n-2 t-1}+t K_{1}\right)\left(2 \leq t \leq \frac{n-1}{2}\right)$.

Theorem 1.3 (Xu, Zhai and Wang [15]). Let $G$ be a connected graph of order $n \geq 6 t+3$ with $\delta \geq t \geq 2$. If

$$
e(G) \geq\binom{ n-t}{2}+t^{2}+2
$$

then $G$ is Hamilton-connected unless $C_{n+1}(G) \in\left\{L_{n}^{t}, N_{n}^{t}, M_{n}^{t}\right\}$.
Inspired by the ideas from the conjecture by Erdős and Hajnal [6] and the result on Hamilton-connected graphs by Xu , Zhai and Wang [15], we first show that an $(n+k-1)$ closed non- $k$-leaf-connected graph $G$ must contain a large clique if its number of edges is large enough. Using the key approach and typical spectral techniques, we present a best possible condition based upon the size to guarantee a graph to be $k$-leaf-connected as follows. Our main result not only improves the result of Theorem 1.2, but also extends the result on Hamilton-connected graphs in Theorem 1.3.

Theorem 1.4. Let $G$ be a connected graph of order $n \geq k+17$ and minimum degree $\delta \geq k+1$, where $k \geq 2$. If

$$
e(G) \geq\binom{ n-3}{2}+3 k+5
$$

then $G$ is $k$-leaf-connected unless $C_{n+k-1}(G) \in\left\{K_{k} \vee\left(K_{n-k-2}+K_{2}\right), K_{3} \vee\left(K_{n-5}+2 K_{1}\right), K_{4} \vee\right.$ $\left.\left(K_{n-7}+3 K_{1}\right)\right\}$.

## 2 Preliminaries

We will present in this section some important results that will be used in our subsequent arguments. Gurgel and Wakabayashi [9] proved a sufficient condition in terms of the degree sequence for a graph to be $k$-leaf-connected.

Lemma 2.1 (Gurgel and Wakabayashi [9]). Let $k$ and $n$ be such that $2 \leq k \leq n-3$. Let $G$ be a graph with degree sequence $d_{1} \leq d_{2} \leq \cdots \leq d_{n}$. Suppose that there is no integer $i$ with $k \leq i \leq \frac{n+k-2}{2}$ such that $d_{i-k+1} \leq i$ and $d_{n-i} \leq n-i+k-2$. Then $G$ is $k$-leaf-connected.

Lemma 2.2 (Gurgel and Wakabayashi [9]). Let $G$ be a graph and $k$ be an integer with $2 \leq k \leq n-1$. Then $G$ is $k$-leaf-connected if and only if the $(n+k-1)$-closure $C_{n+k-1}(G)$ of $G$ is $k$-leaf-connected.

An important upper bound on the spectral radius $\rho(G)$ is as follows.

Lemma 2.3 (Hong, Shu and Fang [11], Nikiforov [12]). Let $G$ be a graph with minimum degree $\delta$. Then

$$
\rho(G) \leq \frac{\delta-1}{2}+\sqrt{2 e(G)-\delta n+\frac{(\delta+1)^{2}}{4}} .
$$

The following observation is very useful when we use the above upper bound on $\rho(G)$.
Proposition 2.1 (Hong, Shu and Fang [11], Nikiforov [12]). For graph $G$ with $2 e(G) \leq$ $n(n-1)$, the function

$$
f(x)=\frac{x-1}{2}+\sqrt{2 e(G)-n x+\frac{(x+1)^{2}}{4}}
$$

is decreasing with respect to $x$ for $0 \leq x \leq n-1$.
Feng and Yu [7] proved an upper bound on $q(G)$, which has been widely used in the literature.

Lemma 2.4 (Feng and Yu [7]). Let $G$ be a connected graph on $n$ vertices and $e(G)$ edges. Then

$$
q(G) \leq \frac{2 e(G)}{n-1}+n-2
$$

Let $M$ be the following $n \times n$ matrix

$$
M=\left(\begin{array}{cccc}
M_{1,1} & M_{1,2} & \cdots & M_{1, m} \\
M_{2,1} & M_{2,2} & \cdots & M_{2, m} \\
\vdots & \vdots & \ddots & \vdots \\
M_{m, 1} & M_{m, 2} & \cdots & M_{m, m}
\end{array}\right) \text {, }
$$

whose rows and columns are partitioned into subsets $X_{1}, X_{2}, \ldots, X_{m}$ of $\{1,2, \ldots, n\}$. The quotient matrix $R(M)$ of the matrix $M$ (with respect to the given partition) is the $m \times$ $m$ matrix whose entries are the average row sums of the blocks $M_{i, j}$ of $M$. The above partition is called equitable if each block $M_{i, j}$ of $M$ has constant row (and column) sum.

Lemma 2.5 (Brouwer and Haemers [4], Godsil and Royle [8], Haemers [10]). Let M be a real symmetric matrix and let $R(M)$ be its equitable quotient matrix. Then the eigenvalues of the quotient matrix $R(M)$ are eigenvalues of $M$. Furthermore, if $M$ is nonnegative and irreducible, then the spectral radius of the quotient matrix $R(M)$ equals to the spectral radius of $M$.

## 3 Proof of Theorem 1.4

Before presenting our main result, we first show that an $(n+k-1)$-closed non- $k$-leafconnected graph $G$ must contain a large clique if its number of edges is large enough. We denote by $\omega(G)$ the clique number of $G$. Let $\left(d_{1}, d_{2}, \ldots, d_{n}\right)$ be the degree sequence of $G$, where $d_{1} \leq d_{2} \leq \cdots \leq d_{n}$.

