J. DIFFERENTIAL GEOMETRY
45 (1997) 287-334

AN INDEX THEOREM FOR FAMILIES OF DIRAC OPERATORS ON ODD-DIMENSIONAL MANIFOLDS WITH BOUNDARY

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Abstract

For a family of Dirac operators, acting on Hermitian Clifford modules over the odd-dimensional compact manifolds with boundary which are the fibres of a fibration with compact base, we compute the Chern character of the index, in K^1 of the base. Although we assume a product decomposition near the boundary, we make no assumptions on invertibility of the boundary family and instead obtain a family of self-adjoint Fredholm operators by choice of an auxiliary family of projections respecting the $\mathbb{Z}_2$ decomposition of bundles over the boundary. In case the boundary family is invertible, this projection can be taken to be the Atiyah-Patodi-Singer projection and the resulting formula is as conjectured by Bismut and Cheeger. The derivation of the index formula is effected by the combination of the superconnection formalism of Quillen and Bismut, the calculus of b-pseudodifferential operators and suspension.

Introduction

Let $\phi : M \longrightarrow B$ be a fibration of Riemannian manifolds, with B compact and with fibres diffeomorphic to a fixed *odd-dimensional* compact manifold with boundary X. Suppose that the fibres carry smoothly varying spin structures and that the Riemannian metrics on the fibres have smoothly varying product decompositions near the boundary. Let $\eth = \eth_z$ be, for $z \in B$, the associated family of Dirac operators and let $\eth_0 = \eth_{0,z}$ be the boundary family. If $\eth_{0,z}$ is invertible for each $z \in B$, the Atiyah-Patodi-Singer boundary condition makes $\eth_z$ into a

Received July 5, 1994, and, in revised form, October 31, 1995.
Research of the first author was supported in part by the National Science Foundation under grant 9306389-DMS, and that of the second author was supported in part by *M.U.R.S.T.* (Italy)

continuous family of self-adjoint Fredholm operators and thus, following Atiyah and Singer in [8], defines an element $\operatorname{Ind}(\eth) \in K^1(B)$. Under this assumption of invertibility of the boundary family a formula for the Chern character $\operatorname{Ch}(\operatorname{Ind}(\eth)) \in H^{\mathrm{odd}}(B)$ was conjectured by Bismut and Cheeger in [12]. In this paper, using ideas similar to those used in [24] for the even-dimensional case, we prove such a formula without making any assumptions on the boundary family and for the Dirac operator of general Hermitian Clifford modules with unitary Clifford connections.

To explain how we define a continuous family of self-adjoint Fredholm operators consider, for simplicity, the spinor bundle but with no invertibility assumptions on the boundary family. Observe first that the restriction of the spinor bundle to the boundary of the fibration is $\mathbb{Z}_2$-graded, $S_{\partial M} = S_0^+ \oplus S_0^-$. Let σ, defined by Clifford multiplication in the normal direction, be the parity operator on $S_{\partial M}$:

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The boundary operator $\eth_0$ is odd with respect to this $\mathbb{Z}_2$ grading and self-adjoint

$$\sigma\eth_0 + \eth_0\sigma = 0,\ \eth_0 = \eth_0^*. \tag{1}$$

For any $\mathbb{Z}_2$-graded vector bundle L on a fibration $\psi : M' \longrightarrow B$, with fibres diffeomorphic to a fixed closed compact manifold Y, and for any family of elliptic, self-adjoint, $\mathbb{Z}_2$-graded odd differential operators A (i.e., satisfying (1)) we introduce the notion of a $\mathrm{Cl}(1)$ spectral section P. This is a spectral section as in [24] (a family of generalized Atiyah-Patodi-Singer projections, see Definition 1 of §2) with the additional property that

$$\sigma P + P\sigma = \sigma.$$

We then prove that a $\mathrm{Cl}(1)$ spectral section exists for such a family A if and only if the virtual bundle $\operatorname{Ind}(A) = [\operatorname{null}(A^+)] - [\operatorname{null}(A^-)]$ vanishes in $K^0(B)$. Since, by cobordism invariance, it is always the case that $\operatorname{Ind}(\eth_0) = 0$ in $K^0(B)$ there does exist a $\mathrm{Cl}(1)$ spectral section P for the boundary family $\eth_0$. The choice of a spectral section fixes a self-adjoint boundary condition, varying smoothly with the base point,

$$\eth u = f \text{ in } M,\ P(u \restriction \partial M) = 0; \tag{2}$$

this also carries over to the case of an Hermitian Clifford module E.

