AN INDEX OF TRACKING FOR LONGITUDINAL DATA

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ABSTRACT

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The concept of tracking is defined for a specified time span as the maintenance of relative (or peer) ranking within the risk factor distribution. Methods to appropriately evaluate tracking have, to date, been inadequate. An index is proposed, to quantify the degree of tracking, based on the estimated growth curve parameters. This index, a generalized U-statistic, is an extension of the usual probability of concordance to the probability of concordant risk factor responses over a fixed time interval. The asymptotic normality of the index is established, for both the identically distributed and non-identically distributed cases, based on Hoeffding (1948). An estimate of the variance is obtained by the method of structural components (Sen, 1960). Methods of modelling the tracking phenomenon are discussed, including one which is based on empirical Bayes procedures applied to a polynomial growth curve model. Monte Carlo methods simulate censored observations, and the index of tracking is examined in the presence of censoring. By progressively truncating the kernel of this generalized U-statistic, the index can be viewed as a Gaussian process in time. By weighted least squares modelling of the progressively truncated statistics, the untruncated parameter value can be predicted. Finally, applications are presented which estimate the index of tracking for two longitudinal studies, the Thousand Aviator Study, and the Bogalusa Heart Study.

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CHAPTER I

INTRODUCTION AND REVIEW OF LITERATURE

1.1 Introduction

Variations over time in risk factors of cardiovascular disease have long been of interest to epidemiologists and clinicians. In particular, those individuals at the extremes of the risk factor distributions are suspect as potential heart and vascular disease candidates. It would be quite useful to identify these individuals as early in life as possible. This is true whether the risk factor is obesity (Garn and Cole, 1980), elevated serum lipid levels, or hypertension. Since investigations of hypertension predominate in the literature, this discussion will focus on the identification of individuals with consistently high blood pressure levels.

1.2 Review of the Epidemiologic Literature

As early as 1934, Jenss examined the relationship between blood pressure and age. Previous cross-sectional studies relied on comparisons of average blood pressures across age groups, a deficiency which lead to the use of repeated measurements in longitudinal studies based on more homogeneous groups. Jenss investigated individual trends in blood pressure by estimating the slope of a linear regression on age for each of the U. S. Army officers followed for a minimum of seven years. She concluded that systolic blood pressure increases at an accelerated rate with age, not at a uniform rate as widely believed at that time.

More specifically, "there is some indication of a decline in average pressure in the late twenties and early thirties. After this age there is a continuous rise in pressure, which occurs first at an accelerated and then at a retarded rate." The variance of blood pressure showed an increase with age, resulting in an almost constant lower limit (95% confidence limit) of systolic blood pressure. A secondary analysis involved the correlation between trend (slope) of systolic blood pressure and initial blood pressure level.

Similarly, Diehl and Hesdorffer (1933) suggested the need for follow-up studies to determine the importance of transient elevations of blood pressure. They studied 155 University of Minnesota students for five to ten years, and concluded that transient elevations in blood pressure indicated a greater risk of hypertension in later years. The increased risk of hypertension was directly related both to the variability and absolute level of blood pressure.

Although blood pressure had been shown to increase with age in numerous previous studies, Harlan et al. (1962) demonstrated that, when the influence of other confounding factors was removed, blood pressure was not significantly affected by age throughout their 18 year investigation. In examining the distribution both systolic and diastolic blood pressure, they expressed an interest in "whether there was a tendency for each individual to maintain his relative standing within the frequency distribution curves throughout the study." Such a tendency was noted, however, the authors suggested that the ranking was also influenced by both environmental and genetic conditions. Groups of individuals at the upper and lower tails of the blood pressure distribution at each examination were contrasted. Evidence of excessive weight gain and family

history of cardiovascular disease were considered. Harlan et al. concluded that the effect of weight gain on blood pressure was strongest among individuals at the extremes of the blood pressure distribution. Harlan et al. were among the earliest to refer to the tendency to maintain relative ranking across serial examinations. This concept will be labelled and more precisely defined later.

The conclusion that blood pressure was not significantly affected by age, when adjusted for confounding factors, was echoed by Miall and Lovell (1967), based on their longitudinal surveys of two Welsch communities. They investigated the relationship between change in blood pressure level and age. Showing an association, they stressed that this was not evidence of a causal relationship. This relationship appeared to be positive, i.e., the rate of increase was directly associated with the attained level of blood pressure. There also appeared to be an age effect on blood pressure, since the majority of changes in blood pressure were positive and increased in magnitude with higher attained levels of blood pressure.

Interest in the tendency to maintain peer ranking of blood pressure through middle age was sustained by Oberman et al. (1967). Their study population consisted of 1056 normotensive aviation cadets and instructors, followed for 24 years. To investigate longitudinal trends, the initial cohort was classified into quintiles, and patterns in succeeding examinations for each quintile were noted. Particularly for the quintiles defined by the 1951 examinations, relative quintile positions were maintained in subsequent examinations, despite the occurrence of regression to the mean. At the time of the 1951 examinations, the majority of the cohort were still in their mid-thirties, implying a

predisposition early in life to a particular relative position in the blood pressure distribution. This study provides one of the applications presented in Chapter IV.

Employing data from the Framingham Heart Study, Feinleib et al. (1969) addressed the problem of regression to the mean. Their results, stratified by age and systolic blood pressure at the third examination, demonstrated that the regression to the mean phenomenon applied irrespective of the time axis, i.e. regression to the mean was evidenced from Exam 3 to Exam 4, and from Exam 3 to Exam 2 as well. Basing the remainder of the analysis on Exam 3 through Exam 8, a positive linear trend in blood pressure on time was highly significant. When this trend was examined within each age cohort, the slopes increased as the age of the cohort increased, implying at least the quadratic model inferred earlier by Jenss. The question of dependence of the trend in blood pressure was redefined as the estimation of the correlation between the estimated slope ($\hat{\beta}_i$) and the estimated true initial level ($\hat{\alpha}_i$). Feinleib et al. correlated estimates of these parameters based on mutually exclusive sets of observations, and concluded that trends in blood pressure were independent of the initial blood pressure level.

In addition to determining the existence of familial aggregation of blood pressure, Zinner et al. (1974) discussed the relation between initial blood pressure scores and the distributional patterns of follow-up scores. Based on blood pressure scores in standard deviation units of 2-14 year old children and their follow-up scores four years later, there was a tendency for blood pressure scores to follow the distributional patterns established by the initial scores. They concluded that "stratification of blood pressure within peer groups begins and is

detectable in childhood." Continued measurements on this study group were suggested as a means of assessing the strength and duration of the tendency to maintain peer rankings, and to identify potential essential hypertensives.

Rosner et al. (1977) introduced the term "tracking correlation", defined as "the correlation coefficient between blood pressure readings on the same person taken at two different times." They reported age and sex specific correlations, both factors were shown to be significant in a weighted two-way analysis of variance. In addition, the correlations showed a step-function rise at age 20, rather than the expected continuous trend. The time interval between measurements also affected the correlation.

Few significant correlations could be found among the very young. Levine et al. (1978) studied infants from the second day of life through one year of age. Small positive correlations were found for both systolic and diastolic blood pressure at six months and one year, reflecting both genetic and environmental influences. Levine et al. concluded that casual infant blood pressure measurements cannot be used to identify potential hypertensive adults.

The Muscatine Study (Clarke et al., 1978) analyzed repeated measurements on 820 school children over a six year period. For purposes of this study, tracking was defined as "the phenomenon of children maintaining their rank within their age-sex groups", then the tracking of several risk factors within this sample was examined. The tendency to maintain rank ordering was dependent upon initial blood pressure quintile, and was particularly strong in the extreme percentiles. For each of the risk factors, the probability of remaining in the upper quintile was

estimated. The probability of a systolic blood pressure reading in the upper quintile for four consecutive measurements (a six year time span) was estimated at 0.17, and the corresponding estimate for diastolic blood pressure was 0.09. Clarke et al. advocated further longitudinal studies of children relating blood pressure to potential cardiovascular disease.

The Chatham Blood Pressure Study (Darby and Fearn, 1979), involving primary school children in Britain, presented evidence to support the hypothesis that blood pressures track. In the process, both population and individual trends in blood pressure were modelled. The authors concluded with an interesting discussion of the resulting variance components. "Total variation in the observations changes little with age, and this accords with the small size of σ^2_β compared with σ^2_α and σ^2_e....Thus there will be little crossing over of the lines fitted to individuals in the age range under observation. Hence the evidence is that mean blood pressures 'track'." Rabkin et al. (1979) investigated changes in blood pressure after entry into a study in relation to subsequent ischemic heart disease status. They stressed the use of repeated measurements at different points in time rather than a single examination, and that measurements prior to the most recent may contribute to risk assessment.

The Bogalusa Heart Study (Voors et al., 1979) focused specifically on whether "tracking" occurred in both systolic and diastolic blood pressure in children originally in the extreme blood pressure deciles. Tracking was defined here as the extent to which a child with relatively high blood pressure levels will presist with relatively high levels. After adjusting for regression to the mean, the average shifts from one

exam to the next in blood pressure levels for these extremes could be quantified. The authors referred to this method as quantifying the tracking, however, they also referred to the intra-child partial correlation coefficient as a measure of tracking. An additional component of the Bogalusa Heart Study, involving infants, provides an application which is presented in Chapter IV.

A review of the epidemiologic and medical literature indicates the evolution of the concept of "tracking". What was originally defined as a "tendency" has become a more refined phenomenon in recent years. Oberman (1967) said that tracking occurs when "individuals maintain their rank in the distribution of systolic blood pressure through middle age." Of course, the evidence of tracking is not solely confined to a specific risk factor, nor to a specific age range. Later definitions, such as Rosner's or Levine's, linked the concept of tracking with product-moment correlations. They defined the tracking correlation as "the correlation coefficient between blood pressure readings on the same person taken at two different times." Perfect, or complete, tracking of a risk factor over a given time span is then defined as the maintenance of relative (or peer) ranking. Although this conceptual definition addresses the issue of a fixed length of time, tracking is also a function of the initial age, or of the starting point of that fixed time span. Relative ranking, and thus tracking, can also be affected by censoring due to losses to follow-up, particularly when the censoring is related to mortality. In another context, tracking may be defined as the ability to predict a future response.

Tracking can be seen as an indication of the ability to more precisely predict a subsequent response given a set of earlier responses.

Such an enhanced prediction ability would imply the capability of early detection of potential heart and vascular disease candidates. For example, from Voors et al. (1979), "the higher the degree of tracking the more likely that primary hypertension begins early in life."

The various definitions of tracking lend themselves to several specific analyses to demonstrate the occurrence of tracking. Feinleib (1969) summarized several methods of analysis of the relationship between change in blood pressure and initial level. After accounting for regression to the mean, the regression coefficients of initial blood pressure on later blood pressure measurements were examined. Other similar techniques involved correlating change in blood pressure with initial level, or correlating the difference between two blood pressure measurements with their average. A final method presented estimated a linear regression of blood pressure on time, then correlated independent estimates of initial value and clope. Levine, Rosner, Zinner and Clarke also employed regression/correlation techniques.

A slightly different technique involved the classification of individuals into blood pressure percentiles, then the tendency to remain in these percentiles, i.e. to retain group rankings (or ORDAC scores, Priore, 1964), was assessed with particular interest in the extreme percentiles. This method of analysis was reported by Oberman, Zinner, Levine and Clarke. One method, unique in this literature review to Clarke, relied on survivorship analysis techniques. Survival, in this context, was defined as remaining in the upper quintile. The probability of remaining in the upper quintile for six years was estimated. Unlike correlation analysis, which estimated the six year correlation based on the initial and final blood pressure measurements, the six year survival

probability was based on those individuals who remained in the upper quintile for all interim measurements, and on the information from those individuals who remained in the upper quintile for less than six years.

1.3 Review of the Statistical Literature

Longitudinal studies involving multiple observations on each subject preclude the usual assumption of independent responses. Elston and Grizzle (1962), Rao (1965), and Grizzle and Allen (1969) all examined a model which does not rely on this assumption. The linear model is expressed as

$$\underset{\sim}{X} = \underset{\sim}{B}\,\underset{\sim}{\zeta}\,\underset{\sim}{A} + \underset{\sim}{E} ,$$

where $\underset{\sim}{X}$ is the matrix of p responses for each of N individuals. $\underset{\sim}{B}$ is the within individual design matrix, and $\underset{\sim}{A}$ is the within group design matrix. Both $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are known matrices of full rank. The columns of $\underset{\sim}{X}$ are assumed to be independently distributed as p-dimensional multivariate normal vectors with a common, unknown covariance matrix. $\underset{\sim}{\zeta}$ is the matrix of parameters to be estimated. The parameter estimates are allowed to vary from group to group, however, the degree of the polynomial remains the same for all groups. $\underset{\sim}{A} = \underset{\sim}{I}(N)$ in the case of estimating a separate polynomial for each individual.

Rao presented a systematic approach to polynomial growth curves employing orthogonal polynomial regression coefficients. This approach also included determining the necessary polynomial degree, and the use of higher order coefficients (or a subset) as covariates. Grizzle and Allen's examples illustrated the selection, by analyses of covariance, of a subset of covariates which may yield improved parameter estimates over those based on the entire set of covariates, and the estimation and hypothesis testing in this growth curve model.

Often longitudinal studies have, among other objectives, the need to predict a future response, or to predict q responses for an individual, given the first (p-q) responses. Lee and Geisser (1972), and Geisser (1970, 1974, 1975) considered this problem, particularly in the context of the growth curve model where prediction is given greater emphasis than estimation. Geisser (1970) discussed the Bayesian justification for Rao's adjusted growth curve parameter estimates, and the problem of prediction of future observations from the generalized growth model. Lee and Geisser referred to the two prediction problems mentioned above as conditional and partial prediction, respectively. In predicting $\underset{\sim}{V} = (\underset{\sim}{V}(1), \underset{\sim}{V}(2))'$, a set of future observations on k individuals, $E(\underset{\sim}{V})$ is estimated from the predictive distribution. The conditional prediction of $\underset{\sim}{V}(2)$ given $\underset{\sim}{V}(1)$ is approximated.

James and Stein (1961) showed that the usual maximum likelihood estimators are inadmissable when the number of parameters being estimated exceeds two. When k parameters, $\theta_1,\ldots,\theta_k$ (k>2), are to be estimated,

$$x_i \text{ given } \theta_i \sim N(\theta_i,1), \quad i = 1,\ldots,k ,$$

and the maximum likelihood estimator $\delta_i(x) = x_i$ has risk function

$$R(\theta,\delta) = E\ L(\theta,\delta(x)) = 1$$

(using squared error loss). The James-Stein estimator of θ_i is

$$\delta_i(x) = (1-(k-2)/(\sum_{j=1}^{k} x_j^2))x_i, \quad i,j = 1,\ldots,k , \qquad (1.3.1)$$

and $R(\theta,\delta) < 1$ for all values of θ. When $\sum_{j=1}^{k} x_j^2 < (k-2)$, the estimator

$$\delta_i^+(x) = \left(1-(\min(1,k-2)/(\sum_{j=1}^{k} x_j^2))\right)x_i, \quad i,j = 1,\ldots,k ,$$

uniformly improves on the James-Stein rule (1.3.1). $\delta_i^+(x)$ is known as

the "positive part" version of the James-Stein estimator.

The concept of exchangeability discussed by Lindley and Smith (1972) was used to address the problem of combining estimation problems (see Efron and Morris, 1973). The dilemma involves combining non-related or possible related estimation problems, or applying James-Stein estimators separately to each estimation problem. The example presented considers combining left- and right-handed batters to estimate their batting averages, or estimate batting averages for the two groups separately. A class of compromise estimators is introduced which would allow the data to dictate the degree to which the groups are combined for estimation. Methods for determining interval estimates for unknown parameters were derived (Morris, 1976), using formal Bayes theory.

Rao (1975) incorporated the empirical Bayes procedures of James and Stein, and of Efron and Morris, to the problem of simultaneous estimation of a vector of parameters in a linear model. Consider k models of the form

$$\underset{\sim}{Y}_i = \underset{\sim\sim}{X}\underset{\sim}{\beta}_i + \underset{\sim}{\varepsilon}_i, \quad i = 1,\ldots,k ,$$

and assume the following expectations, and dispersion matrices,

$$E(\underset{\sim}{\varepsilon}_i|\underset{\sim}{\beta}_i) = \underset{\sim}{0} , \quad D(\underset{\sim}{\varepsilon}_i|\underset{\sim}{\beta}_i) = \sigma^2\underset{\sim}{V}$$

$$E(\underset{\sim}{\beta}_i) = \underset{\sim}{\beta} , \quad D(\underset{\sim}{\beta}_i) = \underset{\sim}{F}$$

$$\operatorname{cov}(\underset{\sim}{\beta}_i,\underset{\sim}{\beta}_j) = \underset{\sim}{0} , \quad i \neq j .$$

These are separate linear models for each of k individuals, where $\underset{\sim}{Y}_i$ is an $(n \times 1)$ vector of responses, e.g., sequential blood pressure determinations for a given individual. $\underset{\sim}{X}$ and $\underset{\sim}{V}$ are known matrices of full rank. Then estimating the unknown parameter vector $\underset{\sim}{\beta}_i$ is equivalent to

minimizing the mean square error,

$$E(p\underset{\sim}{\beta}_i - \underset{\sim}{a}_0 - \underset{\sim}{a}_1'\underset{\sim}{Y}_i)^2 ,$$

for a given value of p. When σ^2, $\underset{\sim}{\beta}$, and $\underset{\sim}{F}$ are known, let $\underset{\sim}{\beta}_i^{(\ell)}$ denote the ordinary least square estimator of $\underset{\sim}{\beta}_i$.

$$\underset{\sim}{\beta}_i^{(\ell)} = (\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X})^{-1}\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{Y}_i .$$

Then the Bayes estimate is

$$\underset{\sim}{\beta}_i^{(b)} = \underset{\sim}{\beta} + \underset{\sim}{F}(\underset{\sim}{F}+\sigma^2(\underset{\sim}{X}'\underset{\sim}{V}\underset{\sim}{X})^{-1})^{-1}(\underset{\sim}{\beta}_i^{(\ell)}-\underset{\sim}{\beta}) ,$$

i.e., a weighted linear combination of the least square estimator and the mean of a prior distribution of $\underset{\sim}{\beta}_i$. This Bayes estimator has several desirable properties. It has minimum mean dispersion error matrix, and the compound loss,

$$E \sum_{i=1}^{k} (\underset{\sim}{p}'\underset{\sim}{\beta}_i - \underset{\sim}{p}'\underset{\sim}{\beta}_i^{(b)})^2,$$

and is minimum compared to any other set of linear estimators.

When σ^2, $\underset{\sim}{\beta}$, and $\underset{\sim}{F}$ are unknown, the empirical Bayes estimator of $\underset{\sim}{\beta}_i$ is given by

$$\underset{\sim}{\beta}_i^{(e)} = \underset{\sim}{\beta}_i^{(\ell)} - c\underset{\sim}{W}\underset{\sim}{U}\underset{\sim}{B}^{-1}(\underset{\sim}{\beta}_i^{(\ell)}-\underset{\sim}{\beta}_*), \quad i = 1,\ldots,k$$

where

$$k\underset{\sim}{\beta}_* = \sum_{i=1}^{k} \underset{\sim}{\beta}_i^{(\ell)} ,$$

$$k(n-m)\sigma_*^2 = \sum_{i=1}^{k} (\underset{\sim}{Y}_i'\underset{\sim}{V}^{-1}\underset{\sim}{Y}_i - \underset{\sim}{Y}_i'\underset{\sim}{V}^{-1}\underset{\sim}{X}\underset{\sim}{\beta}_i^{(\ell)}) = W ,$$

$$(k-1)(\underset{\sim}{F}+\sigma_*^2\underset{\sim}{U}) = \sum_{i=1}^{k} (\underset{\sim}{\beta}_i^{(\ell)}-\underset{\sim}{\beta}_*)(\underset{\sim}{\beta}_i^{(\ell)}-\underset{\sim}{\beta}_*)' = \underset{\sim}{B} ,$$

$$c = (k-m-2)/(kn-km+2).$$

Rao showed that if $\underset{\sim}{\beta}_i$ and $\underset{\sim}{\varepsilon}_i$ have multivariate normal distributions, then W and $\underset{\sim}{B}$ are independently distributed as

$$W \sim \sigma^2 \chi^2_{(kn-km)},$$

$$\underset{\sim}{B} \sim W_m(k-1, F+\sigma^2\underset{\sim}{U}),$$

and the compound loss is

$$E \sum_{i=1}^{k} (\underset{\sim}{\beta}_i^{(e)} - \underset{\sim}{\beta}_i)(\underset{\sim}{\beta}_i^{(e)} - \underset{\sim}{\beta}_i)' = k\sigma^2\underset{\sim}{U} - \frac{\sigma^4 k(n-m)(k-m-2)}{k(n-m)+2} \underset{\sim}{U}(\underset{\sim}{F}+\sigma^2\underset{\sim}{U})^{-1}\underset{\sim}{U} \quad (k \geq m+2).$$

This expected compound loss for the empirical Bayes estimator is smaller than that of the least squares estimator. Recently Rao and Shinozaki (1978) have investigated the bias and mean square error of the James-Stein estimator. The authors urge caution in using these estimators, and they give limits for parameter values which are more precisely estimated by James-Stein estimators than by the usual unbiased estimators.

Blomqvist (1977) presented a special case of growth curves, an example of the relation between initial blood pressure value and change in the presence of random errors in the dependent variable. Let the observed pairs $(x_{i1}, t_1), \ldots, (x_{ik}, t_k)$ from a random sample of size n follow the linear regression:

$$x_{ij} = m_i + b_i t_j + \varepsilon_{ij}, \quad (j=1,\ldots,k,\ i=1,\ldots,n) \ ,$$

or

$$\underset{\sim}{X} = \underset{\sim\sim}{T}\underset{\sim}{\beta}_i + \underset{\sim}{E}_i, \quad (i=1,\ldots,n) \ ,$$

where

$$\underset{\sim}{E}_i \sim N_k(\underset{\sim}{0}, \sigma^2\underset{\sim}{I}) \ ,$$

and $\underset{\sim}{E}_i$ and $\underset{\sim}{\beta}_i$ are independent. Then assume that the true initial value $m_i \sim N(\mu, \sigma_m^2)$ and that the regression of true slope on true initial value

is given by

$$b_i = \beta + \theta(m_i - \mu) + \delta_i, \ (i=1,\ldots,n) \ ,$$

where the δ_i are independent and identically distributed $N(0,\sigma_b^2)$. The regression coefficient, θ, of true slope on true initial value is the primary parameter to be estimated. Blomqvist gives this adjusted estimator as

$$\tilde{\theta} = \frac{(\hat{\theta} + \lambda)}{(1 - \lambda)} \ ,$$

where $\hat{\theta}$ is the usual least squares regression coefficient computed from the estimated (m_i,b_i) and $\lambda = \hat{\sigma}^2/\hat{\sigma}_m^2$. Thus the maximum likelihood estimates of the true values are obtained by adjusting the estimates based on observed values. Davis (1978) proposed an empirical Bayes estimator for θ in the above situation. When λ is known, and under the full Bayes model, this is identical to Blomqvist.

1.4 Definitions and Known Results

A U-statistic will be defined in the sequel which will rely on established results. These theorems and definitions will be summarized here.

Definition 1.4.1 Hoeffding (1948):

Let $\theta = \theta(F)$ be a functional defined for all distribution functions $F \in \mathcal{F}$. θ is estimable if it is possible, given a sufficiently large but finite random sample of observations, to construct an estimator $\phi(x_1,\ldots,x_n)$ which is unbiased for θ for all F, i.e.,

$$\int\ldots\int \phi(x_1,\ldots,x_n)dF(x_1)\ldots dF(x_n) = \theta(F) \ .$$

Definition 1.4.2:

The degree, m, of an estimable parameter is defined to be the smallest sample size for which the parameter has an unbiased estimator.

Definition 1.4.3:

An estimator based on the minimum sample size is called a kernel of $\theta(F)$ if it is symmetric in its m arguments. The requirement of symmetry is not prohibitive since a symmetric kernel of $\theta(F)$ can be constructed

$$\Phi_0(x_1,\ldots,x_m) = \frac{1}{m!} \Sigma\, \Phi(x_{i_1},\ldots,x_{i_m}) ,$$

where the summation is over all permutations $(i_1,\ldots,i_m)$ of $(i,\ldots,m)$, i.e., the average of m! unbiased estimators of θ.

Definition 1.4.4:

The U-statistic for estimating θ is

$$U = \binom{n}{m}^{-1} \Sigma\, \Phi_0(x_{i_1},\ldots,x_{i_m}) ,$$

where the summation is over all $\binom{n}{m}$ combinations $(i_1,\ldots,i_m)$ of m integers $(1,\ldots,m)$. Since

$$E_\theta\{\Phi_0(X_1,\ldots,X_m)\} = \theta ,$$

$$E_\theta\{U(X_1,\ldots,X_n)\} = \int\ldots\int U(x_1,\ldots,x_n)dF_{(x_1)}\cdots dF_{(x_n)} = \theta .$$

So the U-statistic is an unbiased estimator of θ.

Theorem 1.4.1:

Fraser (1957) showed that under certain general conditions, U is the uniformly minimum variance unbiased estimator of θ. U is also a consistent estimator of θ. It is unique when $\mathcal{F}$ is equal to the class of all continuous distribution functions (Lehmann, 1951).

Theorem 1.4.2:

Hoeffding stated that in the case of independent, identically distributed random variables, the existence of

$$E\{\Phi^2(X_1,\ldots,X_m)\}$$

is sufficient for the asymptotic normality of U.

Sen (1960), in proving the structural convergence of the U-statistic, defined the j-th component of $U(X_1,\ldots,X_n)$ by

$$V_j = \binom{n-1}{m-1}^{-1} \Sigma\, \Phi(x_j, X_{\alpha_1},\ldots,X_{\alpha_{m-1}}) ,$$

where the summation extends over $1 \le \alpha_1 < \ldots < \alpha_{m-1} \le n$, and $\alpha_i \neq j$ for all $i = 1,\ldots,m-1$, such that $U_n = \frac{1}{n} \sum_{j=1}^{n} V_j$.

Theorem 1.4.3:

Sen's Proposition 1 proved that

$$S^2 = \frac{1}{n-1} \sum_{j=1}^{n} [V_j - U(X_1,\ldots,X_n)]^2$$

converges in probability to ζ_1, where

$$\Phi_c(x_1,\ldots,x_c) = E_\theta[\Phi(x_1,\ldots,x_c, X_{c+1},\ldots,X_m)] ,$$

$$\psi_c(x_1,\ldots,x_c) = \Phi_c(x_1,\ldots,x_c) - \theta ,$$

and

$$\zeta_c = E_\theta\{\psi_c(X_1,\ldots,X_c)\}^2 .$$

Theorem 1.4.4:

In Proposition 2, Sen proved that $t = \sqrt{n}\ (U-\theta)/ms$ asymptotically has a standard normal distribution.

Since the U-statistic which will be proposed is a function of independent, but not necessarily identically distributed random variables,

similar limit theorems will be relied upon which do not require the assumption of identically distributed random variables. Hoeffding (1948) has shown the asymptotic normality of this U-statistic.

Theorem 1.4.5:

When the distributions, $F_\nu(x)$, of $X_1,\ldots,X_n$ are different, let

$$\theta_{\alpha_1,\ldots,\alpha_m} = E\{\Phi(X_{\alpha_1},\ldots,X_{\alpha_m})\}$$

$$\psi_{c(\alpha_1,\ldots,\alpha_m)\beta_1,\ldots,\beta_{m-c}}(x_1,\ldots,x_c) = E\{\Phi(x_1,\ldots,x_c,X_{\beta_1},\ldots,X_{\beta_{m-c}})\}$$

$$-\theta_{\alpha_1,\ldots,\alpha_m,\beta_1,\ldots,\beta_{m-c}},$$

and

$$\zeta_{c(\alpha_1,\ldots,\alpha_c)\beta_1,\ldots,\beta_{m-c};\gamma_1,\ldots,\gamma_{m-c}} =$$

$$E\{\psi_{c(\alpha_1,\ldots,\alpha_c)\beta_1,\ldots,\beta_{m-c}}\psi_{c(\alpha_1,\ldots,\alpha_c)\gamma_1,\ldots,\gamma_{m-c}}\} .$$

Then there is a number A such that for every n=1,2,... , the conditions for asymptotic normality are:

$$\int\ldots\int\Phi^2(x_1\ldots x_m)dF_{\alpha_1}(x_1)\ldots dF_{\alpha_m}(x_m) < A, \quad (1\le\alpha_1<\ldots<\alpha_m\le n), \tag{1.4.1}$$

$$E|\overline{\psi}_{1(\nu)}^{\,3}(X_\nu)| < \infty, \quad (\nu=1,2,\ldots,n), \tag{1.4.2}$$

$$\lim_{n\to\infty}\sum_{\nu=1}^{n} E|\overline{\psi}_{1(\nu)}^{\,3}(X_\nu)| \Big/ \{\sum_{\nu=1}^{n} E[\overline{\psi}_{1(\nu)}^{\,2}(X_\nu)]\}^{3/2} = 0 . \tag{1.4.3}$$

Theorem 1.4.6:

Hoeffding presents a second set of conditions which, although more restrictive, are easier to apply in establishing asymptotic normality. (1.4.1), (1.4.2), and (1.4.3) can be replaced by

$$\int\cdots\int|\Phi^{3}_{(x_1,\ldots,x_m)}|dF_{\alpha_1}(x_m) < C,\ \alpha_i=1,2,\ldots,\ i=1,\ldots,m\ , \tag{1.4.4}$$

and

$$\zeta_{1(\nu)\alpha_1,\ldots,\alpha_m;\beta_1,\ldots,\beta_{m-1}} > D,\ \text{for}\ \ C>0,\ D>0, \tag{1.4.5}$$

and

$$1 \le \alpha_1<\ldots<\alpha_{m-1},\ 1 \le \beta_1<\ldots<\beta_{m-1}\ .$$

Theorem 1.4.7:

DeLong (1979) has shown for the two sample generalized U-statistic that the process

$$U^{*}_{n}(t) = \sqrt{n}\ \ (U_n(T) - \theta(T))\quad 0<t<1 \tag{1.4.6}$$

is tight in D[0,1] with respect to the uniform metric, given the following conditions:

1) F and G are continuous distribution functions.

2) $\lim_{N\to\infty} \frac{m}{N} = \lambda,\quad 0<\lambda<1$ (1.4.7)

3) $\phi_T(X_1,\ldots,X_{k1},Y_1,\ldots,Y_{k2})$ can be written

$$\phi_T = \phi_{1,T} - \phi_{2,T} \tag{1.4.8}$$

where

$$E\phi_{i,T} = \theta_i(T),\quad i=1,2 \tag{1.4.9}$$

and with probability 1, ϕ_{iT} is increasing in T for i=1,2.

4) There exists a strictly increasing continuous function H on [0,1] such that

$$\mu_2(S,T) \le H(t) - H(s)\ , \tag{1.4.10}$$

$$\mu_4(S,T) \le H(t) - H(s)\ , \tag{1.4.11}$$

$$\theta_i(T) - \theta_i(s) \le H(t) - H(s)\ . \tag{1.4.12}$$

CHAPTER II

AN INDEX OF TRACKING

2.1 Introduction

Condensing the epidemiologic literature to date leads to the formulation of the concept of tracking. Within this chapter, tracking will be explicitly defined, and an estimable parameter will be proposed as an index of tracking. An estimator of this index will be developed based on established theory. Finally, the properties, central moments and limiting distribution of the estimator will be derived in both the independent and identically distributed, and non-identically distributed cases.

2.2 Definition of Tracking

A literature review indicates the evolution of an epidemiologic concept, tracking. This phenomenon has become more precisely defined in recent years. According to Oberman (1967), tracking occurs when "individuals maintain their rank in the distribution of systolic blood pressure through middle age." Though Oberman's definition is specific to his purpose, in general, evidence of tracking is not exclusive to a specific risk factor, or to a specific age range.

Later tracking definitions, such as Rosner's (1977) or Levine's (1978), linked the concept of tracking with product-moment correlations. They defined tracking correlation as "the correlation coefficient between blood pressure readings on the same person taken at two different times."

Reliance on correlation coefficients neglects the longitudinal nature of the data, by ignoring intervening observations. A definition of tracking must overcome such a deficiency by incorporating all available data.

Definition 2.2.1:

Perfect, or complete, tracking of a specific risk factor is the maintenance of relative or peer ranking over a given time span.

This conceptual definition will form the basis for the sequel. Although this definition addresses the issue of a fixed length of time, tracking is also a function of the initial age, or the age at the beginning of follow-up. Relative ranking, and thus tracking, can also be affected by censored data due to losses to follow-up, particularly when the losses are related to mortality. This situation will be discussed in detail in a later section.

Intuitively, perfect tracking implies that responses from two randomly selected individuals would be concordant throughout the time span. This is the ultimate extension of the percentile ranking technique of Clarke and others, where individuals are ranked separately. The remainder of this chapter will be devoted to translating this conceptual definition of tracking to a meaningful mathematical definition.

2.3 Definition of the Index of Tracking

In longitudinal studies, serial measurements on n individuals are recorded at times 1,2,...,k. The resulting k-dimensional vectors lend themselves to quantification of the tracking exhibited.

Let $\underset{\sim}{Y}_1, \underset{\sim}{Y}_2, \ldots, \underset{\sim}{Y}_n$ ($\underset{\sim}{Y} = (Y^{(1)}, Y^{(2)}, \ldots, Y^{(k)})$) be n independent and identically distributed random vectors, having a cumulative distribution function (c.d.f.) $F(\underset{\sim}{Y})$. Then the concept of tracking may be described

as a group of individual curves which will not intersect if perfect tracking occurs, as in Figure 2.3.1. For the remainder of this discussion assume the true curve for the i-th individual has the functional form $f(t,\underset{\sim}{\beta}_i)$, which can be approximated by a polynomial in time, then define the index of tracking as follows.

Definition 2.3.1:

If $f(t,\underset{\sim}{\beta}_i)$ and $f(t,\underset{\sim}{\beta}_j)$ are the curves of two randomly chosen individuals, an index of tracking in the interval $[T_1,T_2]$ is:

$$\gamma(T_1,T_2) = P\{f(t,\underset{\sim}{\beta}_i) \geq f(t,\underset{\sim}{\beta}_j) \quad \text{for all } t \in [T_1,T_2]$$

$$\text{or} \quad f(t,\underset{\sim}{\beta}_i) \leq f(t,\underset{\sim}{\beta}_j) \quad \text{for all } t \in [T_1,T_2]\} \ ,$$

where $\underset{\sim}{\beta}_i$ is the vector of functional parameters specific to the i-th individual.

