AN IMPROVEMENT TO
A BEREZIN-LI-YAU TYPE INEQUALITY

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This paper is dedicated to Professor Evans M. Harrell

ABSTRACT. In this article we improve a lower bound for $\sum_{j=1}^{k} \beta_j$ (a Berezin-Li-Yau type inequality) that appeared in an earlier paper of Harrell and Yolcu. Here β_j denotes the jth eigenvalue of the Klein Gordon Hamiltonian $H_{0,\Omega} = |p|$ when restricted to a bounded set $\Omega \subset \mathbb{R}^n$. $H_{0,\Omega}$ can also be described as the generator of the Cauchy stochastic process with a killing condition on $\partial\Omega$. To do this, we adapt the proof of Melas, who improved the estimate for the bound of $\sum_{j=1}^{k} \lambda_j$, where λ_j denotes the jth eigenvalue of the Dirichlet Laplacian on a bounded domain in $\mathbb{R}^d$.

1. Introduction

In this article, we consider the pseudodifferential operator $H_{0,\Omega} := \sqrt{-\Delta}$ restricted to an open bounded set Ω in $\mathbb{R}^d$. This operator is sometimes called the fractional Laplacian with power $\frac{1}{2}$ (cf. [2] and [3]). We note that $H_{0,\Omega}$ is the generator of the Cauchy stochastic process with a killing condition on $\partial\Omega$ (cf. [2], [3]). Let β_k denote the kth eigenvalue of $H_{0,\Omega}$ and u_k denote the corresponding normalized eigenfunction. Then the eigenvalues β_j satisfy

$$0 < \beta_1 < \beta_2 \leq \beta_3 \leq \cdots \leq \beta_j \leq \cdots \rightarrow \infty,$$

where each eigenvalue is repeated according to its multiplicity. Throughout this article, $|\Omega|$ denotes the volume of the set Ω. Consider the quadratic form

$$Q(\varphi) = \int_{\Omega} \overline{\varphi}\sqrt{-\Delta}\,\varphi, \qquad \varphi \in C_c^{\infty}(\Omega),$$

where $\sqrt{-\Delta}$ is calculated for $\mathbb{R}^d$. Note that the quadratic form Q is defined on a dense subset $C_c^{\infty}(\Omega)$ of $L^2(\Omega)$. By using the Fubini theorem and the definition of the Fourier transform and its inverse, it is easy to see that Q is positive and symmetric. Since $\Omega \subset \mathbb{R}^d$ is non-empty, bounded and open, the operator $H_{0,\Omega}$ is defined as the Friedrichs extension ([1]) of the quadratic form Q on $L^2(\Omega)$. Notice that $H_{0,\Omega}$ is the unique minimal positive operator extending Q.

To show the analogy between the Dirichlet Laplacian and $H_{0,\Omega}$, we first mention similar results for the Dirichlet Laplacian. Let λ_j be the eigenvalues of the Dirichlet Laplacian on Ω. One such result is the so-called Li-Yau inequality proved by P. Li and S.-T. Yau. In [12], they proved that

$$(1.1) \qquad \sum_{j=1}^{k} \lambda_j \geq \frac{dC_d}{d+2} |\Omega|^{-2/d} k^{1+2/d},$$

where $C_d = 4\pi\Gamma(1+d/2)^{2/d}$.

As mentioned in [11], (1.1) can be obtained by a Legendre transform of an earlier result by Berezin [5]. Hence, instead of calling it the Li-Yau inequality, we prefer the Berezin-Li-Yau inequality.

A.D. Melas improved the bound in the Berezin-Li-Yau inequality (1.1) in [14] and proved that

$$(1.2) \qquad \sum_{j=1}^{k} \lambda_j \geq \frac{dC_d}{d+2} |\Omega|^{-2/d} k^{1+2/d} + M_d k \frac{|\Omega|}{I(\Omega)},$$

where the constant M_d depends only on the dimension. Here $I(\Omega)$ is the moment of inertia, which is defined as $I(\Omega) = \min\limits_{u \in \mathbb{R}^d} \int_{\Omega} |x-u|^2 dx$.