To compute the index of this family of self-adjoint Fredholm operators we follow the idea of Atiyah, Patodi and Singer in replacing the incomplete metric on the fibres by a complete metric with cylindrical end. In fact we prefer to think of the index theorem as in this 'category' of exact b-metrics, as in [24]. A Cl(1) spectral section for $\eth_0$ fixes a class of (families of) $\mathbb{Z}_2$-graded finite rank smoothing operators, A^0_P, with the property that $(\eth_0 + A^0_P)_z$ is invertible for any $z \in B$. One can think of A^0_P as a trivializing perturbation for $\eth_0$ corresponding to the fact that $\mathrm{Ind}(\eth_0) = 0$ in $K^0(B)$. The b-calculus allows us to use this perturbation to define an index class $\mathrm{Ind}(\eth, P) \in K^1(B)$. Although there is no completely natural choice of A^0_P, the index class is independent of the particular choice of trivializing family corresponding to a fixed spectral section and it is shown in §4 to be equal to the index class of the elliptic boundary problem (2). As in the even-dimensional case, we prove a relative index theorem showing that the difference of the index classes $\mathrm{Ind}(\eth, P_1) - \mathrm{Ind}(\eth, P_2)$, for two choices of Cl(1) spectral section, to be the class in $K^1(B)$ corresponding to the formal difference of P_1 and P_2. For fixed P_2, as P_1 varies over Cl(1) spectral sections for $\eth_0$ the formal difference classes exhaust $K^1(B)$; see Proposition 12.

The main result of this paper is a formula for the Chern character of $\mathrm{Ind}(\eth, P)$. The global boundary term is given by the differential form on B

$$\eta_{\mathrm{odd},P} = \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{STr}_{\partial M} \left(\frac{d\widetilde{\mathbb{B}}_u}{du} e^{-(\widetilde{\mathbb{B}}_u)^2} \right) du, \tag{3}$$

where

$$\widetilde{\mathbb{B}}_u = u^{\frac{1}{2}}(\eth_0 + \chi(u)A^0_P) + \mathbb{B}_{[1]} + u^{-\frac{1}{2}}\mathbb{B}_{[2]}$$

is the rescaled perturbed Bismut superconnection on the boundary fibration. The cut-off function $\chi \in \mathcal{C}^\infty(\mathbb{R})$, with $\chi(u) = 0$ for $u < 1$ and $\chi(u) = 1$ for $u > 2$, is introduced to ensure convergence at $u = 0$. The supertrace appearing in (3) is the natural one defined by the $\mathbb{Z}_2$-grading of E_0. The differential form $\eta_{\mathrm{odd},P}$ is well defined, up to an exact form, independently of the particular choice of the trivializing family A^0_P. For a family of Dirac operators on odd-dimensional manifolds with boundary as above and for a choice of a Cl(1) spectral section P, for the boundary family $\eth_0$, we establish the index formula

$$\begin{aligned}\mathrm{Ch}(\mathrm{Ind}(\eth, P)) = \;& (2\pi i)^{-\frac{n+1}{2}} \int_{M/B} \widehat{A}(M/B) \cdot \mathrm{Ch}'(E) \\ & -\frac{1}{2}\eta_{\mathrm{odd},P} \text{ in } H^{\mathrm{odd}}(B)\end{aligned} \tag{4}$$

in the context of Dirac operators associated to Hermitian Clifford modules, where $\mathrm{Ch}'(E)$ is the Chern character of the twisting curvature.