Obviously, if all the curves in the population are parallel, then $\gamma(T_1,T_2) = 1$, as in Figure 2.3.1. However, $\gamma(T_1,T_2) = 1$ does not imply that the curves are parallel, since they may intersect outside the interval $[T_1,T_2]$, recall the concept of maintenance of relative rank. Note that for an extended interval, i.e., $T_3>T_2$,

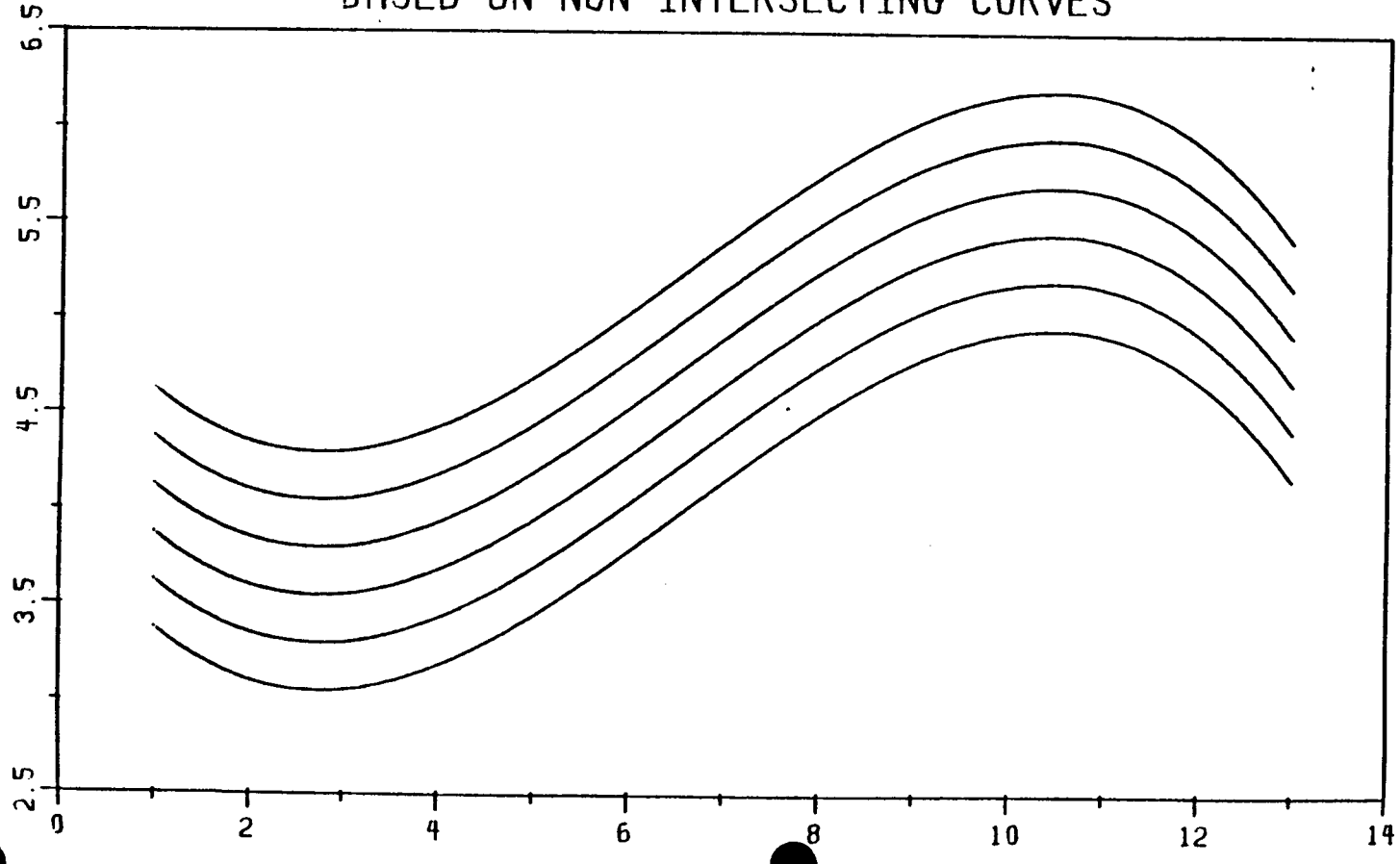
$$\gamma(T_1,T_3) \leq \gamma(T_1,T_2) \tag{2.3.1}$$

If none of the curves in the population are parallel (in a pairwise sense), then $\gamma(T_1,\infty) = 0$. No tracking occurs when $\gamma(T_1,T_2) \leq 1/2$. When the probability of two randomly chosen curves $f(t,\underset{\sim}{\beta}_i)$ crossing is at least 1/2 within the time span, the information on relative rank at any one point in time has no bearing on relative rank at a later time, thus no tracking occurs.

FIGURE 2.3.1

EXAMPLE OF PERFECT TRACKING

BASED ON NON-INTERSECTING CURVES



The probability that a pair of vectors drawn at random from the population is concordant is γ. Since $\gamma(F)$ is an estimable parameter of degree 2, let $\phi(\underset{\sim}{Y}_i, \underset{\sim}{Y}_j)$ be an unbiased estimator of γ which is symmetric in its arguments. In practice, $f(t, \underset{\sim}{\beta}_i)$ is not observed, but a series of observations y_{ij} which are assumed to be of the form

$$y_{it} = f(t, \underset{\sim}{\beta}_i) + e_{it} ,$$

where $E(e_{it} | \underset{\sim}{\beta}_i) = 0$, $Var(e_{it} | \underset{\sim}{\beta}_i) = \sigma_e^2$, $Cov(e_{it}, e_{it'} | \underset{\sim}{\beta}_i) = \sigma_{tt'}$. If the functional form is known, standard statistical methods can be used to estimate $\underset{\sim}{\beta}_i$. Then define the following U-statistic for describing the tracking exhibited in the sample.

Definition 2.3.2:

The symmetric kernel of degree 2 is:

$$\phi(\underset{\sim}{Y}_i, \underset{\sim}{Y}_j) = \begin{cases} 1 \text{ if } f(t, \hat{\underset{\sim}{\beta}}_i) \geq f(t, \hat{\underset{\sim}{\beta}}_j) \text{ for all } t \in [T_1, T_2] \\ \quad \text{or } f(t, \hat{\underset{\sim}{\beta}}_i) \leq f(t, \hat{\underset{\sim}{\beta}}_j) \text{ for all } t \in [T_1, T_2] \\ 0 \text{ otherwise} \end{cases} \qquad (2.3.2)$$

Clearly, $E_F\{\phi(\underset{\sim}{Y}_i, \underset{\sim}{Y}_j)\} = \gamma(F) = \gamma$ for all F. The corresponding U-statistic is:

$$\tilde{\gamma} = U_n = U(\underset{\sim}{Y}_1, \ldots, \underset{\sim}{Y}_n) = \binom{n}{2}^{-1} \sum_{1 \leq \alpha_1 < \alpha_2 \leq n} \phi(\underset{\sim}{Y}_{\alpha_1}, \underset{\sim}{Y}_{\alpha_2}) , \qquad (2.3.3)$$

which is an unbiased estimator of $\gamma(F)$.

For example, assume that individual responses follow a linear regression on time, n linear models:

$$\underset{\sim}{Y}_i = \underset{\sim\sim}{X} \underset{\sim}{\beta}_i + \underset{\sim}{\varepsilon}_i ,$$

with the assumptions on the expectations and dispersion matrices:

$$E(\underset{\sim}{\beta}_i) = \underset{\sim}{\beta}, \quad D(\underset{\sim}{\beta}_i) = \underset{\sim}{F} ,$$

$$E(\underset{\sim}{\varepsilon}_i | \underset{\sim}{\beta}_i) = \underset{\sim}{0}, \quad D(\underset{\sim}{\varepsilon}_i | \underset{\sim}{\beta}_i) = \sigma^2_{\varepsilon}\underset{\sim}{V}, \quad \text{cov}(\underset{\sim}{\varepsilon}_i, \underset{\sim}{\varepsilon}_j) = 0 .$$

Further assume that σ^2_α, σ^2_β, and σ^2_ε are known and $\underset{\sim}{V}=\underset{\sim}{I}$, also that α_i and $\underset{\sim}{\beta}_i$ are independent, $\underset{\sim}{F}$ diagonal. Then $\gamma(F)$ is equivalent to the probability of concordance for the endpoint responses, by definition. Since Kendall's Tau is defined as

$$\tau = P - Q ,$$

where P and Q are the probabilities of concordance and discordance, respectively, the corresponding sample values are

$$t = p - q .$$

Given that $p + q = 1$,

$$t = p - (1-p) = 2p - 1 . \qquad (2.3.4)$$

Kendall (1962) has shown that for samples from a normal population,

$$E(t) = \frac{2}{\pi} \sin^{-1}\rho \qquad (2.3.5)$$

then from (2.3.4) and (2.3.5),

$$E(p) = \frac{1}{2} + \frac{1}{\pi} \sin^{-1}\rho \qquad (2.3.6)$$

or equivalently,

$$E(\gamma(F)) = \frac{1}{2} + \frac{1}{\pi} \sin^{-1}\rho . \qquad (2.3.7)$$

Let $\underset{\sim}{X} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $\underset{\sim}{F} = \begin{bmatrix} \sigma^2_\alpha & 0 \\ 0 & \sigma^2_\beta \end{bmatrix}$ in the model as stated above, then the correlation coefficient of the endpoint responses is

$$\rho = (\sigma^2_\alpha - \sigma^2_\beta)/(\sigma^2_\alpha + \sigma^2_\beta + \sigma^2_\varepsilon) . \qquad (2.3.8)$$

The expected value of interest results from substituting (2.3.8) in (2.3.7),

$$E(\gamma(F)) = \frac{1}{2} + \frac{1}{\pi} \sin^{-1} \left(\frac{\sigma_\alpha^2 - \sigma_\beta^2}{\sigma_\alpha^2 + \sigma_\beta^2 + \sigma_\varepsilon^2} \right) . \qquad (2.3.9)$$

For specified values of σ_α^2, σ_β^2, and σ_ε^2, assuming $\underset{\sim}{V}=\underset{\sim}{I}$, the expected value of $\gamma(F)$ is given in Table 2.3.1, and plotted in Figure 2.3.2. Note that $\rho=1$ if and only if $\sigma_\beta^2=\sigma_\varepsilon^2=0$, thus $\gamma=1$ if and only if $\sigma_\beta^2=\sigma_\varepsilon^2=0$, since both σ_β^2 and σ_ε^2 must be non-negative. This implies that perfect tracking occurs when the intercept is random, and the slope is constant and there is no variability in measurement. Similarly, $\rho=0$ implies $\gamma=1/2$, which occurs under two conditions, 1) σ_ε^2 is large; or 2) $\sigma_\alpha^2=\sigma_\beta^2$. So that even in the case where tracking occurs, if σ_ε^2 is large, the data will not support the hypothesis that tracking occurs, H_o: $\gamma>1/2$. In addition, considering the case where $\sigma_\beta^2>\sigma_\alpha^2$, when σ_ε^2 is large, γ will increase to 1/2 which also indicates that values $\gamma\leq 1/2$ imply no tracking. The second condition, $\sigma_\alpha^2=\sigma_\beta^2$, implies that when the variability is equal among intercepts and among slopes, no tracking occurs, $\gamma=1/2$. Note that the values of γ, $0.5\leq\gamma\leq 1.0$, correspond to non-negative correlations. These values of γ correspond to Rosner's tracking correlations.

Following the same derivation that lead to (2.3.8), when $\underset{\sim}{F}$ is not diagonal, $\sigma_{\alpha\beta}\neq 0$, then the correlation coefficient of the responses at the endpoints of the time span of a linear model is

$$\rho = \frac{\sigma_\alpha^2 - \sigma_\beta^2}{\sqrt{\sigma_\alpha^2+\sigma_\beta^2+\sigma_\varepsilon^2-2\sigma_{\alpha\beta}}\ \sqrt{\sigma_\alpha^2+\sigma_\beta^2+\sigma_\varepsilon^2+2\sigma_{\alpha\beta}}} ,$$

again $\gamma(F) = 1/2 + 1/\pi \sin^{-1}\rho$. By specifying additional values for $\sigma_{\alpha\beta}$, the population values of $\gamma(F)$ are given in Table 2.3.2.

TABLE 2.3.1

INDEX OF TRACKING FOR LINEAR MODELS

ASSUMING NORMALITY AND DIAGONAL COVARIANCE STRUCTURE

RHO = (SA - SB)/(SA + SB + SE)
GAMMA = .5 + INV(PHI)*ARCSIN(RHO)

SA	SB	SE	RHO	GAMMA	SA	SB	SE	RHO	GAMMA	SA	SB	SE	RHO	GAMMA
0	0	0	.	.	2	0	0	1.0000	1.00000	4	0	0	1.0000	1.00000
0	0	1	0.0000	0.50000	2	0	1	0.6667	0.73228	4	0	1	0.8000	0.79517
0	0	2	0.0000	0.50000	2	0	2	0.5000	0.66667	4	0	2	0.6667	0.73228
0	0	3	0.0000	0.50000	2	0	3	0.4000	0.63099	4	0	3	0.5714	0.69361
0	0	4	0.0000	0.50000	2	0	4	0.3333	0.60817	4	0	4	0.5000	0.66667
0	0	5	0.0000	0.50000	2	0	5	0.2857	0.59223	4	0	5	0.4444	0.64660
0	1	0	-1.0000	0.00000	2	1	0	0.3333	0.60817	4	1	0	0.6000	0.70483
0	1	1	-0.5000	0.33333	2	1	1	0.2500	0.58043	4	1	1	0.5000	0.66667
0	1	2	-0.3333	0.39183	2	1	2	0.2000	0.56409	4	1	2	0.4286	0.64098
0	1	3	-0.2500	0.41957	2	1	3	0.1667	0.55330	4	1	3	0.3750	0.62236
0	1	4	-0.2000	0.43591	2	1	4	0.1429	0.54563	4	1	4	0.3333	0.60817
0	1	5	-0.1667	0.44670	2	1	5	0.1250	0.53989	4	1	5	0.3000	0.59699
0	2	0	-1.0000	0.00000	2	2	0	0.0000	0.50000	4	2	0	0.3333	0.60817
0	2	1	-0.6667	0.26772	2	2	1	0.0000	0.50000	4	2	1	0.2857	0.59223
0	2	2	-0.5000	0.33333	2	2	2	0.0000	0.50000	4	2	2	0.2500	0.58043
0	2	3	-0.4000	0.36901	2	2	3	0.0000	0.50000	4	2	3	0.2222	0.57133
0	2	4	-0.3333	0.39183	2	2	4	0.0000	0.50000	4	2	4	0.2000	0.56409
0	2	5	-0.2857	0.40777	2	2	5	0.0000	0.50000	4	2	5	0.1818	0.55820
0	3	0	-1.0000	0.00000	2	3	0	-0.2000	0.43591	4	3	0	0.1429	0.54563
0	3	1	-0.7500	0.23005	2	3	1	-0.1667	0.44670	4	3	1	0.1250	0.53989
0	3	2	-0.6000	0.29517	2	3	2	-0.1429	0.45437	4	3	2	0.1111	0.53544
0	3	3	-0.5000	0.33333	2	3	3	-0.1250	0.46011	4	3	3	0.1000	0.53188
0	3	4	-0.4286	0.35902	2	3	4	-0.1111	0.46456	4	3	4	0.0909	0.52898
0	3	5	-0.3750	0.37764	2	3	5	-0.1000	0.46812	4	3	5	0.0833	0.52656
0	4	0	-1.0000	0.00000	2	4	0	-0.3333	0.39183	4	4	0	0.0000	0.50000
0	4	1	-0.8000	0.20483	2	4	1	-0.2857	0.40777	4	4	1	0.0000	0.50000
0	4	2	-0.6667	0.26772	2	4	2	-0.2500	0.41957	4	4	2	0.0000	0.50000
0	4	3	-0.5714	0.30639	2	4	3	-0.2222	0.42867	4	4	3	0.0000	0.50000
0	4	4	-0.5000	0.33333	2	4	4	-0.2000	0.43591	4	4	4	0.0000	0.50000
0	4	5	-0.4444	0.35340	2	4	5	-0.1818	0.44180	4	4	5	0.0000	0.50000
0	5	0	-1.0000	0.00000	2	5	0	-0.4286	0.35902	4	5	0	-0.1111	0.46456
0	5	1	-0.8333	0.18643	2	5	1	-0.3750	0.37764	4	5	1	-0.1000	0.46812
0	5	2	-0.7143	0.24675	2	5	2	-0.3333	0.39183	4	5	2	-0.0909	0.47102
0	5	3	-0.6250	0.28510	2	5	3	-0.3000	0.40301	4	5	3	-0.0833	0.47344
0	5	4	-0.5556	0.31251	2	5	4	-0.2727	0.41207	4	5	4	-0.0769	0.47549
0	5	5	-0.5000	0.33333	2	5	5	-0.2500	0.41957	4	5	5	-0.0714	0.47724
1	0	0	1.0000	1.00000	3	0	0	1.0000	1.00000	5	0	0	1.0000	1.00000
1	0	1	0.5000	0.66667	3	0	1	0.7500	0.76995	5	0	1	0.8333	0.81357
1	0	2	0.3333	0.60817	3	0	2	0.6000	0.70483	5	0	2	0.7143	0.75325
1	0	3	0.2500	0.58043	3	0	3	0.5000	0.66667	5	0	3	0.6250	0.71490
1	0	4	0.2000	0.56409	3	0	4	0.4286	0.64098	5	0	4	0.5556	0.68749
1	0	5	0.1667	0.55330	3	0	5	0.3750	0.62236	5	0	5	0.5000	0.66667
1	1	0	0.0000	0.50000	3	1	0	0.5000	0.66667	5	1	0	0.6667	0.73228
1	1	1	0.0000	0.50000	3	1	1	0.4000	0.63099	5	1	1	0.5714	0.69361
1	1	2	0.0000	0.50000	3	1	2	0.3333	0.60817	5	1	2	0.5000	0.66667
1	1	3	0.0000	0.50000	3	1	3	0.2857	0.59223	5	1	3	0.4444	0.64660
1	1	4	0.0000	0.50000	3	1	4	0.2500	0.58043	5	1	4	0.4000	0.63099
1	1	5	0.0000	0.50000	3	1	5	0.2222	0.57133	5	1	5	0.3636	0.61846
1	2	0	-0.3333	0.39183	3	2	0	0.2000	0.56409	5	2	0	0.4286	0.64098
1	2	1	-0.2500	0.41957	3	2	1	0.1667	0.55330	5	2	1	0.3750	0.62236
1	2	2	-0.2000	0.43591	3	2	2	0.1429	0.54563	5	2	2	0.3333	0.60817
1	2	3	-0.1667	0.44670	3	2	3	0.1250	0.53989	5	2	3	0.3000	0.59699
1	2	4	-0.1429	0.45437	3	2	4	0.1111	0.53544	5	2	4	0.2727	0.58793
1	2	5	-0.1250	0.46011	3	2	5	0.1000	0.53188	5	2	5	0.2500	0.58043
1	3	0	-0.5000	0.33333	3	3	0	0.0000	0.50000	5	3	0	0.2500	0.58043
1	3	1	-0.4000	0.36901	3	3	1	0.0000	0.50000	5	3	1	0.2222	0.57133
1	3	2	-0.3333	0.39183	3	3	2	0.0000	0.50000	5	3	2	0.2000	0.56409
1	3	3	-0.2857	0.40777	3	3	3	0.0000	0.50000	5	3	3	0.1818	0.55820
1	3	4	-0.2500	0.41957	3	3	4	0.0000	0.50000	5	3	4	0.1667	0.55330
1	3	5	-0.2222	0.42867	3	3	5	0.0000	0.50000	5	3	5	0.1538	0.54917
1	4	0	-0.6000	0.29517	3	4	0	-0.1429	0.45437	5	4	0	0.1111	0.53544
1	4	1	-0.5000	0.33333	3	4	1	-0.1250	0.46011	5	4	1	0.1000	0.53188
1	4	2	-0.4286	0.35902	3	4	2	-0.1111	0.46456	5	4	2	0.0909	0.52898
1	4	3	-0.3750	0.37764	3	4	3	-0.1000	0.46812	5	4	3	0.0833	0.52656
1	4	4	-0.3333	0.39183	3	4	4	-0.0909	0.47102	5	4	4	0.0769	0.52451
1	4	5	-0.3000	0.40301	3	4	5	-0.0833	0.47344	5	4	5	0.0714	0.52276
1	5	0	-0.6667	0.26772	3	5	0	-0.2500	0.41957	5	5	0	0.0000	0.50000
1	5	1	-0.5714	0.30639	3	5	1	-0.2222	0.42867	5	5	1	0.0000	0.50000
1	5	2	-0.5000	0.33333	3	5	2	-0.2000	0.43591	5	5	2	0.0000	0.50000
1	5	3	-0.4444	0.35340	3	5	3	-0.1818	0.44180	5	5	3	0.0000	0.50000
1	5	4	-0.4000	0.36901	3	5	4	-0.1667	0.44670	5	5	4	0.0000	0.50000
1	5	5	-0.3636	0.38154	3	5	5	-0.1538	0.45083	5	5	5	0.0000	0.50000

FIGURE 2.3.2
INDEX OF TRACKING FOR LINEAR MODELS
ASSUME NORMALITY AND DIAGONAL COVARIANCE $_{SE=0}$

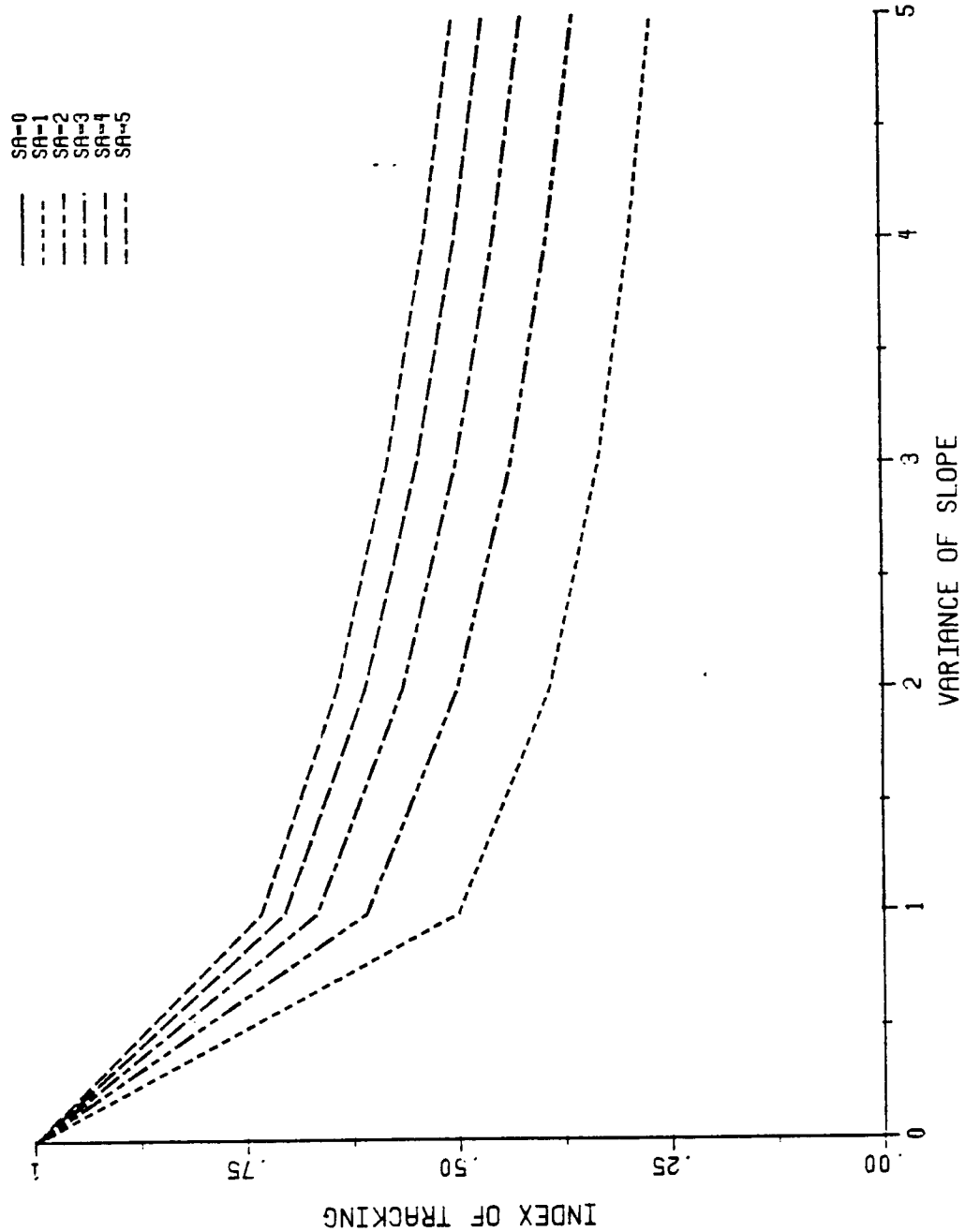


FIGURE 2.3.2
INDEX OF TRACKING FOR LINEAR MODELS
ASSUME NORMALITY AND DIAGONAL COVARIANCE
SE=1

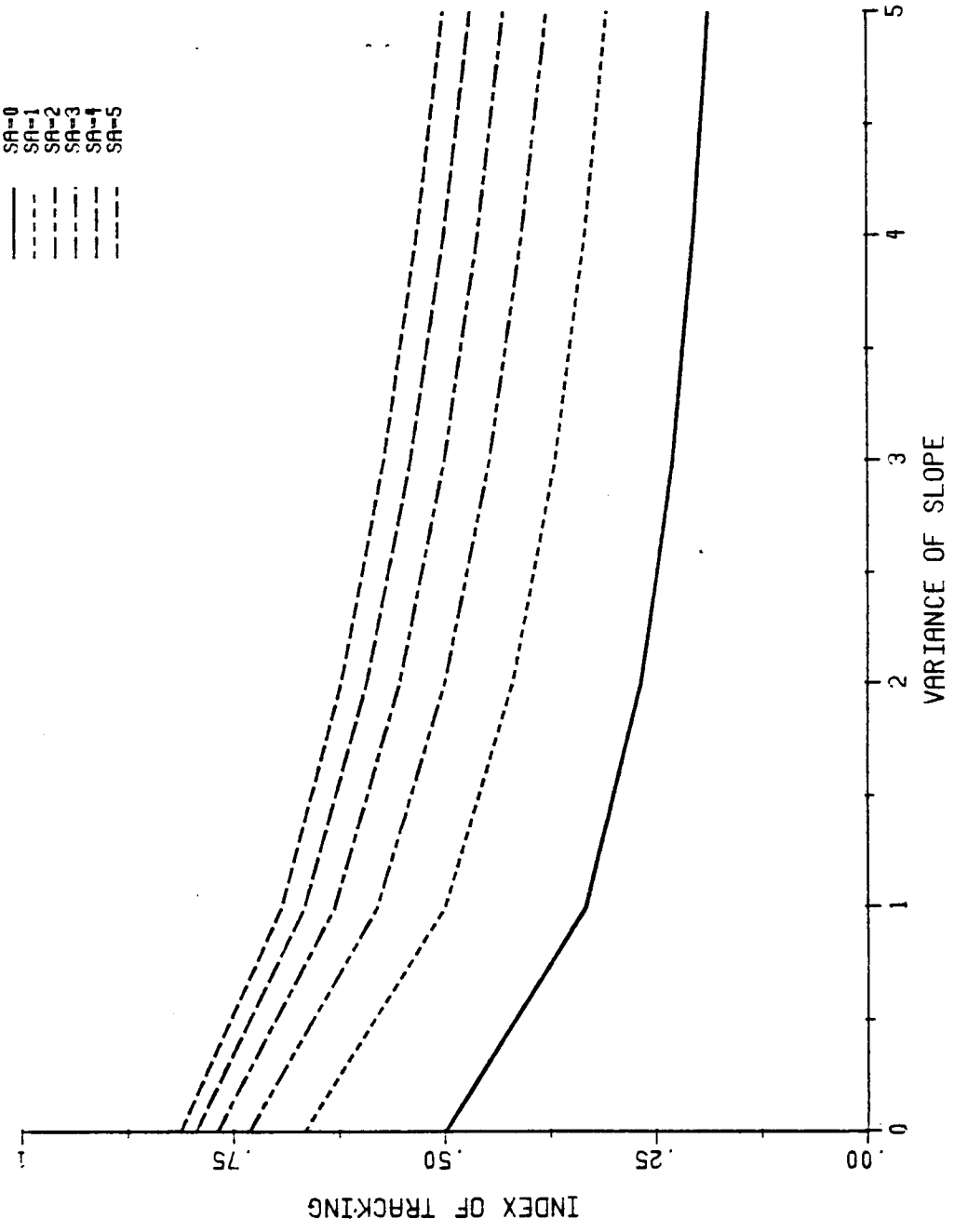


FIGURE 2.3.2
INDEX OF TRACKING FOR LINEAR MODELS
ASSUME NORMALITY AND DIAGONAL COVARIANCE
σ_E^2

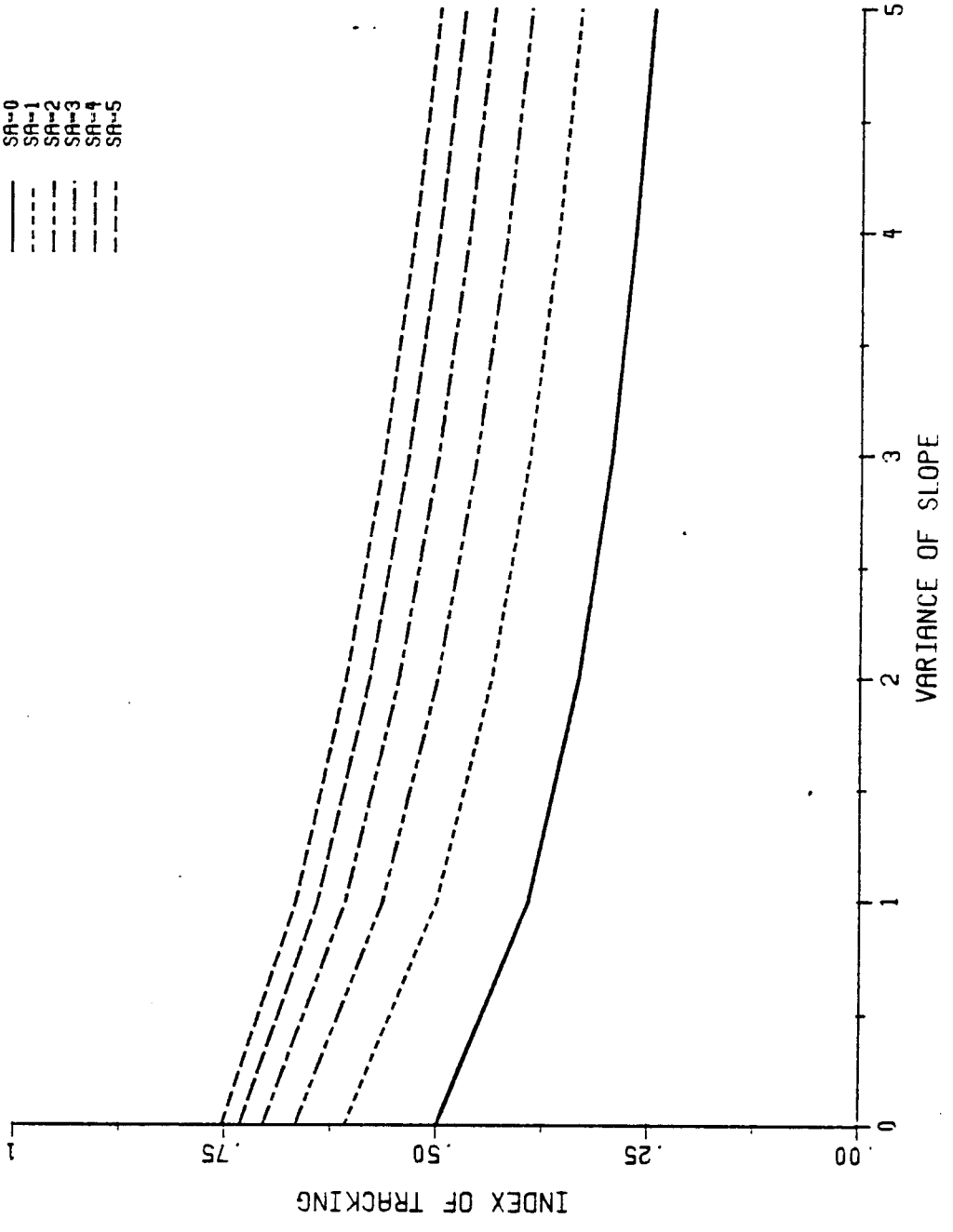


FIGURE 2.3.2
INDEX OF TRACKING FOR LINEAR MODELS
ASSUME NORMALITY AND DIAGONAL COVARIANCE
$SE=3$

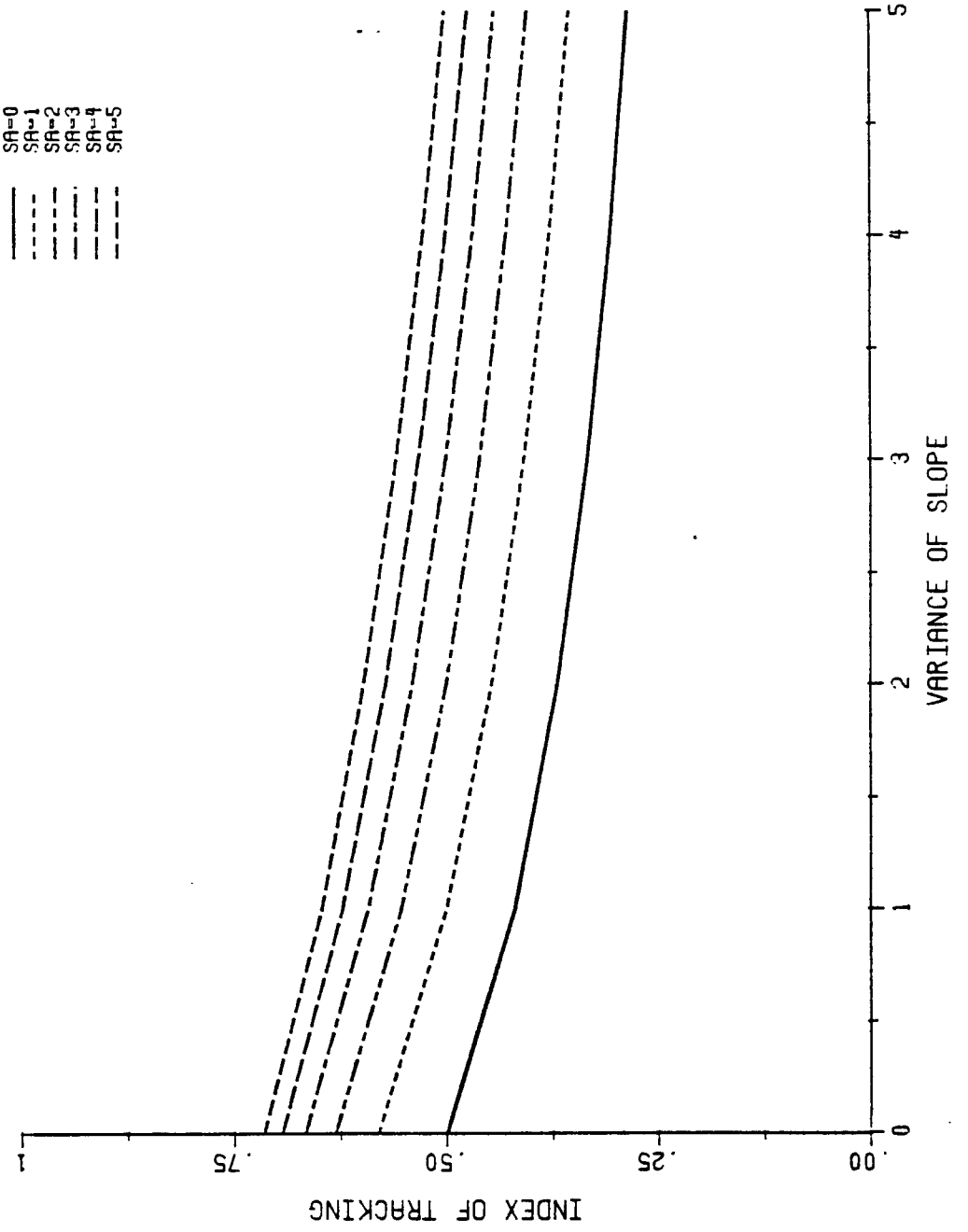
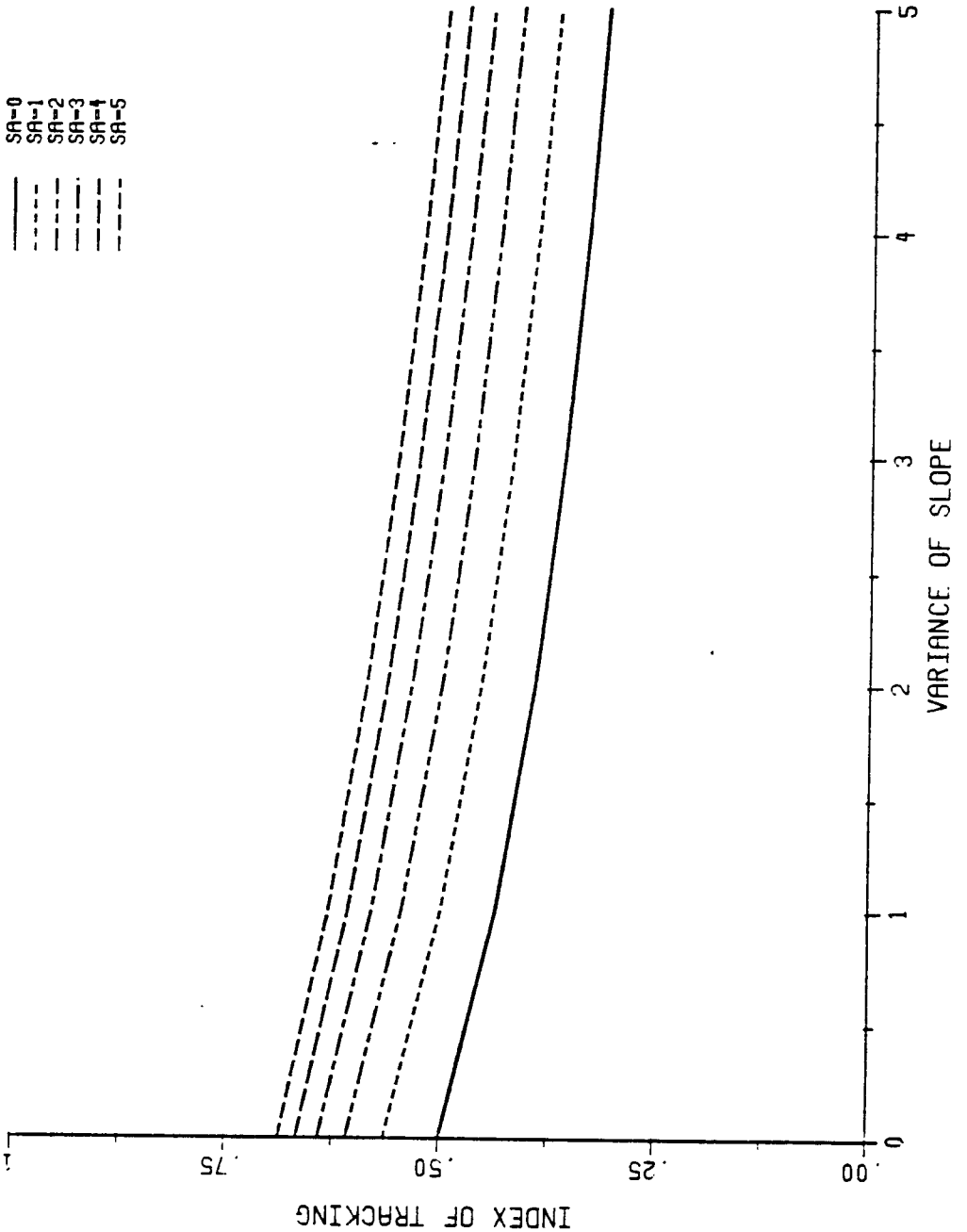