Lemma 3.1. Let $G$ be an ( $n+k-1$ )-closed non- $k$-leaf-connected graph of order $n \geq k+17$ with $\delta \geq k+1$ and $k \geq 2$. If

$$
e(G) \geq\binom{ n-3}{2}+3 k+5
$$

then $\omega(G)=n-2$ unless $G \cong K_{4} \vee\left(K_{n-7}+3 K_{1}\right)$.
Proof. Note that $\delta \geq k+1$. First we claim that $\omega(G) \leq n-2$. Otherwise, suppose that $\omega(G) \geq n-1$, then $G$ contains an ( $n-1$ )-clique, and hence for any two vertices $u, v \in V(G)$, we always have $d_{G}(u)+d_{G}(v) \geq n+k-1$. If there exists two vertices $u v \notin E(G)$, then $d_{G}(u)+d_{G}(v) \leq n+k-2$ since $G$ is an $(n+k-1)$-closed graph, a contradiction. Hence any two vertices of $G$ are adjacent. That is, $G \cong K_{n}$, and obviously $G$ is $k$-leaf-connected, a contradiction.

Let $\left(d_{1}, d_{2}, \ldots, d_{n}\right)$ be the degree sequence of $G$ with $d_{1} \leq d_{2} \leq \cdots \leq d_{n}$. Note that $G$ is not $k$-leaf-connected. By Lemma 2.1, there exists an integer $i$ with $k \leq i \leq \frac{n+k-2}{2}$ such that $d_{i-k+1} \leq i$ and $d_{n-i} \leq n-i+k-2$. Then we have

$$
\begin{aligned}
e(G) & =\frac{1}{2} \sum_{j=1}^{n} d_{j} \\
& =\frac{1}{2}\left(\sum_{j=1}^{i-k+1} d_{j}+\sum_{j=i-k+2}^{n-i} d_{j}+\sum_{j=n-i+1}^{n} d_{j}\right) \\
& \leq \frac{1}{2}[(i-k+1) i+(n-2 i+k-1)(n-i+k-2)+i(n-1)] \\
& =\binom{n-3}{2}+3 k+5+\frac{f_{1}(i)}{2},
\end{aligned}
$$

where

$$
f_{1}(i)=3 i^{2}-(2 n+4 k-5) i+(2 k+4) n+k^{2}-9 k-20 .
$$

By the assumption $e(G) \geq\binom{ n-3}{2}+3 k+5$, then we have $f_{1}(i) \geq 0$. Note that $k+1 \leq \delta \leq$ $d_{i-k+1} \leq i \leq \frac{n+k-2}{2}$. We shall divide the proof into the following three cases.

## Case 1. $k+3 \leq i \leq \frac{n+k-2}{2}$.

Since $f_{1}^{\prime \prime}(i)=6>0$, then $f_{1}(i)$ is a concave function on $i$. For $n \geq k+17$, we have

$$
\begin{gathered}
f_{1}(k+3)=-2 n+2 k+22<0, \\
\text { and } \quad f_{1}\left(\frac{n+k-2}{2}\right)=-\frac{n^{2}}{4}+\frac{k+11}{2} n-\frac{k^{2}}{4}-\frac{11 k}{2}-22<0 .
\end{gathered}
$$

This implies that $f_{1}(i)<0$, a contradiction.
Case 2. $i=k+2$.
Then the corresponding degree sequence of $G$ is
$\underbrace{d_{1} \leq d_{2} \leq d_{3} \leq k+2}_{V_{1}}, \underbrace{d_{4} \leq d_{5} \leq \cdots \leq d_{n-k-2} \leq n-4}_{V_{2}}, \underbrace{d_{n-k-1} \leq d_{n-k} \leq \cdots \leq d_{n} \leq n-1}_{V_{3}}$.
According to the above degree sequence, we divide $V(G)$ into three parts: $V_{1}, V_{2}$ and $V_{3}$.


Fig. 1: Graph $K_{k+2} \vee\left(K_{n-k-5}+3 K_{1}\right)$.

Claim 1. There is no vertex of degree less than $\frac{n+k-1}{2}$ in $V_{2}$.
Proof. Suppose that there exists a vertex of degree less than $\frac{n+k-1}{2}$ in $V_{2}$. Then

$$
\begin{aligned}
e(G) & =\frac{1}{2} \sum_{j=1}^{n} d_{j} \\
& <\frac{1}{2}\left[3(k+2)+(n-k-6)(n-4)+(k+2)(n-1)+\frac{n+k-1}{2}\right] \\
& =\binom{n-3}{2}+3 k+5-\frac{n-k-11}{4} \\
& \leq\binom{ n-3}{2}+3 k+5-\frac{3}{2} \\
& <e(G)
\end{aligned}
$$

a contradiction, since $n \geq k+17$.
By Claim 1, it follows that $d_{G}(u)+d_{G}(v) \geq n+k-1$ for any two different vertices $u, v \in V_{2} \cup V_{3}$. Note that $G$ is $(n+k-1)$-closed. Then $V_{2} \cup V_{3}$ is a clique of $G$, and hence

$$
\omega(G) \geq\left|V_{2} \cup V_{3}\right| \geq(n-k-5)+(k+2)=n-3 .
$$

Recall that $\omega(G) \leq n-2$. Then we have

$$
n-3 \leq \omega(G) \leq n-2
$$

If $\omega(G)=n-2$, then $d_{3} \geq n-3$. Note that $d_{3} \leq k+2$. Then $n \leq k+5$, which contradicts $n \geq k+17$. Thus, we have $\omega(G)=n-3$. Let $C=V_{2} \cup V_{3}$. Note that $|C|=n-3$. Then $C$ is a maximum clique of $G$, and $V(G)=V_{1} \cup C$. Notice that $k+1 \leq \delta \leq d_{G}(v) \leq k+2$ for each $v \in V_{1}$. Let $V_{1}=\left\{v_{1}, v_{2}, v_{3}\right\}$ and $V_{1}^{*}=\left\{v_{i} \in V_{1} \mid d_{G}\left(v_{i}\right)=k+2\right\}$.
Claim 2. $\left|V_{1}^{*}\right| \geq 2$.
Proof. Suppose, to the contrary, that $\left|V_{1}^{*}\right| \leq 1$. Note that $k+1 \leq d_{G}\left(v_{i}\right) \leq k+2$ for any $v_{i} \in V_{1}$. Then

$$
e(G) \leq e(C)+\sum_{i=1}^{3} d_{G}\left(v_{i}\right) \leq\binom{ n-3}{2}+2(k+1)+(k+2)=\binom{n-3}{2}+3 k+4<e(G)
$$

a contradiction.