An improvement to the last inequality (1.2) has recently been studied by many authors (cf. [10], [15], [9]). More precisely, in [10], H. Kovařík, S. Vugalter and T. Weidl improved (1.2) when $d = 2$ and assuming geometric properties of the boundary of Ω. Their proof is ingenious but somewhat intricate, and they first state and prove their result in the case of polygons, then in the case of general domains. Moreover, their result has a second term that has the order of k as in the asymptotic behavior of the sum on the left hand side of (1.1):

$$(1.3) \quad \sum_{j=1}^{k} \lambda_j = \frac{dC_d}{d+2} |\Omega|^{-2/d} k^{1+2/d} + \tilde{C}_d \frac{|\partial\Omega|}{|\Omega|^{1+1/d}} k^{1+1/d} + o(k^{1+1/d}) \text{ as } k \to \infty.$$

As stated in [10], the correction term in (1.2) is of larger order than k, which appears in the asymptotics of (1.1).

The Riesz mean of order σ is defined as

$$R_\sigma(z) = \sum_j (z - \lambda_j)_+^\sigma.$$

Another analogous result is given in [15], where T. Weidl found a Berezin type bound for the Riesz mean $R_\sigma(z)$ when $\sigma > 3/2$. The second term in this bound is similar to the second term in the asymptotics of $R_\sigma(z)$, up to a constant. His method utilizes sharp Lieb-Thirring inequalities for operator-valued potentials.

A natural question is how this approach can be adapted to the case of Klein-Gordon operators. This article answers this question and improves the Berezin-Li-Yau type bound in [8]. We follow the basic strategy of [14], with some important differences of detail.

We first state the analogue of the Weyl asymptotic formula and the Berezin-Li-Yau type inequality in the case of Klein-Gordon operators $H_{0,\Omega}$. In [8], E. Harrell and the author provided a different proof for the following asymptotic formula in [6]:

Theorem 1.1 (Analogue of the Weyl asymptotic formula). *As $k \to \infty$,*

(1.4) $$\beta_k \sim \tilde{C}_d |\Omega|^{-1/d} k^{1/d},$$

where $\tilde{C}_d = \sqrt{4\pi}\, \Gamma(1 + d/2)^{1/d}$.

This theorem can be proved by adapting a proof of the Weyl asymptotic formula for the Laplacian.

The analogue of the Berezin-Lie-Yau inequality shown in [8] reads:

Theorem 1.2 (Analogue of the Berezin-Lie-Yau inequality). *The eigenvalues β_k of $H_{0,\Omega}$ satisfy*

(1.5) $$\sum_{j=1}^{k} \beta_k \geq \frac{d\tilde{C}_d}{d+1} |\Omega|^{-1/d} k^{1+1/d}.$$

As in the original Li-Yau paper [12], the main tool used in the proof of this theorem is a generalization of the lemma which is attributed to Hörmander in [12]. This result is also sharp in the sense of the Weyl asymptotic formula as in the case of the Laplacian.

2. Statement and proof of the theorem

The main result of this paper is given below.

Theorem 2.1. *For $k \geq 1$ and the bounded set Ω,*

(2.1) $$\sum_{j=1}^{k} \beta_j \geq \frac{d\tilde{C}_d}{d+1} |\Omega|^{-1/d} k^{1+1/d} + \tilde{M}_d \frac{|\Omega|^{1+1/d}}{I(\Omega)} k^{1-1/d},$$

where $\tilde{C}_d = \sqrt{4\pi}\, \Gamma(1 + d/2)^{1/d}$ and the constant $\tilde{M}_d$ depends only on the dimension d.

Observe that, in (1.2), the power of k in the first term is $1 + 2/d$ while in (2.1) the corresponding power is $1 + 1/d$. This is not surprising because the Klein-Gordon operator can be viewed as the square root of the Laplacian in $\mathbb{R}^d$. Also, the improvement in (1.2) consists of $|\Omega|/I(\Omega)$ and in (2.1) we have $|\Omega|^{1+1/d}/I(\Omega)$. Moreover, the difference between the powers of the k terms on the right hand side of (2.1) is $2/d$ as in (1.2).

It would be useful to compare the result with an expected two-term asymptotic expansion for the sum of the eigenvalues of $H_{0,\Omega}$ such as the formula (1.3). To the knowledge of the author, this has not been done before in the literature. However, R. Bañuelos, T. Kulczyski and B. Siudeja explored a similar result for the partition function $Z_\Omega(t) = \sum_{j=1}^{\infty} e^{-\beta_j t}$. See [4] for more details.

First, we will state and prove the following lemma, which is the crucial step in proving the theorem.