FIGURE 2.3.2
INDEX OF TRACKING FOR LINEAR MODELS
ASSUME NORMALITY AND DIAGONAL COVARIANCE
SF=4



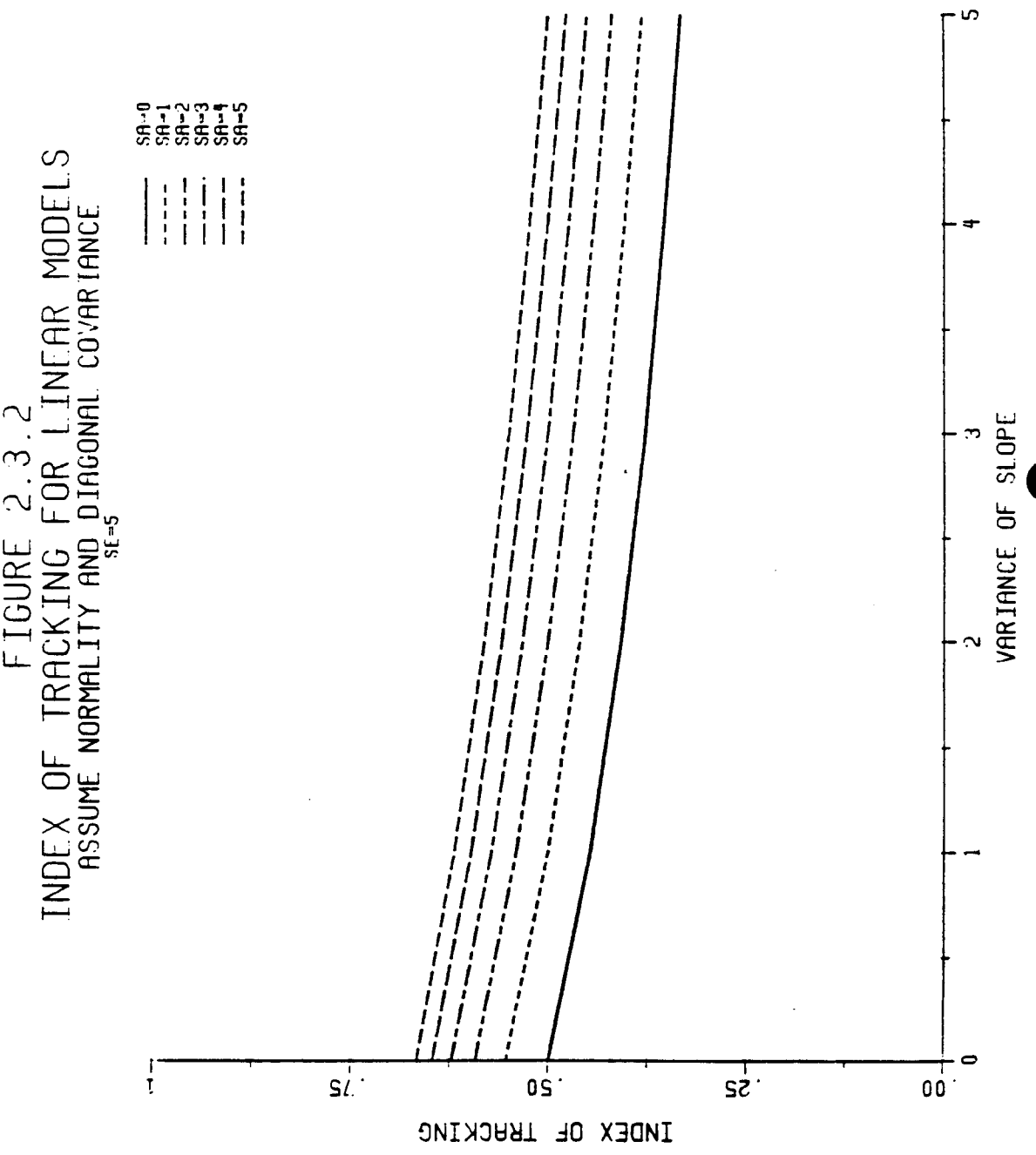


FIGURE 2.3.2
INDEX OF TRACKING FOR LINEAR MODELS
ASSUME NORMALITY AND DIAGONAL COVARIANCE
SE=5
SA=0
SA=1
SA=2
SA=3
SA=4
SA=5
INDEX OF TRACKING
.00
.25
.50
.75
1
0
1
2
3
4
5
VARIANCE OF SLOPE

TABLE 2.3.2

INDEX OF TRACKING FOR LINEAR MODELS

ASSUMING NORMALITY AND GENERAL COVARIANCE STRUCTURE

$$RHO = (SA - SB)/SQRT(SA + SB - 2*SAB + SE)*SQRT(SA + SB + 2*SAB + SE)$$

$$GAMMA = .5 + INV(PHI)*ARCSIN(RHO)$$

SAB=1

SA	SB	SE	RHO	GAMMA	SA	SB	SE	RHO	GAMMA	SA	SB	SE	RHO	GAMMA
0	0	0	.	.	2	0	0	.	.	4	0	0	.	.
0	0	1	.	.	2	0	1	0.8944	0.85242	4	0	1	0.8729	0.83774
0	0	2	.	.	2	0	2	0.5774	0.69591	4	0	2	0.7071	0.75000
0	0	3	0.0000	0.50000	2	0	3	0.4364	0.64376	4	0	3	0.5963	0.70336
0	0	4	0.0000	0.50000	2	0	4	0.3536	0.61503	4	0	4	0.5164	0.67273
0	0	5	0.0000	0.50000	2	0	5	0.2981	0.59637	4	0	5	0.4558	0.65066
0	1	0	.	.	2	1	0	0.4472	0.64758	4	1	0	0.6547	0.72718
0	1	1	.	.	2	1	1	0.2887	0.59321	4	1	1	0.5303	0.67793
0	1	2	-0.4472	0.35242	2	1	2	0.2182	0.57002	4	1	2	0.4472	0.64758
0	1	3	-0.2887	0.40679	2	1	3	0.1768	0.55657	4	1	3	0.3873	0.62659
0	1	4	-0.2182	0.42998	2	1	4	0.1491	0.54763	4	1	4	0.3419	0.61106
0	1	5	-0.1768	0.44343	2	1	5	0.1291	0.54121	4	1	5	0.3062	0.59905
0	2	0	.	.	2	2	0	0.0000	0.50000	4	2	0	0.3536	0.61503
0	2	1	-0.8944	0.14758	2	2	1	0.0000	0.50000	4	2	1	0.2981	0.59637
0	2	2	-0.5774	0.30409	2	2	2	0.0000	0.50000	4	2	2	0.2582	0.58313
0	2	3	-0.4364	0.35624	2	2	3	0.0000	0.50000	4	2	3	0.2279	0.57319
0	2	4	-0.3536	0.38497	2	2	4	0.0000	0.50000	4	2	4	0.2041	0.56543
0	2	5	-0.2981	0.40363	2	2	5	0.0000	0.50000	4	2	5	0.1849	0.55920
0	3	0	.	.	2	3	0	-0.2182	0.42998	4	3	0	0.1491	0.54763
0	3	1	-0.8660	0.16667	2	3	1	-0.1768	0.44343	4	3	1	0.1291	0.54121
0	3	2	-0.6547	0.27282	2	3	2	-0.1491	0.45237	4	3	2	0.1140	0.53635
0	3	3	-0.5303	0.32207	2	3	3	-0.1291	0.45879	4	3	3	0.1021	0.53254
0	3	4	-0.4472	0.35242	2	3	4	-0.1140	0.46365	4	3	4	0.0925	0.52947
0	3	5	-0.3873	0.37341	2	3	5	-0.1021	0.46746	4	3	5	0.0845	0.52693
0	4	0	.	.	2	4	0	-0.3536	0.38497	4	4	0	0.0000	0.50000
0	4	1	-0.8729	0.16226	2	4	1	-0.2981	0.40363	4	4	1	0.0000	0.50000
0	4	2	-0.7071	0.25000	2	4	2	-0.2582	0.41687	4	4	2	0.0000	0.50000
0	4	3	-0.5963	0.29664	2	4	3	-0.2279	0.42681	4	4	3	0.0000	0.50000
0	4	4	-0.5164	0.32727	2	4	4	-0.2041	0.43457	4	4	4	0.0000	0.50000
0	4	5	-0.4558	0.34934	2	4	5	-0.1849	0.44080	4	4	5	0.0000	0.50000
0	5	0	.	.	2	5	0	-0.4472	0.35242	4	5	0	-0.1140	0.46365
0	5	1	-0.8839	0.15492	2	5	1	-0.3873	0.37341	4	5	1	-0.1021	0.46746
0	5	2	-0.7454	0.23228	2	5	2	-0.3419	0.38894	4	5	2	-0.0925	0.47053
0	5	3	-0.6455	0.27665	2	5	3	-0.3062	0.40095	4	5	3	-0.0845	0.47307
0	5	4	-0.5698	0.30702	2	5	4	-0.2774	0.41054	4	5	4	-0.0778	0.47519
0	5	5	-0.5103	0.32953	2	5	5	-0.2535	0.41840	4	5	5	-0.0722	0.47701
1	0	0	.	.	3	0	0	.	.	5	0	0	.	.
1	0	1	.	.	3	0	1	0.8660	0.83333	5	0	1	0.8839	0.84508
1	0	2	0.4472	0.64758	3	0	2	0.6547	0.72718	5	0	2	0.7454	0.76772
1	0	3	0.2887	0.59321	3	0	3	0.5303	0.67793	5	0	3	0.6455	0.72335
1	0	4	0.2182	0.57002	3	0	4	0.4472	0.64758	5	0	4	0.5698	0.69298
1	0	5	0.1768	0.55657	3	0	5	0.3873	0.62659	5	0	5	0.5103	0.67047
1	1	0	.	.	3	1	0	0.5774	0.69591	5	1	0	0.7071	0.75000
1	1	1	0.0000	0.50000	3	1	1	0.4364	0.64376	5	1	1	0.5963	0.70336
1	1	2	0.0000	0.50000	3	1	2	0.3536	0.61503	5	1	2	0.5164	0.67273
1	1	3	0.0000	0.50000	3	1	3	0.2981	0.59637	5	1	3	0.4558	0.65066
1	1	4	0.0000	0.50000	3	1	4	0.2582	0.58313	5	1	4	0.4082	0.63386
1	1	5	0.0000	0.50000	3	1	5	0.2279	0.57319	5	1	5	0.3698	0.62057
1	2	0	-0.4472	0.35242	3	2	0	0.2182	0.57002	5	2	0	0.4472	0.64758
1	2	1	-0.2887	0.40679	3	2	1	0.1768	0.55657	5	2	1	0.3873	0.62659
1	2	2	-0.2182	0.42998	3	2	2	0.1491	0.54763	5	2	2	0.3419	0.61106
1	2	3	-0.1768	0.44343	3	2	3	0.1291	0.54121	5	2	3	0.3062	0.59905
1	2	4	-0.1491	0.45237	3	2	4	0.1140	0.53635	5	2	4	0.2774	0.58946
1	2	5	-0.1291	0.45879	3	2	5	0.1021	0.53254	5	2	5	0.2535	0.58160
1	3	0	-0.5774	0.30409	3	3	0	0.0000	0.50000	5	3	0	0.2582	0.58313
1	3	1	-0.4364	0.35624	3	3	1	0.0000	0.50000	5	3	1	0.2279	0.57319
1	3	2	-0.3536	0.38497	3	3	2	0.0000	0.50000	5	3	2	0.2041	0.56543
1	3	3	-0.2981	0.40363	3	3	3	0.0000	0.50000	5	3	3	0.1849	0.55920
1	3	4	-0.2582	0.41687	3	3	4	0.0000	0.50000	5	3	4	0.1690	0.55406
1	3	5	-0.2279	0.42681	3	3	5	0.0000	0.50000	5	3	5	0.1557	0.54976
1	4	0	-0.6547	0.27282	3	4	0	-0.1491	0.45237	5	4	0	0.1140	0.53635
1	4	1	-0.5303	0.32207	3	4	1	-0.1291	0.45879	5	4	1	0.1021	0.53254
1	4	2	-0.4472	0.35242	3	4	2	-0.1140	0.46365	5	4	2	0.0925	0.52947
1	4	3	-0.3873	0.37341	3	4	3	-0.1021	0.46746	5	4	3	0.0845	0.52693
1	4	4	-0.3419	0.38894	3	4	4	-0.0925	0.47053	5	4	4	0.0778	0.52481
1	4	5	-0.3062	0.40095	3	4	5	-0.0845	0.47307	5	4	5	0.0722	0.52299
1	5	0	-0.7071	0.25000	3	5	0	-0.2582	0.41687	5	5	0	0.0000	0.50000
1	5	1	-0.5963	0.29664	3	5	1	-0.2279	0.42681	5	5	1	0.0000	0.50000
1	5	2	-0.5164	0.32727	3	5	2	-0.2041	0.43457	5	5	2	0.0000	0.50000
1	5	3	-0.4558	0.34934	3	5	3	-0.1849	0.44080	5	5	3	0.0000	0.50000
1	5	4	-0.4082	0.36614	3	5	4	-0.1690	0.44594	5	5	4	0.0000	0.50000
1	5	5	-0.3698	0.37943	3	5	5	-0.1557	0.45024	5	5	5	0.0000	0.50000

TABLE 2.3.2

INDEX OF TRACKING FOR LINEAR MODELS

ASSUMING NORMALITY AND GENERAL COVARIANCE STRUCTURE

$$RHO = (SA - SB)/SQRT(SA + SB - 2*SAB + SE)*SQRT(SA + SB + 2*SAB + SE)$$

$$GAMMA = .5 + INV(PHI)*ARCSIN(RHO)$$

SAB=2

SA	SB	SE	RHO	GAMMA	SA	SB	SE	RHO	GAMMA	SA	SB	SE	RHO	GAMMA
0	0	0	.	.	2	0	0	.	.	4	0	0	.	.
0	0	1	.	.	2	0	1	.	.	4	0	1	.	.
0	0	2	.	.	2	0	2	.	.	4	0	2	0.8944	0.85242
0	0	3	.	.	2	0	3	0.6667	0.73228	4	0	3	0.6963	0.74518
0	0	4	.	.	2	0	4	0.4472	0.64758	4	0	4	0.5774	0.69591
0	0	5	0.0000	0.50000	2	0	5	0.3482	0.61319	4	0	5	0.4961	0.66525
0	1	0	.	.	2	1	0	.	.	4	1	0	1.0000	1.00000
0	1	1	.	.	2	1	1	.	.	4	1	1	0.6708	0.73406
0	1	2	.	.	2	1	2	0.3333	0.60817	4	1	2	0.5222	0.67490
0	1	3	.	.	2	1	3	0.2236	0.57178	4	1	3	0.4330	0.64255
0	1	4	-0.3333	0.39183	2	1	4	0.1741	0.55569	4	1	4	0.3721	0.62136
0	1	5	-0.2236	0.42822	2	1	5	0.1443	0.54611	4	1	5	0.3273	0.60615
0	2	0	.	.	2	2	0	.	.	4	2	0	0.4472	0.64758
0	2	1	.	.	2	2	1	0.0000	0.50000	4	2	1	0.3482	0.61319
0	2	2	.	.	2	2	2	0.0000	0.50000	4	2	2	0.2887	0.59321
0	2	3	-0.6667	0.26772	2	2	3	0.0000	0.50000	4	2	3	0.2481	0.57980
0	2	4	-0.4472	0.35242	2	2	4	0.0000	0.50000	4	2	4	0.2182	0.57002
0	2	5	-0.3482	0.38681	2	2	5	0.0000	0.50000	4	2	5	0.1952	0.56253
0	3	0	.	.	2	3	0	-0.3333	0.39183	4	3	0	0.1741	0.55569
0	3	1	.	.	2	3	1	-0.2236	0.42822	4	3	1	0.1443	0.54611
0	3	2	-1.0000	0.00000	2	3	2	-0.1741	0.44431	4	3	2	0.1240	0.53958
0	3	3	-0.6708	0.26594	2	3	3	-0.1443	0.45389	4	3	3	0.1091	0.53480
0	3	4	-0.5222	0.32510	2	3	4	-0.1240	0.46042	4	3	4	0.0976	0.53111
0	3	5	-0.4330	0.35745	2	3	5	-0.1091	0.46520	4	3	5	0.0884	0.52817
0	4	0	.	.	2	4	0	-0.4472	0.35242	4	4	0	0.0000	0.50000
0	4	1	.	.	2	4	1	-0.3482	0.38681	4	4	1	0.0000	0.50000
0	4	2	-0.8944	0.14758	2	4	2	-0.2887	0.40679	4	4	2	0.0000	0.50000
0	4	3	-0.6963	0.25482	2	4	3	-0.2481	0.42020	4	4	3	0.0000	0.50000
0	4	4	-0.5774	0.30409	2	4	4	-0.2182	0.42998	4	4	4	0.0000	0.50000
0	4	5	-0.4961	0.33475	2	4	5	-0.1952	0.43747	4	4	5	0.0000	0.50000
0	5	0	.	.	2	5	0	-0.5222	0.32510	4	5	0	-0.1240	0.46042
0	5	1	.	.	2	5	1	-0.4330	0.35745	4	5	1	-0.1091	0.46520
0	5	2	-0.8704	0.16387	2	5	2	-0.3721	0.37864	4	5	2	-0.0976	0.46889
0	5	3	-0.7217	0.24337	2	5	3	-0.3273	0.39385	4	5	3	-0.0884	0.47183
0	5	4	-0.6202	0.28706	2	5	4	-0.2928	0.40542	4	5	4	-0.0808	0.47424
0	5	5	-0.5455	0.31632	2	5	5	-0.2652	0.41457	4	5	5	-0.0745	0.47625
1	0	0	.	.	3	0	0	.	.	5	0	0	.	.
1	0	1	.	.	3	0	1	.	.	5	0	1	.	.
1	0	2	.	.	3	0	2	1.0000	1.00000	5	0	2	0.8704	0.83613
1	0	3	.	.	3	0	3	0.6708	0.73406	5	0	3	0.7217	0.75663
1	0	4	0.3333	0.60817	3	0	4	0.5222	0.67490	5	0	4	0.6202	0.71294
1	0	5	0.2236	0.57178	3	0	5	0.4330	0.64255	5	0	5	0.5455	0.68368
1	1	0	.	.	3	1	0	.	.	5	1	0	0.8944	0.85242
1	1	1	.	.	3	1	1	0.6667	0.73228	5	1	1	0.6963	0.74518
1	1	2	.	.	3	1	2	0.4472	0.64758	5	1	2	0.5774	0.69591
1	1	3	0.0000	0.50000	3	1	3	0.3482	0.61319	5	1	3	0.4961	0.66525
1	1	4	0.0000	0.50000	3	1	4	0.2887	0.59321	5	1	4	0.4364	0.64376
1	1	5	0.0000	0.50000	3	1	5	0.2481	0.57980	5	1	5	0.3904	0.62765
1	2	0	.	.	3	2	0	0.3333	0.60817	5	2	0	0.5222	0.67490
1	2	1	.	.	3	2	1	0.2236	0.57178	5	2	1	0.4330	0.64255
1	2	2	-0.3333	0.39183	3	2	2	0.1741	0.55569	5	2	2	0.3721	0.62136
1	2	3	-0.2236	0.42822	3	2	3	0.1443	0.54611	5	2	3	0.3273	0.60615
1	2	4	-0.1741	0.44431	3	2	4	0.1240	0.53958	5	2	4	0.2928	0.59458
1	2	5	-0.1443	0.45389	3	2	5	0.1091	0.53480	5	2	5	0.2652	0.58543
1	3	0	.	.	3	3	0	0.0000	0.50000	5	3	0	0.2887	0.59321
1	3	1	-0.6667	0.26772	3	3	1	0.0000	0.50000	5	3	1	0.2481	0.57980
1	3	2	-0.4472	0.35242	3	3	2	0.0000	0.50000	5	3	2	0.2182	0.57002
1	3	3	-0.3482	0.38681	3	3	3	0.0000	0.50000	5	3	3	0.1952	0.56253
1	3	4	-0.2887	0.40679	3	3	4	0.0000	0.50000	5	3	4	0.1768	0.55657
1	3	5	-0.2481	0.42020	3	3	5	0.0000	0.50000	5	3	5	0.1617	0.55169
1	4	0	-1.0000	0.00000	3	4	0	-0.1741	0.44431	5	4	0	0.1240	0.53958
1	4	1	-0.6708	0.26594	3	4	1	-0.1443	0.45389	5	4	1	0.1091	0.53480
1	4	2	-0.5222	0.32510	3	4	2	-0.1240	0.46042	5	4	2	0.0976	0.53111
1	4	3	-0.4330	0.35745	3	4	3	-0.1091	0.46520	5	4	3	0.0884	0.52817
1	4	4	-0.3721	0.37864	3	4	4	-0.0976	0.46889	5	4	4	0.0808	0.52576
1	4	5	-0.3273	0.39385	3	4	5	-0.0884	0.47183	5	4	5	0.0745	0.52375
1	5	0	-0.8944	0.14758	3	5	0	-0.2887	0.40679	5	5	0	0.0000	0.50000
1	5	1	-0.6963	0.25482	3	5	1	-0.2481	0.42020	5	5	1	0.0000	0.50000
1	5	2	-0.5774	0.30409	3	5	2	-0.2182	0.42998	5	5	2	0.0000	0.50000
1	5	3	-0.4961	0.33475	3	5	3	-0.1952	0.43747	5	5	3	0.0000	0.50000
1	5	4	-0.4364	0.35624	3	5	4	-0.1768	0.44343	5	5	4	0.0000	0.50000
1	5	5	-0.3904	0.37235	3	5	5	-0.1617	0.44831	5	5	5	0.0000	0.50000

TABLE 2.3.2

INDEX OF TRACKING FOR LINEAR MODELS

ASSUMING NORMALITY AND GENERAL COVARIANCE STRUCTURE

$$RHO = (SA - SB)/SQRT(SA + SB - 2*SAB + SE)*SQRT(SA + SB + 2*SAB + SE)$$

$$GAMMA = .5 + INV(PHI)*ARCSIN(RHO)$$

SAB=3

SA	SB	SE	RHO	GAMMA	SA	SB	SE	RHO	GAMMA	SA	SB	SE	RHO	GAMMA
0	0	0	.	.	2	0	0	.	.	4	0	0	.	.
0	0	1	.	.	2	0	1	.	.	4	0	1	.	.
0	0	2	.	.	2	0	2	.	.	4	0	2	.	.
0	0	3	.	.	2	0	3	.	.	4	0	3	.	.
0	0	4	.	.	2	0	4	.	.	4	0	4	0.7559	0.77281
0	0	5	.	.	2	0	5	0.5547	0.68717	4	0	5	0.5963	0.70336
0	1	0	.	.	2	1	0	.	.	4	1	0	.	.
0	1	1	.	.	2	1	1	.	.	4	1	1	.	.
0	1	2	.	.	2	1	2	.	.	4	1	2	0.8321	0.81283
0	1	3	.	.	2	1	3	.	.	4	1	3	0.5669	0.69188
0	1	4	.	.	2	1	4	0.2774	0.59946	4	1	4	0.4472	0.64758
0	1	5	.	.	2	1	5	0.1890	0.56052	4	1	5	0.3750	0.62236
0	2	0	.	.	2	2	0	.	.	4	2	0	.	.
0	2	1	.	.	2	2	1	.	.	4	2	1	0.5547	0.68717
0	2	2	.	.	2	2	2	.	.	4	2	2	0.3780	0.62338
0	2	3	.	.	2	2	3	0.0000	0.50000	4	2	3	0.2981	0.59637
0	2	4	.	.	2	2	4	0.0000	0.50000	4	2	4	0.2500	0.58043
0	2	5	-0.5547	0.31283	2	2	5	0.0000	0.50000	4	2	5	0.2169	0.56960
0	3	0	.	.	2	3	0	.	.	4	3	0	0.2774	0.58946
0	3	1	.	.	2	3	1	.	.	4	3	1	0.1890	0.56052
0	3	2	.	.	2	3	2	-0.2774	0.41054	4	3	2	0.1491	0.54763
0	3	3	.	.	2	3	3	-0.1890	0.43948	4	3	3	0.1250	0.53989
0	3	4	-0.8321	0.18717	2	3	4	-0.1491	0.45237	4	3	4	0.1085	0.53459
0	3	5	-0.5669	0.30812	2	3	5	-0.1250	0.46011	4	3	5	0.0962	0.53068
0	4	0	.	.	2	4	0	.	.	4	4	0	0.0000	0.50000
0	4	1	.	.	2	4	1	-0.5547	0.31283	4	4	1	0.0000	0.50000
0	4	2	.	.	2	4	2	-0.3780	0.37662	4	4	2	0.0000	0.50000
0	4	3	.	.	2	4	3	-0.2981	0.40363	4	4	3	0.0000	0.50000
0	4	4	-0.7559	0.22719	2	4	4	-0.2500	0.41957	4	4	4	0.0000	0.50000
0	4	5	-0.5963	0.29664	2	4	5	-0.2169	0.43040	4	4	5	0.0000	0.50000
0	5	0	.	.	2	5	0	-0.8321	0.18717	4	5	0	-0.1491	0.45237
0	5	1	.	.	2	5	1	-0.5669	0.30812	4	5	1	-0.1250	0.46011
0	5	2	.	.	2	5	2	-0.4472	0.35242	4	5	2	-0.1085	0.46541
0	5	3	-0.9449	0.10615	2	5	3	-0.3750	0.37764	4	5	3	-0.0962	0.46932
0	5	4	-0.7454	0.23228	2	5	4	-0.3254	0.39450	4	5	4	-0.0867	0.47236
0	5	5	-0.6250	0.28510	2	5	5	-0.2887	0.40679	4	5	5	-0.0791	0.47481
1	0	0	.	.	3	0	0	.	.	5	0	0	.	.
1	0	1	.	.	3	0	1	.	.	5	0	1	.	.
1	0	2	.	.	3	0	2	.	.	5	0	2	.	.
1	0	3	.	.	3	0	3	.	.	5	0	3	0.9449	0.89385
1	0	4	.	.	3	0	4	0.8321	0.81283	5	0	4	0.7454	0.76772
1	0	5	.	.	3	0	5	0.5669	0.69188	5	0	5	0.6250	0.71490
1	1	0	.	.	3	1	0	.	.	5	1	0	.	.
1	1	1	.	.	3	1	1	.	.	5	1	1	.	.
1	1	2	.	.	3	1	2	.	.	5	1	2	0.7559	0.77281
1	1	3	.	.	3	1	3	0.5547	0.68717	5	1	3	0.5963	0.70336
1	1	4	.	.	3	1	4	0.3780	0.62338	5	1	4	0.5000	0.66667
1	1	5	0.0000	0.50000	3	1	5	0.2981	0.59637	5	1	5	0.4339	0.64285
1	2	0	.	.	3	2	0	.	.	5	2	0	0.8321	0.81283
1	2	1	.	.	3	2	1	.	.	5	2	1	0.5669	0.69188
1	2	2	.	.	3	2	2	0.2774	0.58946	5	2	2	0.4472	0.64758
1	2	3	.	.	3	2	3	0.1890	0.56052	5	2	3	0.3750	0.62236
1	2	4	-0.2774	0.41054	3	2	4	0.1491	0.54763	5	2	4	0.3254	0.60550
1	2	5	-0.1890	0.43948	3	2	5	0.1250	0.53989	5	2	5	0.2887	0.59321
1	3	0	.	.	3	3	0	.	.	5	3	0	0.3780	0.62338
1	3	1	.	.	3	3	1	0.0000	0.50000	5	3	1	0.2981	0.59637
1	3	2	.	.	3	3	2	0.0000	0.50000	5	3	2	0.2500	0.58043
1	3	3	-0.5547	0.31283	3	3	3	0.0000	0.50000	5	3	3	0.2169	0.56960
1	3	4	-0.3780	0.37662	3	3	4	0.0000	0.50000	5	3	4	0.1925	0.56164
1	3	5	-0.2981	0.40363	3	3	5	0.0000	0.50000	5	3	5	0.1734	0.55548
1	4	0	.	.	3	4	0	-0.2774	0.41054	5	4	0	0.1491	0.54763
1	4	1	.	.	3	4	1	-0.1890	0.43948	5	4	1	0.1250	0.53989
1	4	2	-0.8321	0.18717	3	4	2	-0.1491	0.45237	5	4	2	0.1085	0.53459
1	4	3	-0.5669	0.30812	3	4	3	-0.1250	0.46011	5	4	3	0.0962	0.53068
1	4	4	-0.4472	0.35242	3	4	4	-0.1085	0.46541	5	4	4	0.0867	0.52764
1	4	5	-0.3750	0.37764	3	4	5	-0.0962	0.46932	5	4	5	0.0791	0.52519
1	5	0	.	.	3	5	0	-0.3780	0.37662	5	5	0	0.0000	0.50000
1	5	1	-1.1094	.	3	5	1	-0.2981	0.40363	5	5	1	0.0000	0.50000
1	5	2	-0.7559	0.22719	3	5	2	-0.2500	0.41957	5	5	2	0.0000	0.50000
1	5	3	-0.5963	0.29664	3	5	3	-0.2169	0.43040	5	5	3	0.0000	0.50000
1	5	4	-0.5000	0.33333	3	5	4	-0.1925	0.43836	5	5	4	0.0000	0.50000
1	5	5	-0.4339	0.35715	3	5	5	-0.1734	0.44452	5	5	5	0.0000	0.50000

When the individual responses are assumed to follow a quadratic regression on time,

$$\underset{\sim}{Y}_i = X\underset{\sim}{\beta}_i + \underset{\sim}{\varepsilon}_i, \text{ where } \underset{\sim}{\beta}_i = \begin{bmatrix} \alpha_i \\ \beta_i \\ \delta_i \end{bmatrix}$$

with the same assumptions as in the linear case, consider

$\underset{\sim}{T}_{ij} = \underset{\sim}{Y}_i - \underset{\sim}{Y}_j = X\underset{\sim}{\beta}_i - X\underset{\sim}{\beta}_j = X\underset{\sim}{\beta}_{ij}$, where

$$\underset{\sim}{\beta}_{ij} = \begin{bmatrix} \alpha_i - \alpha_j \\ \beta_i - \beta_j \\ \delta_i - \delta_j \end{bmatrix} .$$

Without loss of generality,

$$\underset{\sim}{T}_{ij} = \underset{\sim}{Y}_i - \underset{\sim}{Y}_j \quad \text{if } \delta_i < \delta_j$$

$$= \underset{\sim}{Y}_j - \underset{\sim}{Y}_i \quad \text{if } \delta_i > \delta_j \quad ,$$

then max $\underset{\sim}{T}_{ij}$ occurs at $X_{2ij} = -\beta_{ij}/2\delta_{ij}$, $\delta_i \neq \delta_j$.

$$\gamma = P\{T_{ij} > 0 \text{ for all } X \in [X_1, X_3] \text{ or}$$

$$T_{ij} < 0 \text{ for all } X \in [X_1, X_3]\}$$

$$= 2P\{T_{ij} > 0 \text{ for all } X \in [X_1, X_3]\} \text{ by symmetry.}$$

If $X_2 = -\beta_{ij}/2\delta_{ij} \notin [X_1, X_3]$, then

$$P\{T_{ij} > 0 \text{ for all } X \in [X_1, X_3]\} =$$

$$P\{T_{ij}(X_1) > 0, T_{ij}(X_3) > 0\} \quad ,$$

which has been discussed for the linear case.