Define $C^{*}=\left\{v \in C \mid N_{G}(v) \cap V_{1} \neq \emptyset\right\}$.
Claim 3. $\left|C^{*}\right|=k+2$.
Proof. By the definition of $C^{*}$, we know that $d_{G}(v) \geq n-3$ for each $v \in C^{*}$. Then $d_{G}(v)+d_{G}\left(v_{i}\right) \geq(n-3)+(k+2)=n+k-1$ for any $v \in C^{*}$ and $v_{i} \in V_{1}^{*}$. Note that $G$ is $(n+k-1)$-closed. It follows that each vertex of $C^{*}$ is adjacent to each vertex of $V_{1}^{*}$. Combining Claim 2, we have $d_{G}(v) \geq d_{C}(v)+\left|V_{1}^{*}\right| \geq(n-4)+2=n-2$ for each $v \in C^{*}$. Therefore, $d_{G}(v)+d_{G}\left(v_{i}\right) \geq(n-2)+(k+1)=n+k-1$ for any $v \in C^{*}$ and $v_{i} \in V_{1}$. Then each vertex of $V_{1}$ is adjacent to each vertex of $C^{*}$, which implies that $\left|C^{*}\right| \leq d_{G}\left(v_{i}\right) \leq k+2$, where $v_{i} \in V_{1}$.

On the other hand, let $e\left(V_{1}, C\right)$ denote the number of edges between $V_{1}$ and $C$. Notice that $e\left(V_{1}, C\right)=e\left(V_{1}, C^{*}\right)=\left|V_{1}\right|\left|C^{*}\right|=3\left|C^{*}\right|$ and $e\left(V_{1}\right)=\frac{1}{2}\left(\sum_{v_{i} \in V_{1}} d_{G}\left(v_{i}\right)-3\left|C^{*}\right|\right) \leq \frac{3\left(k+2-\left|C^{*}\right|\right)}{2}$. Then

$$
e(G)=e(C)+e\left(V_{1}, C^{*}\right)+e\left(V_{1}\right) \leq\binom{ n-3}{2}+\frac{3\left(k+2+\left|C^{*}\right|\right)}{2} .
$$

Combining the assumption $e(G) \geq\binom{ n-3}{2}+3 k+5$, we have $\left|C^{*}\right| \geq k+2$. Therefore, $\left|C^{*}\right|=k+2$.

Recall that $d_{G}\left(v_{i}\right) \leq k+2$ for each $v_{i} \in V_{1}$. According to Claim 3, $V_{1}$ is an independent set. This implies that $G \cong K_{k+2} \vee\left(K_{n-k-5}+3 K_{1}\right)$ (see Fig. 1). Define

$$
L=V\left(K_{k+2}\right), \quad M=V\left(K_{n-k-5}\right) \text { and } N=V\left(3 K_{1}\right) .
$$

Notice that the vertices of $N$ are only adjacent to those of $L$. When $k \geq 3$, for any $S \subseteq V(G)$ with $|S|=k$, we always find a spanning tree $T$ (see Fig. 2) such that $S$ is precisely the set of leaves (labeled by red vertices) of $T$. Hence $K_{k+2} \vee\left(K_{n-k-5}+3 K_{1}\right)$ is $k$-leaf-connected, which contradicts the assumption. However, $K_{4} \vee\left(K_{n-7}+3 K_{1}\right)$ is not 2-leaf-connected. Therefore, $G \cong K_{4} \vee\left(K_{n-7}+3 K_{1}\right)$.

Case 3. $i=k+1$.
Then the degree sequence of $G$ is given by

$$
\underbrace{d_{1}=d_{2}=k+1}_{V_{1}}, \underbrace{d_{3} \leq d_{4} \leq \cdots \leq d_{n-k-1} \leq n-3}_{V_{2}}, \underbrace{d_{n-k} \leq d_{n-k+1} \leq \cdots \leq d_{n} \leq n-1}_{V_{3}} .
$$

Claim 4. There are at most three vertices of degree less than $\frac{n+k-1}{2}$ in $V_{2}$.
Proof. Assume that there exist four vertices of degree less than $\frac{n+k-1}{2}$ in $V_{2}$. Then we have

$$
\begin{aligned}
e(G) & =\frac{1}{2} \sum_{j=1}^{n} d_{j} \\
& <\frac{1}{2}\left[2(k+1)+(n-k-7)(n-3)+(k+1)(n-1)+4 \cdot \frac{n+k-1}{2}\right] \\
& =\binom{n-3}{2}+3 k+4, \\
& <e(G)
\end{aligned}
$$

a contradiction.


Fig. 2: (a). $k$ vertices are chosen from $M ;(b)$. One of $k$ vertices belongs to $L$, and the rest belong to $M ;(c)$. At least two vertices come from $L$, and the rest come from $M$; (d). One vertex is from $N$, and the remaining vertices come from $L \cup M$; (e). Two vertices belong to $N$, and the remaining vertices come from $L \cup M$. (f). Three vertices belong to $N$, and the remaining vertices come from $L \cup M$.