To prove this formula we extend to manifolds with boundary the suspension argument used by Bismut and Freed in [14]. When dealing with elliptic boundary problems, the suspension, in the obvious analytic sense, of the Dirac operator with Atiyah-Patodi-Singer boundary condition does *not* give the Atiyah-Patodi-Singer boundary condition for the suspended Dirac family. This prevents a trivial reduction to the even-dimensional case. For the related reason that the suspension of a cone is not a simple cone, the operation of suspension has not been successfully integrated into the approach of Bismut and Cheeger ([10] and [12]) in which a metrically incomplete cone (see also [18] and [19] for the analytic background) is attached to the manifold with boundary. This has prevented the application of such a method in the odd-dimensional case. It is a feature of our approach via the b-calculus that the analysis of suspension on manifolds with boundary is relatively straightforward. This can be stated succinctly as the fact that for the complete problem replacing (3) the boundary condition is just the L^2 condition on the domain of the operator and that L^2 is preserved under suspension.

We remark that if the boundary family $\eth_0$ is invertible then, in the case of exact b-metrics, no perturbation is needed, each

$$\eth_z : H^1_{\mathrm{b}}(M_z; S) \longrightarrow L^2_{\mathrm{b}}(M_z; S) \tag{5}$$

being Fredholm and self-adjoint. For the Chern character of the index class $\mathrm{Ind}(\eth) \in K^1(B)$, defined by (5), we obtain the index formula proposed by Bismut and Cheeger

$$\mathrm{Ch}(\mathrm{Ind}(\eth)) = (2\pi i)^{-\frac{n+1}{2}} \int_{M/B} \widehat{A}(M/B) - \frac{1}{2}\eta_{\mathrm{odd}} \text{ in } H^{\mathrm{odd}}(B). \tag{6}$$

Here

$$\eta_{\mathrm{odd}} = \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{STr}_{\partial M}\left(\frac{d\mathbb{B}_u}{du} e^{-(\mathbb{B}_u)^2}\right) du, \tag{7}$$

and

$$\mathbb{B}_u = u^{\frac{1}{2}}\eth_0 + \mathbb{B}_{[1]} + u^{-\frac{1}{2}}\mathbb{B}_{[2]}$$

is the rescaled Bismut superconnection of the spinor bundle $S_{\partial M} = S_0^+ \oplus S_0^-$.

In §1 the decomposition of Dirac operators on products is discussed. The extension of the notion of a spectral section from [24] to the odd-dimensional, Cl(1), case is described in §2. Suspension in the context of K-theory is discussed in §3 and used in §4 to prove the equality of the index classes in the incomplete and complete (exact b-metric) senses. Suspension at the level of Dirac operators is in §5; this passage from odd to even-dimensional cases is our basic tool. The boundary behaviour of the suspended Dirac operator is treated in §6. In §7 the relative index theorem, giving the change of the index class under the change in spectral section, is reduced to the even-dimensional theorem from [24]. The suspension of the superconnection is examined in §8 and used to define the odd eta form in §9. In the last two sections the index formula, (4), is derived from the index formula of [24], first in the case of an invertible boundary family and then, in §11, in the general case. The main result is stated precisely at the end of the paper.

1. Product decompositions

For Clifford algebras we shall use the convention of [23] and demand that for any two covectors α and β

$$\operatorname{cl}(\alpha)\operatorname{cl}(\beta) + \operatorname{cl}(\beta)\operatorname{cl}(\alpha) = 2\langle \alpha, \beta \rangle. \tag{1.1}$$

Product decompositions of Clifford modules and the associated Dirac operators arise here in a neighbourhood of a boundary in both the odd- and even-dimensional cases. The even-dimensional case is well known, so we suppose that X is an odd-dimensional manifold with boundary equipped with an exact b-metric. Let $x \in \mathcal{C}^\infty(X)$ be a distinguished defining function for the boundary, meaning that the metric takes the form

$$g = (\frac{dx}{x})^2 + g', \tag{1.2}$$

where g' is a smooth 2-tensor inducing a metric, h, on the boundary.