If $X_{2_{ij}} = -\beta_{ij}/2\delta_{ij} \in [X_1, X_3]$, then

$$P\{T_{ij} > 0 \text{ for all } X \in [X_1, X_3]\} =$$

$$P\{T_{ij}(X_1) > 0,\ T_{ij}(X_{2ij}) > 0,\ T_{ij}(X_3) > 0\} =$$

$$\frac{1}{2} - \frac{1}{4\pi}\left(\cos^{-1}\rho_{12_{ij}} + \cos^{-1}\rho_{13} + \cos^{-1}\rho_{2_{ij}3}\right),$$

assuming trivariate normality (Johnson and Kotz, 1972). For $\underset{\sim}{F}$ diagonal,

$$\rho_{12_{ij}} = \frac{\sigma_\alpha^2 + X_1 X_{2_{ij}}\sigma_\beta^2 + X_1^2 X_{2_{ij}}^2 \sigma_\delta^2}{\sqrt{\sigma_\alpha^2 + X_1^2\sigma_\beta^2 + X_1^4\sigma_\delta^2 + \sigma_\varepsilon^2}\ \sqrt{\sigma_\alpha^2 + X_{2_{ij}}^2\sigma_\beta^2 + X_{2_{ij}}^4\sigma_\beta^2 + \sigma_\varepsilon^2}},$$

with similar definitions for ρ_{13} and $\rho_{2_{ij}3}$. Therefore

$$\gamma = P\{\max T_{ij} \notin [X_1, X_3]\} * 2 * P\{T_{ij}(X_1) > 0,\ T_{ij}(X_3) > 0 \mid \max T_{ij} \notin [X_1, X_3]\}$$

$$+ P\{\max T_{ij} \in [X_1, X_3]\} * 2 * P\{T_{ij}(X_1) > 0,\ T_{ij}(X_2) > 0,\ T_{ij}(X_3) > 0 \mid \max T_{ij} \in [X_1, X_3]\}$$

$$= P\{\max T_{ij} \notin [X_1, X_3]\} * \left\{\frac{1}{2} + \frac{1}{\pi}\sin^{-1}\rho_{13}\right\} +$$

$$P\{\max T_{ij} \in [X_1, X_3]\} * \left\{1 - \frac{1}{2\pi}\left(\cos^{-1}\rho_{12_{ij}} + \cos^{-1}\rho_{13} + \cos^{-1}\rho_{2_{ij}3}\right)\right\}.$$

Rather than proceed analytically to evaluate $\gamma(F)$ for the quadratic model, which would require specification of the distribution of the random variable $X_{2_{ij}} = -\beta_{ij}/2\delta_{ij}$, the expected value of $\gamma(F)$ can be computed based on simulated vectors $\underset{\sim}{\beta}_i$ and $\underset{\sim}{\varepsilon}_i$, with predetermined covariance structures. Simulations of the expected value of $\gamma(F)$ for the quadratic model with specified parameters are summarized in Table 2.3.3. The development for higher degree models is similar, relying on the higher dimensional orthant probabilities.

Thus the mathematical expression of tracking is consistent with the conceptual definition. In mathematical terms, tracking occurs when the variability of the higher degree terms, or the variability of the

TABLE 2.3.3

INDEX OF TRACKING FOR QUADRATIC MODELS

ASSUMING NORMALITY AND DIAGONAL COVARIANCE STRUCTURE

σ_α^2	σ_β^2	σ_δ^2	σ_ε^2	γ
1.0000	1.0000	1.0000	1.0000	0.51106
1.0000	1.0000	1.0000	3.0000	0.48745
1.0000	1.0000	3.0000	1.0000	0.53240
1.0000	1.0000	3.0000	3.0000	0.48886
1.0000	3.0000	1.0000	1.0000	0.42128
1.0000	3.0000	1.0000	3.0000	0.43494
1.0000	3.0000	3.0000	1.0000	0.46483
1.0000	3.0000	3.0000	3.0000	0.45731
3.0000	1.0000	1.0000	1.0000	0.61423
3.0000	1.0000	1.0000	3.0000	0.56414
3.0000	1.0000	3.0000	1.0000	0.60130
3.0000	1.0000	3.0000	3.0000	0.54852
3.0000	3.0000	1.0000	1.0000	0.52176
3.0000	3.0000	1.0000	3.0000	0.51031
3.0000	3.0000	3.0000	1.0000	0.53275
3.0000	3.0000	3.0000	3.0000	0.51613
0.8000	0.2000	0.2000	0.0001	0.66485
0.4000	0.6000	0.2000	0.0001	0.46415

shape of the curve, is small relative to the variability of the intercepts. Darby and Fearn (1979) came to essentially the same conclusion, without the additional capability given here of quantifying the tracking exhibited.

2.4 Central Moments of the Index

Theorem 2.4.1:

For the symmetric kernel $\phi(\underset{\sim}{Y}_i, \underset{\sim}{Y}_j)$ defined in (2.3.1),

$$U_n = \binom{n}{2}^{-1} \sum_{1 \le \alpha_1 < \alpha_2 \le n} \phi(\underset{\sim}{Y}_{\alpha_1}, \underset{\sim}{Y}_{\alpha_2}).$$

$$E_F\{U_n\} = \gamma(F) \quad \text{for all } F,\ n \ge m = 2.$$

Proof:

$$E_F\{U_n\} = E_F\{\binom{n}{2}^{-1} \sum_{1 \le \alpha_1 < \alpha_2 \le n} \phi(\underset{\sim}{Y}_{\alpha_1}, \underset{\sim}{Y}_{\alpha_2})\}$$

$$= \binom{n}{2}^{-1} \sum_{1 \le \alpha_1 < \alpha_2 \le n} E_F\{\phi(\underset{\sim}{Y}_{\alpha_1}, \underset{\sim}{Y}_{\alpha_2})\}$$

$$= \binom{n}{2}^{-1} \sum_{1 \le \alpha_1 < \alpha_2 \le n} \gamma(F) = \gamma(F) .$$

Now following Hoeffding (1948), define

$$\phi_0 = E_F\{\phi(\underset{\sim}{Y}_i, \underset{\sim}{Y}_j)\} = \gamma(F) \quad i \ne j = 1,2,\ldots,n .$$

$$\phi_1(\underset{\sim}{y}_i) = E_F\{\phi(\underset{\sim}{y}_i, \underset{\sim}{Y}_j)\} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \phi(\underset{\sim}{y}_i, \underset{\sim}{y}_j) dF(y_j^{(1)}, \ldots, y_j^{(k)})$$

$$= \int_{y_i^{(1)}}^{\infty} \cdots \int_{y_i^{(k)}}^{\infty} dF(y_j^{(1)}, \ldots, y_j^{(k)}) + \int_{-\infty}^{y_i^{(1)}} \cdots \int_{-\infty}^{y_i^{(k)}} dF(y_j^{(1)}, \ldots, y_j^{(k)}).$$

$$\phi_2(\underset{\sim}{y}_i, \underset{\sim}{y}_j) = E_F\{\phi(\underset{\sim}{y}_i, \underset{\sim}{y}_j)\} = \phi(\underset{\sim}{y}_i, \underset{\sim}{y}_j) .$$

$$\psi_0 = \phi_0 - \gamma(F) = E_F\{\phi(\underset{\sim}{Y}_i, \underset{\sim}{Y}_j)\} - \gamma(F) = \gamma(F) - \gamma(F) = 0 .$$

$$\psi_1(\underset{\sim}{y}_i) = \phi_1(\underset{\sim}{y}_i) - \gamma(F) \qquad (2.4.1)$$

$$= \int_{y_i^{(1)}}^{\infty} \cdots \int_{y_i^{(k)}}^{\infty} dF(y_j^{(1)},\ldots,y_j^{(k)}) + \int_{-\infty}^{y_i^{(1)}} \cdots \int_{-\infty}^{y_i^{(k)}} dF(y_j^{(1)},\ldots,y_j^{(k)}) - \gamma(F).$$

$$\psi_2(\underset{\sim}{y}_i,\underset{\sim}{y}_j) = \phi_2(\underset{\sim}{y}_i,\underset{\sim}{y}_j) - \gamma(F) .$$

$$\zeta_0 = E_F\{\psi_0^2\} = 0 .$$

$$\zeta_1 = E_F\{\psi_1^2(\underset{\sim}{Y}_i)\} = E\{\phi_1^2(\underset{\sim}{Y}_i)\} - \gamma^2 . \qquad (2.4.2)$$

$$\zeta_2 = E_F\{\psi_2^2(\underset{\sim}{Y}_i,\underset{\sim}{Y}_j)\} = E\{\phi_2^2(\underset{\sim}{Y}_i,\underset{\sim}{Y}_j)\} - \gamma^2 .$$

If the variance of U exists, according to Hoeffding, it is equal to

$$\sigma^2(U) = \binom{n}{m}^{-1} \sum_{c=1}^{m} \binom{m}{c}\binom{n-m}{m-c}\zeta_c ,$$

where m is the degree of the functional.

In this case, the functional $\gamma(F)$ is of degree m=2, so

$$\sigma^2(U) = \binom{n}{2}^{-1} \sum_{c=1}^{2} \binom{2}{c}\binom{n-2}{2-c}\zeta_c$$

$$= \frac{2}{n(n-1)} [2\zeta_1(n-2)+\zeta_2] .$$

In the bivariate case, when $Y^{(1)}$ and $Y^{(2)}$ are independent, the probability of concordance $\gamma = 1/2$. Then

$$\phi_1(y_i) = \int_{y_i^{(1)}}^{\infty} \int_{y_i^{(2)}}^{\infty} dF(y_j^{(1)},y_j^{(2)}) + \int_{-\infty}^{y_i^{(2)}} \int_{-\infty}^{y_i^{(2)}} dF(y_j^{(1)},y_j^{(2)})$$

$$= 1 - F(y_i^{(1)},\infty) - F(\infty,y_i^{(2)}) + 2F(y_i^{(1)},y_i^{(2)})$$

$$= 1 - F(y_i^{(1)}) - F(y_i^{(2)}) + 2F(y_i^{(1)})F(y_i^{(2)}) .$$

$$\zeta_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[1-F(y_i^{(1)})-F(y_i^{(2)})+2F(y^{(1)},F(y^{(2)})]^2 dF(y_i^{(1)})dF(y_i^{(2)})-\gamma^2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[1-F(y_i^{(1)}]^2[1-F(y_i^{(2)}]^2 dF(y_i^{(1)})dF(y_i^{(2)})$$

$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(y_i^{(1)})^2(F(y_i^{(2)})^2 dF(y_i^{(1)})dF(y_i^{(2)})$$

$$+ 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[1-F(y_i^{(1)})][1-F(y_i^{(2)})]F(y_i^{(1)})F(y_i^{(2)})dF(y_i^{(1)})dF(y_i^{(2)})-\gamma^2$$

$$= \int_{-\infty}^{\infty}[1-F(y_i^{(1)})]^2 dF(y_i^{(1)}) \int_{-\infty}^{\infty}[1-F(y_i^{(2)})]^2 dF(y_i^{(2)})$$

$$+ \int_{-\infty}^{\infty}F(y_i^{(1)})^2 dF(y_i^{(1)}) \int_{-\infty}^{\infty}F(y_i^{(2)})^2 dF(y_i^{(2)})$$

$$+ 2\int_{-\infty}^{\infty}[1-F(y_i^{(1)})]F(y_i^{(1)})dF(y_i^{(1)}) \int_{-\infty}^{\infty}[1-F(y_i^{(2)})]F(y_i^{(2)})dF(y_i^{(2)})-\gamma^2$$

$$= \frac{1}{9} + \frac{1}{9} + 2\left(\frac{1}{36}\right) - \left(\frac{1}{2}\right)^2 = \frac{1}{36} \quad .$$

$$\zeta_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(y_i^{(1)},y_i^{(2)})dF(y_i^{(1)},y_i^{(2)}) + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(y_j^{(1)},y_j^{(2)})dF(y_j^{(1)},y_j^{(2)})-\gamma^2$$

$$= \int_{-\infty}^{\infty}F(y_i^{(1)})dF(y_i^{(1)})\int_{-\infty}^{\infty}F(y_i^{(2)})dF(y_i^{(2)})+\int_{-\infty}^{\infty}F(y_j^{(1)})dF(y_j^{(1)})\int_{-\infty}^{\infty}F(y_j^{(2)})dF(y_j^{(2)})-\gamma^2$$

$$= \frac{1}{4} + \frac{1}{4} - \left(\frac{1}{2}\right)^2 = \frac{1}{4} \quad .$$

Therefore

$$\sigma^2(U_n) = \frac{2}{n(n-1)}\,[2\zeta_1(n-2)+\zeta_2]$$

$$= \frac{2}{n(n-1)}\,\left[2\left(\frac{1}{36}\right)(n-2) + \frac{1}{4}\right] = \frac{2n+5}{18n(n-1)} \quad .$$

When $\gamma \neq 1/2$, the variance of the U-statistic, in the bivariate case, is bounded.

Theorem 2.4.2:

In the bivariate case, for the symmetric kernel given in (2.3.1), and the U-statistic defined in (2.3.2),

$$\sigma^2(U_n) \leq \frac{32(1-\gamma)\gamma}{n} \quad . \tag{2.4.3}$$

Proof:

Kendall (1962) gives the upper bound to the variance of the sample estimator t of Kendall's τ as:

$$\sigma^2(t) \leq \frac{2}{n}(1-\tau^2).$$

Since $\tau = 2\gamma-1$, and $U_n = 2t-1$, then

$$\begin{aligned}\sigma^2(U_n) &= 4\sigma^2(t) \\ &\leq 4\left(\frac{2}{n}\right)[1-(2\gamma-1)^2] \\ &= \frac{8}{n}[1-4\gamma^2+4\gamma-1] \\ &= \frac{32\gamma(1-\gamma)}{n} \quad .\end{aligned}$$

The preceding results can be extended to the case when the response vectors have different distributions. Let $F_{\alpha}(\underset{\sim}{Y})$ be the c.d.f. of $\underset{\sim}{Y}_{\alpha}$, $\alpha = 1,2,\ldots,n$, and let

$$\gamma_{\alpha_1,\alpha_2} = E\{\phi(\underset{\sim}{Y}_{\alpha_1},\underset{\sim}{Y}_{\alpha_2})\},\ \alpha_1 \neq \alpha_2 = 1,\ldots,n.$$

Then, following Hoeffding's notation,

$$\psi_{1(\alpha_1,\beta_1)}(\underset{\sim}{y}_{\alpha_1}) = E\{\phi(\underset{\sim}{y}_{\alpha_1},\underset{\sim}{Y}_{\beta_1})\} - \gamma_{\alpha_1\beta_1},\ \alpha_1 \neq \beta_1 = 1,\ldots,n, \tag{2.4.4}$$

$$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\phi(\underset{\sim}{y}_{\alpha_1},\underset{\sim}{y}_{\beta_1})dF_{\beta_1}(y_{\beta_1}^{(1)}\ldots y_{\beta_1}^{(k)}) - \gamma_{\alpha_1\beta_1}$$

$$= \int_{y_{\alpha_1}^{(1)}}^{\infty}\cdots\int_{y_{\alpha_1}^{(k)}}^{\infty} dF_{\beta_1}(y_{\beta_1}^{(1)}\ldots y_{\beta_1}^{(k)}) + \int_{-\infty}^{y_{\alpha_1}^{(1)}}\cdots\int_{-\infty}^{y_{\alpha_1}^{(k)}} dF_{\beta_1}(y_{\beta_1}^{(1)}\ldots y_{\beta_1}^{(k)}) - \gamma_{\alpha_1\beta_1} \quad .$$

$$\psi_{2(\alpha_1,\alpha_2)}(y_{\alpha_1},y_{\alpha_2}) = E\{\phi(\underset{\sim}{y}_{\alpha_1},\underset{\sim}{y}_{\alpha_2})\} - \gamma_{\alpha_1,\alpha_2}$$

$$= \phi(\underset{\sim}{y}_{\alpha_1},\underset{\sim}{y}_{\alpha_2}) - \gamma_{\alpha_1,\alpha_2} \quad .$$

$$\zeta_{1(\alpha_1)\beta_1;\gamma_1} = E\{\psi_{1(\alpha_1)\beta_1}(\underset{\sim}{Y}_{\alpha_1})\psi_{1(\alpha_1)\gamma_1}(\underset{\sim}{Y}_{\alpha_1})\}, \ \alpha_1 \neq \beta_1 \neq \gamma_1 = 1,\ldots,n$$

$$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\psi_{1(\alpha_1)\beta_1}(\underset{\sim}{y}_{\alpha_1})\psi_{1(\alpha_1)\gamma_1}(\underset{\sim}{y}_{\alpha_1})dF_{\alpha_1}(\underset{\sim}{y}^{(1)}_{\alpha_1}\ldots\underset{\sim}{y}^{(k)}_{\alpha_1}) \quad .$$

$$\zeta_{2(\alpha_1,\alpha_2)} = E\{\psi^2_{2(\alpha_1\alpha_2)}(\underset{\sim}{Y}_{\alpha_1},\underset{\sim}{Y}_{\alpha_2})\}$$

$$= E\{\phi^2(\underset{\sim}{Y}_{\alpha_1},\underset{\sim}{Y}_{\alpha_2})\} - \gamma^2_{\alpha_1,\alpha_2}$$

$$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\phi^2(\underset{\sim}{y}_{\alpha_1},\underset{\sim}{y}_{\alpha_2})dF_{\alpha_1}(\underset{\sim}{y}^{(1)}_{\alpha_1}\ldots\underset{\sim}{y}^{(k)}_{\alpha_1})dF_{\alpha_2}(\underset{\sim}{y}^{(1)}_{\alpha_2}\ldots\underset{\sim}{y}^{(k)}_{\alpha_2}) - \gamma^2_{\alpha_1,\alpha_2}$$

$$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{-\infty}^{y^{(1)}_{\alpha_2}}\cdots\int_{-\infty}^{y^{(k)}_{\alpha_2}} dF_{\alpha_1}(y^{(1)}_{\alpha_1}\ldots y^{(k)}_{\alpha_1})dF_{\alpha_2}(y^{(1)}_{\alpha_2}\ldots y^{(k)}_{\alpha_2})$$

$$+ \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{y^{(1)}_{\alpha_2}}^{\infty}\cdots\int_{y^{(k)}_{\alpha_2}}^{\infty} dF_{\alpha_1}(y^{(1)}_{\alpha_1}\ldots y^{(k)}_{\alpha_1})dF_{\alpha_2}(y^{(1)}_{\alpha_2}\ldots y^{(k)}_{\alpha_2}) - \gamma^2_{\alpha_1,\alpha_2}.$$

Then, again from Hoeffding (1948), the variance of U (degree m=2) is:

$$\sigma^2(U_n) = \binom{n}{2}^{-1}\sum_{c=1}^{2}\binom{2}{c}\binom{n-2}{2-c}\zeta_{c,n} \ ,$$

where

$$\zeta_{c,n} = \frac{c!(2-c)!(2-c)!}{n(n-1)\ldots(n+c-3)}\sum_{S}\zeta_{c(\alpha_1\ldots\alpha_c)\beta_1;\gamma_1},$$

where the summation S extends over

$$1\leq\alpha_1<\alpha_2\leq n, \ 1\leq\beta_1\leq n, \ 1\leq\gamma_1\leq n, \ \alpha_1\neq\alpha_2, \ \beta_1\neq\gamma_1\neq\alpha_1 \quad .$$

Then

$$\zeta_{1,n} = \frac{1}{n(n-1)(n-2)}\sum_{S}\zeta_{1(\alpha_1)\beta_1;\gamma_1} = \frac{1}{2(n-2)}\binom{n}{2}\sum_{S}\zeta_{1(\alpha_1)\beta_i;\gamma_1} \ ,$$

$$\zeta_{2,n} = \frac{2}{n(n-1)} \sum_{S} \zeta_{2(\alpha_1,\alpha_2)} = \binom{n}{2} \sum_{S} \zeta_{2(\alpha_1,\alpha_2)} .$$

Therefore,

$$\sigma^2(U_n) = \binom{n}{2}^{-1}[\binom{2}{2}\binom{n-2}{2-2}\zeta_{2,n} + \binom{2}{1}\binom{n-2}{2-1}\zeta_{1,n}]$$

$$= \binom{n}{2}^{-1}[\zeta_{2,n} + 2(n-2)\zeta_{1,n}]$$

$$= \sum_{S} \zeta_{2(\alpha_1,\alpha_2)} + \sum_{S} \zeta_{1(\alpha_1)\beta_1;\gamma_1} = \sum_{S} [\zeta_{2(\alpha_1,\alpha_2)} + \zeta_{1(\alpha_1)\beta_1;\gamma_1}].$$

A consistent estimator for the variance of U_n has been demonstrated by Sen's property of structural convergence (Sen, 1960). Let

$$V_j = \binom{n-1}{m-1}^{-1} \sum_{i} \phi(\underset{\sim}{y}_{\alpha_i}, \underset{\sim}{y}_j) ,$$

where the summation extends over $1 \le \alpha_1 \le n$, and $\alpha_i \ne j$, for all $i = 1,\ldots,m-1$, such that

$$U_n = \frac{1}{n} \sum_{j=1}^{n} V_j .$$

Then based on these n identically distributed and asymptotically uncorrelated linear components, a consistent estimator of $n*\mathrm{Var}(\gamma[T_1,T_2])$ is

$$s^2 = \frac{1}{n-1} \sum_{j=1}^{n} [V_j - U_n]^2 .$$

2.5 Asymptotic Distribution of the Index

Theorem 2.5.1:

Let $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n$ be n independent, identically distributed random k-vectors, $\underset{\sim}{Y}_i = (Y_i^{(1)},\ldots,Y_i^{(k)})$, i=1,...,n, having a c.d.f. $F(\underset{\sim}{Y})$.

Let $\phi(\underset{\sim}{Y}_i,\underset{\sim}{Y}_j)$, as defined in (2.3.2), be a real-valued function not involving n, ϕ being symmetric in both its vector arguments.

$$U_n = \binom{n}{2}^{-1} \sum_{1\le\alpha_1<\alpha_2\le n} \phi(\underset{\sim}{Y}_{\alpha_1}, \underset{\sim}{Y}_{\alpha_2}) \ .$$

Then $\sqrt{n}\ (U_n-\gamma)$ tends, as $n\to\infty$, to the univariate normal density function with zero mean and variance $4\zeta_1$, as defined in (2.4.2).

Proof:

Since $E_F\{\phi(\underset{\sim}{Y}_i,\underset{\sim}{Y}_j)\} = \gamma$, and $0\le E_F\{\phi(Y_i,Y_j)\} \le 1$, and

$$E_F\{\phi^2(\underset{\sim}{Y}_i,\underset{\sim}{Y}_j)\} = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \phi^2(\underset{\sim}{y}_i,\underset{\sim}{y}_j)dF(y_i^{(1)},\ldots,y_i^{(k)})dF(y_j^{(1)},\ldots,y_j^{(k)})$$

$$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{y_i^{(1)}}^{\infty}\cdots\int_{y_i^{(k)}}^{\infty} dF(y_j^{(1)},\ldots,y_j^{(k)})dF(y_i^{(1)},\ldots,y_i^{(k)})$$

$$+ \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{-\infty}^{y_i^{(1)}}\cdots\int_{-\infty}^{y_i^{(k)}} dF(y_j^{(1)},\ldots,y_j^{(k)})dF(y_i^{(1)},\ldots,y_i^{(k)}) < \infty,$$

then $\sqrt{n}\ (U_n-\gamma)$ tends, as n approaches ∞, to the univariate normal density function with mean zero and variance $4\zeta_1$, by Hoeffding (1948) Theorem 7.1,

$$\lim_{n\to\infty} P_F\left\{ \frac{\sqrt{n}(U_n-\gamma)}{2\zeta_1^{1/2}} \le y \right\} = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}}\ e^{-t^2/2}\ dt.$$

Theorem 2.5.2:

Let $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n$ be n independent random k-vectors, having distributions which are not all identical. Let $F_\alpha(\underset{\sim}{Y})$ be the c.d.f. of $\underset{\sim}{Y}_\alpha$, $\alpha = 1,\ldots,n$. Let $\phi(\underset{\sim}{Y}_i,\underset{\sim}{Y}_j)$ be a function symmetric in its 2 vector arguments, which does not involve n, and let

$$\overline{\psi}_{1(\nu)}(\underset{\sim}{y}_\nu) = \binom{n-1}{2-1}^{-1} \sum_{(\neq\nu)}' \psi_{1(\nu)\alpha_1}(\underset{\sim}{y}_1), \ \nu=1,\ldots,n \ ,$$

where $\psi_{1(\nu)\alpha_1}$ is as defined in (2.4.1), and the summation extends over

all subscripts $1 \le \alpha_1 \le n$, $\alpha_1 \neq \nu$, $\nu=1,\ldots,n$. Then, as $n \to \infty$, the limiting distribution of $[U_n - E(U_n)]/\sigma(U_n)$ is normal with mean zero and variance one.

Proof:

As a result of Hoeffding (1948) Theorem 8.1, the asymptotic normality of $[U_n - E(U_n)]/\sigma(U_n)$ follows from

$$\text{i)} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \phi^2(\underset{\sim}{y}_i, \underset{\sim}{y}_j) dF(\underset{\sim}{y}_i) dF(\underset{\sim}{y}_j) \le 1 , \tag{2.5.1}$$

$$\text{ii)} \; E|\bar{\psi}^3_{1(\nu)}(\underset{\sim}{Y}_\nu)| \le 1 , \tag{2.5.2}$$

$$\text{iii)} \lim_{n\to\infty} \sum_{\nu=1}^{n} E|\bar{\psi}^3_{1(\nu)}(\underset{\sim}{Y}_\nu)| / \{\sum_{\nu=1}^{n} E[\bar{\psi}^2_{1(\nu)}(\underset{\sim}{Y}_{(\nu)})]\}^{3/2} = 0 . \tag{2.5.3}$$

The first two conditions, (2.5.1) and (2.5.2), follow immediately. To show (2.5.3), assume

$$E\bar{\psi}^2_{1(\nu)}(\underset{\sim}{Y}_\nu) = 0 \quad \text{for all} \quad \nu=1,\ldots,n .$$

Therefore, $E[\frac{1}{n-1} \sum_{\alpha_i \neq \nu} \psi_{1(\nu)\alpha_i}(\underset{\sim}{Y}_\nu)]^2 = 0$, by definition,

$E[\sum_{\alpha_i \neq \nu} \psi_{1(\nu)\alpha_i}(\underset{\sim}{Y}_{(\nu)})]^2 = 0$, and

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \phi(\underset{\sim}{y}_\nu, \underset{\sim}{y}_{\alpha_i}) dF_{\alpha_i}(\underset{\sim}{y}_{\alpha_i}) - \gamma_{\nu,\alpha_i}] dF_{\nu(\underset{\sim}{y}_\nu)} = 0,$$

which implies $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \phi(\underset{\sim}{y}_\nu, \underset{\sim}{y}_{\alpha_i}) dF_{\alpha_i}(\underset{\sim}{y}_{\alpha_i}) \equiv \gamma_{\nu,\alpha_i}$ for all $\nu=1,\ldots,n$, $\alpha_i=1,\ldots,n$, $\nu \neq \alpha_i$ which cannot hold. Thus,

$$E\bar{\psi}^2_{1(\nu)\alpha_i}(\underset{\sim}{Y}_\nu) \neq 0 \quad \text{for all} \quad \nu=1,\ldots,n ,$$

and

$$\lim_{n\to\infty} \sum_{\alpha_i \neq \nu} E\bar{\psi}^2_{1(\nu)}(\underset{\sim}{Y}_\nu) \to \infty ,$$

which implies

$$\lim_{n\to\infty} \sum_{\nu=1}^{n} E|\bar{\psi}_{1(\nu)}^{3}(\underset{\sim}{Y}_{\nu})|/\{\sum_{\nu=1}^{n} E[\bar{\psi}_{1(\nu)}^{2}(\underset{\sim}{Y}_{(\nu)})]\}^{3/2} = 0 .$$

Let $\underset{\sim}{\gamma}(F) = \gamma_1(F),\ldots,\gamma_s(F))$ be a vector of functionals, and let the corresponding vector of U-statistics from a sample of size n be $\underset{\sim}{U}_n = (U_n^{(1)},\ldots,U_n^{(s)})$, each of degree m_i, i=1,...,s. Puri and Sen (1971) have shown that, under certain regularity conditions,

$$(n^{1/2})[U_n^{(i)}-\gamma_i(F)]/m_i, \quad i=1,\ldots,s ,$$

have a joint asymptotic distribution which is multivariate normal with mean vector zero, and covariance matrix, $\underset{\sim}{\Sigma}$, with elements

$$((\sigma_{ij})) = E_F\{\psi_1^{(i)}(\underset{\sim}{y})\psi_1^{(j)}(\underset{\sim}{y}) , \quad i,j=1,\ldots,s \qquad (2.5.4)$$

where $\psi_1^{(i)}(\underset{\sim}{Y})$ is given for a specific i in (2.4.1). Given this asymptotic multinormal distribution, linear contrasts of the U-statistics can be formed, and based on the normality of these contrasts, hypotheses can be tested.

For example, as outlined in Davis and Quade (1968), let $\underset{\sim}{\gamma}(F) = (\gamma_1(F),\gamma_2(F))'$ and $\underset{\sim}{U}_n = (U_n^{(1)},U_n^{(2)})'$ with

$$\underset{\sim}{\Sigma} = \begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{bmatrix}$$

as defined in (2.5.4). Also, let $\underset{\sim}{C} = (1\text{-}1)$, then the statistic

$$z = \frac{\underset{\sim\sim}{cU}_n}{(\underset{\sim\sim\sim}{c\Sigma c'})^{1/2}} = \frac{U_n^{(1)}-U_n^{(2)}}{(\sigma_{11}+\sigma_{22}-2\sigma_{12})^{1/2}}$$

tests the hypothesis H_o: $\gamma_1(F) = \gamma_2(F)$.

Based on Sen (1960), a consistent estimator, $\hat{\underset{\sim}{\Sigma}}$, of $\underset{\sim}{\Sigma}$ can be obtained by the structural components method. Then replacing $\underset{\sim}{\Sigma}$ by its estimator, for large n, the limiting distribution of

$$z = \frac{\underset{\sim\sim}{c}\underset{n}{U}}{(\underset{\sim\sim\sim}{c}\hat{\underset{\sim}{\Sigma}}\underset{\sim\sim}{c}')^{1/2}}$$

is still the standard normal distribution.

CHAPTER III

MODELLING THE TRACKING PHENOMENON

3.1 Introduction

In defining the index of tracking, a functional form

$$\underset{\sim}{Y}_i = f(t,\underset{\sim}{\beta}_i)$$

was assumed for the responses for an individual. Now various methods of modelling the response vector will be examined. These include the usual least squares technique, plus Bayes and empirical Bayes methods. In addition, a method of modelling the tracking phenomenon in the presence of censored observations is discussed.

3.2 Polynomial Growth Curves

When studying change or growth of biological processes or body measurements, the data are responses at different time points from a number of individuals. Observations from one individual are assumed to be independent of those from every other individual; however, random deviations of any one individual's responses from expected responses cannot be assumed uncorrelated. Methods have been established for estimating individual response curves under these conditions, assuming only that the deviations from the true response curves follow a multivariate normal distribution.

Consider for each of n individuals a regression model of the form

$$\underset{\sim}{Y}_i = \underset{\sim}{X}\underset{\sim}{\beta}_i + \underset{\sim}{\varepsilon}_i, \quad i=1,\ldots,n \ , \qquad (3.2.1)$$

where $\underset{\sim}{Y}_i$ is the vector of observations made on the i-th individual, $\underset{\sim}{X}$ is a known matrix of full rank, both $\underset{\sim}{\beta}_i$ and $\underset{\sim}{\varepsilon}_i$ are unobservable random vectors. In addition, assume expectations and dispersion matrices:

$$E(\underset{\sim}{\varepsilon}_i|\underset{\sim}{\beta}_i) = 0,\ D(\underset{\sim}{\varepsilon}_i|\underset{\sim}{\beta}_i) = \sigma^2_{\varepsilon}\underset{\sim}{V}\ , \tag{3.2.2}$$

$$E(\underset{\sim}{\beta}_i) = \underset{\sim}{\beta},\ D(\underset{\sim}{\beta}_i) = \underset{\sim}{F},\ \underset{\sim}{V} \text{ known},\ \underset{\sim}{F} \text{ positive definite.} \tag{3.2.3}$$

Also assume

$$\text{cov}(\underset{\sim}{\beta}_i,\underset{\sim}{\beta}_j) = 0,\ i \neq j, \tag{3.2.4}$$

which corresponds to intra-individual independence. As a consequence of the above,

$$D(\underset{\sim}{Y}_i) = \underset{\sim\sim\sim}{XFX'} + \sigma^2_{\varepsilon}\underset{\sim}{V}\ .$$

Note that this is identical to the model of Grizzle and Allen (1969) or Potthoff and Roy (1964) where individuals are ungrouped, i.e., the design matrix across individuals $\underset{\sim}{A} = \underset{\sim}{I}$ (Grizzle and Allen notation) or $\underset{\sim}{P} = \underset{\sim}{I}$ (Potthoff and Roy notation). Each individual polynomial model is assumed to be of the same degree.

Then the best linear unbiased estimator of $\underset{\sim}{\beta}_i$ is the usual least squares estimator

$$\underset{\sim}{\beta}_i^{(\ell)} = (\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X})^{-1}\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{Y}_i\ .$$

If the model is adequate and the observations follow a multivariate normal distribution as assumed, a statistic proportional to

$$F = (\underset{\sim}{Y}_i'\underset{\sim}{V}^{-1}\underset{\sim}{Y}_i - \underset{\sim}{Y}_i'\underset{\sim}{V}^{-1}\underset{\sim\sim}{X\beta}_i^{(\ell)})n/(n-1)$$

(the sum of squares for error from the regression model) follows the F-distribution. This statistic allows for testing the null hypothesis of an adequate model.

Estimation of the parameters and confidence bands for the mean response curve for all individuals by several methods was discussed by Elston and Grizzle (1962). In addition, hypotheses of the form

$$H_o: \underset{\sim}{C}\underset{\sim}{\beta}_i = 0$$

can be tested under the assumed model.