Let $V_{2}^{*}=\left\{v \in V_{2} \left\lvert\, d_{G}(v) \geq \frac{n+k-1}{2}\right.\right\}$. By Claim 4, we have $\left|V_{2}^{*}\right| \geq\left|V_{2}\right|-3=n-k-6>0$. It is clear that $d_{G}(u)+d_{G}(v) \geq n+k-1$ for any $u, v \in V_{2}^{*} \cup V_{3}$. Note that $G$ is an $(n+k-1)$ closed graph. This implies that $V_{2}^{*} \cup V_{3}$ is a clique of $G$, and hence $\omega(G) \geq\left|V_{2}^{*} \cup V_{3}\right| \geq$ $(n-k-6)+(k+1)=n-5$. Note that $\omega(G) \leq n-2$. Then we have

$$
n-5 \leq \omega(G) \leq n-2
$$

Define $C=V_{2}^{*} \cup V_{3}$.
Claim 5. $C$ is a maximum clique of $G$.
Proof. By the definition of $V_{2}^{*}$, we know that $d_{G}(u)<\frac{n+k-1}{2} \leq n-9<n-5$ for any $u \in V_{1} \cup\left(V_{2} \backslash V_{2}^{*}\right)$, since $n \geq k+17$. Hence there exists at least one vertex $v \in C$ such that $u v \notin E(G)$ for any $u \in V_{1} \cup\left(V_{2} \backslash V_{2}^{*}\right)$, and thus $u \notin C$. This implies that $C$ is a maximum clique of $G$.

Next let $\omega(G)=\omega$ for short.
Claim 6. $d_{G}(u) \leq n+k-\omega-1$ for each $u \in V_{2} \backslash V_{2}^{*}$.
Proof. Suppose, to the contrary, that $d_{G}(u) \geq n+k-\omega$ for each $u \in V_{2} \backslash V_{2}^{*}$. Then $d_{G}(u)+$ $d_{G}(v) \geq(n+k-\omega)+(\omega-1)=n+k-1$ for $u \in V_{2} \backslash V_{2}^{*}$ and $v \in C$. Note that $G$ is an ( $n+k-1$ )-closed graph. Then $u$ is adjacent to every vertex of $C$, and hence $C \cup\{u\}$ is a larger clique, which contradicts Claim 5.

Notice that $\left|V_{2} \backslash V_{2}^{*}\right|=n-\left|V_{1}\right|-\left|V_{2}^{*} \cup V_{3}\right|=n-\omega-2$. Hence by Claim 6, we obtain

$$
\sum_{u \in V_{2} \backslash V_{2}^{*}} d_{G}(u) \leq(n-\omega-2)(n+k-\omega-1) .
$$

Then we have

$$
\begin{aligned}
e(G) & \leq \sum_{u \in V_{1}} d_{G}(u)+\sum_{u \in V_{2} \backslash V_{2}^{*}} d_{G}(u)+e\left(V_{2}^{*} \cup V_{3}\right) \\
& \leq 2(k+1)+(n-\omega-2)(n+k-\omega-1)+\binom{\omega}{2} \\
& =\frac{3}{2} \omega^{2}-\left(2 n+k-\frac{5}{2}\right) \omega+n^{2}+k n-3 n+4 \\
& \triangleq f_{2}(\omega) .
\end{aligned}
$$

Note that $f_{2}(\omega)$ is a concave function on $\omega$. If $n-5 \leq \omega(G) \leq n-3$, then

$$
e(G) \leq \max \left\{f_{2}(n-5), f_{2}(n-3)\right\}=\binom{n-3}{2}+3 k+4<e(G) .
$$

a contradiction. Therefore, $\omega(G)=n-2$. This completes the proof.
Remark 3.1. The sufficient condition in terms of edge in Lemma 3.1 is best possible. Let $G \cong K_{3} \vee\left(K_{n-6}+K_{2}+K_{1}\right)$. Note that $C_{n+1}(G)=G$. Then $G$ is not 2 -leaf-connected and $e(G)=\binom{n-3}{2}+10$. However, $\omega(G)=n-3$.

Using the above technical Lemma 3.1, we will present the proof of Theorem 1.4.
Proof of Theorem 1.4. Suppose, to the contrary, that $G$ is not $k$-leaf-connected, where $n \geq k+17, \delta \geq k+1$ and $k \geq 2$. Let $H=C_{n+k-1}(G)$. By Lemma 2.2, $H$ is not $k$-leaf-connected. Note that $G \subseteq H$. By the assumption $e(G) \geq\binom{ n-3}{2}+3 k+5$, then $e(H) \geq\binom{ n-3}{2}+3 k+5$. By Lemma 3.1, either $\omega(H)=n-2$ or $H \cong K_{4} \vee\left(K_{n-7}+3 K_{1}\right)$.

Assume that $\omega(H)=n-2$. Next we will characterize the structure of $H$. Let $C$ be an ( $n-2$ )-clique of $H$ and $F$ be a subgraph of $H$ induced by $V(H) \backslash C$, and let $V(F)=\left\{v_{1}, v_{2}\right\}$.

Claim 7. $d_{H}\left(v_{i}\right)=k+1$ for each $v_{i} \in V(F)$.
Proof. Suppose there exists a vertex $v_{i} \in V(F)$ with $d_{H}\left(v_{i}\right) \geq k+2$. Then $d_{H}\left(v_{i}\right)+d_{H}(v) \geq$ $(k+2)+(n-3)=n+k-1$ for any $v \in C$. Recall that $H=C_{n+k-1}(G)$. Then $v_{i}$ is adjacent to vertex $v$. Note that $v$ is an arbitrary vertex of $C$. Hence $v_{i}$ is adjacent to all vertices of $C$. This implies that $\omega(H) \geq n-1$, a contradiction.