Lemma 2.2. *Let $d \geq 2$ and $\varphi : [0, \infty) \to [0, \infty)$ be a decreasing, absolutely continuous function. Assume that*

(2.2) $$0 \leq -\varphi'(x) \leq m, \quad x > 0.$$

Then,

$$\int_0^\infty x^d \varphi(x)dx \;\geq\; \frac{1}{d+1}\left(d\int_0^\infty x^{d-1}\varphi(x)dx\right)^{1+1/d}\varphi(0)^{-1/d}$$

$$(2.3) \qquad\qquad +\; \frac{\varphi(0)^{2+1/d}}{6m^2(d^2-1)}\left(d\int_0^\infty x^{d-1}\varphi(x)dx\right)^{1-1/d}.$$

Proof. Let us first define

$$(2.4) \qquad\qquad \eta(x) = \frac{1}{\varphi(0)}\,\varphi\left(\frac{\varphi(0)}{m}\,x\right).$$

Then $\eta(0) = 1$ and $0 \leq -\eta'(x) \leq 1$. To ease the notation, define $f(x) := -\eta'(x)$ for $x \geq 0$. Hence, $0 \leq f(x) \leq 1$ for $x > 0$ and $\int_0^\infty f(x)dx = \eta(0) = 1$. Now, define

$$(2.5) \qquad A := \int_0^\infty x^{d-1}\eta(x)dx \qquad \text{and} \qquad B := \int_0^\infty x^d\eta(x)dx.$$

Assume that $B < +\infty$, as otherwise the result is immediate. Thus, we can find a sequence $\{R_j\}$ such that $R_j \to \infty$ and $R_j^{d+1}\eta(R_j) \to 0$ as $j \to \infty$. Then, using integration by parts we get

$$\int_0^\infty x^d f(x)dx = Ad \qquad \text{and} \qquad \int_0^\infty x^{d+1}f(x)dx \leq (d+1)B.$$

By the intermediate value theorem, there exist an $\alpha \geq 0$ such that

$$(2.6) \qquad\qquad \int_\alpha^{\alpha+1} x^{d-1}dx = (Ad)^{1-1/d}$$

and

$$(2.7) \qquad\qquad \int_\alpha^{\alpha+1} x^{d+1}dx \leq \int_0^\infty x^{d+1}f(x)dx \leq (d+1)B.$$

As we shall see later, the key point in the proof of the lemma is the inequality

$$(2.8) \qquad (d-1)x^{d+1} - (d+1)y^2x^{d-1} + 2y^{d+1} \geq 2y^{d-1}(x-y)^2$$

for $y > 0$ and $x \geq 0$. The proof of (2.8) is straightforward. Indeed, first divide both sides by y^{d+1}. Then, by setting $\tau = x/y$ we get the polynomial

$$\begin{aligned}
g(\tau) \;&:=\; (d-1)\tau^{d+1} - (d+1)\tau^{d-1} - 2\tau^2 + 4\tau \\
&=\; \left((\tau-1)^2\tau\right)\left(\sum_{k=0}^{d-3}(2k+4)\tau^k + (d-1)\tau^{d-2}\right).
\end{aligned}$$

An induction on d gives the second equality, which leads to $g(\tau) \geq 0$. Now, integrating (2.8) from α to $\alpha+1$ and using (2.6) and (2.7) we get

$$\begin{aligned}
(d+1)(d-1)B - (d+1)y^2(Ad)^{1-1/d} + 2y^{d+1} \;&\geq\; 2y^{d-1}\int_\alpha^{\alpha+1}(x-y)^2dx \\
&\geq\; 2y^{d-1}\int_{-1/2}^{1/2}s^2ds \\
&=\; \frac{y^{d-1}}{6}.
\end{aligned}$$

Choosing $y = (Ad)^{1/d}$ yields

$$B \geq \frac{1}{d+1}(Ad)^{1+1/d} + \frac{1}{6(d^2-1)}(Ad)^{1-1/d},$$

or, equivalently,

$$\int_0^\infty x^d \eta(x)dx \;\geq\; \frac{1}{d+1}\left(d\int_0^\infty x^{d-1}\eta(x)dx\right)^{1+1/d}$$
$$+\frac{1}{6(d^2-1)}\left(d\int_0^\infty x^{d-1}\eta(x)dx\right)^{1-1/d},$$

which together with (2.4) gives

$$\int_0^\infty x^d\varphi(x)dx \;\geq\; \frac{1}{d+1}\left(d\int_0^\infty x^{d-1}\varphi(x)dx\right)^{1+1/d}\varphi(0)^{-1/d}$$

$$(2.9) \qquad\qquad +\frac{\varphi(0)^{2+1/d}}{6m^2(d^2-1)}\left(d\int_0^\infty x^{d-1}\varphi(x)dx\right)^{1-1/d},$$

concluding the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Let us now prove Theorem 2.1 by using the lemma above.