The assumed within individual model (3.2.1) does not preclude, nor does the sequel, the incorporation of covariates, fixed for the individual. Discussions of growth curve estimation with covariates have been presented in detail by Grizzle and Allen (1969), and others. In this context, the design matrix $\underset{\sim}{X}$ and the vector $\underset{\sim}{\beta}$ are augmented to include continuous and/or categorical covariate values. For example, when modelling an individual's blood pressure response vector, the design matrix can include weight or quetelet values in addition to the r+1 polynomial coefficients that correspond to the r-th degree polynomial model assumed. The discussions and applications in the sequel are presented without covariates, with the understanding that covariates could be included.

In order to compare estimates of the index of tracking with the corresponding population values as given in Tables 2.3.1 and 2.3.2, consider observations, Y_i, equally spaced on $X\varepsilon[-1,1]$, then

$$\begin{aligned}\underset{\sim}{T}_{ij} &= \underset{\sim}{Y}_i - \underset{\sim}{Y}_j \\ &= \underset{\sim}{X}\underset{\sim}{\beta}_i + \underset{\sim}{\varepsilon}_i - \underset{\sim}{X}\underset{\sim}{\beta}_j - \underset{\sim}{\varepsilon}_j \\ &= \underset{\sim}{X}(\underset{\sim}{\beta}_i - \underset{\sim}{\beta}_j) + (\underset{\sim}{\varepsilon}_i - \underset{\sim}{\varepsilon}_j) = \underset{\sim}{X}\underset{\sim}{\beta}_{ij} + \underset{\sim}{\varepsilon}_{ij} ,\end{aligned}$$

where $\underset{\sim}{\beta}_{ij} = \underset{\sim}{\beta}_i - \underset{\sim}{\beta}_j$ and $\underset{\sim}{\varepsilon}_{ij} = \underset{\sim}{\varepsilon}_i - \underset{\sim}{\varepsilon}_j$.

Then from (3.2.1)-(3.2.4), and assuming $\underset{\sim}{V} = \underset{\sim}{I}$,

$$D(\underset{\sim}{T}_{ij}) = 2[\underset{\sim}{X}(\underset{\sim}{F}+\sigma^2_\varepsilon(\underset{\sim}{X}'\underset{\sim}{X})^{-1})\underset{\sim}{X}'+\sigma^2_\varepsilon\underset{\sim}{I}] .$$

In the linear case, the predicted endpoints of each individual's linear model are sufficient to estimate the index of tracking, therefore,

$$\underset{\sim}{X} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

and

$$\gamma(-1,1) = P\{T_{ij}>0 \quad \text{for all} \quad X\varepsilon[-1,1]$$

$$\text{or} \qquad T_{ij}<0 \quad \text{for all} \quad X\varepsilon[-1,1]\} \ .$$

Equivalently, by symmetry,

$$\gamma(-1,1) = 2P\{T_{ij}>0 \quad \text{for all} \quad X\varepsilon[-1,1]\} \ .$$

As in Section 2.3, with linear models the critical relation is

$$\gamma = \frac{1}{2} + \frac{1}{\pi} \sin^{-1}\rho \ .$$

In the case of $\underset{\sim}{F}$ diagonal,

$$\rho = \frac{\sigma_\alpha^2-\sigma_\beta^2}{\sigma_\alpha^2+\sigma_\beta^2+2\sigma_\varepsilon^2} \ ,$$

and when $\underset{\sim}{F}$ is not diagonal,

$$\rho = \frac{\sigma_\alpha^2 - \sigma_\beta^2}{\sqrt{\sigma_\alpha^2+\sigma_\beta^2+2\sigma_\varepsilon^2-2\sigma_{\alpha\beta}}\ \sqrt{\sigma_\alpha^2+\sigma_\beta^2+2\sigma_\varepsilon^2+2\sigma_{\alpha\beta}}}$$

The estimated correlation coefficient and estimated index of tracking are given for diagonal $\underset{\sim}{F}$ and for general $\underset{\sim}{F}$ in Tables 3.2.1 and 3.2.2, respectively. Note that, in the range of interest for tracking, based on least squares estimates, $\hat{\gamma} \leq \gamma$, with equality when $\sigma_\alpha^2 = \sigma_\beta^2$, or when $\sigma_\varepsilon^2 = 0$. However, in the range $0 \leq \gamma \leq 1/2$, the least squares estimates result in $\hat{\gamma} \geq \gamma$. So that relying on least squares estimates results in a slight bias toward the null, $\gamma = 1/2$, not unlike the bias

TABLE 3.2.1

INDEX OF TRACKING FOR LINEAR MODELS

LEAST SQUARES ESTIMATES ASSUMING NORMALITY AND DIAGONAL COVARIANCE STRUCTURE

RHO = (SA - SB)/(SA + SB + 2*SE)
GAMMA = .5 + INV(PHI)*ARCSIN(RHO)

SA	SB	SE	RHO	GAMMA	SA	SB	SE	RHO	GAMMA	SA	SB	SE	RHO	GAMMA
0	0	0	.	.	2	0	0	1.0000	1.00000	4	0	0	1.0000	1.00000
0	0	1	0.0000	0.50000	2	0	1	0.5000	0.66667	4	0	1	0.6667	0.73228
0	0	2	0.0000	0.50000	2	0	2	0.3333	0.60817	4	0	2	0.5000	0.66667
0	0	3	0.0000	0.50000	2	0	3	0.2500	0.58043	4	0	3	0.4000	0.63099
0	0	4	0.0000	0.50000	2	0	4	0.2000	0.56409	4	0	4	0.3333	0.60817
0	0	5	0.0000	0.50000	2	0	5	0.1667	0.55330	4	0	5	0.2857	0.59223
0	1	0	-1.0000	0.00000	2	1	0	0.3333	0.60817	4	1	0	0.6000	0.70483
0	1	1	-0.3333	0.39183	2	1	1	0.2000	0.56409	4	1	1	0.4286	0.64098
0	1	2	-0.2000	0.43591	2	1	2	0.1429	0.54563	4	1	2	0.3333	0.60817
0	1	3	-0.1429	0.45437	2	1	3	0.1111	0.53544	4	1	3	0.2727	0.58793
0	1	4	-0.1111	0.46456	2	1	4	0.0909	0.52898	4	1	4	0.2308	0.57412
0	1	5	-0.0909	0.47102	2	1	5	0.0769	0.52451	4	1	5	0.2000	0.56409
0	2	0	-1.0000	0.00000	2	2	0	0.0000	0.50000	4	2	0	0.3333	0.60817
0	2	1	-0.5000	0.33333	2	2	1	0.0000	0.50000	4	2	1	0.2500	0.58043
0	2	2	-0.3333	0.39183	2	2	2	0.0000	0.50000	4	2	2	0.2000	0.56409
0	2	3	-0.2500	0.41957	2	2	3	0.0000	0.50000	4	2	3	0.1667	0.55330
0	2	4	-0.2000	0.43591	2	2	4	0.0000	0.50000	4	2	4	0.1429	0.54563
0	2	5	-0.1667	0.44670	2	2	5	0.0000	0.50000	4	2	5	0.1250	0.53989
0	3	0	-1.0000	0.00000	2	3	0	-0.2000	0.43591	4	3	0	0.1429	0.54563
0	3	1	-0.6000	0.29517	2	3	1	-0.1429	0.45437	4	3	1	0.1111	0.53544
0	3	2	-0.4286	0.35902	2	3	2	-0.1111	0.46456	4	3	2	0.0909	0.52898
0	3	3	-0.3333	0.39183	2	3	3	-0.0909	0.47102	4	3	3	0.0769	0.52451
0	3	4	-0.2727	0.41207	2	3	4	-0.0769	0.47549	4	3	4	0.0667	0.52124
0	3	5	-0.2308	0.42588	2	3	5	-0.0667	0.47876	4	3	5	0.0588	0.51873
0	4	0	-1.0000	0.00000	2	4	0	-0.3333	0.39183	4	4	0	0.0000	0.50000
0	4	1	-0.6667	0.26772	2	4	1	-0.2500	0.41957	4	4	1	0.0000	0.50000
0	4	2	-0.5000	0.33333	2	4	2	-0.2000	0.43591	4	4	2	0.0000	0.50000
0	4	3	-0.4000	0.36901	2	4	3	-0.1667	0.44670	4	4	3	0.0000	0.50000
0	4	4	-0.3333	0.39183	2	4	4	-0.1429	0.45437	4	4	4	0.0000	0.50000
0	4	5	-0.2857	0.40777	2	4	5	-0.1250	0.46011	4	4	5	0.0000	0.50000
0	5	0	-1.0000	0.00000	2	5	0	-0.4286	0.35902	4	5	0	-0.1111	0.46456
0	5	1	-0.7143	0.24675	2	5	1	-0.3333	0.39183	4	5	1	-0.0909	0.47102
0	5	2	-0.5556	0.31251	2	5	2	-0.2727	0.41207	4	5	2	-0.0769	0.47549
0	5	3	-0.4545	0.34980	2	5	3	-0.2308	0.42588	4	5	3	-0.0667	0.47876
0	5	4	-0.3846	0.37433	2	5	4	-0.2000	0.43591	4	5	4	-0.0588	0.48127
0	5	5	-0.3333	0.39183	2	5	5	-0.1765	0.44353	4	5	5	-0.0526	0.48324
1	0	0	1.0000	1.00000	3	0	0	1.0000	1.00000	5	0	0	1.0000	1.00000
1	0	1	0.3333	0.60817	3	0	1	0.6000	0.70483	5	0	1	0.7143	0.75325
1	0	2	0.2000	0.56409	3	0	2	0.4286	0.64098	5	0	2	0.5556	0.68749
1	0	3	0.1429	0.54563	3	0	3	0.3333	0.60817	5	0	3	0.4545	0.65020
1	0	4	0.1111	0.53544	3	0	4	0.2727	0.58793	5	0	4	0.3846	0.62567
1	0	5	0.0909	0.52898	3	0	5	0.2308	0.57412	5	0	5	0.3333	0.60817
1	1	0	0.0000	0.50000	3	1	0	0.5000	0.66667	5	1	0	0.6667	0.73228
1	1	1	0.0000	0.50000	3	1	1	0.3333	0.60817	5	1	1	0.5000	0.66667
1	1	2	0.0000	0.50000	3	1	2	0.2500	0.58043	5	1	2	0.4000	0.63099
1	1	3	0.0000	0.50000	3	1	3	0.2000	0.56409	5	1	3	0.3333	0.60817
1	1	4	0.0000	0.50000	3	1	4	0.1667	0.55330	5	1	4	0.2857	0.59223
1	1	5	0.0000	0.50000	3	1	5	0.1429	0.54563	5	1	5	0.2500	0.58043
1	2	0	-0.3333	0.39183	3	2	0	0.2000	0.56409	5	2	0	0.4286	0.64098
1	2	1	-0.2000	0.43591	3	2	1	0.1429	0.54563	5	2	1	0.3333	0.60817
1	2	2	-0.1429	0.45437	3	2	2	0.1111	0.53544	5	2	2	0.2727	0.58793
1	2	3	-0.1111	0.46456	3	2	3	0.0909	0.52898	5	2	3	0.2308	0.57412
1	2	4	-0.0909	0.47102	3	2	4	0.0769	0.52451	5	2	4	0.2000	0.56409
1	2	5	-0.0769	0.47549	3	2	5	0.0667	0.52124	5	2	5	0.1765	0.55647
1	3	0	-0.5000	0.33333	3	3	0	0.0000	0.50000	5	3	0	0.2500	0.58043
1	3	1	-0.3333	0.39183	3	3	1	0.0000	0.50000	5	3	1	0.2000	0.56409
1	3	2	-0.2500	0.41957	3	3	2	0.0000	0.50000	5	3	2	0.1667	0.55330
1	3	3	-0.2000	0.43591	3	3	3	0.0000	0.50000	5	3	3	0.1429	0.54563
1	3	4	-0.1667	0.44670	3	3	4	0.0000	0.50000	5	3	4	0.1250	0.53989
1	3	5	-0.1429	0.45437	3	3	5	0.0000	0.50000	5	3	5	0.1111	0.53544
1	4	0	-0.6000	0.29517	3	4	0	-0.1429	0.45437	5	4	0	0.1111	0.53544
1	4	1	-0.4286	0.35902	3	4	1	-0.1111	0.46456	5	4	1	0.0909	0.52898
1	4	2	-0.3333	0.39183	3	4	2	-0.0909	0.47102	5	4	2	0.0769	0.52451
1	4	3	-0.2727	0.41207	3	4	3	-0.0769	0.47549	5	4	3	0.0667	0.52124
1	4	4	-0.2308	0.42588	3	4	4	-0.0667	0.47876	5	4	4	0.0588	0.51873
1	4	5	-0.2000	0.43591	3	4	5	-0.0588	0.48127	5	4	5	0.0526	0.51676
1	5	0	-0.6667	0.26772	3	5	0	-0.2500	0.41957	5	5	0	0.0000	0.50000
1	5	1	-0.5000	0.33333	3	5	1	-0.2000	0.43591	5	5	1	0.0000	0.50000
1	5	2	-0.4000	0.36901	3	5	2	-0.1667	0.44670	5	5	2	0.0000	0.50000
1	5	3	-0.3333	0.39183	3	5	3	-0.1429	0.45437	5	5	3	0.0000	0.50000
1	5	4	-0.2857	0.40777	3	5	4	-0.1250	0.46011	5	5	4	0.0000	0.50000
1	5	5	-0.2500	0.41957	3	5	5	-0.1111	0.46456	5	5	5	0.0000	0.50000

TABLE 3.2.2

INDEX OF TRACKING FOR LINEAR MODELS

LEAST SQUARES ESTIMATES ASSUMING NORMALITY AND GENERAL COVARIANCE STRUCTURE

$$RHO = (SA-SB)/SQRT(SA+SB+2*SE-2*SAB)*SQRT(SA+SB+2*SE+2*SAB)$$
$$GAMMA = .5 + INV(PHI)*ARCSIN(PHO)$$
$$SAB=1$$

SA	SB	SE	PHO	GAMMA	SA	SB	SE	RHO	GAMMA	SA	SB	SE	RHO	GAMMA
0	0	0	.	.	2	0	0	.	.	4	0	0	.	.
0	0	1	.	.	2	0	1	0.5774	0.69591	4	0	1	0.7071	0.75000
0	0	2	0.0000	0.50000	2	0	2	0.3536	0.61503	4	0	2	0.5164	0.67273
0	0	3	0.0000	0.50000	2	0	3	0.2582	0.58313	4	0	3	0.4082	0.63386
0	0	4	0.0000	0.50000	2	0	4	0.2041	0.56543	4	0	4	0.3381	0.60977
0	0	5	0.0000	0.50000	2	0	5	0.1690	0.55406	4	0	5	0.2887	0.59321
0	1	0	.	.	2	1	0	0.4472	0.64758	4	1	0	0.6547	0.72718
0	1	1	-0.4472	0.35242	2	1	1	0.2182	0.57002	4	1	1	0.4472	0.64758
0	1	2	-0.2182	0.42998	2	1	2	0.1491	0.54763	4	1	2	0.3419	0.61106
0	1	3	-0.1491	0.45237	2	1	3	0.1140	0.53635	4	1	3	0.2774	0.58946
0	1	4	-0.1140	0.46365	2	1	4	0.0925	0.52947	4	1	4	0.2335	0.57503
0	1	5	-0.0925	0.47053	2	1	5	0.0778	0.52481	4	1	5	0.2018	0.56468
0	2	0	.	.	2	2	0	0.0000	0.50000	4	2	0	0.3536	0.61503
0	2	1	-0.5774	0.30409	2	2	1	0.0000	0.50000	4	2	1	0.2582	0.58313
0	2	2	-0.3536	0.38497	2	2	2	0.0000	0.50000	4	2	2	0.2041	0.56543
0	2	3	-0.2582	0.41687	2	2	3	0.0000	0.50000	4	2	3	0.1690	0.55406
0	2	4	-0.2041	0.43457	2	2	4	0.0000	0.50000	4	2	4	0.1443	0.54611
0	2	5	-0.1690	0.44594	2	2	5	0.0000	0.50000	4	2	5	0.1260	0.54021
0	3	0	.	.	2	3	0	-0.2182	0.42998	4	3	0	0.1491	0.54763
0	3	1	-0.6547	0.27282	2	3	1	-0.1491	0.45237	4	3	1	0.1140	0.53635
0	3	2	-0.4472	0.35242	2	3	2	-0.1140	0.46365	4	3	2	0.0925	0.52947
0	3	3	-0.3419	0.38894	2	3	3	-0.0925	0.47053	4	3	3	0.0778	0.52481
0	3	4	-0.2774	0.41054	2	3	4	-0.0778	0.47519	4	3	4	0.0673	0.52143
0	3	5	-0.2335	0.42497	2	3	5	-0.0673	0.47857	4	3	5	0.0592	0.51887
0	4	0	.	.	2	4	0	-0.3536	0.38497	4	4	0	0.0000	0.50000
0	4	1	-0.7071	0.25000	2	4	1	-0.2582	0.41687	4	4	1	0.0000	0.50000
0	4	2	-0.5164	0.32727	2	4	2	-0.2041	0.43457	4	4	2	0.0000	0.50000
0	4	3	-0.4082	0.36614	2	4	3	-0.1690	0.44594	4	4	3	0.0000	0.50000
0	4	4	-0.3381	0.39023	2	4	4	-0.1443	0.45389	4	4	4	0.0000	0.50000
0	4	5	-0.2887	0.40679	2	4	5	-0.1260	0.45979	4	4	5	0.0000	0.50000
0	5	0	.	.	2	5	0	-0.4472	0.35242	4	5	0	-0.1140	0.46365
0	5	1	-0.7454	0.23228	2	5	1	-0.3419	0.38894	4	5	1	-0.0925	0.47053
0	5	2	-0.5698	0.30702	2	5	2	-0.2774	0.41054	4	5	2	-0.0778	0.47519
0	5	3	-0.4623	0.34704	2	5	3	-0.2335	0.42497	4	5	3	-0.0673	0.47857
0	5	4	-0.3892	0.37273	2	5	4	-0.2018	0.43532	4	5	4	-0.0592	0.48113
0	5	5	-0.3363	0.39081	2	5	5	-0.1777	0.44313	4	5	5	-0.0529	0.48315
1	0	0	.	.	3	0	0	.	.	5	0	0	.	.
1	0	1	0.4472	0.64758	3	0	1	0.6547	0.72718	5	0	1	0.7454	0.76772
1	0	2	0.2182	0.57002	3	0	2	0.4472	0.64758	5	0	2	0.5698	0.69298
1	0	3	0.1491	0.54763	3	0	3	0.3419	0.61106	5	0	3	0.4623	0.65296
1	0	4	0.1140	0.53635	3	0	4	0.2774	0.58946	5	0	4	0.3892	0.62727
1	0	5	0.0925	0.52947	3	0	5	0.2335	0.57503	5	0	5	0.3363	0.60919
1	1	0	.	.	3	1	0	0.5774	0.69591	5	1	0	0.7071	0.75000
1	1	1	0.0000	0.50000	3	1	1	0.3536	0.61503	5	1	1	0.5164	0.67273
1	1	2	0.0000	0.50000	3	1	2	0.2582	0.58313	5	1	2	0.4082	0.63386
1	1	3	0.0000	0.50000	3	1	3	0.2041	0.56543	5	1	3	0.3381	0.60977
1	1	4	0.0000	0.50000	3	1	4	0.1690	0.55406	5	1	4	0.2987	0.59321
1	1	5	0.0000	0.50000	3	1	5	0.1443	0.54611	5	1	5	0.2520	0.58108
1	2	0	-0.4472	0.35242	3	2	0	0.2182	0.57002	5	2	0	0.4472	0.64758
1	2	1	-0.2182	0.42998	3	2	1	0.1491	0.54763	5	2	1	0.3419	0.61106
1	2	2	-0.1491	0.45237	3	2	2	0.1140	0.53635	5	2	2	0.2774	0.58946
1	2	3	-0.1140	0.46365	3	2	3	0.0925	0.52947	5	2	3	0.2335	0.57503
1	2	4	-0.0925	0.47053	3	2	4	0.0778	0.52481	5	2	4	0.2018	0.56468
1	2	5	-0.0778	0.47519	3	2	5	0.0673	0.52143	5	2	5	0.1777	0.55687
1	3	0	-0.5774	0.30409	3	3	0	0.0000	0.50000	5	3	0	0.2582	0.58313
1	3	1	-0.3536	0.38497	3	3	1	0.0000	0.50000	5	3	1	0.2041	0.56543
1	3	2	-0.2582	0.41687	3	3	2	0.0000	0.50000	5	3	2	0.1690	0.55406
1	3	3	-0.2041	0.43457	3	3	3	0.0000	0.50000	5	3	3	0.1443	0.54611
1	3	4	-0.1690	0.44594	3	3	4	0.0000	0.50000	5	3	4	0.1260	0.54021
1	3	5	-0.1443	0.45389	3	3	5	0.0000	0.50000	5	3	5	0.1118	0.53566
1	4	0	-0.6547	0.27282	3	4	0	-0.1491	0.45237	5	4	0	0.1140	0.53635
1	4	1	-0.4472	0.35242	3	4	1	-0.1140	0.46365	5	4	1	0.0925	0.52947
1	4	2	-0.3419	0.38894	3	4	2	-0.0925	0.47053	5	4	2	0.0778	0.52481
1	4	3	-0.2774	0.41054	3	4	3	-0.0778	0.47519	5	4	3	0.0673	0.52143
1	4	4	-0.2335	0.42497	3	4	4	-0.0673	0.47857	5	4	4	0.0592	0.51887
1	4	5	-0.2018	0.43532	3	4	5	-0.0592	0.48113	5	4	5	0.0529	0.51685
1	5	0	-0.7071	0.25000	3	5	0	-0.2582	0.41687	5	5	0	0.0000	0.50000
1	5	1	-0.5164	0.32727	3	5	1	-0.2041	0.43457	5	5	1	0.0000	0.50000
1	5	2	-0.4082	0.36614	3	5	2	-0.1690	0.44594	5	5	2	0.0000	0.50000
1	5	3	-0.3381	0.39023	3	5	3	-0.1443	0.45389	5	5	3	0.0000	0.50000
1	5	4	-0.2887	0.40679	3	5	4	-0.1260	0.45979	5	5	4	0.0000	0.50000
1	5	5	-0.2520	0.41892	3	5	5	-0.1118	0.46434	5	5	5	0.0000	0.50000

TABLE 3.2.2

INDEX OF TRACKING FOR LINEAR MODELS

LEAST SQUARES ESTIMATES ASSUMING NORMALITY AND GENERAL COVARIANCE STRUCTURE

RHO = (SA-SB)/SQRT(SA+SB+2*SE-2*SAB)*SQRT(SA+SB+2*SE+2*SAB)
GAMMA = .5 + INV(PHI)*ARCSIN(RHO)
SAB=2

SA	SB	SE	RHO	GAMMA	SA	SB	SE	RHO	GAMMA	SA	SB	SE	RHO	GAMMA
0	0	0	.	.	2	0	0	.	.	4	0	0	.	.
0	0	1	.	.	2	0	1	.	.	4	0	1	0.8944	0.85242
0	0	2	.	.	2	0	2	0.4472	0.64758	4	0	2	0.5774	0.69591
0	0	3	0.0000	0.50000	2	0	3	0.2887	0.59321	4	0	3	0.4364	0.64376
0	0	4	0.0000	0.50000	2	0	4	0.2182	0.57002	4	0	4	0.3536	0.61503
0	0	5	0.0000	0.50000	2	0	5	0.1768	0.55657	4	0	5	0.2981	0.59637
0	1	0	.	.	2	1	0	.	.	4	1	0	1.0000	1.00000
0	1	1	.	.	2	1	1	0.3333	0.60817	4	1	1	0.5222	0.67490
0	1	2	-0.3333	0.39183	2	1	2	0.1741	0.55569	4	1	2	0.3721	0.62136
0	1	3	-0.1741	0.44431	2	1	3	0.1240	0.53958	4	1	3	0.2928	0.59458
0	1	4	-0.1240	0.46042	2	1	4	0.0976	0.53111	4	1	4	0.2425	0.57798
0	1	5	-0.0976	0.46889	2	1	5	0.0808	0.52576	4	1	5	0.2075	0.56654
0	2	0	.	.	2	2	0	.	.	4	2	0	0.4472	0.64758
0	2	1	.	.	2	2	1	0.0000	0.50000	4	2	1	0.2887	0.59321
0	2	2	-0.4472	0.35242	2	2	2	0.0000	0.50000	4	2	2	0.2182	0.57002
0	2	3	-0.2887	0.40679	2	2	3	0.0000	0.50000	4	2	3	0.1768	0.55657
0	2	4	-0.2182	0.42998	2	2	4	0.0000	0.50000	4	2	4	0.1491	0.54763
0	2	5	-0.1768	0.44343	2	2	5	0.0000	0.50000	4	2	5	0.1291	0.54121
0	3	0	.	.	2	3	0	-0.3333	0.39183	4	3	0	0.1741	0.55569
0	3	1	-1.0000	0.00000	2	3	1	-0.1741	0.44431	4	3	1	0.1240	0.53958
0	3	2	-0.5222	0.32510	2	3	2	-0.1240	0.46042	4	3	2	0.0976	0.53111
0	3	3	-0.3721	0.37864	2	3	3	-0.0976	0.46889	4	3	3	0.0808	0.52576
0	3	4	-0.2928	0.40542	2	3	4	-0.0808	0.47424	4	3	4	0.0692	0.52204
0	3	5	-0.2425	0.42202	2	3	5	-0.0692	0.47796	4	3	5	0.0605	0.51928
0	4	0	.	.	2	4	0	-0.4472	0.35242	4	4	0	0.0000	0.50000
0	4	1	-0.8944	0.14758	2	4	1	-0.2887	0.40679	4	4	1	0.0000	0.50000
0	4	2	-0.5774	0.30409	2	4	2	-0.2182	0.42998	4	4	2	0.0000	0.50000
0	4	3	-0.4364	0.35624	2	4	3	-0.1768	0.44343	4	4	3	0.0000	0.50000
0	4	4	-0.3536	0.38497	2	4	4	-0.1491	0.45237	4	4	4	0.0000	0.50000
0	4	5	-0.2981	0.40363	2	4	5	-0.1291	0.45879	4	4	5	0.0000	0.50000
0	5	0	.	.	2	5	0	-0.5222	0.32510	4	5	0	-0.1240	0.46042
0	5	1	-0.8704	0.16387	2	5	1	-0.3721	0.37864	4	5	1	-0.0976	0.46889
0	5	2	-0.6202	0.28706	2	5	2	-0.2928	0.40542	4	5	2	-0.0808	0.47424
0	5	3	-0.4880	0.33775	2	5	3	-0.2425	0.42202	4	5	3	-0.0692	0.47796
0	5	4	-0.4042	0.36754	2	5	4	-0.2075	0.43346	4	5	4	-0.0605	0.48072
0	5	5	-0.3459	0.38759	2	5	5	-0.1816	0.44188	4	5	5	-0.0538	0.48285
1	0	0	.	.	3	0	0	.	.	5	0	0	.	.
1	0	1	.	.	3	0	1	1.0000	1.00000	5	0	1	0.8704	0.83613
1	0	2	0.3333	0.60817	3	0	2	0.5222	0.67490	5	0	2	0.6202	0.71294
1	0	3	0.1741	0.55569	3	0	3	0.3721	0.62136	5	0	3	0.4880	0.66225
1	0	4	0.1240	0.53958	3	0	4	0.2928	0.59458	5	0	4	0.4042	0.63246
1	0	5	0.0976	0.53111	3	0	5	0.2425	0.57798	5	0	5	0.3459	0.61241
1	1	0	.	.	3	1	0	.	.	5	1	0	0.8944	0.85242
1	1	1	.	.	3	1	1	0.4472	0.64758	5	1	1	0.5774	0.69591
1	1	2	0.0000	0.50000	3	1	2	0.2887	0.59321	5	1	2	0.4364	0.64376
1	1	3	0.0000	0.50000	3	1	3	0.2182	0.57002	5	1	3	0.3536	0.61503
1	1	4	0.0000	0.50000	3	1	4	0.1768	0.55657	5	1	4	0.2981	0.59637
1	1	5	0.0000	0.50000	3	1	5	0.1491	0.54763	5	1	5	0.2582	0.58313
1	2	0	.	.	3	2	0	0.3333	0.60817	5	2	0	0.5222	0.67490
1	2	1	-0.3333	0.39183	3	2	1	0.1741	0.55569	5	2	1	0.3721	0.62136
1	2	2	-0.1741	0.44431	3	2	2	0.1240	0.53958	5	2	2	0.2928	0.59458
1	2	3	-0.1240	0.46042	3	2	3	0.0976	0.53111	5	2	3	0.2425	0.57798
1	2	4	-0.0976	0.46889	3	2	4	0.0808	0.52576	5	2	4	0.2075	0.56654
1	2	5	-0.0808	0.47424	3	2	5	0.0692	0.52204	5	2	5	0.1816	0.55812
1	3	0	.	.	3	3	0	0.0000	0.50000	5	3	0	0.2887	0.59321
1	3	1	-0.4472	0.35242	3	3	1	0.0000	0.50000	5	3	1	0.2182	0.57002
1	3	2	-0.2887	0.40679	3	3	2	0.0000	0.50000	5	3	2	0.1768	0.55657
1	3	3	-0.2182	0.42998	3	3	3	0.0000	0.50000	5	3	3	0.1491	0.54763
1	3	4	-0.1768	0.44343	3	3	4	0.0000	0.50000	5	3	4	0.1291	0.54121
1	3	5	-0.1491	0.45237	3	3	5	0.0000	0.50000	5	3	5	0.1140	0.53635
1	4	0	-1.0000	0.00000	3	4	0	-0.1741	0.44431	5	4	0	0.1240	0.53958
1	4	1	-0.5222	0.32510	3	4	1	-0.1240	0.46042	5	4	1	0.0976	0.53111
1	4	2	-0.3721	0.37864	3	4	2	-0.0976	0.46889	5	4	2	0.0808	0.52576
1	4	3	-0.2928	0.40542	3	4	3	-0.0808	0.47424	5	4	3	0.0692	0.52204
1	4	4	-0.2425	0.42202	3	4	4	-0.0692	0.47796	5	4	4	0.0605	0.51928
1	4	5	-0.2075	0.43346	3	4	5	-0.0605	0.48072	5	4	5	0.0538	0.51715
1	5	0	-0.8944	0.14758	3	5	0	-0.2887	0.40679	5	5	0	0.0000	0.50000
1	5	1	-0.5774	0.30409	3	5	1	-0.2182	0.42998	5	5	1	0.0000	0.50000
1	5	2	-0.4364	0.35624	3	5	2	-0.1768	0.44343	5	5	2	0.0000	0.50000
1	5	3	-0.3536	0.38497	3	5	3	-0.1491	0.45237	5	5	3	0.0000	0.50000
1	5	4	-0.2981	0.40363	3	5	4	-0.1291	0.45879	5	5	4	0.0000	0.50000
1	5	5	-0.2582	0.41687	3	5	5	-0.1140	0.46365	5	5	5	0.0000	0.50000

TABLE 3.2.2

INDEX OF TRACKING FOR LINEAR MODELS

LEAST SQUARES ESTIMATES ASSUMING NORMALITY AND GENERAL COVARIANCE STRUCTURE

RHO = (SA-SB)/SQRT(SA+SB+2*SE-2*SAB)*SQRT(SA+SB+2*SE+2*SAB)
GAMMA = .5 + INV(PHI)*ARCSIN(RHO)
SAB=3

SA	SB	SE	RHO	GAMMA	SA	SB	SE	RHO	GAMMA	SA	SB	SE	RHO	GAMMA
0	0	0	.	.	2	0	0	.	.	4	0	0	.	.
0	0	1	.	.	2	0	1	.	.	4	0	1	.	.
0	0	2	.	.	2	0	2	.	.	4	0	2	0.7559	0.77281
0	0	3	.	.	2	0	3	0.3780	0.62338	4	0	3	0.5000	0.66667
0	0	4	0.0000	0.50000	2	0	4	0.2500	0.58043	4	0	4	0.3849	0.62576
0	0	5	0.0000	0.50000	2	0	5	0.1925	0.56164	4	0	5	0.3162	0.60242
0	1	0	.	.	2	1	0	.	.	4	1	0	.	.
0	1	1	.	.	2	1	1	.	.	4	1	1	0.8321	0.81283
0	1	2	.	.	2	1	2	0.2774	0.58946	4	1	2	0.4472	0.64758
0	1	3	-0.2774	0.41054	2	1	3	0.1491	0.54763	4	1	3	0.3254	0.60550
0	1	4	-0.1491	0.45237	2	1	4	0.1085	0.53459	4	1	4	0.2601	0.58377
0	1	5	-0.1085	0.46541	2	1	5	0.0867	0.52764	4	1	5	0.2182	0.57002
0	2	0	.	.	2	2	0	.	.	4	2	0	.	.
0	2	1	.	.	2	2	1	.	.	4	2	1	0.3780	0.62338
0	2	2	.	.	2	2	2	0.0000	0.50000	4	2	2	0.2500	0.58043
0	2	3	-0.3780	0.37662	2	2	3	0.0000	0.50000	4	2	3	0.1925	0.56164
0	2	4	-0.2500	0.41957	2	2	4	0.0000	0.50000	4	2	4	0.1581	0.55054
0	2	5	-0.1925	0.43836	2	2	5	0.0000	0.50000	4	2	5	0.1348	0.54305
0	3	0	.	.	2	3	0	.	.	4	3	0	0.2774	0.58946
0	3	1	.	.	2	3	1	-0.2774	0.41054	4	3	1	0.1491	0.54763
0	3	2	-0.8321	0.18717	2	3	2	-0.1491	0.45237	4	3	2	0.1085	0.53459
0	3	3	-0.4472	0.35242	2	3	3	-0.1085	0.46541	4	3	3	0.0867	0.52764
0	3	4	-0.3254	0.39450	2	3	4	-0.0867	0.47236	4	3	4	0.0727	0.52317
0	3	5	-0.2601	0.41623	2	3	5	-0.0727	0.47683	4	3	5	0.0629	0.52003
0	4	0	.	.	2	4	0	.	.	4	4	0	0.0000	0.50000
0	4	1	.	.	2	4	1	-0.3780	0.37662	4	4	1	0.0000	0.50000
0	4	2	-0.7559	0.22719	2	4	2	-0.2500	0.41957	4	4	2	0.0000	0.50000
0	4	3	-0.5000	0.33333	2	4	3	-0.1925	0.43836	4	4	3	0.0000	0.50000
0	4	4	-0.3849	0.37424	2	4	4	-0.1581	0.44946	4	4	4	0.0000	0.50000
0	4	5	-0.3162	0.39758	2	4	5	-0.1348	0.45695	4	4	5	0.0000	0.50000
0	5	0	.	.	2	5	0	-0.8321	0.18717	4	5	0	-0.1491	0.45237
0	5	1	.	.	2	5	1	-0.4472	0.35242	4	5	1	-0.1085	0.46541
0	5	2	-0.7454	0.23228	2	5	2	-0.3254	0.39450	4	5	2	-0.0867	0.47236
0	5	3	-0.5423	0.31754	2	5	3	-0.2601	0.41623	4	5	3	-0.0727	0.47683
0	5	4	-0.4336	0.35726	2	5	4	-0.2182	0.42998	4	5	4	-0.0629	0.47997
0	5	5	-0.3637	0.38151	2	5	5	-0.1886	0.43960	4	5	5	-0.0555	0.48233
1	0	0	.	.	3	0	0	.	.	5	0	0	.	.
1	0	1	.	.	3	0	1	.	.	5	0	1	.	.
1	0	2	.	.	3	0	2	0.8321	0.81283	5	0	2	0.7454	0.76772
1	0	3	0.2774	0.58946	3	0	3	0.4472	0.64758	5	0	3	0.5423	0.68246
1	0	4	0.1491	0.54763	3	0	4	0.3254	0.60550	5	0	4	0.4336	0.64274
1	0	5	0.1085	0.53459	3	0	5	0.2601	0.58377	5	0	5	0.3637	0.61849
1	1	0	.	.	3	1	0	.	.	5	1	0	.	.
1	1	1	.	.	3	1	1	.	.	5	1	1	0.7559	0.77281
1	1	2	.	.	3	1	2	0.3780	0.62338	5	1	2	0.5000	0.66667
1	1	3	0.0000	0.50000	3	1	3	0.2500	0.58043	5	1	3	0.3849	0.62576
1	1	4	0.0000	0.50000	3	1	4	0.1925	0.56164	5	1	4	0.3162	0.60242
1	1	5	0.0000	0.50000	3	1	5	0.1581	0.55054	5	1	5	0.2697	0.58692
1	2	0	.	.	3	2	0	.	.	5	2	0	0.8321	0.81283
1	2	1	.	.	3	2	1	0.2774	0.58946	5	2	1	0.4472	0.64758
1	2	2	-0.2774	0.41054	3	2	2	0.1491	0.54763	5	2	2	0.3254	0.60550
1	2	3	-0.1491	0.45237	3	2	3	0.1085	0.53459	5	2	3	0.2601	0.58377
1	2	4	-0.1085	0.46541	3	2	4	0.0867	0.52764	5	2	4	0.2182	0.57002
1	2	5	-0.0867	0.47236	3	2	5	0.0727	0.52317	5	2	5	0.1886	0.56040
1	3	0	.	.	3	3	0	.	.	5	3	0	0.3780	0.62338
1	3	1	.	.	3	3	1	0.0000	0.50000	5	3	1	0.2500	0.58043
1	3	2	-0.3780	0.37662	3	3	2	0.0000	0.50000	5	3	2	0.1925	0.56164
1	3	3	-0.2500	0.41957	3	3	3	0.0000	0.50000	5	3	3	0.1581	0.55054
1	3	4	-0.1925	0.43836	3	3	4	0.0000	0.50000	5	3	4	0.1348	0.54305
1	3	5	-0.1581	0.44946	3	3	5	0.0000	0.50000	5	3	5	0.1179	0.53760
1	4	0	.	.	3	4	0	-0.2774	0.41054	5	4	0	0.1491	0.54763
1	4	1	-0.8321	0.18717	3	4	1	-0.1491	0.45237	5	4	1	0.1085	0.53459
1	4	2	-0.4472	0.35242	3	4	2	-0.1085	0.46541	5	4	2	0.0867	0.52764
1	4	3	-0.3254	0.39450	3	4	3	-0.0867	0.47236	5	4	3	0.0727	0.52317
1	4	4	-0.2601	0.41623	3	4	4	-0.0727	0.47683	5	4	4	0.0629	0.52003
1	4	5	-0.2182	0.42998	3	4	5	-0.0629	0.47997	5	4	5	0.0555	0.51767
1	5	0	.	.	3	5	0	-0.3780	0.37662	5	5	0	0.0000	0.50000
1	5	1	-0.7559	0.22719	3	5	1	-0.2500	0.41957	5	5	1	0.0000	0.50000
1	5	2	-0.5000	0.33333	3	5	2	-0.1925	0.43836	5	5	2	0.0000	0.50000
1	5	3	-0.3849	0.37424	3	5	3	-0.1581	0.44946	5	5	3	0.0000	0.50000
1	5	4	-0.3162	0.39758	3	5	4	-0.1348	0.45695	5	5	4	0.0000	0.50000
1	5	5	-0.2697	0.41306	3	5	5	-0.1179	0.46240	5	5	5	0.0000	0.50000

toward $\rho = 0$ encountered when estimating ρ, in the presence of measurement error. This bias toward the null can be seen in Figure 3.2.1.