Claim 8. $N_{H}\left(v_{1}\right) \cap C=N_{H}\left(v_{2}\right) \cap C$.
Proof. Without loss of generality, assume that a vertex $v$ of $C$ is adjacent to $v_{1}$ of $F$, then $d_{H}(v) \geq n-2$. Therefore, $d_{H}(v)+d_{H}\left(v_{2}\right) \geq(n-2)+(k+1)=n+k-1$. Note that $H=C_{n+k-1}(G)$. Then $v$ is also adjacent to vertex $v_{2}$. Hence $N_{H}\left(v_{1}\right) \cap C=N_{H}\left(v_{2}\right) \cap C$.

Let $\left|N_{H}\left(v_{i}\right) \cap C\right|=t$. Note that $|V(F)|=2$. By Claim 7, we know that $d_{H}\left(v_{i}\right)=k+1$. Then $t \geq k$. On the other hand, $t \leq d_{H}\left(v_{i}\right)=k+1$. Hence $k \leq t \leq k+1$. Next, we will discuss the following two cases.
Case 1. $t=k$.
Then $H \cong K_{k} \vee\left(K_{n-k-2}+K_{2}\right)$. Note that $G-V\left(K_{k}\right)$ is not connected. Then $G$ has no spanning tree such that $V\left(K_{k}\right)$ is precisely the set of leaves, and this implies that $G$
is not $k$-leaf-connected. Note that $e(H)=\binom{n-2}{2}+2 k+1>\binom{n-3}{2}+3 k+5$. Hence $H \cong$ $K_{k} \vee\left(K_{n-k-2}+K_{2}\right)$.
Case 2. $t=k+1$.
Then $H \cong K_{k+1} \vee\left(K_{n-k-3}+2 K_{1}\right)$. By Theorem 1.5 in [1], we know that $K_{k+1} \vee\left(K_{n-k-3}+\right.$ $\left.2 K_{1}\right)$ is $k$-leaf-connected for $k \geq 3$, a contradiction. However, $K_{3} \vee\left(K_{n-5}+2 K_{1}\right)$ is not 2-leaf-connected. Notice that $e(H)=\binom{n-2}{2}+6>\binom{n-3}{2}+11$. Therefore, $H \cong K_{3} \vee\left(K_{n-5}+2 K_{1}\right)$.

By the above proof, we have $H=C_{n+k-1}(G) \in\left\{K_{k} \vee\left(K_{n-k-2}+K_{2}\right), K_{3} \vee\left(K_{n-5}+\right.\right.$ $\left.2 K_{1}\right), K_{4} \vee\left(K_{n-7}+3 K_{1}\right)$, as desired.

## 4 Applications

As applications, we will provide sufficient spectral conditions to guarantee a graph to be $k$-leaf-connected. The following lemmas are used in the sequel.
Lemma 4.1. Let $H \cong K_{k} \vee\left(K_{n-k-2}+K_{2}\right)$.
(i) If $n \geq 2 k+8$, then $\rho(H)>\frac{k}{2}+\sqrt{n^{2}-(k+8) n+\frac{k^{2}}{4}+7 k+23}$.
(ii) If $n \geq 3 k+10$, then $q(H)>2 n-8+\frac{6 k+16}{n-1}$.
(iii) If $n \geq 3 k+9$, then $\rho(\bar{H})<\sqrt{\frac{(n-k)(3 n-3 k-11)}{n}}$.

Proof. (i) Note that $K_{n-2}$ is a proper subgraph of $H$. Then for $n \geq 2 k+8$, we have

$$
\rho(H)>\rho\left(K_{n-2}\right)=n-3>\frac{k}{2}+\sqrt{n^{2}-(k+8) n+\frac{k^{2}}{4}+7 k+23} .
$$

(ii) For $n \geq 3 k+10$, by a direct calculation, we obtain that

$$
q(H)>q\left(K_{n-2}\right)=2 n-6>2 n-8+\frac{6 k+16}{n-1}
$$

(iii) Obviously, $\bar{H} \cong k K_{1} \cup\left[(n-k-2) K_{1} \vee 2 K_{1}\right]$. For $n \geq 3 k+9$, we have

$$
\rho(\bar{H})=\rho\left(K_{2, n-k-2}\right)=\sqrt{2(n-k-2)}<\sqrt{\frac{(n-k)(3 n-3 k-11)}{n}},
$$

as desired.
Lemma 4.2. Let $H \cong K_{3} \vee\left(K_{n-5}+2 K_{1}\right)$.
(i) If $n \geq 9$, then $\rho(H)>1+\sqrt{n^{2}-10 n+38}$.
(ii) If $n \geq 10$, then $q(H)>2 n-8+\frac{28}{n-1}$.
(iii) If $n \geq 17$, then $\rho(\bar{H})<\sqrt{\frac{(n-2)(3 n-17)}{n}}$.