Let E be a Clifford module for the Clifford bundle of ${}^bT^*X$. Over the boundary the bundle ${}^bT^*X$ decomposes orthogonally as follows:

$$ {}^bT^*_{\partial X}X = \mathbb{R}(\frac{dx}{x}) \oplus T^*\partial X. \tag{1.3}$$

As in [24] we fix a Clifford action of $T^*\partial X$ on $E_{\partial X}$ by

$$ \mathrm{cl}_\partial(\eta) = i\,\mathrm{cl}(\frac{dx}{x})\,\mathrm{cl}(\eta) \tag{1.4}$$

for each $\eta \in T^*\partial X$. If $\sigma = \mathrm{cl}(\frac{dx}{x})$ then $\sigma^2 = 1$ and we obtain a decomposition of the Clifford module:

$$ E_{\partial X} = E_0^+ \oplus E_0^-, \tag{1.5}$$

where $E_0^\pm$ are respectively the ± 1 eigenspaces of the action of $\frac{dx}{x}$. Assuming that $\dim X > 1$ these two bundles have the same rank, since Clifford multiplication by any non-zero element of $T^*\partial X$ gives an isomorphism between the fibres at any point.

Proposition 1. *If E is an Hermitian Clifford module with (true) unitary Clifford connection for a smoothly varying family of exact b-metrics on the odd-dimensional compact manifolds with boundary forming the fibres of a fibration with compact base, the indicial operator of the associated Dirac operator is*

$$ I(\eth) = \sigma\left(x(\frac{1}{i}\frac{\partial}{\partial x}) + \frac{1}{i}\eth_0 \right) \tag{1.6}$$

with $\eth_0 : \mathcal{C}^\infty(\partial X; E_0) \longrightarrow \mathcal{C}^\infty(\partial X; E_0)$ the Dirac operator associated to the boundary Clifford action (1.4) and the induced graded unitary Clifford connection. Moreover

$$ (\eth_0)^* = \eth_0, \quad \sigma\eth_0 = -\eth_0\sigma, \quad \sigma^* = \sigma. \tag{1.7}$$

Proof. This follows directly from the definition of the Dirac operator.

From (1.7) we deduce that

$$ \eth_0 = \begin{pmatrix} 0 & \eth_0^- \\ \eth_0^+ & 0 \end{pmatrix} $$

with $\eth_0^- = (\eth_0^+)^*$ and $\eth_0^\pm : \mathcal{C}^\infty(\partial X; E_0^\pm) \longrightarrow \mathcal{C}^\infty(\partial X; E_0^\mp)$. The Dirac operator $\eth$ is Fredholm on the natural Sobolev spaces of the metric provided the indicial operator (see [23]) is invertible. This in turn is equivalent to the invertibility of the indicial family

$$I(\eth, \lambda) = \sigma(\lambda + \frac{1}{i}\eth_0) \tag{1.8}$$

for $\lambda \in \mathbb{R}$. Since $\eth_0 = (\eth_0)^*$ this is certainly invertible for each real $\lambda \neq 0$. Thus $\eth$ is Fredholm as a map

$$\eth : H^1_{\mathrm{b}}(X; E) \longrightarrow L^2_{\mathrm{b}}(X; E) \tag{1.9}$$

if and only if the boundary Dirac operators

$$\eth_0^{\pm} : \mathcal{C}^\infty(\partial X; E_0^{\pm}) \longrightarrow \mathcal{C}^\infty(\partial X; E_0^{\mp})$$

are invertible.