3.3 Bayes and Empirical Bayes Estimators

The next task is, according to Geisser (1970), to "study the estimation problem of the growth model through Bayesian spectacles." Bayes and empirical Bayes procedures have often been proposed as simultaneous estimators of parameters from several linear models (Geisser, 1970, Efron and Morris, 1971-1973, James and Stein, 1961, Lindley and Smith, 1972). Rao (1975) has shown that with respect to quadratic loss function, empirical Bayes estimators are superior to least squares estimators.

When σ_ε^2, $\underset{\sim}{\beta}$ and $\underset{\sim}{F}$ are known, this prior information can improve the simultaneous estimation of $\underset{\sim}{\beta}_i$, i=1,...,n, particularly in the light of the fact that least squares estimators have been shown inadmissable in dimensions greater than two (James and Stein, 1961). For i=1,...,n,

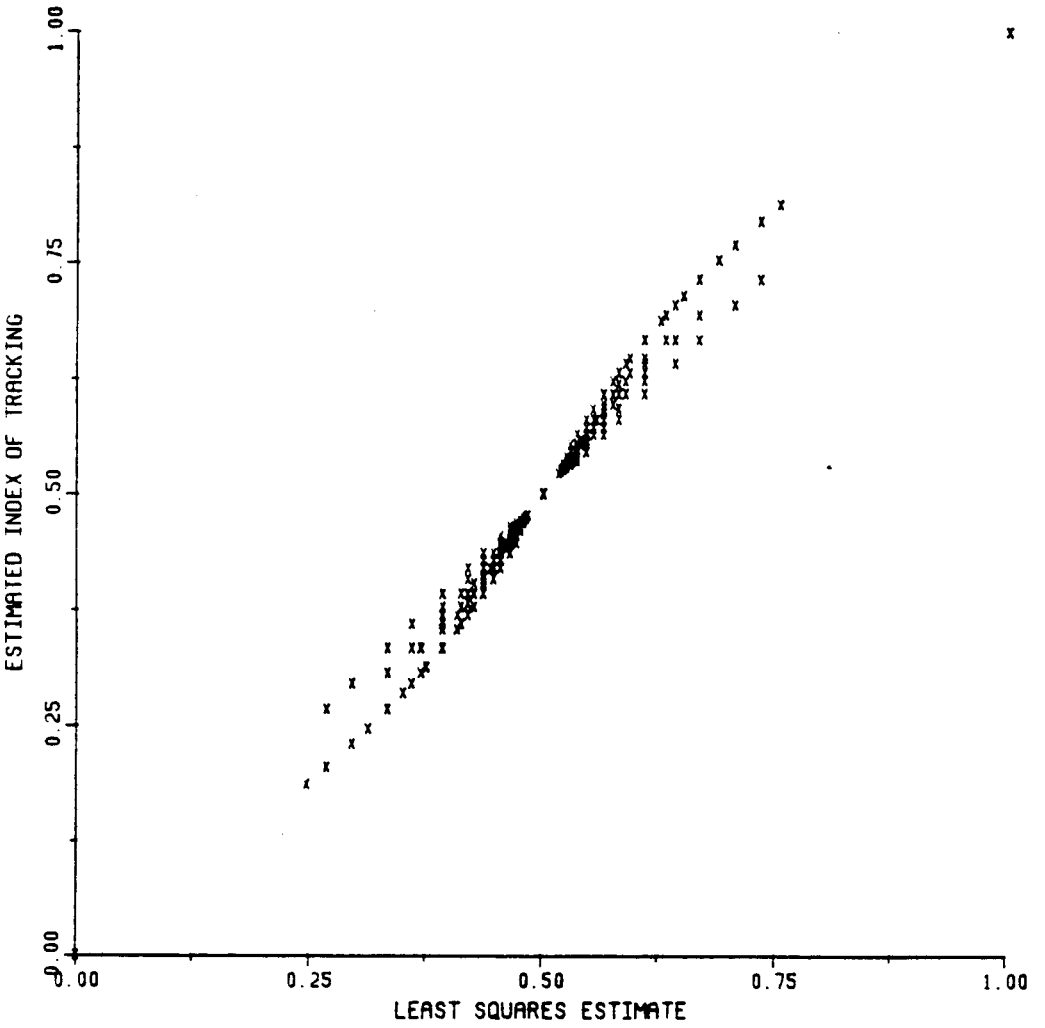
$$\underset{\sim}{\beta}_i^{(b)} = \underset{\sim}{\beta}+\underset{\sim\sim}{F}\underset{\sim\sim}{X}'(\underset{\sim\sim\sim}{X}\underset{\sim\sim}{F}\underset{\sim\sim}{X}'+\sigma_\varepsilon^2\underset{\sim}{V})^{-1}(\underset{\sim}{Y}_i-\underset{\sim\sim}{X}\underset{\sim}{\beta}) \quad , \tag{3.3.1}$$

$$= (\underset{\sim}{I}-\sigma_\varepsilon^2(\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X})^{-1}(\underset{\sim}{F}+\sigma_\varepsilon^2(\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X})^{-1})^{-1})\underset{\sim}{\beta}_i^{(\ell)}+\sigma_\varepsilon^2(\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X})^{-1}(\underset{\sim}{F}+\sigma_\varepsilon^2(\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X})^{-1})^{-1}\underset{\sim}{\beta} \tag{3.3.2}$$

is the estimator of $\underset{\sim}{\beta}_i$, where the prior distribution is described by (3.2.3). Note that $\underset{\sim}{\beta}_i^{(b)}$ is a weighted linear combination of the least squares estimator, $\underset{\sim}{\beta}_i^{(\ell)}$, and its prior mean, $\underset{\sim}{\beta}$, with weights inversely proportional to the variances of $\underset{\sim}{Y}_i$ and $\underset{\sim}{\beta}_i$. As mentioned in Chapter I, the Bayes estimator has minimum mean dispersion error matrix, and for a given $\underset{\sim}{p}$ vector, the mean square error is minimum, and the compound loss

$$E\sum_{i=1}^{n}(\underset{\sim}{p}'\underset{\sim}{\beta}_i-\underset{\sim}{p}'\underset{\sim}{\beta}_i^{(b)})^2 \quad ,$$

FIGURE 3.2.1
ESTIMATED INDEX OF TRACKING
VS. LEAST SQUARES ESTIMATE



is minimal compared to any other linear estimators. However, the optimality of the Bayes estimator over the least square estimator is not uniform over all values of $\underset{\sim}{\beta}_i$. This property led to the suggestion of a compromise between the Bayes estimator and the maximum likelihood, or least squares, estimator, suggested by Efron and Morris (1971, 1972) and others. This compromise estimator will be discussed briefly in Chapter VI.

Estimates of the index of tracking based on Bayes estimates of the individual linear models can also be compared to the population values. The dispersion matrix corresponding to the estimator (3.3.1), assuming $\underset{\sim}{V} = \underset{\sim}{I}$, is:

$$D(\underset{\sim}{\beta}_i^{(b)}) = [\underset{\sim}{I}-\sigma_\varepsilon^2(\underset{\sim}{X}'\underset{\sim}{X})^{-1}(\underset{\sim}{F}+\sigma_\varepsilon^2(\underset{\sim}{X}'\underset{\sim}{X})^{-1})^{-1}](\underset{\sim}{F}+\sigma_\varepsilon^2(\underset{\sim}{X}'\underset{\sim}{X})^{-1})[\underset{\sim}{I}-\sigma_\varepsilon^2(\underset{\sim}{X}'\underset{\sim}{X})^{-1}(\underset{\sim}{F}+\sigma_\varepsilon^2(\underset{\sim}{X}'\underset{\sim}{X})^{-1})^{-1}]$$

$$+ \sigma_\varepsilon^2(\underset{\sim}{X}'\underset{\sim}{X})^{-1}(\underset{\sim}{F}+\sigma_\varepsilon^2(\underset{\sim}{X}'\underset{\sim}{X})^{-1})^{-1}\underset{\sim}{F}(\underset{\sim}{F}+\sigma_\varepsilon^2(\underset{\sim}{X}'\underset{\sim}{X})^{-1})^{-1}(\underset{\sim}{X}'\underset{\sim}{X})^{-1}\sigma_\varepsilon^2 . \quad (3.3.3)$$

Estimates of the index of tracking based on Bayes estimates of the individual linear models can also be compared to the population values as given in Tables 2.3.1 and 2.3.2. Estimates are presented for both the diagonal and general covariance cases, given specific parameter values, in Tables 3.3.1 and 3.3.2.

The empirical Bayes estimators, in the case when σ_ε^2, $\underset{\sim}{\beta}$ and $\underset{\sim}{F}$ are not known, is

$$\underset{\sim}{\beta}_i^{(e)} = \underset{\sim}{\beta}_i^{(\ell)}+cW(\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X})^{-1}\underset{\sim}{B}^{-1}(\underset{\sim}{\beta}_i^{(\ell)}-\underset{\sim}{\beta}_*), \quad i=1,\ldots,n ,$$

based on the usual unbiased estimators of the unknown parameters.

$$n\underset{\sim}{\beta}_* = \sum_{i=1}^{n} \underset{\sim}{\beta}_i^{(\ell)}$$

$$n(k-m)\sigma_*^2 = \sum_{i=1}^{n} (\underset{\sim i\sim}{Y}\underset{\sim}{V}^{-1}\underset{\sim i}{Y}-\underset{\sim i\sim}{Y}\underset{\sim}{V}^{-1}\underset{\sim\sim}{X}\underset{\sim i}{\beta}^{(\ell)}) = W$$

$$(n-1)(\underset{\sim}{F}_*+\sigma_*^2(\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X})^{-1}) = \sum_{i=1}^{n} (\underset{\sim}{\beta}_i^{(\ell)}-\underset{\sim}{\beta}_*)(\underset{\sim}{\beta}_i^{(\ell)}-\underset{\sim}{\beta}_*)' = \underset{\sim}{B} ,$$

TABLE 3.3.1

INDEX OF TRACKING FOR LINEAR MODELS

BAYES ESTIMATES ASSUMING NORMALITY AND DIAGONAL COVARIANCE STRUCTURE

RHO = CALCULATED FROM (3.3.3)
GAMMA = .5 + INV(PHI)*ARCSIN(RHO)

SA	SB	SZ	RHO	GAMMA	SA	SB	SZ	RHO	GAMMA	SA	SB	SZ	RHO	GAMMA
0	0	0	.	.	2	0	0	1.0000	1.00000	4	0	0	1.0000	1.00000
0	0	1	0.0000	0.50000	2	0	1	0.6269	0.71566	4	0	1	0.7828	0.78623
0	0	2	0.0000	0.50000	2	0	2	0.4375	0.64414	4	0	2	0.6269	0.71566
0	0	3	0.0000	0.50000	2	0	3	0.3348	0.60868	4	0	3	0.5166	0.67282
0	0	4	0.0000	0.50000	2	0	4	0.2727	0.58793	4	0	4	0.4375	0.64414
0	0	5	0.0000	0.50000	2	0	5	0.2315	0.57436	4	0	5	0.3791	0.62378
0	1	0	-1.0000	0.00000	2	1	0	0.3333	0.60817	4	1	0	0.6000	0.70483
0	1	1	-0.4375	0.35586	2	1	1	0.2609	0.58403	4	1	1	0.5252	0.67602
0	1	2	-0.2727	0.41207	2	1	2	0.1871	0.55991	4	1	2	0.4272	0.64049
0	1	3	-0.2021	0.43521	2	1	3	0.1423	0.54547	4	1	3	0.3512	0.61422
0	1	4	-0.1628	0.44795	2	1	4	0.1150	0.53670	4	1	4	0.2958	0.59558
0	1	5	-0.1373	0.45615	2	1	5	0.0973	0.53101	4	1	5	0.2551	0.58210
0	2	0	-1.0000	0.00000	2	2	0	0.0000	0.50000	4	2	0	0.3333	0.60817
0	2	1	-0.6269	0.28434	2	2	1	0.0000	0.50000	4	2	1	0.3063	0.59908
0	2	2	-0.4375	0.35586	2	2	2	0.0000	0.50000	4	2	2	0.2609	0.58403
0	2	3	-0.3348	0.39132	2	2	3	0.0000	0.50000	4	2	3	0.2198	0.57055
0	2	4	-0.2727	0.41207	2	2	4	0.0000	0.50000	4	2	4	0.1871	0.55991
0	2	5	-0.2315	0.42564	2	2	5	0.0000	0.50000	4	2	5	0.1618	0.55174
0	3	0	-1.0000	0.00000	2	3	0	-0.2000	0.43591	4	3	0	0.1429	0.54563
0	3	1	-0.7247	0.24197	2	3	1	-0.1793	0.44261	4	3	1	0.1343	0.54289
0	3	2	-0.5493	0.31490	2	3	2	-0.1472	0.45299	4	3	2	0.1183	0.53775
0	3	3	-0.4375	0.35586	2	3	3	-0.1203	0.46162	4	3	3	0.1023	0.53261
0	3	4	-0.3631	0.38173	2	3	4	-0.1003	0.46803	4	3	4	0.0885	0.52821
0	3	5	-0.3109	0.39936	2	3	5	-0.0856	0.47272	4	3	5	0.0773	0.52463
0	4	0	-1.0000	0.00000	2	4	0	-0.3333	0.39183	4	4	0	0.0000	0.50000
0	4	1	-0.7828	0.21377	2	4	1	-0.3063	0.40092	4	4	1	0.0000	0.50000
0	4	2	-0.6269	0.28434	2	4	2	-0.2609	0.41597	4	4	2	0.0000	0.50000
0	4	3	-0.5166	0.32718	2	4	3	-0.2198	0.42945	4	4	3	0.0000	0.50000
0	4	4	-0.4375	0.35586	2	4	4	-0.1871	0.44009	4	4	4	0.0000	0.50000
0	4	5	-0.3791	0.37622	2	4	5	-0.1618	0.44826	4	4	5	0.0000	0.50000
0	5	0	-1.0000	0.00000	2	5	0	-0.4286	0.35902	4	5	0	-0.1111	0.46456
0	5	1	-0.8210	0.19341	2	5	1	-0.4000	0.36901	4	5	1	-0.1068	0.46593
0	5	2	-0.6828	0.26075	2	5	2	-0.3498	0.38624	4	5	2	-0.0978	0.46881
0	5	3	-0.5782	0.30375	2	5	3	-0.3018	0.40242	4	5	3	-0.0878	0.47202
0	5	4	-0.4987	0.33380	2	5	4	-0.2616	0.41576	4	5	4	-0.0783	0.47505
0	5	5	-0.4375	0.35586	2	5	5	-0.2292	0.42638	4	5	5	-0.0700	0.47771
1	0	0	1.0000	1.00000	3	0	0	1.0000	1.00000	5	0	0	1.0000	1.00000
1	0	1	0.4375	0.64414	3	0	1	0.7247	0.75803	5	0	1	0.8210	0.80659
1	0	2	0.2727	0.58793	3	0	2	0.5493	0.68510	5	0	2	0.6828	0.73925
1	0	3	0.2021	0.56479	3	0	3	0.4375	0.64414	5	0	3	0.5782	0.69625
1	0	4	0.1628	0.55205	3	0	4	0.3631	0.61827	5	0	4	0.4987	0.66620
1	0	5	0.1373	0.54385	3	0	5	0.3109	0.60064	5	0	5	0.4375	0.64414
1	1	0	0.0000	0.50000	3	1	0	0.5000	0.66667	5	1	0	0.6667	0.73228
1	1	1	0.0000	0.50000	3	1	1	0.4206	0.63817	5	1	1	0.5985	0.70422
1	1	2	0.0000	0.50000	3	1	2	0.3253	0.60547	5	1	2	0.5039	0.66812
1	1	3	0.0000	0.50000	3	1	3	0.2582	0.58313	5	1	3	0.4258	0.64002
1	1	4	0.0000	0.50000	3	1	4	0.2128	0.56827	5	1	4	0.3656	0.61914
1	1	5	0.0000	0.50000	3	1	5	0.1814	0.55805	5	1	5	0.3194	0.60347
1	2	0	-0.3333	0.39183	3	2	0	0.2000	0.56409	5	2	0	0.4286	0.64098
1	2	1	-0.2609	0.41597	3	2	1	0.1793	0.55739	5	2	1	0.4000	0.63099
1	2	2	-0.1871	0.44009	3	2	2	0.1472	0.54701	5	2	2	0.3498	0.61376
1	2	3	-0.1423	0.45453	3	2	3	0.1203	0.53838	5	2	3	0.3018	0.59758
1	2	4	-0.1150	0.46330	3	2	4	0.1003	0.53197	5	2	4	0.2616	0.58424
1	2	5	-0.0973	0.46899	3	2	5	0.0856	0.52728	5	2	5	0.2292	0.57362
1	3	0	-0.5000	0.33333	3	3	0	0.0000	0.50000	5	3	0	0.2500	0.58043
1	3	1	-0.4206	0.36183	3	3	1	0.0000	0.50000	5	3	1	0.2377	0.57641
1	3	2	-0.3253	0.39453	3	3	2	0.0000	0.50000	5	3	2	0.2137	0.56854
1	3	3	-0.2582	0.41687	3	3	3	0.0000	0.50000	5	3	3	0.1883	0.56031
1	3	4	-0.2128	0.43173	3	3	4	0.0000	0.50000	5	3	4	0.1657	0.55297
1	3	5	-0.1814	0.44195	3	3	5	0.0000	0.50000	5	3	5	0.1465	0.54680
1	4	0	-0.6000	0.29517	3	4	0	-0.1429	0.45437	5	4	0	0.1111	0.53544
1	4	1	-0.5252	0.32398	3	4	1	-0.1343	0.45711	5	4	1	0.1068	0.53407
1	4	2	-0.4272	0.35951	3	4	2	-0.1183	0.46225	5	4	2	0.0978	0.53119
1	4	3	-0.3512	0.38578	3	4	3	-0.1023	0.46739	5	4	3	0.0878	0.52798
1	4	4	-0.2958	0.40442	3	4	4	-0.0885	0.47179	5	4	4	0.0783	0.52495
1	4	5	-0.2551	0.41790	3	4	5	-0.0773	0.47537	5	4	5	0.0700	0.52229
1	5	0	-0.6667	0.26772	3	5	0	-0.2500	0.41957	5	5	0	0.0000	0.50000
1	5	1	-0.5985	0.29578	3	5	1	-0.2377	0.42359	5	5	1	0.0000	0.50000
1	5	2	-0.5039	0.33188	3	5	2	-0.2137	0.43146	5	5	2	0.0000	0.50000
1	5	3	-0.4258	0.35998	3	5	3	-0.1883	0.43969	5	5	3	0.0000	0.50000
1	5	4	-0.3656	0.38086	3	5	4	-0.1657	0.44703	5	5	4	0.0000	0.50000
1	5	5	-0.3194	0.39653	3	5	5	-0.1465	0.45320	5	5	5	0.0000	0.50000

TABLE 3.3.2

INDEX OF TRACKING FOR LINEAR MODELS

BAYES ESTIMATES ASSUMING NORMALITY AND DIAGONAL COVARIANCE STRUCTURE

RHO = CALCULATED FROM (3.3.3)
GAMMA = .5 + INV(PHI)*ARCSIN(RHO)
SAB= 1

SA	SB	SE	RHO	GAMMA	SA	SB	SE	RHO	GAMMA	SA	SB	SE	RHO	GAMMA
0	0	0	.	.	2	0	0	.	.	4	0	0	.	.
0	0	1	.	.	2	0	1	.	.	4	0	1	.	.
0	0	2	.	.	2	0	2	0.8944	0.85242	4	0	2	0.7071	0.75000
0	0	3	.	.	2	0	3	0.4572	0.65113	4	0	3	0.5618	0.68388
0	0	4	.	.	2	0	4	0.3352	0.60881	4	0	4	0.4677	0.65491
0	0	5	0.0000	0.50000	2	0	5	0.2702	0.58710	4	0	5	0.4011	0.63138
0	1	0	.	.	2	1	0	0.4472	0.64758	4	1	0	0.6547	0.72718
0	1	1	.	.	2	1	1	0.2869	0.59262	4	1	1	0.5508	0.68568
0	1	2	.	.	2	1	2	0.1952	0.56253	4	1	2	0.4375	0.64414
0	1	3	-0.6748	0.26424	2	1	3	0.1480	0.54727	4	1	3	0.3570	0.61620
0	1	4	-0.2708	0.41272	2	1	4	0.1200	0.53827	4	1	4	0.3002	0.59706
0	1	5	-0.1897	0.43924	2	1	5	0.1016	0.53240	4	1	5	0.2590	0.58340
0	2	0	.	.	2	2	0	0.0000	0.50000	4	2	0	0.3536	0.61503
0	2	1	.	.	2	2	1	0.0000	0.50000	4	2	1	0.3194	0.60348
0	2	2	-0.8944	0.14758	2	2	2	-0.0000	0.50000	4	2	2	0.2671	0.58607
0	2	3	-0.4572	0.34887	2	2	3	0.0000	0.50000	4	2	3	0.2228	0.57152
0	2	4	-0.3352	0.39119	2	2	4	0.0000	0.50000	4	2	4	0.1888	0.56045
0	2	5	-0.2702	0.41290	2	2	5	0.0000	0.50000	4	2	5	0.1631	0.55214
0	3	0	.	.	2	3	0	-0.2182	0.42998	4	3	0	0.1491	0.54763
0	3	1	.	.	2	3	1	-0.1897	0.43924	4	3	1	0.1389	0.54437
0	3	2	-0.6963	0.25482	2	3	2	-0.1513	0.45166	4	3	2	0.1209	0.53857
0	3	3	-0.5058	0.33121	2	3	3	-0.1221	0.46104	4	3	3	0.1036	0.53303
0	3	4	-0.4048	0.36736	2	3	4	-0.1013	0.46768	4	3	4	0.0892	0.52844
0	3	5	-0.3396	0.38972	2	3	5	-0.0865	0.47244	4	3	5	0.0778	0.52478
0	4	0	.	.	2	4	0	-0.3536	0.38497	4	4	0	0.0000	0.50000
0	4	1	.	.	2	4	1	-0.3194	0.39652	4	4	1	0.0000	0.50000
0	4	2	-0.7071	0.25000	2	4	2	-0.2671	0.41393	4	4	2	0.0000	0.50000
0	4	3	-0.5618	0.31012	2	4	3	-0.2228	0.42848	4	4	3	0.0000	0.50000
0	4	4	-0.4677	0.34509	2	4	4	-0.1888	0.43955	4	4	4	0.0000	0.50000
0	4	5	-0.4011	0.36862	2	4	5	-0.1631	0.44786	4	4	5	0.0000	0.50000
0	5	0	.	.	2	5	0	-0.4472	0.35242	4	5	0	-0.1140	0.46365
0	5	1	-0.9526	0.09842	2	5	1	-0.4131	0.36446	4	5	1	-0.1092	0.46518
0	5	2	-0.7350	0.23720	2	5	2	-0.3567	0.38389	4	5	2	-0.0994	0.46831
0	5	3	-0.6108	0.29084	2	5	3	-0.3054	0.40121	4	5	3	-0.0887	0.47171
0	5	4	-0.5218	0.32525	2	5	4	-0.2637	0.41507	4	5	4	-0.0789	0.47486
0	5	5	-0.4550	0.34965	2	5	5	-0.2307	0.42591	4	5	5	-0.0703	0.47759
1	0	0	.	.	3	0	0	.	.	5	0	0	.	.
1	0	1	.	.	3	0	1	.	.	5	0	1	0.9526	0.90158
1	0	2	.	.	3	0	2	0.6963	0.74518	5	0	2	0.7350	0.76280
1	0	3	0.6748	0.73576	3	0	3	0.5058	0.66879	5	0	3	0.6108	0.70916
1	0	4	0.2708	0.58728	3	0	4	0.4048	0.63264	5	0	4	0.5218	0.67475
1	0	5	0.1897	0.56076	3	0	5	0.3396	0.61028	5	0	5	0.4550	0.65035
1	1	0	.	.	3	1	0	0.5774	0.69591	5	1	0	0.7071	0.75000
1	1	1	0.0000	0.50000	3	1	1	0.4494	0.64836	5	1	1	0.6200	0.71288
1	1	2	0.0000	0.50000	3	1	2	0.3352	0.60881	5	1	2	0.5138	0.67178
1	1	3	0.0000	0.50000	3	1	3	0.2640	0.58504	5	1	3	0.4315	0.64203
1	1	4	0.0000	0.50000	3	1	4	0.2176	0.56983	5	1	4	0.3698	0.62057
1	1	5	0.0000	0.50000	3	1	5	0.1857	0.55945	5	1	5	0.3229	0.60467
1	2	0	-0.4472	0.35242	3	2	0	0.2182	0.57002	5	2	0	0.4472	0.64758
1	2	1	-0.2869	0.40738	3	2	1	0.1897	0.56076	5	2	1	0.4131	0.63554
1	2	2	-0.1952	0.43747	3	2	2	0.1513	0.54834	5	2	2	0.3567	0.61611
1	2	3	-0.1480	0.45273	3	2	3	0.1221	0.53896	5	2	3	0.3054	0.59879
1	2	4	-0.1200	0.46173	3	2	4	0.1013	0.53232	5	2	4	0.2637	0.58493
1	2	5	-0.1016	0.46760	3	2	5	0.0865	0.52756	5	2	5	0.2307	0.57409
1	3	0	-0.5774	0.30409	3	3	0	0.0000	0.50000	5	3	0	0.2582	0.58313
1	3	1	-0.4494	0.35164	3	3	1	0.0000	0.50000	5	3	1	0.2441	0.57851
1	3	2	-0.3352	0.39119	3	3	2	0.0000	0.50000	5	3	2	0.2176	0.56981
1	3	3	-0.2640	0.41496	3	3	3	0.0000	0.50000	5	3	3	0.1906	0.56103
1	3	4	-0.2176	0.43017	3	3	4	0.0000	0.50000	5	3	4	0.1669	0.55338
1	3	5	-0.1857	0.44055	3	3	5	0.0000	0.50000	5	3	5	0.1473	0.54705
1	4	0	-0.6547	0.27282	3	4	0	-0.1491	0.45237	5	4	0	0.1140	0.53635
1	4	1	-0.5508	0.31432	3	4	1	-0.1389	0.45563	5	4	1	0.1092	0.53482
1	4	2	-0.4375	0.35586	3	4	2	-0.1209	0.46143	5	4	2	0.0994	0.53169
1	4	3	-0.3570	0.38380	3	4	3	-0.1036	0.46697	5	4	3	0.0887	0.52829
1	4	4	-0.3002	0.40294	3	4	4	-0.0892	0.47156	5	4	4	0.0789	0.52514
1	4	5	-0.2590	0.41660	3	4	5	-0.0778	0.47522	5	4	5	0.0703	0.52241
1	5	0	-0.7071	0.25000	3	5	0	-0.2582	0.41687	5	5	0	0.0000	0.50000
1	5	1	-0.6200	0.28712	3	5	1	-0.2441	0.42149	5	5	1	0.0000	0.50000
1	5	2	-0.5138	0.32822	3	5	2	-0.2176	0.43019	5	5	2	0.0000	0.50000
1	5	3	-0.4315	0.35797	3	5	3	-0.1906	0.43897	5	5	3	0.0000	0.50000
1	5	4	-0.3698	0.37943	3	5	4	-0.1669	0.44662	5	5	4	0.0000	0.50000
1	5	5	-0.3229	0.39533	3	5	5	-0.1473	0.45295	5	5	5	0.0000	0.50000