Proof. (i) Let $R(A)$ be an equitable quotient matrix of the adjacency matrix $A(H)$ with respect to the partition $\left(V\left(K_{3}\right), V\left(K_{n-5}\right), V\left(2 K_{1}\right)\right)$. In the proof of Theorem 4.2 [1], we known that the characteristic polynomial of $R(A)$ is $P_{R(A)}(x)=x^{3}-(n-4) x^{2}-(n+3) x+$ $6 n-36$, and $P_{R(A)}(x)$ is a monotonically increasing function on $\left[\frac{n-4+\sqrt{n^{2}-5 n+25}}{3},+\infty\right)$. Note that $\rho(H)=\lambda_{1}(R(A))>\frac{n-4+\sqrt{n^{2}-5 n+25}}{3}$ and

$$
1+\sqrt{n^{2}-10 n+38}>\frac{n-4+\sqrt{n^{2}-5 n+25}}{3}
$$

By Maple, $P_{R(A)}\left(1+\sqrt{n^{2}-10 n+38}\right)<0=P_{R(A)}(\rho(H))$ for $n \geq 9$. This implies that $\rho(H)>1+\sqrt{n^{2}-10 n+38}$.
(ii) Let $R(Q)$ be an equitable quotient matrix of the signless Laplacian matrix $Q(H)$ with respect to the partition $\left(V\left(K_{3}\right), V\left(K_{n-5}\right), V\left(2 K_{1}\right)\right)$. In the proof of Theorem 4.7 [1], the characteristic polynomial of $R(Q)$ is $P_{R(Q)}(x)=x^{3}-(3 n-5) x^{2}+\left(2 n^{2}-n-24\right) x-6 n^{2}+$ $42 n-72$, and $P_{R(Q)}(x)$ is a monotonically increasing function on $\left[\frac{3 n-5+\sqrt{3 n^{2}-27 n+97}}{3},+\infty\right)$. Note that $q(H)>\frac{3 n-5+\sqrt{3 n^{2}-27 n+97}}{3}$ and

$$
2 n-8+\frac{28}{n-1}>\frac{3 n-5+\sqrt{3 n^{2}-27 n+97}}{3}
$$

By a simple calculation, we have $P_{R(Q)}\left(2 n-8+\frac{28}{n-1}\right)<0=P_{R(Q)}(q(H))$ for $n \geq 10$. Hence, $q(H)>2 n-8+\frac{28}{\frac{n-1}{H}}$.
(iii) We have $\bar{H} \cong 3 K_{1} \cup\left[(n-5) K_{1} \vee K_{2}\right]$. Let $R C(A)$ be an equitable quotient matrix of the adjacency matrix $A(\bar{H})$ with respect to the partition $\left(V\left(3 K_{1}\right), V\left((n-5) K_{1}\right), V\left(K_{2}\right)\right)$. One can see that

$$
R C(A)=\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 2 \\
0 & n-5 & 1
\end{array}\right)
$$

Then the characteristic polynomial of $R C(A)$ is given by $P_{R C(A)}(x)=x\left(x^{2}-x-2 n+10\right)$. By a direct calculation, $\rho(\bar{H})=\frac{1+\sqrt{8 n-39}}{2}<\sqrt{\frac{(n-2)(3 n-17)}{n}}$ for $n \geq 17$.

Lemma 4.3. Let $H \cong K_{4} \vee\left(K_{n-7}+3 K_{1}\right)$.
(i) If $n \geq 9$, then $\rho(H)<1+\sqrt{n^{2}-10 n+38}$.
(ii) If $n \geq 9$, then $q(H)<2 n-8+\frac{28}{n-1}$.
(iii) If $n \geq 7$, then $\rho(\bar{H})>\sqrt{\frac{(n-2)(3 n-17)}{n}}$.

Proof. (i) Let $R(A)$ be an equitable quotient matrix of the adjacency matrix $A(H)$ with respect to the partition $\left(V\left(K_{4}\right), V\left(K_{n-7}\right), V\left(3 K_{1}\right)\right)$. One can see that

$$
R(A)=\left(\begin{array}{ccc}
3 & n-7 & 3 \\
4 & n-8 & 0 \\
4 & 0 & 0
\end{array}\right)
$$

Then the characteristic polynomial of $R(A)$ is given by $P_{R(A)}(x)=x^{3}-(n-5) x^{2}-(n+$ 8) $x+12 n-96$. By Lemma 2.5 , we know that $\rho(H)=\lambda_{1}(R(A))$ is the largest root of the equation $P_{R(A)}(x)=0$. Let $P_{R(A)}^{\prime}(x)=3 x^{2}-2(n-5) x-n-8=0$. We can solve this equation to obtain that

$$
x_{1}=\frac{n-5-\sqrt{n^{2}-7 n+49}}{3} \text { and } x_{2}=\frac{n-5+\sqrt{n^{2}-7 n+49}}{3} .
$$

Then $P_{R(A)}(x)$ is a monotonically increasing function on $\left[x_{2},+\infty\right)$. Note that $\rho(H)=$ $\lambda_{1}(R(A))>x_{2}$ and $1+\sqrt{n^{2}-10 n+38}>x_{2}$. By Maple, $P_{R(A)}\left(1+\sqrt{n^{2}-10 n+38}\right)>$ $0=P_{R(A)}(\rho(H))$ for $n \geq 9$. This implies that $\rho(H)<1+\sqrt{n^{2}-10 n+38}$.
(ii) Let $R(Q)$ be an equitable quotient matrix of the signless Laplacian matrix $Q(H)$ with respect to the partition $\left(V\left(K_{4}\right), V\left(K_{n-7}\right), V\left(3 K_{1}\right)\right)$. Then

$$
R(Q)=\left(\begin{array}{ccc}
n+2 & n-7 & 3 \\
4 & 2 n-12 & 0 \\
4 & 0 & 4
\end{array}\right)
$$

Then the characteristic polynomial of $R(Q)$ is given by $P_{R(Q)}(x)=x^{3}-3(n-2) x^{2}+\left(2 n^{2}-\right.$ 48) $x-8 n^{2}+72 n-160$. By Lemma 2.5, we have $q(H)=\lambda_{1}(R(Q))$ is the largest root of the equation $P_{R(Q)}(x)=0$. Let $P_{R(Q)}^{\prime}(x)=3 x^{2}-6(n-2) x+2 n^{2}-48=0$. The two roots $x_{1}$ and $x_{2}$ of this equation are as follows:

$$
x_{1}=\frac{3 n-6-\sqrt{3 n^{2}-36 n+180}}{3} \text { and } x_{2}=\frac{3 n-6+\sqrt{3 n^{2}-36 n+180}}{3}
$$