Consider now a fibration of compact manifolds $\phi : M \longrightarrow B$ with fibres, M_z for $z \in B$, diffeomorphic to a fixed odd-dimensional manifold with boundary X as above. Let E be a Hermitian Clifford module for the vertical b-cotangent bundle ${}^bT^*(M/B)$ endowed with a fibre (true) unitary Clifford connection. Let $\eth = \eth_z$, for $z \in B$, be the associated family of Dirac operators and assume that each operator of the boundary family $\eth_0$ is invertible. The discussion above shows that in this case the family of operators $\eth$ defines a continuous family of self-adjoint Fredholm operators

$$\eth_z : H^1_{\mathrm{b}}(M_z; E_z) \longrightarrow L^2_{\mathrm{b}}(M_z; E_z) \tag{1.10}$$

and thus an element $\mathrm{Ind}(\eth)$ of $K^1(B)$. In the general case we need to deform the Dirac operator to get such a Fredholm family.

2. Cl(1) spectral sections

The notion of a spectral section for a family of self-adjoint operators was introduced in [24]. For an odd, $\mathbb{Z}_2$-graded, elliptic differential operator

$$A = \begin{pmatrix} 0 & A^- \\ A^+ & 0 \end{pmatrix}, \ A^- = (A^+)^*, \tag{2.1}$$

acting on a superbundle $L = L^+ \oplus L^-$ we refine this notion to that of a Cl(1) spectral section. If σ is the parity operator

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.2}$$

then, for a self-adjoint operator, the existence of the block decomposition (2.1) is equivalent to the anticommutation condition

$$\sigma A + A\sigma = 0. \tag{2.3}$$

Notice that it follows from (2.3) that if u is an eigenfunction of A with eigenvalue λ, then σu is an eigenfunction of A with eigenvalue $-\lambda$.

If A is invertible, we define its Atiyah-Patodi-Singer projection to be the orthogonal projection onto the span of the eigenfunctions corresponding to the positive eigenvalues of A. If this projection is P_0, then $\sigma P_0 \sigma$ is the orthogonal projection onto the negative eigenspaces of A so

$$P_0 + \sigma P_0 \sigma = \mathrm{Id}\,.$$

In the general case we need a smooth family of projections with this property.

Definition 1. For a family, A_z, of odd $\mathbb{Z}_2$-graded self-adjoint elliptic differential operators (of positive order and acting on the compact fibres of a fibration) a $\mathrm{Cl}(1)$ spectral section is a spectral section, P_z, for A, i.e., a family of self-adjoint projections such that for some $R > 0$

$$\begin{aligned} Au = \lambda u,\ \lambda > R &\implies Pu = u, \\ Au = \lambda u,\ \lambda < -R &\implies Pu = 0, \end{aligned} \tag{2.4}$$

with the anticommutation property

$$\sigma P + P\sigma = \sigma. \tag{2.5}$$

Notice that the additional condition (2.5) can also be written as

$$\sigma P \sigma = \mathrm{Id} - P. \tag{2.6}$$

Proposition 2. *A family of odd $\mathbb{Z}_2$-graded self-adjoint elliptic differential operators, A, admits a $\mathrm{Cl}(1)$ spectral section if and only if its index vanishes in K^0 of the base of the fibration.*

Proof. The K^1 index of the whole self-adjoint family A vanishes, as can be seen from the fact that it is homotopic through self-adjoint families to

$$\begin{pmatrix} B & A^- \\ A^+ & -B \end{pmatrix},$$

which is invertible if B is a first-order positive elliptic pseudodifferential operator. By Proposition 1 of [24], A admits a spectral section. In fact, as shown in Proposition 2 of [24], if $r > 0$ is preassigned, there is a spectral section, P', such that $P'u = 0$ if u is an eigenfunction with eigenvalue less than r; in particular P' annihilates the null space of A.

It follows from (2.3) that the family of self-adjoint projections $P'' = \sigma P' \sigma$ annihilates all eigenfunctions of A corresponding to eigenvalues $\lambda > -r$, and P' and P'' commute. Since, on each fibre, it is contained in the finite dimensional space spanned by the eigenfunctions of A, with eigenvalues in the range $[-r, r]$ the null space of the projection $P' + P''$ is a finite dimensional bundle, denoted N. Moreover σ acts as an involution on N which therefore splits into the sum of the ± 1 eigenspaces:

$$N = N^+ \oplus N^-. \tag{2.7}$$

The properties of P' imply that

$$\begin{aligned} A =& \quad A' + A'', \quad \text{where} \\ A' =& \quad P'AP' + P''AP'' \\ =& \quad (P' + P'')A(P' + P''), \end{aligned} \tag{2.8}$$

with both terms on the right $\mathbb{Z}_2$-graded. Since A' is invertible on the range of $P' + P''$ and A'' is finite dimensional, N, as a superbundle, represents the index of A in $K^0(B)$.