TABLE 3.3.2

INDEX OF TRACKING FOR LINEAR MODELS

BAYES ESTIMATES ASSUMING NORMALITY AND DIAGONAL COVARIANCE STRUCTURE

RHO = CALCULATED FROM (3.3.3)
GAMMA = .5 + INV(PHI)*ARCSIN(RHO)
SAB= 2

SA	SB	SE	RHO	GAMMA	SA	SB	SE	RHO	GAMMA	SA	SB	SE	RHO	GAMMA
0	0	0	.	.	2	0	0	.	.	4	0	0	.	.
0	0	1	.	.	2	0	1	.	.	4	0	1	.	.
0	0	2	.	.	2	0	2	.	.	4	0	2	.	.
0	0	3	.	.	2	0	3	.	.	4	0	3	.	.
0	0	4	.	.	2	0	4	.	.	4	0	4	0.8944	0.85242
0	0	5	.	.	2	0	5	.	.	4	0	5	0.5808	0.69727
0	1	0	.	.	2	1	0	.	.	4	1	0	1.0000	1.00000
0	1	1	.	.	2	1	1	.	.	4	1	1	0.6533	0.72660
0	1	2	.	.	2	1	2	.	.	4	1	2	0.4891	0.66268
0	1	3	.	.	2	1	3	0.3394	0.61023	4	1	3	0.3913	0.62797
0	1	4	.	.	2	1	4	0.1854	0.55937	4	1	4	0.3263	0.60581
0	1	5	.	.	2	1	5	0.1390	0.54439	4	1	5	0.2802	0.59041
0	2	0	.	.	2	2	0	.	.	4	2	0	0.4472	0.64758
0	2	1	.	.	2	2	1	0.0000	0.50000	4	2	1	0.3675	0.61979
0	2	2	.	.	2	2	2	0.0000	0.50000	4	2	2	0.2869	0.59262
0	2	3	.	.	2	2	3	0.0000	0.50000	4	2	3	0.2326	0.57473
0	2	4	.	.	2	2	4	0.0000	0.50000	4	2	4	0.1952	0.56253
0	2	5	.	.	2	2	5	0.0000	0.50000	4	2	5	0.1682	0.55379
0	3	0	.	.	2	3	0	-0.3333	0.39183	4	3	0	0.1741	0.55569
0	3	1	.	.	2	3	1	-0.2298	0.42619	4	3	1	0.1556	0.54972
0	3	2	.	.	2	3	2	-0.1652	0.44716	4	3	2	0.1291	0.54121
0	3	3	.	.	2	3	3	-0.1293	0.45872	4	3	3	0.1077	0.53435
0	3	4	.	.	2	3	4	-0.1066	0.46601	4	3	4	0.0915	0.52916
0	3	5	-0.7255	0.24159	2	3	5	-0.0909	0.47102	4	3	5	0.0793	0.52528
0	4	0	.	.	2	4	0	-0.4472	0.35242	4	4	0	0.0000	0.50000
0	4	1	.	.	2	4	1	-0.3675	0.38021	4	4	1	-0.0000	0.50000
0	4	2	.	.	2	4	2	-0.2869	0.40738	4	4	2	0.0000	0.50000
0	4	3	.	.	2	4	3	-0.2326	0.42527	4	4	3	-0.0000	0.50000
0	4	4	-0.8944	0.14758	2	4	4	-0.1952	0.43747	4	4	4	0.0000	0.50000
0	4	5	-0.5808	0.30273	2	4	5	-0.1682	0.44621	4	4	5	0.0000	0.50000
0	5	0	.	.	2	5	0	-0.5222	0.32510	4	5	0	-0.1240	0.46042
0	5	1	.	.	2	5	1	-0.4589	0.34825	4	5	1	-0.1172	0.46261
0	5	2	.	.	2	5	2	-0.3786	0.37639	4	5	2	-0.1045	0.46668
0	5	3	.	.	2	5	3	-0.3168	0.39740	4	5	3	-0.0917	0.47076
0	5	4	-0.7270	0.24093	2	5	4	-0.2708	0.41272	4	5	4	-0.0807	0.47429
0	5	5	-0.5686	0.30748	2	5	5	-0.2360	0.42416	4	5	5	-0.0715	0.47724
1	0	0	.	.	3	0	0	.	.	5	0	0	.	.
1	0	1	.	.	3	0	1	.	.	5	0	1	.	.
1	0	2	.	.	3	0	2	.	.	5	0	2	.	.
1	0	3	.	.	3	0	3	.	.	5	0	3	.	.
1	0	4	.	.	3	0	4	.	.	5	0	4	0.7270	0.75907
1	0	5	.	.	3	0	5	0.7255	0.75841	5	0	5	0.5686	0.69252
1	1	0	.	.	3	1	0	.	.	5	1	0	0.8944	0.85242
1	1	1	.	.	3	1	1	.	.	5	1	1	0.6919	0.74325
1	1	2	.	.	3	1	2	0.4472	0.64758	5	1	2	0.5512	0.68585
1	1	3	.	.	3	1	3	0.3215	0.60418	5	1	3	0.4571	0.65110
1	1	4	.	.	3	1	4	0.2558	0.58233	5	1	4	0.3898	0.62747
1	1	5	0.0000	0.50000	3	1	5	0.2139	0.56861	5	1	5	0.3398	0.61035
1	2	0	.	.	3	2	0	0.3333	0.60817	5	2	0	0.5222	0.67490
1	2	1	.	.	3	2	1	0.2298	0.57381	5	2	1	0.4589	0.65175
1	2	2	.	.	3	2	2	0.1652	0.55284	5	2	2	0.3786	0.62361
1	2	3	-0.3394	0.38977	3	2	3	0.1293	0.54128	5	2	3	0.3168	0.60260
1	2	4	-0.1854	0.44063	3	2	4	0.1066	0.53399	5	2	4	0.2708	0.58728
1	2	5	-0.1390	0.45561	3	2	5	0.0909	0.52898	5	2	5	0.2360	0.57584
1	3	0	.	.	3	3	0	0.0000	0.50000	5	3	0	0.2887	0.59321
1	3	1	.	.	3	3	1	0.0000	0.50000	5	3	1	0.2664	0.58585
1	3	2	-0.4472	0.35242	3	3	2	0.0000	0.50000	5	3	2	0.2301	0.57391
1	3	3	-0.3215	0.39582	3	3	3	0.0000	0.50000	5	3	3	0.1974	0.56324
1	3	4	-0.2558	0.41767	3	3	4	0.0000	0.50000	5	3	4	0.1708	0.55465
1	3	5	-0.2139	0.43139	3	3	5	-0.0000	0.50000	5	3	5	0.1498	0.54787
1	4	0	-1.0000	0.00000	3	4	0	-0.1741	0.44431	5	4	0	0.1240	0.53958
1	4	1	-0.6533	0.27340	3	4	1	-0.1556	0.45028	5	4	1	0.1172	0.53739
1	4	2	-0.4891	0.33732	3	4	2	-0.1291	0.45879	5	4	2	0.1045	0.53332
1	4	3	-0.3913	0.37203	3	4	3	-0.1077	0.46565	5	4	3	0.0917	0.52924
1	4	4	-0.3263	0.39419	3	4	4	-0.0915	0.47082	5	4	4	0.0807	0.52571
1	4	5	-0.2802	0.40959	3	4	5	-0.0793	0.47472	5	4	5	0.0715	0.52276
1	5	0	-0.8944	0.14758	3	5	0	-0.2887	0.40679	5	5	0	0.0000	0.50000
1	5	1	-0.6919	0.25675	3	5	1	-0.2664	0.41415	5	5	1	0.0000	0.50000
1	5	2	-0.5512	0.31415	3	5	2	-0.2301	0.42609	5	5	2	0.0000	0.50000
1	5	3	-0.4571	0.34890	3	5	3	-0.1974	0.43676	5	5	3	0.0000	0.50000
1	5	4	-0.3898	0.37253	3	5	4	-0.1708	0.44535	5	5	4	0.0000	0.50000
1	5	5	-0.3398	0.38965	3	5	5	-0.1498	0.45213	5	5	5	0.0000	0.50000

TABLE 3.3.2

INDEX OF TRACKING FOR LINEAR MODELS

BAYES ESTIMATES ASSUMING NORMALITY AND DIAGONAL COVARIANCE STRUCTURE

RHO = CALCULATED FROM (3.3.3)
GAMMA = .5 + INV(PHI)*ARCSIN(RHO)
SAB= 3

SA	SB	SZ	RHO	GAMMA	SA	SB	SZ	RHO	GAMMA	SA	SB	SZ	RHO	GAMMA
0	0	0	.	.	2	0	0	.	.	4	0	0	.	.
0	0	1	.	.	2	0	1	.	.	4	0	1	.	.
0	0	2	.	.	2	0	2	.	.	4	0	2	.	.
0	0	3	.	.	2	0	3	.	.	4	0	3	.	.
0	0	4	.	.	2	0	4	.	.	4	0	4	.	.
0	0	5	.	.	2	0	5	.	.	4	0	5	.	.
0	1	0	.	.	2	1	0	.	.	4	1	0	.	.
0	1	1	.	.	2	1	1	.	.	4	1	1	.	.
0	1	2	.	.	2	1	2	.	.	4	1	2	.	.
0	1	3	.	.	2	1	3	.	.	4	1	3	.	.
0	1	4	.	.	2	1	4	.	.	4	1	4	0.9516	0.90059
0	1	5	.	.	2	1	5	.	.	4	1	5	0.4632	0.65330
0	2	0	.	.	2	2	0	.	.	4	2	0	.	.
0	2	1	.	.	2	2	1	.	.	4	2	1	0.6283	0.71623
0	2	2	.	.	2	2	2	.	.	4	2	2	0.3651	0.61895
0	2	3	.	.	2	2	3	.	.	4	2	3	0.2754	0.58881
0	2	4	.	.	2	2	4	.	.	4	2	4	0.2240	0.57192
0	2	5	.	.	2	2	5	0.0000	0.50000	4	2	5	0.1898	0.56079
0	3	0	.	.	2	3	0	.	.	4	3	0	0.2774	0.58946
0	3	1	.	.	2	3	1	.	.	4	3	1	0.1967	0.56303
0	3	2	.	.	2	3	2	.	.	4	3	2	0.1453	0.54642
0	3	3	.	.	2	3	3	-0.2609	0.41599	4	3	3	0.1161	0.53704
0	3	4	.	.	2	3	4	-0.1550	0.45045	4	3	4	0.0971	0.53095
0	3	5	.	.	2	3	5	-0.1193	0.46195	4	3	5	0.0836	0.52665
0	4	0	.	.	2	4	0	.	.	4	4	0	0.0000	0.50000
0	4	1	.	.	2	4	1	-0.6283	0.28377	4	4	1	-0.0000	0.50000
0	4	2	.	.	2	4	2	-0.3651	0.38105	4	4	2	0.0000	0.50000
0	4	3	.	.	2	4	3	-0.2754	0.41119	4	4	3	0.0000	0.50000
0	4	4	.	.	2	4	4	-0.2240	0.42808	4	4	4	0.0000	0.50000
0	4	5	.	.	2	4	5	-0.1898	0.43921	4	4	5	0.0000	0.50000
0	5	0	.	.	2	5	0	-0.8321	0.18717	4	5	0	-0.1491	0.45237
0	5	1	.	.	2	5	1	-0.5622	0.30995	4	5	1	-0.1349	0.45694
0	5	2	.	.	2	5	2	-0.4261	0.35988	4	5	2	-0.1142	0.46357
0	5	3	.	.	2	5	3	-0.3458	0.38760	4	5	3	-0.0971	0.46906
0	5	4	.	.	2	5	4	-0.2918	0.40573	4	5	4	-0.0838	0.47330
0	5	5	.	.	2	5	5	-0.2529	0.41863	4	5	5	-0.0735	0.47658
1	0	0	.	.	3	0	0	.	.	5	0	0	.	.
1	0	1	.	.	3	0	1	.	.	5	0	1	.	.
1	0	2	.	.	3	0	2	.	.	5	0	2	.	.
1	0	3	.	.	3	0	3	.	.	5	0	3	.	.
1	0	4	.	.	3	0	4	.	.	5	0	4	.	.
1	0	5	.	.	3	0	5	.	.	5	0	5	.	.
1	1	0	.	.	3	1	0	.	.	5	1	0	.	.
1	1	1	.	.	3	1	1	.	.	5	1	1	.	.
1	1	2	.	.	3	1	2	.	.	5	1	2	.	.
1	1	3	.	.	3	1	3	.	.	5	1	3	0.8447	0.82024
1	1	4	.	.	3	1	4	.	.	5	1	4	0.5431	0.68274
1	1	5	.	.	3	1	5	.	.	5	1	5	0.4308	0.64178
1	2	0	.	.	3	2	0	.	.	5	2	0	0.8321	0.81283
1	2	1	.	.	3	2	1	.	.	5	2	1	0.5622	0.69005
1	2	2	.	.	3	2	2	.	.	5	2	2	0.4261	0.64012
1	2	3	.	.	3	2	3	0.2609	0.58401	5	2	3	0.3458	0.61240
1	2	4	.	.	3	2	4	0.1550	0.54955	5	2	4	0.2918	0.59427
1	2	5	.	.	3	2	5	0.1193	0.53805	5	2	5	0.2529	0.58137
1	3	0	.	.	3	3	0	.	.	5	3	0	0.3780	0.62338
1	3	1	.	.	3	3	1	0.0000	0.50000	5	3	1	0.3176	0.60287
1	3	2	.	.	3	3	2	0.0000	0.50000	5	3	2	0.2539	0.58171
1	3	3	.	.	3	3	3	-0.0000	0.50000	5	3	3	0.2098	0.56727
1	3	4	.	.	3	3	4	-0.0000	0.50000	5	3	4	0.1785	0.55713
1	3	5	.	.	3	3	5	0.0000	0.50000	5	3	5	0.1554	0.54967
1	4	0	.	.	3	4	0	-0.2774	0.41054	5	4	0	0.1491	0.54763
1	4	1	.	.	3	4	1	-0.1967	0.43697	5	4	1	0.1349	0.54306
1	4	2	.	.	3	4	2	-0.1453	0.45358	5	4	2	0.1142	0.53643
1	4	3	.	.	3	4	3	-0.1161	0.46296	5	4	3	0.0971	0.53094
1	4	4	-0.9516	0.09941	3	4	4	-0.0971	0.46905	5	4	4	0.0838	0.52670
1	4	5	-0.4632	0.34670	3	4	5	-0.0836	0.47335	5	4	5	0.0735	0.52342
1	5	0	.	.	3	5	0	-0.3780	0.37662	5	5	0	0.0000	0.50000
1	5	1	.	.	3	5	1	-0.3176	0.39713	5	5	1	0.0000	0.50000
1	5	2	.	.	3	5	2	-0.2539	0.41829	5	5	2	0.0000	0.50000
1	5	3	-0.8447	0.17976	3	5	3	-0.2098	0.43273	5	5	3	0.0000	0.50000
1	5	4	-0.5431	0.31726	3	5	4	-0.1785	0.44287	5	5	4	0.0000	0.50000
1	5	5	-0.4308	0.35822	3	5	5	-0.1554	0.45033	5	5	5	0.0000	0.50000

where $W \sim \sigma_\varepsilon^2 \chi^2(k_n - m_n)$, $\underset{\sim}{B} \sim W_m(n-1, F + \sigma_\varepsilon^2(\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X})^{-1})$, when $\underset{\sim}{\beta}_i$ and ε_i are assumed to have multinormal distributions. This estimator is a weighted linear combination of the least squares estimators. Rao (1975) has shown that, although the mean dispersion error matrix, or compound loss for the empirical Bayes estimators is larger than that of the Bayes estimators due to estimating the unknown σ_ε^2, $\underset{\sim}{\beta}$ and $\underset{\sim}{F}$, the compound loss for the empirical Bayes estimator is still less than that of the least squares estimator,

$$E \sum_{i=1}^{n} (\underset{\sim}{\beta}_i^{(\ell)} - \underset{\sim}{\beta}_i)(\underset{\sim}{\beta}_i^{(\ell)} - \underset{\sim}{\beta}_i)' > E \sum_{i=1}^{n} (\underset{\sim}{\beta}_i^{(e)} - \underset{\sim}{\beta}_i)(\underset{\sim}{\beta}_i^{(e)} - \underset{\sim}{\beta}_i)'$$

$$> E \sum_{i=1}^{n} (\underset{\sim}{\beta}_i^{(b)} - \underset{\sim}{\beta}_i)(\underset{\sim}{\beta}_i^{(b)} - \underset{\sim}{\beta}_i)' .$$

Given the relation of the compound losses, the Bayes estimator is preferable. A further comparison of the expected value of the index of tracking (Table 2.3.1) with the corresponding least squares (Table 3.2.1) and Bayes (Table 3.3.1) estimates for fixed underlying parameters yields the same results; however, the differences between the Bayes and least squares results are so slight as to be of no practical difference. A few specific comparisons are given in Table 3.3.3.

In every instance the Bayes estimate is the more accurate estimate, so that when a point estimate of the index is desired, the Bayes estimate would be preferable. However, when the tracking hypothesis is the main focus, either estimate would yield the same conclusion. Therefore, for hypothesis testing with complete data, the least squares estimates will be employed throughout the sequel, specifically when examining the tracking hypotheses for the Thousand Aviator Study and the Bogalusa Heart Study.

TABLE 3.3.3

COMPARISON OF LEAST SQUARES AND BAYES ESTIMATES

SA	SB	SE	Expected Value	L. SQ. Estimate	Bayes Estimate
1	2	1	0.41957	0.43591	0.41597
1	2	2	0.43591	0.45437	0.44009
1	2	3	0.44670	0.46456	0.45453
1	2	4	0.45437	0.47102	0.46330
1	2	5	0.46011	0.47549	0.46899
5	1	1	0.69361	0.66667	0.70422
5	1	2	0.66667	0.63099	0.66812
5	1	3	0.64660	0.60817	0.64002
5	1	4	0.63099	0.59223	0.61914
5	1	5	0.61846	0.58043	0.60347

3.4 Censored Observations

As defined earlier, tracking is a function of relative ranking over time. Since relative ranks are affected by losses to follow-up the quantification of tracking is also affected by censored observations. Losses to follow-up may occur due to migration or drop-outs for other reasons, but it is of particular concern when the losses are related to mortality, that is, non-random. For example, when quantifying the degree of tracking of blood pressures over an extended period of time, those individuals with higher levels of blood pressure are at greater risk of cardiovascular disease death, and thus are at greater risk of censoring.

Rao (1975) provides a method for predicting the last r components of a response vector for an individual. Given the model and the assumptions as outlined in Section 3.2, where σ^2_{ε}, $\underset{\sim}{\beta}$, and $\underset{\sim}{F}$ are known, the response vector for individual j, the design matrix, and the dispersion matrix are partitioned as follows:

$$\underset{\sim}{Y}_j = \begin{bmatrix} \underset{\sim}{u}_1 \\ \underset{\sim}{u}_2 \end{bmatrix}, \quad \underset{\sim}{X} = \begin{bmatrix} \underset{\sim}{X}_1 \\ \underset{\sim}{X}_2 \end{bmatrix}, \quad \underset{\sim}{V} = \begin{bmatrix} \underset{\sim}{V}_1 & \underset{\sim}{V}_2 \\ \underset{\sim}{V}_2' & \underset{\sim}{V}_3 \end{bmatrix} \tag{3.4.1}$$

where $\underset{\sim}{u}_1$ is the vector of the first (k-r) responses, $\underset{\sim}{V}_1$ is (k-r)x(k-r) matrix, $\underset{\sim}{V}_3$ is rxr matrix, and $\underset{\sim}{X}$ is partitioned similarly. Then,

$$E(\underset{\sim}{u}_2) = \underset{\sim}{X}_2\underset{\sim}{\beta} + (\underset{\sim}{X}_1\underset{\sim}{F}\underset{\sim}{X}_2' + \sigma^2_{\varepsilon}\underset{\sim}{V}_2)'(\underset{\sim}{X}_1\underset{\sim}{F}\underset{\sim}{X}_1' + \sigma^2_{\varepsilon}\underset{\sim}{V}_1)^{-1}(\underset{\sim}{u}_1 - \underset{\sim}{X}_1\underset{\sim}{\beta}) , \tag{3.4.2}$$

with prediction variance

$$\sigma^2_{\varepsilon}\underset{\sim}{V}_3 + \underset{\sim}{X}_2'\underset{\sim}{F}\underset{\sim}{X}_2 + (\underset{\sim}{X}_1\underset{\sim}{F}\underset{\sim}{X}_2' + \sigma^2_{\varepsilon}\underset{\sim}{V}_2)'(\underset{\sim}{X}_1\underset{\sim}{F}\underset{\sim}{X}_1' + \sigma^2_{\varepsilon}\underset{\sim}{V}_1)^{-1}(\underset{\sim}{X}_1\underset{\sim}{F}\underset{\sim}{X}_2' + \sigma^2_{\varepsilon}\underset{\sim}{V}_2) . \tag{3.4.3}$$

To illustrate the estimation of the index of tracking in the presence of censored observations, 100 5-dimensional observations were simulated such that

$$\underset{\sim}{Y}_i = \underset{\sim\sim}{X}\underset{\sim}{\beta}_i + \underset{\sim}{\varepsilon}_i, \quad i=1,\ldots,100 ,$$

where $\underset{\sim}{\beta}_i$ and $\underset{\sim}{\varepsilon}_i$ multinormal, with expectations and dispersion matrices as given in (3.2.2)-(3.2.4). The two multivariate normal vectors, $\underset{\sim}{\beta}_i$ and $\underset{\sim}{\varepsilon}_i$, were generated by the International Mathematical and Statistical Libraries (IMSL) subroutine GGNSM, which generates multinormal vectors with a specified covariance structure using the triangular factorization method. The matrices

$$\underset{\sim}{X} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1/2 & 1/4 \\ 1 & 0 & 0 \\ 1 & 1/2 & 1/4 \\ 1 & 1 & 1 \end{bmatrix}, \quad \underset{\sim}{\beta} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} ,$$

were fixed throughout the simulation. The covariance structures were fixed to give a high degree of tracking,

$$\underset{\sim}{F} = \begin{bmatrix} .8 & 0 & 0 \\ 0 & .2 & 0 \\ 0 & 0 & .2 \end{bmatrix}, \quad \sigma_\varepsilon^2 = 0.0001, \ \underset{\sim}{V} = \underset{\sim}{I}_{(5)} , \tag{3.4.4}$$

for the first simulation. Next, the same covariance structures were fixed to give a lower degree of tracking,

$$\underset{\sim}{F} = \begin{bmatrix} .4 & 0 & 0 \\ 0 & .6 & 0 \\ 0 & 0 & .2 \end{bmatrix}, \quad \sigma_\varepsilon^2 = 0.0001, \ \underset{\sim}{V} = \underset{\sim}{I}_{(5)} , \tag{3.4.5}$$

for the second simulation, and for the last simulation,

$$\underset{\sim}{F} = \begin{bmatrix} .4 & 0 & 0 \\ 0 & .6 & 0 \\ 0 & 0 & .2 \end{bmatrix}, \ \sigma_\varepsilon^2 \underset{\sim}{V} = \begin{bmatrix} .001 & 0 & 0 & 0 & 0 \\ 0 & .005 & 0 & 0 & 0 \\ 0 & 0 & .01 & 0 & 0 \\ 0 & 0 & 0 & .05 & 0 \\ 0 & 0 & 0 & 0 & .1 \end{bmatrix}. \tag{3.4.6}$$

Estimates of $\gamma[1,5]$ were computed based on the $\underset{\sim}{Y}_i$'s, and on the $\hat{\underset{\sim}{Y}}_i$'s, predicted responses from a least squares quadratic model.

A proportion of the observations were then censored as a function of both a random indicator and a logistic risk function based on initial response, i.e., $Y^{(1)}$. The uniform (0,1) pseudo random number was generated by the IMSL subroutine GGUBS. The arbitrarily chosen logistic risk function,

$$\text{Risk}_{(i)} = (1 + \exp(-0.9264-2.0174*Y_i^{(1)}))^{-1} ,$$

is plotted in Figure 3.4.1. This function was chosen to assign a higher probability of censoring to observations with an elevated initial response, and to effect the desired proportion of censored observations. The censored responses were predicted by (3.4.2). Estimates of $\gamma[-1,1]$ were then computed based on these predictors, and also based solely on the uncensored responses. The results of these simulations are summarized in Table 3.4.1.

These simulations indicate that even in the presence of substantial censoring, an average of over 20% of the total sample censored, Rao's method of predicting the censored responses produces an estimate of the index of tracking which closely approximates the parameter. Also, estimation excluding the censored observations is only viable when the measurement error is minimal.

FIGURE 3.4.1

LOGISTIC RISK FUNCTION

USED IN CENSORING SIMULATION

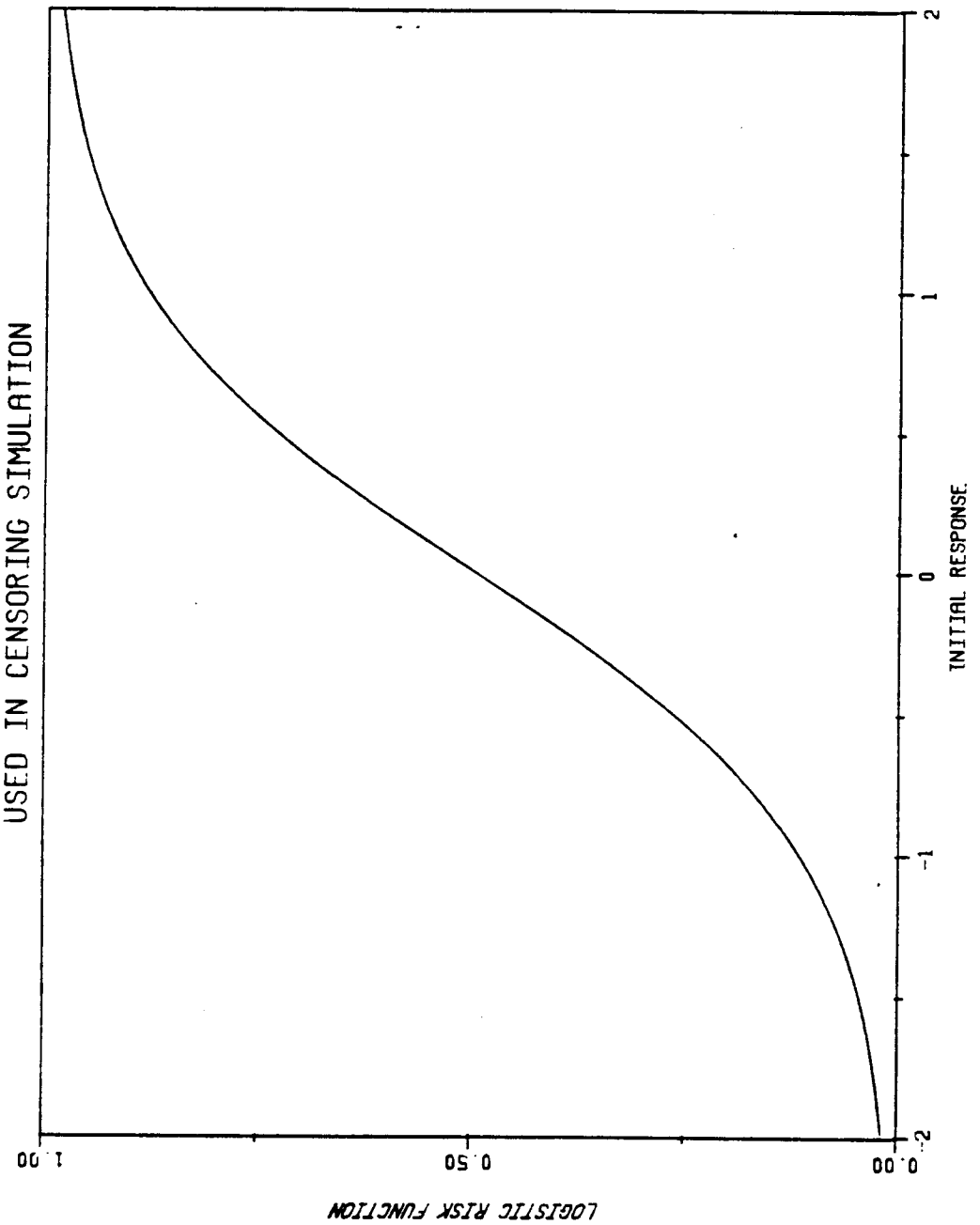


TABLE 3.4.1

SIMULATION OF AN INDEX OF TRACKING ASSUMING A QUADRATIC MODEL

SUMMARY STATISTICS FROM 500 SIMULATED SAMPLES OF 100 OBSERVATIONS

Estimated Parameter	Uncensored Estimate	Censored Estimate	Estimate Excluding Censored Obs.	% Censored at 4th Response	% Censored at 5th Response
		F and $\sigma_\varepsilon^2 V$ given in (3.4.4)			
0.6630	0.6632	0.6634	0.6313	12.4	8.0
		F and $\sigma_\varepsilon^2 V$ given in (3.4.5)			
0.4359	0.4361	0.4362	0.4280	12.4	8.1
		F and $\sigma_\varepsilon^2 V$ given in (3.4.6)			
0.4045	0.4279	0.4312	0.3951	12.4	8.2

CHAPTER IV

THE PROGRESSIVELY TRUNCATED INDEX OF TRACKING

4.1 Introduction

An additional U-statistic can be created by truncating the kernel of a generalized U-statistic at the time point T, resulting in a statistic that is a function of both truncation time and sample size. Then estimators which arise as functions of such U-statistics estimate parameters which vary with time, and represent random processes with specific behaviors.

In some situations, it may no longer be possible or practical to collect all the data, or a preliminary estimate may be desired prior to the completion of the study. In a prospective study, over a 10 year time span, recording yearly responses would allow preliminary estimates during the course of the study. These preliminary estimates would be based on the data at hand, after five years of follow-up, after six years, etc.

The truncated estimate of the parameter of interest can be formed based on the data accumulated to any point in time. The main focus, however, is usually the parameter corresponding to the entire time span, and it is this parameter that will be predicted based on the accumulated portion of the data. With this focus, the progressively truncated estimators follow a continuous path and approach an asymptote with increasing truncation time. Thus, the progressively truncated estimates

and corresponding truncation times provide a series of bivariate observations which can be modelled by a polynomial in truncation time. The full time span parameter would dictate the degree of this polynomial. Extrapolating from this model would result in a prediction of the full time span parameter.

4.2 The Distribution of $U_n(T)$

Let $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n$ be independent and identically distributed random vectors from a population with a continuous distribution function F and let

$$U_n = \binom{n}{2}^{-1} \sum_{1\le i<j\le n} \phi(\underset{\sim}{Y}_i,\underset{\sim}{Y}_j)$$

be a U-statistic of degree 2 with kernel ϕ, as defined in 2.3.1. Then define the truncated estimate of time T, $T\varepsilon[T_1,T_2]$ to be

$$U_n(T) = \binom{n}{2}^{-1} \sum_{1\le i<j\le n} \phi_T(\underset{\sim}{Y}_i,\underset{\sim}{Y}_j) \ , \qquad (4.2.1)$$

where

$$\phi_T(\underset{\sim}{Y}_i,\underset{\sim}{Y}_j) = \begin{cases} 1 & \text{if } f(t,\underset{\sim}{\beta}_i) \ge f(t,\underset{\sim}{\beta}_j) \text{ for all } t\varepsilon[T_1,T] \\ & \text{or } f(t,\underset{\sim}{\beta}_i) \le f(t,\underset{\sim}{\beta}_j) \text{ for all } t\varepsilon[T_1,T] \\ 0 & \text{otherwise} \end{cases} . \qquad (4.2.2)$$

Then, $U_n(T)$ is unbiased for

$$\gamma(T) = \gamma(T_1,T) = P\{f(t,\underset{\sim}{\beta}_i)\le f(t,\underset{\sim}{\beta}_j) \text{ for all } t\varepsilon[T_1,T]$$
$$\text{or } f(t,\underset{\sim}{\beta}_i)\ge f(t,\underset{\sim}{\beta}_j) \text{ for all } t\varepsilon[T_1,T]\}. \qquad (4.2.3)$$

Note that by (2.3.1), $\gamma(T)$ decreases monotonically as T approaches T_2. However, the estimator $U_n(T)$ may not decrease monotonically as T approaches T_2, particularly for small samples sizes. Since $U_n(T)$ is

also a U-statistic, identical to U_n for an abbreviated time span, it inherits all the properties of generalized U-statistics presented in Chapter I. In particular, when $\xi_{T_1}>0$ and $\xi_{T_c}<\infty$, where ξ_{T_c} is the truncated form of (2.4.1), the following statistic has an asymptotically normal distribution:

$$\sqrt{n}\ (U_n(T)-\gamma(T)) \sim N(0,V_T)\ ,$$

where

$$V_T = \frac{1}{n(n-1)}\ [2\zeta_{T_1}(n-2)+\zeta_{T_2}]\ . \tag{4.2.4}$$

Proof:

The asymptotic normality follows immediately from Hoeffding's (1949) Theorem 7.1, the proof in this case is identical to that of Theorem 2.5.1.

Note that, as T approaches T_2, i.e., as truncation time approaches the full time span, V_T coincides with the variance of the full time span statistic.

Now form the process

$$U_n^*(T) = \sqrt{n}\ (U_n(T)-\gamma(T)),\quad T_1<T\le T_2\ ,$$

and compute its covariance structure. The '*' notation will refer to a process rather than a single statistic.

For truncation time S<T,

$$V_{ST} = \mathrm{Cov}(U_n^*(S),U_n^*(T)) = n[EU_n(S)U_n(T)-\gamma(S)\gamma(T)] \tag{4.2.5}$$

$$= n\ [\frac{4}{n^2(n-1)^2}\ \sum_{i\neq j}\ \sum_{k\neq \ell} E\phi_S(Y_i,Y_j)\phi_T(Y_k,Y_\ell)-\gamma(S)\gamma(T)]\ ,$$

where

$$E\phi_S(Y_{\sim i},Y_{\sim j})\phi_T(Y_{\sim k},Y_{\sim \ell}) = \begin{cases} \gamma(S)\gamma(T) & i\neq k,\ j\neq \ell \\ \int\ldots\int dF_S(y_i)dF_S(y_{\sim j})dF_T(y_{\sim i})dF_T(y_{\sim \ell}) & i=k,\ j\neq \ell \\ \int\ldots\int dF_S(y_{\sim i})dF_S(y_{\sim j})dF_T(y_{\sim k})dF_T(y_{\sim j}) & i\neq k,\ j=\ell \\ \gamma(T) & i=k,\ j=\ell \end{cases}$$

4.3 The U-Statistic as a Process

Now investigate these statistics as processes in time. The vector, $U_n^*(T)$, considered as a process in time, converges weakly to a Gaussian process in $D[0,\infty]$, the space of functions on $[0,\infty]$ with only jump discontinuities.

In order to demonstrate weak convergence of a sequence of probability measures on a separable metric space, show

a) weak convergence of the finite-dimensional distributions, and

b) tightness.

Assume the function, $U_n(T)$ has a unique limit as T goes to T_2. The function value at T_2 will be defined as the limiting value, i.e.,

$$\lim_{T\to T_2} \sqrt{n}\ (U_n(T)-\gamma(T)) = \sqrt{n}\ (U_n-\gamma(F))\ ,$$

where $U_n(T)$ is the truncated index of tracking, U_n is the full time span statistic and is an estimator of $\gamma(F)$.

Now, utilizing the results of Delong (1979), the one sample statistic being a special case, and implicitly assuming a one-to-one bicontinuous map $H[0,1] \to [0,\infty]$, this result follows:

Theorem 4.3.1:

Assume F is a strictly increasing continuous distribution function. Then the process $U_n^*(T)$ is weakly convergent in $D[0,1]$ to a Gaussian process with covariance structure given by V_{ST} (4.2.5).

Proof:

Result 2.7.1 of Delong allows the one sample statistic as a special case. Then by satisfying the assumptions of Result 2.7.1, the weak convergence follows. The second assumption,

$$\lim_{N\to\infty} \frac{m}{N} = \lambda, \ 0<\lambda<1 \ ,$$

is irrelevant in the one sample case. Assumption 3 is met by setting $\phi_{1,T} = \phi_T$, and $\phi_{2,T} = 0$. Now define a strictly increasing continuous function H on [0,1] such that, for S<T,

$$\begin{aligned} \mu_2(S,T) &\le H(t) - H(s) \ , \\ \mu_4(S,T) &\le H(t) - H(s) \ , \\ \gamma_i(S)-\gamma_i(T) &\le H(t) - H(s) \ . \end{aligned} \tag{4.3.1}$$

Consider

$$\mu_2(S,T) = E[(\phi_S-\phi_T) - (\gamma(S)-\gamma(T))]^2 \ ,$$

where

$$\phi_S-\phi_T = \begin{cases} 1 & \text{if } \phi_S = 1 \text{ and } \phi_T = 0 \\ 0 & \text{otherwise} \end{cases} \ ,$$

then for any k≠0, $(\phi_S-\phi_T)^k = (\phi_S-\phi_T)$. Therefore,

$$\mu_2(S,T) = [\gamma(S)-\gamma(T)] - [\gamma(S)-\gamma(T)]^2 \ .$$

But $\gamma(S)-\gamma(T)<1$, therefore, $\mu_2(S,T)\le\gamma(S)-\gamma(T)$.

Now for

$$\mu_4(S,T) = E[(\phi_S-\phi_T) - (\gamma(S)-\gamma(T))]^4$$

$$= [\gamma(S)-\gamma(T)]-4[\gamma(S)-\gamma(T)]^2+6[\gamma(S)-\gamma(T)]^3-3[\gamma(S)-\gamma(T)]^4 \ ,$$

$$\mu_4(S,T) = [\gamma(S)-\gamma(T)]-[\gamma(S)-\gamma(T)]^2-3\{[\gamma(S)-\gamma(T)]-[\gamma(S)-\gamma(T)]^2\}^2 \ ,$$

$$= \mu_2(S,T)-3\{\mu_2(S,T)\}^2<\mu_2(S,T)\le\gamma(S)-\gamma(T) \ .$$

F is strictly increasing, and γ is strictly decreasing and is a continuous function, thus (4.3.1) is satisfied by setting $H(x) = 1-\gamma(x)$, and $U_n^*(T)$ is tight in D[0,1] with respect to the uniform metric, and converges weakly to a Gaussian process.