Then $P_{R(Q)}(x)$ is a monotonically increasing function on $\left[x_{2},+\infty\right)$. Note that $q(H)>x_{2}$ and $2 n-8+\frac{28}{n-1}>x_{2}$. By a simple calculation, we have $P_{R(Q)}\left(2 n-8+\frac{28}{n-1}\right)>0=P_{R(Q)}(q(H))$ for $n \geq 9$. Hence $q(H)<2 n-8+\frac{28}{n-1}$.
(iii) It is easy to see that $\bar{H} \cong 4 K_{1} \cup\left[(n-7) K_{1} \vee K_{3}\right]$. Let $R C(A)$ be an equitable quotient matrix of the adjacency matrix $A(\bar{H})$ with respect to the partition $\left(V\left(4 K_{1}\right), V((n-\right.$ 7) $\left.K_{1}\right), V\left(K_{3}\right)$ ). One can see that

$$
R C(A)=\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 3 \\
0 & n-7 & 2
\end{array}\right)
$$

Then the characteristic polynomial of $R C(A)$ is given by $P_{R C(A)}(x)=x\left(x^{2}-2 x-3 n+21\right)$. By a direct calculation, we have $\rho(\bar{H})=1+\sqrt{3 n-20}>\sqrt{\frac{(n-2)(3 n-17)}{n}}$ for $n \geq 7$.

Ao, Liu, Yuan and Li [1] presented sufficient conditions to guarantee a graph to be $k$ -leaf-connected in terms of the (signless Laplacian) spectral radius of $G$ or its complement.

Theorem 4.1 (Ao, Liu, Yuan and Li [1]). Let $G$ be a connected graph of order $n$ and minimum degree $\delta \geq k+1$, where $2 \leq k \leq n-4$. Then
 $\left\{K_{3} \vee 3 K_{1}, K_{4} \vee 4 K_{1}\right\}$.
(ii) If $q(G) \geq 2 n-6+\frac{4 k+6}{n-1}$, then $G$ is $k$-leaf-connected unless $G \cong K_{4} \vee 4 K_{1}$.
(iii) If $\rho(\bar{G}) \leq \sqrt{\frac{(n-k)(2 n-2 k-5)}{n}}$, then $G$ is $k$-leaf-connected.

In this paper, we improve the above result as follows.
Theorem 4.2. Let $G$ be a connected graph of order $n \geq k+17$ and minimum degree $\delta \geq k+1$, where $k \geq 2$. If one of the following holds,
(i) $\rho(G) \geq \frac{k}{2}+\sqrt{n^{2}-(k+8) n+\frac{k^{2}}{4}+7 k+23}$,
(ii) $q(G) \geq 2 n-8+\frac{6 k+16}{n-1}$,
(iii) $\rho(\bar{G}) \leq \sqrt{\frac{(n-k)(3 n-3 k-11)}{n}}$,
then $G$ is $k$-leaf-connected unless $C_{n+k-1}(G) \in\left\{K_{k} \vee\left(K_{n-k-2}+K_{2}\right), K_{3} \vee\left(K_{n-5}+2 K_{1}\right)\right\}$.

Proof. Suppose, to the contrary, that $G$ is not $k$-leaf-connected.
(i) By Lemma 2.3 and Proposition 2.1, we have

$$
\rho(G) \leq \frac{\delta-1}{2}+\sqrt{2 e(G)-\delta n+\frac{(\delta+1)^{2}}{4}} \leq \frac{k}{2}+\sqrt{2 e(G)-(k+1) n+\frac{k^{2}}{4}+k+1} .
$$

Since $\rho(G) \geq \frac{k}{2}+\sqrt{n^{2}-(k+8) n+\frac{k^{2}}{4}+7 k+23}$, we have $e(G) \geq\binom{ n-3}{2}+3 k+5$. Let $H=C_{n+k-1}(G)$. By Theorem 1.4, we have $H \in\left\{K_{k} \vee\left(K_{n-k-2}+K_{2}\right), K_{3} \vee\left(K_{n-5}+2 K_{1}\right), K_{4} \vee\right.$ $\left.\left(K_{n-7}+3 K_{1}\right)\right\}$. Assume that $H \cong K_{4} \vee\left(K_{n-7}+3 K_{1}\right)$. According to (i) of Lemma 4.3, $\rho(G) \leq$ $\rho(H)<1+\sqrt{n^{2}-10 n+38}$, a contradiction. For $H \in\left\{K_{k} \vee\left(K_{n-k-2}+K_{2}\right), K_{3} \vee\left(K_{n-5}+2 K_{1}\right)\right.$ and $n \geq k+17$, by (i) of Lemmas 4.1 and 4.2 , we can not compare completely $\rho(G)$ with $\frac{k}{2}+\sqrt{n^{2}-(k+8) n+\frac{k^{2}}{4}+7 k+23}$. For the brevity of discussion, we have $C_{n+k-1}(G)=$ $H \in\left\{K_{k} \vee\left(K_{n-k-2}+K_{2}\right), K_{3} \vee\left(K_{n-5}+2 K_{1}\right)\right\}$.
(ii) By Lemma 2.4, we have $q(G) \leq \frac{2 e(G)}{n-1}+n-2$. Note that $q(G) \geq 2 n-8+\frac{6 k+16}{n-1}$. Then $e(G) \geq\binom{ n-3}{2}+3 k+5$. Let $H=C_{n+k-1}(G)$. By Theorem 1.4, we have $H \in\left\{K_{k} \vee\left(K_{n-k-2}+\right.\right.$ $\left.\left.K_{2}\right), K_{3} \vee\left(K_{n-5}+2 K_{1}\right), K_{4} \vee\left(K_{n-7}+3 K_{1}\right)\right\}$. Suppose that $H \cong K_{4} \vee\left(K_{n-7}+3 K_{1}\right)$. By (ii) of Lemma 4.3, $q(G) \leq q(H)<2 n-8+\frac{28}{n-1}$, a contradiction. Therefore, $C_{n+k-1}(G)=H \in$ $\left\{K_{k} \vee\left(K_{n-k-2}+K_{2}\right), K_{3} \vee\left(K_{n-5}+2 K_{1}\right)\right\}$.
(iii) Let $H=C_{n+k-1}(G)$. Similar to the proof of Theorem 4.4 in [1], we can obtain that