By assumption the index of A vanishes so N^+ and N^- have the same dimension and are stably isomorphic. Let U be a smooth bundle such that $N^+ \oplus U$ is isomorphic to $N^- \oplus U$. There is an integer q such that any bundle over B of rank at least q contains U as a subbundle. As shown in Proposition 2 of [23] the initial choice of spectral section, P', can be replaced by $\widetilde{P}'$ such that $\operatorname{ran}(\widetilde{P}') \subset \operatorname{ran}(P')$ has arbitrarily large, preassigned, codimension. This new choice replaces $N^\pm$ by $N^\pm \oplus W^\pm$ where A^+ gives an isomorphism between W^+ and W^-. The dimension of $W^\pm$ can be made arbitrarily large, so by an appropriate initial choice we can ensure that N^+ and N^- are bundle isomorphic.

Let $\phi : N^+ \longrightarrow N^-$ be a unitary bundle isomorphism. Using this to write operators on N as 2×2 matrices, the projection

$$P_N = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \tag{2.9}$$

is self-adjoint and satisfies (2.5) on N. Thus we finally have a Cl(1) spectral section:

$$P = P' + P_N. \tag{2.10}$$

For the converse see Lemma 1 below.

As in the non-graded case discussed in [24], a Cl(1) spectral section for a $\mathbb{Z}_2$-graded operator fixes a class of finite rank deformations of the operator.

Lemma 1. *If A is a smooth family of odd $\mathbb{Z}_2$-graded self-adjoint elliptic differential operators of positive order on the fibres of a compact fibration and P is a* Cl(1) *spectral section for A, then there is smooth family, A^0_P, of self-adjoint $\mathbb{Z}_2$-graded finite rank operators, in the span of a finite number of eigenfunctions for A, such that $A+A^0_P$ is invertible and P is the Atiyah-Patodi-Singer projection for $A+A^0_P$; the space of such deformations is connected.*

Proof. As noted in the proof of Proposition 2 a spectral section for A, P', can always be chosen such that $P'AP'$ is strictly positive on the range of P'. As shown in Proposition 2 of [24], it can always be arranged that P' and the given spectral section, P, commute. Set $Q' = P - P'$, $P'' = \sigma P'\sigma$ and $Q'' = \sigma Q'\sigma$ and then consider the decomposition:

$$A = A_1 + A_2,\ A_1 = P'AP' + P''AP''. \tag{2.11}$$

Thus A_1 is $\mathbb{Z}_2$-graded, and A_2 is finite rank and acts in the span of a finite number of eigenspaces of A. The deformed operator

$$A + A^0_P = A_1 + Q' - Q'',\ A^0_P = Q' - Q'' - A_2 \tag{2.12}$$

satisfies the requirements of the Lemma. Different choices clearly give homotopic deformations; in fact the space of such deformations is contractible.

Let us now turn to the special case in which $A = \eth_0$, the boundary family considered in the previous section. The same proof as given in [24] establishes the family version of the cobordism invariance of the index:

Proposition 3. *The analytic index of the boundary family $\eth_0$ is always zero in $K^0(B)$.*

Corollary 1. *Given a family of Dirac operators $\eth$ as in §1 there always exists a* Cl(1) *spectral section P for the boundary family $\eth_0$.*

3. Suspension

In this section we briefly recall the basic properties of suspension.