4.4 Predicting $\gamma(F)$

Since estimates $U_n(T_j)$, $j=1,2,\ldots,k$ are available for time points $T_1,T_2,\ldots,T_k$, and covariance estimates are available using the structural components method (Sen, 1960), the full time span parameter can be predicted based on a weighted least squares estimate of $U_n(T)$ on T. The progressively truncated estimates follow a continuous path, as a function of time, and approach an asymptote. This path can be modelled as a polynomial in time, based on established weighted least squares theory, to allow prediction of the parameter of interest, $\gamma(F)$, the full time span parameter. The model is as follows:

$$\underset{\sim}{\gamma} = \begin{bmatrix} U_n(T_1) \\ \vdots \\ U_n(T_k) \end{bmatrix}, \quad \underset{\sim}{X} = \begin{bmatrix} 1 & T_1 & T_1^2 & \cdots & T_1^p \\ 1 & T_2 & T_2^2 & \cdots & T_2^p \\ \vdots & & & & \\ 1 & T_k & T_k^2 & \cdots & T_k^p \end{bmatrix},$$

$$\underset{\sim}{\Delta}' = [\delta_0 \ldots \delta_r] ,$$

and

$$\underset{\sim}{\gamma} = \underset{\sim\sim}{X\Delta} + \underset{\sim}{\varepsilon} ,$$

assuming

$$\underset{\sim}{\varepsilon} \sim N_k(\underset{\sim}{0},\underset{\sim}{\Sigma}) ,$$

where the ij-th element of $\underset{\sim}{\Sigma}$ is estimated by:

$$((\sigma_{ij})) = \frac{1}{n(m-1)} \sum_{r=1}^{n} [\frac{1}{n} \sum_{s=1}^{n} \phi_{T_i}(\underset{\sim}{Y}_r,\underset{\sim}{Y}_s) - \hat{\gamma}(T_i)] \times [\frac{1}{n} \sum_{s=1}^{n} \phi_{T_j}(\underset{\sim}{Y}_r,\underset{\sim}{Y}_s) - \hat{\gamma}(T_j)] .$$

4.5 An Application of Progressive Truncating

The data presented on 36 dogs by Grizzle and Allen (1969) are used to illustrate this method. In this study, coronary uric potassium (mil equivalents per liter) were measured at two minute intervals during the initial 13 minutes following coronary occlusion. Assuming the responses have the following functional form:

$$y_{it} = f(t,\underset{\sim}{\beta}_i) = \alpha_i + \beta_i t + \varepsilon_{it} ,$$

a linear model in time is estimated for each dog's coronary uric potassium. The major parameter of interest is $\gamma[1,13]$, i.e., the index of tracking within each treatment group for the full experiment, 1 minute through 13 minutes following coronary occlusion. Let

$$\underset{\sim}{\gamma} = \begin{bmatrix} \gamma(1,7) \\ \gamma(1,9) \\ \gamma(1,11) \end{bmatrix} = \begin{bmatrix} U_n(1,7) \\ U_n(1,9) \\ U_n(1,11) \end{bmatrix} , \quad \underset{\sim}{X} = \begin{bmatrix} 1 & 7 \\ 1 & 9 \\ 1 & 11 \end{bmatrix} ,$$

$$\underset{\sim}{\Delta}' = [\alpha \; \beta] ,$$

which results in a model for each treatment group

$$\underset{\sim}{\gamma} = \underset{\sim\sim}{X\Delta} + \underset{\sim}{\varepsilon} .$$

The weighted least squares estimates $\hat{\underset{\sim}{\Delta}}$ for each treatment group can be used to predicted $\gamma[1,13]$, assuming that the model holds through the region of extrapolation. The interim estimates, which constitute the data for the weighted least squares estimation, are given in Table 4.5.1.

The estimated parameters for a linear model for each of the treatment groups, and the predicted full time span index of tracking are presented in Table 4.5.2. For purposes of comparison, the actually estimated values for $\gamma[1,13]$ are also listed. Note that, as indicated

TABLE 4.5.1

ESTIMATES OF INDEX OF TRACKING (AND STANDARD DEVIATIONS)
ASSUMING AN UNDERLYING LINEAR MODEL

Interim Estimate	Group 1 (n=9)	2 (n=10)	3 (n=8)	4 (n=9)
$\hat{\gamma}[1,7]$	.47222 (.00468)	.73333 (.00335)	.64286 (.00656)	.66667 (.00217)
$\hat{\gamma}[1,9]$	.44444 (.00526)	.73333 (.00255)	.67857 (.00565)	.61111 (.00280)
$\hat{\gamma}[1,11]$	.44444 (.00352)	.80000 (.00187)	.53571 (.00565)	.58333 (.00564)
$\hat{\gamma}[1,13]$	.47222 (.00251)	.82222 (.00225)	.53571 (.00419)	.47222 (.00424)

Source: Grizzle and Allen (1969) Biometrics.

TABLE 4.5.2

RESULTS OF WEIGHTED LEAST SQUARES MODELS

Treatment	Sample Size	Intercept	Slope	Predicted* $\hat{Y}[1,13]$	Estimated** $\hat{Y}[1,13]$
1	9	.51145	-.00667	.42474 (.05875)	.47222 (.00251)
2	10	.58881	.01840	.82801 (.16049)	.82222 (.00225)
3	8	.87219	-.02795	.50884 (.39208)	.53571 (.00419)
4	9	.81822	-.02210	.53092 (.06642)	.47222 (.00424)

*Based on observations 1 minute through 11 minutes after occlusion.

**Based on observations 1 minute through 13 minutes after occlusion.

earlier, while the parameter value of the index of tracking in the population does not increase with time, it is possible for the estimated value to increase. See, for example, treatment group 2, where the estimated index not only increases with time, but the results of the weighted least squares estimation yield a positive slope. These results are plotted for each treatment group in Figures 4.5.1-4.5.4.

FIGURE 4.5.1

WEIGHTED LEAST SQUARES PREDICTION

GROUP 1 - CONTROL GROUP

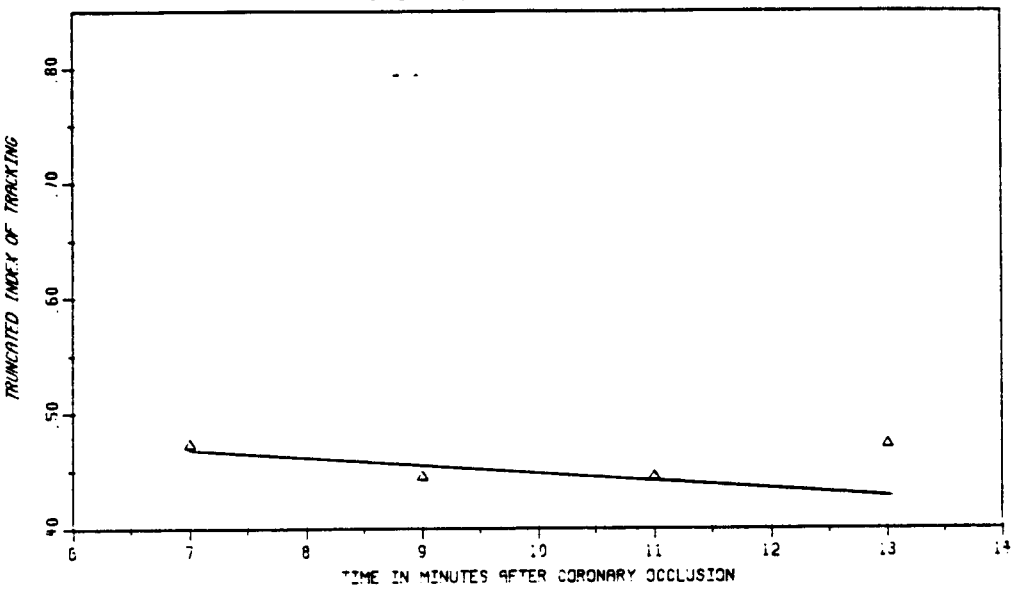


FIGURE 4.5.2

WEIGHTED LEAST SQUARES PREDICTION

GROUP 2 - DENERVATION 3 WEEKS PRIOR TO OCCLUSION

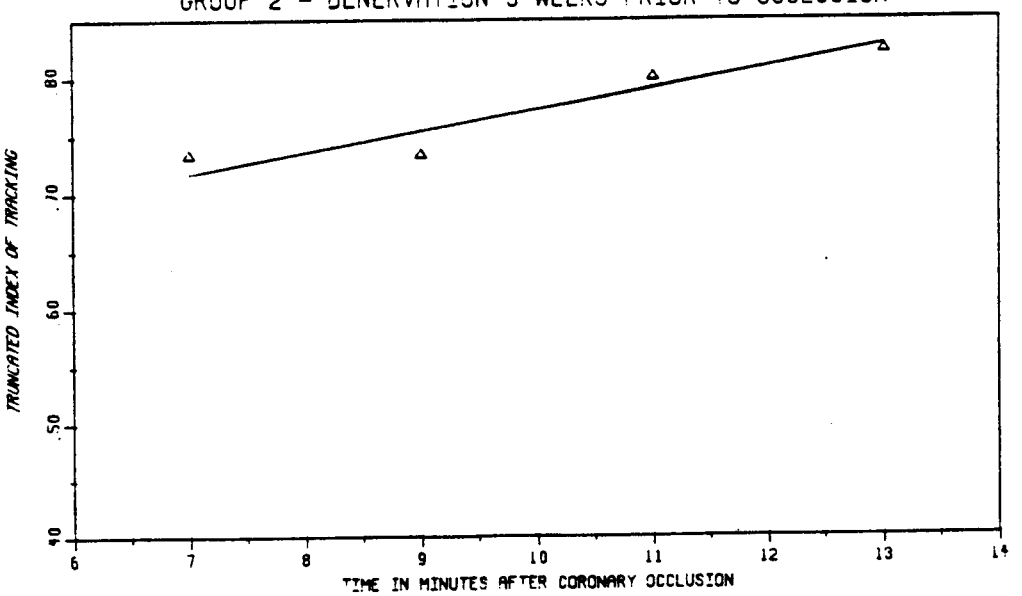


FIGURE 4.5.3

WEIGHTED LEAST SQUARES PREDICTION

GROUP 3 - DENERVATION IMMEDIATELY PRIOR TO OCCLUSION

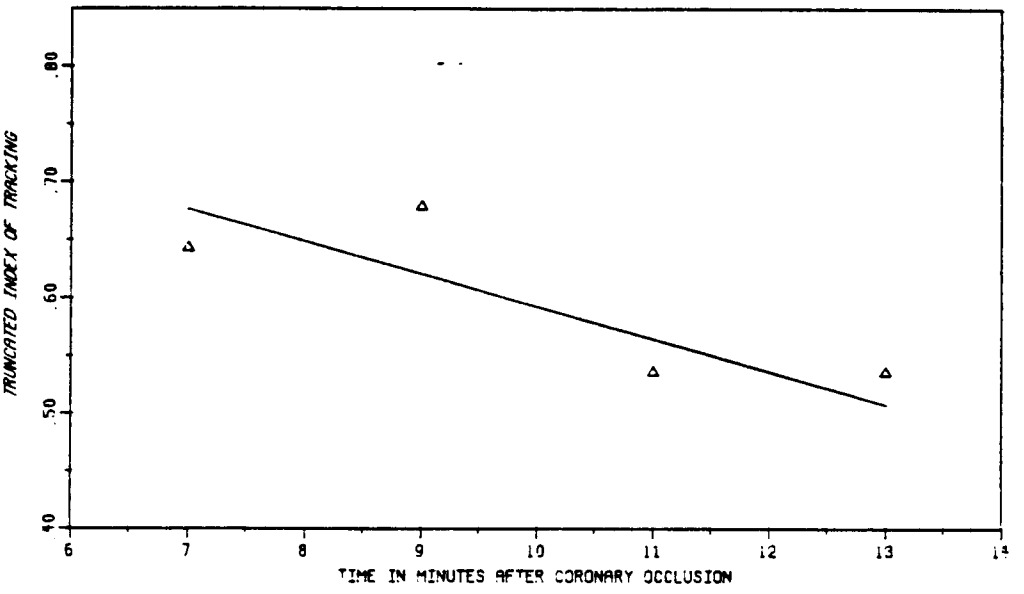
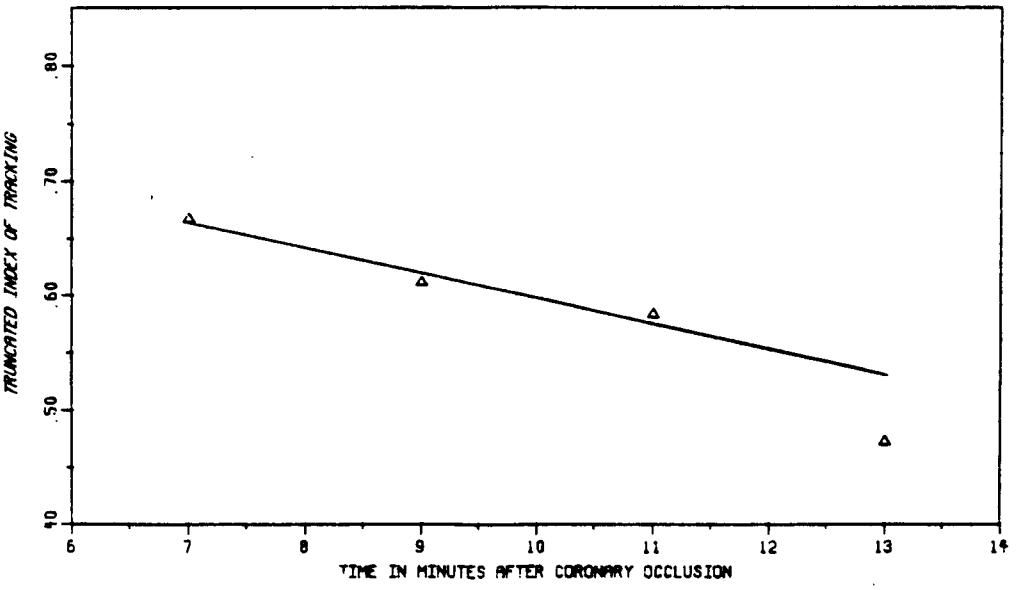


FIGURE 4.5.4

WEIGHTED LEAST SQUARES PREDICTION

GROUP 4 - SYMPATHECTOMY & STELLECTOMY



CHAPTER V

APPLICATIONS FOR LONGITUDINAL DATA

5.1 Introduction

The methods that have been introduced offer a fresh approach to the tracking hypothesis. This chapter will apply these methods, and include estimates of the index of tracking for several risk factors, comparisons with published results and conclusions regarding the tracking hypothesis, and comparisons with published results from similar samples. Two separate longitudinal studies provide these applications, the Thousand Aviator Study, and the Bogalusa Heart Study.

5.2 The Thousand Aviator Study

The Thousand Aviator Study provides a unique source of longitudinal data on naval aviation cadets which permits the examination of biological variations over time not available from cross-sectional studies. The cohort of 1056 white males participating in this study were all selected for training in 1940, their mean age at this time was 24.2 (±2.5) years. Re-examinations occurred in 1952, 1958, 1964, and 1970, spanning their early adult years through middle age. Each individual was initially within 5% of optimal weight (based on current Metropolitan Life tables), with supine systolic blood pressure less than 135 mm, and diastolic blood pressure less than 85 mm. Supine blood pressures were recorded until constant readings were obtained.

The fact that these individuals were physically qualified for naval flight training in 1940 results in a highly selective sample. However, the 30 year follow-up presents a unique opportunity for testing specific tracking hypotheses. According to Harlan (1962), "of identified features at age 24 only one, the relative ranking in blood pressure distribution, emerges as a significant predictor of future high blood pressure." A number of these hypotheses will be examined in detail in the sequel.

Although extensive physical and psychological tests were conducted, only the blood pressure responses will be discussed here. At various years, blood pressures were recorded in any or all of the supine, seated, or standing positions. Pressures were recorded with a standard adult cuff and mercury sphygmomanometer in the course of each examination, with the individual in a supine, seated or standing position. Systolic and diastolic blood pressure levels were recorded corresponding to the first and fifth Korotkoff sounds, respectively, although both fourth and fifth phase changes were recorded, and outlined in Pickering (1955). In an attempt to avoid digit preference in recording, random zero sphygmomanometers were used in 1964, and responses were mechanically recorded in 1970.

Table 5.2.1 summarizes the baseline data recorded on the cadets.

For those years when supine systolic blood pressures were recorded (1940, 1958, 1970) long term trends can be examined. The estimated index of tracking defined in (2.3.3), based on a straight-line model for each individual is $\hat{\gamma}[1940,1970] = 0.5672$ (standard deviation = 0.0208, n=89). This is significantly greater than $\gamma = 1/2$, indicating the tracking of supine systolic blood over a thirty year time span. The magnitude of this estimated index of tracking must be considered in terms of the

TABLE 5.2.1

THOUSAND AVIATOR STUDY

SUMMARY STATISTICS AT BASELINE

Variable	N	Mean	Standard Deviation	Minimum Value	Maximum Value
Age in 1940	982	24.2	2.49	20	49
Weight in 1940 (in lbs.)	1040	158.4	15.75	121	204
Baseline SBP at 2 minutes	770	118.3	8.99	80	150
Baseline SBP at 3 minutes	474	120.6	8.77	82	150
Baseline SBP at 4 minutes	477	120.1	8.82	84	148
Baseline SBP at 5 minutes	774	117.7	8.83	90	144
Tilted SBP at 6 minutes	749	111.4	9.52	80	148
Tilted SBP at 8 minutes	471	112.3	9.56	80	148
Tilted SBP at 8 minutes	767	111.4	9.72	84	152
Tilted Back SBP at 26 minutes	729	119.5	8.99	90	150
Tilted Back SBP at 27 minutes	460	122.3	8.71	98	154
Tilted Back SBP at 28 minutes	752	120.0	8.81	90	150

selective sample of aviators, since these men cannot be considered as representative of a larger, possibly less healthy, population.

The above example, with $\hat{\gamma}[1940,1970] = 0.5672$, and standard deviation = 0.0208, allows a comparison of the estimated variance with the upper bound given in (2.4.3). In this case, $\hat{\sigma}^2(\hat{\gamma}) = (.0208)^2 = .00043$, and the upper bound is

$$\hat{\sigma}^2(\hat{\gamma}) \frac{\leq 32\hat{\gamma}(1-\hat{\gamma})}{n} = 0.08826 .$$

From this comparison it is clear that, given the capability to estimate the variance by Sen's structural components method, the upper bound outlined in Chapter II is of no practical use.

Again, for the individuals with supine systolic blood pressures recorded in 1940, the responses from 2-28 minutes will convey the short term trends. Since supine systolic blood pressures were recorded at seven points in time, the progressive truncation methodology presented in Chapter IV allows the prediction of the index of tracking for the period two minutes through thirty minutes. This prediction is summarized in Table 5.2.2 and Figure 5.2.1.

Based on systolic and diastolic blood pressures recorded in the sitting position in 1950, 1958, and 1970, the hypothesis that tracking of systolic blood pressure is greater than the tracking of diastolic blood pressure can be examined. The asymptotic normality of a linear contrast such as this was discussed in Chapter II.

$\hat{\gamma}(SBP) = 0.5665$, Standard Deviation = 0.0173, n=99,

$\hat{\gamma}(DBP) = 0.5299$, Standard Deviation = 0.0163, n=99,

$H_o: \gamma_{(SBP)} \leq \gamma_{(DBP)}$ vs. $H_o: \gamma_{(SBP)} > \gamma_{(DBP)}$,

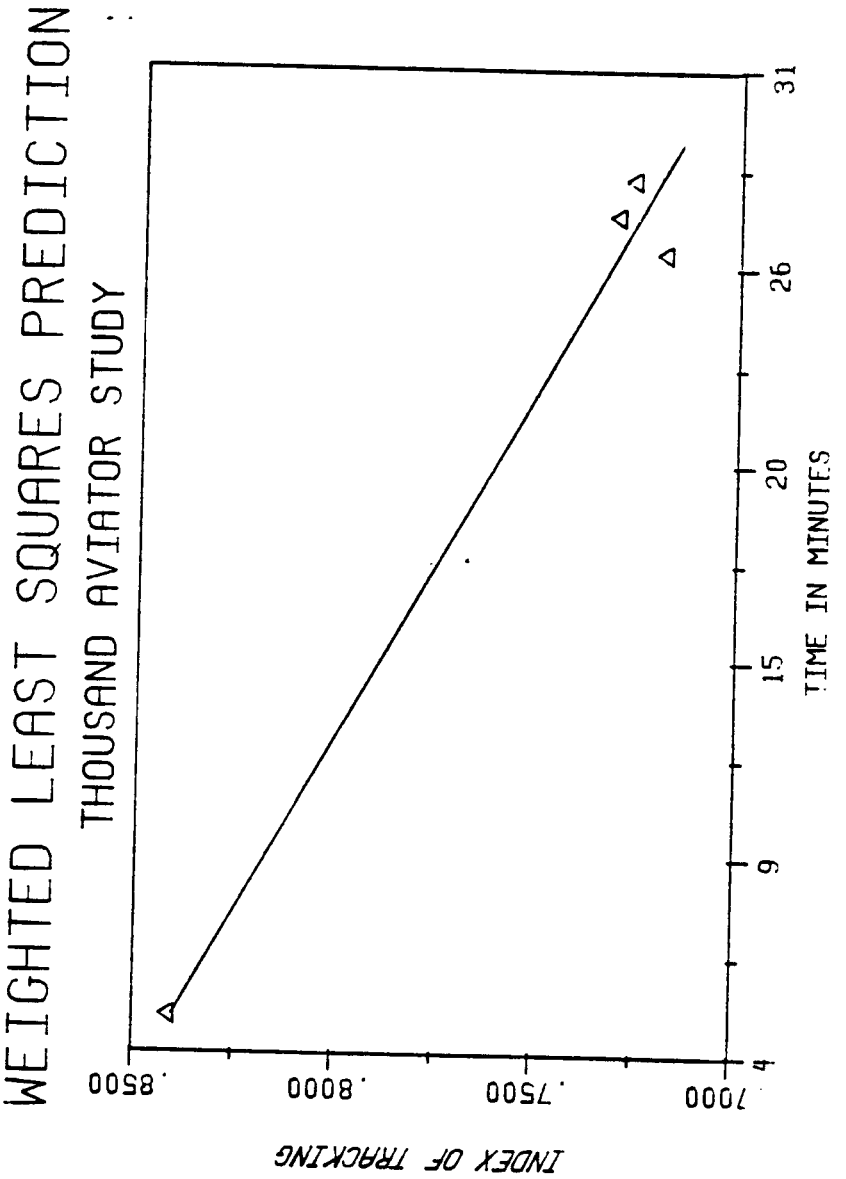
$$z = \frac{0.0366}{0.0119} = 3.0714 .$$

TABLE 5.2.2

THOUSAND AVIATOR STUDY
WEIGHTED LEAST SQUARES PREDICTION OF
SHORT TERM TRENDS IN SUPINE
SYSTOLIC BLOOD PRESSURE
(n=99)

	$\hat{\gamma}$	Standard Deviation
$\hat{\gamma}$[2 min, 5 min]	0.84065	0.00874
$\hat{\gamma}$[2 min, 26 min]	0.71841	0.01547
$\hat{\gamma}$[2 min, 27 min]	0.73036	0.01444
$\hat{\gamma}$[2 min, 28 min]	0.72645	0.01504
Predicted γ[2 min, 30 min]	0.70990	0.01646

FIGURE 5.2.1


WEIGHTED LEAST SQUARES PREDICTION
THOUSAND AVIATOR STUDY
INDEX OF TRACKING
7000
7500
8000
8500
4
9
15
20
26
31
TIME IN MINUTES

Therefore, while neither systolic or diastolic responses exhibit a strong degree of tracking, sitting systolic blood pressure does track more strongly than diastolic in this sample of 99 aviators.

5.3 The Bogalusa Heart Study

The Bogalusa Heart Study collected data on newborns, born between January 1, 1974 and June 30, 1975 to mothers residing in Bogalusa, Louisiana. This investigation parallels that of Levine, et al. (1978), who reported on tracking correlations in infants for the first year of life. Their attention focused on the question of whether tracking correlations were detectable at these early ages.

Complete physical examinations were given, data collection including anthropometric measurements and serum lipids. The children's blood pressures were recorded as the last procedure of the 1-1/2 to 2 hour examination. The randomly assigned observers recorded first and fourth Korotkoff phases, and each of the three observers independently measured three blood pressures per child per examination. The present analysis is limited to the mean response per child per examination, separately by instrument, either the Infrasonde 3000 (Marion Scientific Corporation) or the Arteriosonde 1010 (Roche Medical Electronics).

Blood specimens were collected at birth (from cord blood), at 6 months, 1 year, and 2 years of age. All serum samples were analyzed in a Technicon Auto Analyzer II in the Core Laboratory of SCOR-A according to the protocol developed by the Lipid Research Clinics Program, designated as "standardized" by the Center for Disease Control (CDC) in Atlanta, Georgia.

Blood pressures and lengths were not recorded at the cord blood stage. Lengths were assessed at the 6 month and 1 year stages with the child supine, and at the 2 year stage with the child standing.

Summary statistics by race and sex for each of the four examinations are given in Table 5.3.1.

Although Berenson et al. (1979) suggested that lipids and lipoproteins track at this early stage in life, it was suggested that a prospective study through the first five years of life would be necessary to identify the earliest age at which hyperlipidemia could be identified. The index of tracking estimates through the first two years of life for each of the response variables recorded are given in Table 5.3.2. The estimated index of tracking for each of the lipids and lipoproteins shows no evidence of tracking from birth through two years of age, assuming a quadratic polynomial model for each individual. When the cord blood stage is disregarded, and a straight-line model is assumed, these estimates do indicate early tracking of lipids and lipoproteins. However, at this point, the data are too sparse, four or fewer observations per individual. A five year prospective study, with at least semi-annual observations, would provide adequate data to choose the appropriate polynomial model and estimate the index of tracking. From these estimates it is clear that such a prospective study should begin at 6 months of age, at the earliest.

Weight is the response which tracks most strongly, which is to be expected. Pediatricians, relying on standard growth charts for length and weight, have tacitly assumed that both of these responses track strongly within the pediatric age range.

TABLE 5.3.1

BOGALUSA HEART STUDY
SUMMARY STATISTICS FOR INFANTS JANUARY 1974 TO JUNE 1976
BY AGE, RACE, AND SEX

	WHITE MALES			WHITE FEMALES			BLACK MALES			BLACK FEMALES		
AGE	N	MEAN	STD	N	MEAN	STD	N	MEAN	STD	N	MEAN	STD
					WEIGHT IN KG							
BIRTH	133	3.4	0.6	133	3.2	0.6	84	3.1	0.7	88	3.0	0.6
6 MO	103	8.0	0.9	105	7.3	0.8	70	7.7	1.1	72	7.4	0.9
1 YR	90	10.2	1.0	89	9.5	0.9	66	10.0	1.4	70	9.5	1.1
2 YR	72	12.2	1.1	78	11.9	1.4	60	12.7	1.7	58	12.3	1.5
					TOTAL SERUM CHOLESTEROL (MG/DL)							
BIRTH	129	70.1	17.2	127	75.7	17.3	79	62.3	20.0	84	66.4	16.6
6 MO	96	137.5	24.7	94	137.0	26.2	64	125.9	37.4	58	135.3	29.6
1 YR	85	144.3	22.0	87	147.4	28.4	59	143.1	26.2	60	144.7	30.2
2 YR	63	142.4	25.0	66	145.0	20.4	52	145.2	27.5	49	149.4	26.9
					TRIGLYCERIDES (MG/DL)							
BIRTH	129	43.9	25.5	127	38.6	20.3	79	37.4	18.7	84	37.7	22.1
6 MO	96	103.6	54.4	94	98.8	57.2	64	87.5	39.9	58	80.7	28.4
1 YR	85	94.5	39.7	87	94.7	46.9	59	78.2	40.7	60	76.1	34.0
2 YR	63	69.2	32.6	66	72.8	33.1	52	59.9	22.4	49	59.9	15.9
					ALPHA LIPOPROTEIN							
BIRTH	129	33.9	14.6	126	39.6	14.1	79	33.3	17.1	83	34.1	13.0
6 MO	96	51.9	23.9	94	50.8	19.0	64	49.3	27.3	58	49.1	16.9
1 YR	84	54.3	19.7	86	49.7	18.8	59	56.3	13.4	60	52.0	18.2
2 YR	63	54.9	18.4	66	51.3	17.3	52	57.3	14.9	49	56.4	15.6
					PREBETA LIPOPROTEIN							
BIRTH	127	4.1	4.7	125	3.2	3.6	78	3.5	3.6	82	3.6	4.2
6 MO	94	13.0	8.9	92	11.0	7.8	64	8.3	6.2	58	8.3	5.8
1 YR	78	10.6	8.6	75	10.7	8.2	55	6.7	4.6	59	8.0	6.9
2 YR	62	6.7	6.6	61	5.8	4.6	48	5.5	4.0	48	5.3	3.8

TABLE 5.3.1

BOGALUSA HEART STUDY
SUMMARY STATISTICS FOR INFANTS JANUARY 1974 TO JUNE 1976
BY AGE,RACE, AND SEX

	WHITE MALES			WHITE FEMALES			BLACK MALES			BLACK FEMALES		
AGE	N	MEAN	STD	N	MEAN	STD	N	MEAN	STD	N	MEAN	STD
					BETA LIPOPROTEIN							
BIRTH	127	31.9	10.5	125	32.9	9.9	78	25.3	8.3	82	28.9	10.6
6 MO	94	72.2	18.1	92	75.2	20.9	64	68.3	23.8	58	77.9	23.7
1 YR	78	80.5	18.0	75	86.8	23.4	55	79.9	24.5	59	84.2	22.0
2 YR	62	80.9	19.8	61	86.2	18.5	48	81.5	23.6	48	87.5	18.8
					SYSTOLIC BLOOD PRESSURE - ARTERIOSONDE							
6 MO	103	95.4	12.3	105	95.1	13.6	70	94.0	10.9	72	91.4	10.1
1 YR	89	99.5	10.1	90	98.6	11.0	66	96.4	11.1	70	95.7	10.1
2 YR	73	90.7	11.5	78	90.8	10.8	62	88.6	10.4	61	89.7	9.5
					DIASTOLIC BLOOD PRESSURE - ARTERIOSONDE							
6 MO	103	54.7	10.9	105	54.0	11.7	70	50.9	12.0	72	52.8	8.6
1 YR	89	58.1	7.3	90	59.0	8.9	66	55.9	8.9	70	55.9	9.3
2 YR	73	56.2	7.9	78	56.9	7.4	62	56.3	8.4	61	57.7	7.3
					SYSTOLIC BLOOD PRESSURE - INFRASONDE							
6 MO	103	94.9	12.3	102	93.2	11.9	70	92.6	10.2	72	91.8	10.5
1 YR	89	98.3	11.1	89	97.9	11.7	66	96.3	10.2	70	94.9	10.5
2 YR	72	91.8	13.3	78	89.8	11.1	61	90.9	11.4	62	88.9	9.6
					DIASTOLIC BLOOD PRESSURE - INFRASONDE							
6 MO	103	47.8	9.4	102	47.4	9.2	70	45.5	8.0	72	44.9	10.3
1 YR	89	49.8	9.3	89	52.2	10.0	66	50.1	7.5	70	50.6	10.2
2 YR	72	40.6	10.7	78	43.8	10.8	61	39.7	9.2	62	42.5	11.2

TABLE 5.3.2

ESTIMATED INDEX OF TRACKING

BOGALUSA HEART STUDY

Response Variable	Sample Size	$\hat{\gamma}$[0, 2 yr] (Quadratic Model)	Standard Deviation	$\hat{\gamma}$[6 mo, 2 yr] (Linear Model)	Standard Deviation
α-Lipoprotein	140	0.3371	0.01329	0.54553	0.01356
Pre β-Lipoprotein	140	0.3016	0.01334	0.55694	0.01497
β-Lipoprotein	140	0.4279	0.01566	0.65642	0.01456
Total Cholesterol	160	0.3884	0.01350	0.63255	0.01368
Triglycerides	160	0.3010	0.01331	0.54340	0.01435
Weight	238	0.5243	0.01160	0.74198	0.00822
Systolic BP Arteriosonde	240			0.52326	0.01130
Diastolic BP Arteriosonde	240			0.47469	0.01076
Systolic BP Infrasonde	240			0.52469	0.01089
Diastolic BP Infrasonde	240			0.50049	0.01121

The indices of tracking for each of the four blood pressure responses allow the comparison of instruments by testing the hypothesis:

$$H_o: \gamma(\text{SBP-ARTERIOSONDE}) - \gamma(\text{SBP-INFRASONDE}) = \gamma(\text{DBP-ARTERIOSONDE}) - \gamma(\text{DBP-INFRASONDE}) .$$

The methods for testing hypotheses of this form were presented in Chapter II. The contrast vector is $\underset{\sim}{c} = (1\ -1\ -1\ 1)$ and the vector of parameter estimates is

$$\underset{\sim}{U}_n = \begin{pmatrix} \hat{\gamma}(\text{SBP-ARTERIOSONDE}) \\ \hat{\gamma}(\text{SBP-INFRASONDE}) \\ \hat{\gamma}(\text{DBP-ARTERIOSONDE}) \\ \hat{\gamma}(\text{DBP-INFRASONDE}) \end{pmatrix} = \begin{pmatrix} .52326 \\ .52469 \\ .47469 \\ .50049 \end{pmatrix} .$$

Then

$$z = \frac{\underset{\sim}{c}\underset{\sim}{U}_n}{(\underset{\sim}{c}\underset{\sim}{\Sigma}\underset{\sim}{c}')} = 0.0694 .$$

Thus, there is no difference from one instrument to the other in terms of the estimated index of tracking.

CHAPTER VI

SUMMARY AND SUGGESTIONS FOR FURTHER RESEARCH

6.1 Summary

The concept of tracking has been defined for a specified time span as the maintenance of relative (or peer) ranking within the risk factor distribution. Since methods to appropriately evaluate tracking have been inadequate, a quantifier was introduced. This index of tracking, a generalized U-statistic based on estimated growth curve parameters, is an extension of the usual probability of concordance to the probability of concordant risk factor responses over a fixed time interval. The asymptotic normality of the index was established, for both the identically distributed and non-identically distributed cases based on Hoeffding (1948). An estimate of the variance was obtained by Sen's (1960) method of structural components.

Methods of modelling the tracking phenomenon were discussed. The methods outlined included least squares estimation, Bayes and empirical Bayes procedures. Since losses to follow-up can affect the estimation of an index of tracking, censored observations were simulated by Monte Carlo methods, and the estimated index examined in the presence of censoring. The index was also presented as a Gaussian process in time, by progressively truncating the kernel of the generalized U-statistic. Then, by weighted least squares modelling of the progressively truncated statistics, the untruncated parameter value can be predicted.

Finally, the techniques developed were applied to two longitudinal studies, the Thousand Aviator Study, and the Bogalusa Heart Study.

6.2 Suggestions for Further Research

The techniques discussed in Chapter III are not the only possibilities for modelling the tracking phenomenon. The compromise between the least squares and the Bayes estimators could be used. Efron and Morris (1971-1973) proposed the limited translation estimator specifically for the case when the parameter to be estimated is far from the prior mean. The assumption that the function $f(t,\underset{\sim}{\beta}_i)$ follows a polynomial in time could be replaced, assuming a non-linear model, for example.

The definition of the index of tracking might be made less stringent, still consistent with the conceptual definition of tracking. For example, redefine the kernel to equal the proportion of the time span that two random response vectors are concordant.

Finally, the index of tracking could be applied beyond the limits of the longitudinal study. Rao and Rao (1966) have defined a linked cross-sectional study, a hybrid between conventional longitudinal and cross-sectional studies, which they suggested as the optimum design for studying growth rates and differential growth. The definition of the index of tracking might be modified to address this study design.

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