$$
\rho(\bar{H}) \geq \sqrt{\frac{(n-k) e(\bar{H})}{n}} .
$$

Note that $\bar{H} \subseteq \bar{G}$. Then we have

$$
\rho(\bar{H}) \leq \rho(\bar{G}) \leq \sqrt{\frac{(n-k)(3 n-3 k-11)}{n}}
$$

and therefore,

$$
\sqrt{\frac{(n-k) e(\bar{H})}{n}} \leq \rho(\bar{H}) \leq \rho(\bar{G}) \leq \sqrt{\frac{(n-k)(3 n-3 k-11)}{n}}
$$

It is easy to check that $e(\bar{H}) \leq 3 n-3 k-11$ and

$$
e(H)=\binom{n}{2}-e(\bar{H}) \geq\binom{ n-3}{2}+3 k+5
$$

Applying Theorem 1.4 on $H$, we have $C_{n+k-1}(H)=H \in\left\{K_{k} \vee\left(K_{n-k-2}+K_{2}\right), K_{3} \vee\left(K_{n-5}+\right.\right.$ $\left.\left.2 K_{1}\right), K_{4} \vee\left(K_{n-7}+3 K_{1}\right)\right\}$. Assume that $H \cong K_{4} \vee\left(K_{n-7}+3 K_{1}\right)$. By (iii) of Lemma 4.3, $\rho(\bar{G}) \geq \rho(\bar{H})>\sqrt{\frac{(n-2)(3 n-17)}{n}}$, a contradiction. Hence $C_{n+k-1}(G)=H \in\left\{K_{k} \vee\left(K_{n-k-2}+\right.\right.$ $\left.\left.K_{2}\right), K_{3} \vee\left(K_{n-5}+2 K_{1}\right)\right\}$. This completes the proof of Theorem 4.2.

## Declaration of competing interest

The authors declare that they have no conflict of interest.

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## References

[1] G.Y. Ao, R.F. Liu, J.J. Yuan, R. Li, Improved sufficient conditions for $k$-leafconnected graphs, Discrete Appl. Math. 314 (2022) 17-30.
[2] J.A. Bondy, V. Chvatal, A method in graph theory, Discrete Math. 15 (1976) 111135.
[3] J.A. Bondy, U.S.R. Murty, Graph Theory, Grad. Texts in Math. vol. 244, Springer, New York, 2008.
[4] A.E. Brouwer, W.H. Haemers, Spectra of Graphs, Springer, Berlin, 2011.
[5] M.Z. Chen, X.D. Zhang, The number of edges, spectral radius and Hamiltonconnectedness of graphs, J. Comb. Optim. 35 (2018) 1104-1127.
[6] P. Erdős, A. Hajnal, Ramsey-type theorems, Discrete Appl. Math. 25 (1989) 37-52.
[7] L.H. Feng, G.H. Yu, On three conjectures involving the signless Laplacian spectral radius of graphs, Publ. Inst. Math. (Beograd) 85 (2009) 35-38.
[8] C.D. Godsil, G. Royle, Algebraic Graph Theory, Grad. Texts in Math. vol. 207, Springer, New York, 2001.
[9] M.A. Gurgel, Y. Wakabayashi, On $k$-leaf-connected graphs, J. Combin. Theory Ser. B 41 (1986) 1-16.
[10] W.H. Haemers, Interlacing eigenvalues and graphs, Linear Algebra Appl. 226-228 (1995) 593-616.
[11] Y. Hong, J.L. Shu, K.F. Fang, A sharp upper bound of the spectral radius of graphs, J. Comb. Theory Ser. B 81 (2001) 177-183.
[12] V. Nikiforov, Some inequalities for the largest eigenvalue of a graph, Comb. Probab. Comput. 11 (2002) 179-189.
[13] O. Ore, Note on Hamilton circuits, Amer. Math. Monthly 67 (1960) 55.
[14] J. Wei, Z.F. You, H.J. Lai, Spectral analogues of Erdős' theorem on Hamiltonconnected graphs, Appl. Math. Comput. 340 (2019) 242-250.
[15] Y. Xu, M.Q. Zhai, B. Wang, An improvement of spectral conditions for Hamilton-connected graphs, Linear Multilinear Algebra DOI: 10.1080/03081087.2021.1946465.
[16] G.D. Yu, Y.Z. Fan, Spectral conditions for a graph to be Hamilton-connected, Appl. Mech. Mater. 336-338 (2013) 2329-2334.
[17] Q.N. Zhou, L.G. Wang, Some sufficient spectral conditions on Hamilton-connected and traceable graphs, Linear Multilinear Algebra 65 (2017) 224-234.
[18] Q.N. Zhou, L.G. Wang, Y. Lu, Sufficient conditions for Hamilton-connected graphs in terms of (signless Laplacian) spectral radius, Linear Algebra Appl. 594 (2020) 205-225.
[19] Q.N. Zhou, L.G. Wang, Y. Lu, Signless Laplacian spectral conditions for Hamiltonconnected graphs with large minimum degree, Linear Algebra Appl. 592 (2020) 48-64.


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