Proof. Let the Fourier transform of each eigenfunction u_j corresponding to the jth eigenvalue β_j be denoted by

$$\hat{u}_j(\xi) = \frac{1}{(2\pi)^{d/2}}\int_\Omega e^{-ix\cdot\xi}u_j(x)dx.$$

Since the set of eigenfunctions $\{u_j\}_{j=1}^\infty$ is an orthonormal set, the set of $\{\hat{u}_j(\xi)\}_{j=1}^\infty$ also forms an orthonormal set in $\mathbb{R}^d$ by using Plancherel's theorem. Set

$$F(\xi) := \sum_{j=1}^k |\hat{u}_j(\xi)|^2.$$

Now we will use the decreasing radial rearrangement of $F(\xi)$ and the coarea formula to get the condition in the lemma. Let $F^*(\xi) = \varphi(|\xi|)$ be the decreasing radial rearrangement of F. We may assume that φ is absolutely continuous. Let $\mu(t) = |\{F^*(\xi) > t\}| = |\{F(\xi) > t\}|$. Then, $\mu(\varphi(x)) = \omega_d x^d$, where ω_d denotes the volume of the d-dimensional unit ball. By the coarea formula,

$$\mu(t) = \int_t^{|\Omega|/(2\pi)^d}\int_{\{F=x\}}|\nabla F|^{-1}d\sigma_x dx.$$

Then,

$$(2.10) \qquad\qquad -\mu'(\varphi(x)) = \int_{\{F=\varphi(x)\}}|\nabla F|^{-1}d\sigma_{\varphi(x)}.$$

Next we will estimate $|\nabla F|$:

$$\sum_{j=1}^k |\nabla \hat{u}_j(\xi)|^2 \leq \frac{1}{(2\pi)^d}\int_\Omega |ixe^{-ix\cdot\xi}|^2 dx = \frac{I(\Omega)}{(2\pi)^d},$$

where $I(\Omega)$, the moment of inertia, is defined as

$$I(\Omega) = \min_{u \in \mathbb{R}^d} \int_\Omega |x - u|^2 dx.$$

After translation, we may assume that

$$I(\Omega) = \int_\Omega |x|^2 dx.$$

Observe that for every ξ,

$$(2.11) \quad |\nabla F(\xi)| \leq 2 \left(\sum_{j=1}^k |\hat{u}_j(\xi)|^2 \right)^{1/2} \left(\sum_{j=1}^k |\nabla \hat{u}_j(\xi)|^2 \right)^{1/2} \leq 2(2\pi)^{-d} \sqrt{|\Omega| I(\Omega)}.$$

By letting $m := 2(2\pi)^{-d} \sqrt{|\Omega| I(\Omega)}$ and using (2.11) in (2.10), we obtain

$$
\begin{aligned}
-\mu'(\varphi(x)) &\geq m^{-1} \mathrm{Vol}_{n-1}(\{F = \varphi(x)\}) \\
&\geq m^{-1} d\omega_d x^{d-1}.
\end{aligned}
$$

On the other hand, differentiating $\mu(\varphi(x))$ yields $\mu'(\varphi(x))\varphi'(x) = d\omega_d x^{d-1}$. Thus,

$$(2.12) \qquad\qquad\qquad 0 \leq -\varphi'(x) \leq m,$$

which is the required condition in the lemma. Thus, it remains to prove the theorem by using the lemma. Observe that

$$(2.13) \qquad\qquad\qquad \int_{\mathbb{R}^d} F(\xi)d\xi = k.$$

Observe that because the u_j's form an orthonormal set in $L^2(\Omega)$, by Bessel's inequality,

$$(2.14) \qquad\qquad\qquad 0 \leq F(\xi) \leq \frac{|\Omega|}{(2\pi)^d}.$$

Since

$$\beta_j = \langle u_j, H_{0,\Omega} u_j \rangle = \int_{\mathbb{R}^d} |\xi| |\hat{u}_j(\xi)|^2 d\xi,$$

with the definition of F, we have

$$(2.15) \qquad\qquad\qquad \int_{\mathbb{R}^d} |\xi| F(\xi)d\xi = \sum_{j=1}^k \beta_j.$$

Hence,

$$(2.16) \qquad k = \int_{\mathbb{R}^d} F(\xi)d\xi = \int_{\mathbb{R}^d} F^*(\xi)d\xi = d\omega_d \int_0^\infty x^{d-1}\varphi(x)dx$$

and

$$(2.17) \qquad \sum_{j=1}^k \beta_j = \int_{\mathbb{R}^d} |\xi| F(\xi)d\xi = \int_{\mathbb{R}^d} |\xi| F^*(\xi)d\xi = d\omega_d \int_0^\infty x^d \varphi(x)dx.$$

Equations (2.16), (2.17), when combined with Lemma 2.2, yield

$$(2.18) \quad \sum_{j=1}^k \beta_j \geq \frac{d}{d+1}\omega_d^{-1/d}\varphi(0)^{-1/d}k^{1+1/d} + \frac{d}{6m^2(d^2-1)}\omega_d^{1/d}\varphi(0)^{2+1/d}k^{1-1/d}.$$

Define

$$h(t) = \frac{d}{d+1}\omega_d^{-1/d}k^{1+1/d}t^{-1/d} + \frac{Cd}{m^2(d^2-1)}\omega_d^{1/d}k^{1-1/d}t^{2+1/d},$$

where C is a constant to be chosen later. Observe that the function h is decreasing on the interval

$$0 < t \le \left(\frac{m^2(d-1)k^{2/d}}{C(2d+1)\omega_d^{2/d}}\right)^{d/(d+2)}.$$

Let R be the number such that $|\Omega| = \omega_d R^d$. Then,

$$I(\Omega) \ge \int_{B(R)} |x|^2 dx = \frac{d\omega_d R^{d+2}}{d+2},$$

where $B(R)$ is the ball of radius R. Then,

$$
\begin{aligned}
m &= 2(2\pi)^{-d}\sqrt{|\Omega|I(\Omega)} \\
&\ge 2(2\pi)^{-d}\sqrt{\frac{d}{d+2}\omega_d^{-2/d}|\Omega|^{(2d+2)/d}} \\
&\ge (2\pi)^{-d}\omega_d^{-1/d}|\Omega|^{(d+1)/d}.
\end{aligned}
$$

Choosing $C = \min\left\{\frac{1}{6}, \frac{m^2(d-1)k^{2/d}(2\pi)^{d+2}}{(2d+1)\omega_d^{2/d}|\Omega|^{1+2/d}}\right\}$ will guarantee that

$$\left(\frac{m^2(d-1)k^{2/d}}{C(2d+1)\omega_d^{2/d}}\right)^{d/(d+2)} \ge (2\pi)^{-d}|\Omega|.$$

Hence, the function h is decreasing on $\left(0, (2\pi)^{-d}|\Omega|\right]$. Since $0 < \varphi(0) \le (2\pi)^{-d}|\Omega|$, and h is decreasing, we can replace $\varphi(0)$ in (2.18) with $(2\pi)^{-d}|\Omega|$. Therefore, (2.18) and the fact that $\omega_d = \frac{\pi^{d/2}}{\Gamma(1+d/2)}$ result in the following inequality:

(2.19)
$$\sum_{j=1}^{k}\beta_j \ge \frac{\sqrt{4\pi}dk^{1+1/d}}{d+1}\left(\frac{\Gamma(1+d/2)}{|\Omega|}\right)^{1/d} + \frac{Cdk^{1-1/d}}{8\sqrt{\pi}(d^2-1)(\Gamma(1+d/2))^{1/d}}\frac{|\Omega|^{1+1/d}}{I(\Omega)}.$$

Let $\tilde{M}_d := \frac{Cd}{8\sqrt{\pi}(d^2-1)(\Gamma(1+d/2))^{1/d}}$. Then (2.19) can be written as

(2.20)
$$\sum_{j=1}^{k}\beta_j \ge \frac{d\tilde{C}_d}{d+1}|\Omega|^{-1/d}k^{1+1/d} + \tilde{M}_d\frac{|\Omega|^{1+1/d}}{I(\Omega)}k^{1-1/d},$$

where $\tilde{C}_d = \sqrt{4\pi}\Gamma(1+d/2)^{1/d}$. Recall that the first term on the right of (2.20) is the same bound as in [8]. □

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