Atiyah and Singer in [8] show that for a compact manifold, B, the space, $[B;\widehat{\mathcal{F}}]$, of homotopy classes of continuous maps into the self-adjoint Fredholm operators on a fixed (infinite dimensional) Hilbert space is naturally identified with $K^1(B)$. This identification is obtained by suspension, mapping an homotopy class in $[B;\widehat{\mathcal{F}}]$ to the element of $[B';\mathcal{F}]$ given by

$$\mathrm{Sus}(A) : (0,\pi) \ni t \longmapsto \cos t + iA\sin t, \tag{3.1}$$

where $B' = (0,\pi)\times B$. If A is a family of unbounded self-adjoint operators, with domains forming a Hilbert bundle, then (3.1) should be replaced by

$$\mathrm{Sus}(A) : (0,\pi) \ni t \longmapsto (1+A^2)^{\frac{1}{2}}\cos t + iA\sin t, \tag{3.2}$$

where $(1+A^2)^{\frac{1}{2}}$ is the positive square root, so has the same domain as A. In fact it is shown in [8] that this suspension map induces an homotopy equivalence of $[B;\widehat{\mathcal{F}}]$ onto $[B';\mathcal{F}]$, where for a non-compact manifold the Fredholm operators are required to be invertible outside a compact set of the parameter space. Since the latter space is naturally identified with $K^0(B')$, $B' = (0,\pi)\times B$, we see that, by suspension, any family of self-adjoint Fredholm operators fixes an index class $\mathrm{Ind}(A) = [\mathrm{Sus}(A)]$ in $K^1(B)$.

Notice that if A is a family of formally self-adjoint elliptic differential operators, then we can alternatevely define the associated index class in $K^1(B)$ as the class obtained by suspending as in (3.1) the 0th-order elliptic self-adjoint pseudodifferential family $A/(1+A^2)^{\frac{1}{2}}$. In either case we shall use the notation $[A]$ for the class fixed by A in $[B;\widehat{\mathcal{F}}]$ and the notation $\mathrm{Ind}(A)$ for the class fixed by the suspended family $[\mathrm{Sus}(A)] \in [B';\mathcal{F}] \equiv K^1(B)$.

Recall from [1] the definition of the Chern character homomorphism

$$\mathrm{Ch} : K^1(B) \to H^{\mathrm{odd}}(B,\mathbb{C}) \equiv H^{\mathrm{odd}}(B),$$

(all our cohomology groups will be taken with coefficients in $\mathbb{C}$). The inclusion of $(0,\pi)\times B$ into $\mathbb{S}^1_t\times B$, $\mathbb{S}^1_t = \mathbb{R}/\pi\mathbb{Z}$, defines a natural injective homomorphism $j : K^1(B) \to K^0(\mathbb{S}^1\times B)$ with image equal to $\widetilde{K}^0(\mathbb{S}^1\times B) \equiv \ker(i^* : K^0(\mathbb{S}^1\times B) \to K^0(B))$, i being the inclusion of B into $\mathbb{S}^1\times B$ fixed by the inclusion of the point $p_0 = [0] = [\pi]$ into $\mathbb{S}^1$. On $K^0(\mathbb{S}^1\times B)$ there is a well-defined Chern character homomorphism

$\mathrm{Ch} : K^0(\mathbb{S}^1 \times B) \to H^{\mathrm{even}}(\mathbb{S}^1 \times B)$ with

$$\begin{aligned}\mathrm{Ch}(\widetilde{K}^0(\mathbb{S}^1 \times B)) \subset\ & \widetilde{H}^{\mathrm{even}}(\mathbb{S}^1 \times B) \\ \equiv\ & \ker(i^* : H^{\mathrm{even}}(\mathbb{S}^1 \times B) \to H^{\mathrm{even}}(B)).\end{aligned}$$

If $\Sigma^{-1} : \widetilde{H}^{\mathrm{even}}(\mathbb{S}^1 \times B) \to \widetilde{H}^{\mathrm{odd}}(B) = H^{\mathrm{odd}}(B)$ is the inverse of the suspension isomorphism $\Sigma : \widetilde{H}^*(B) \to \widetilde{H}^*(\mathbb{S}^1 \times B)$ (the map Σ^{-1} is usually referred to as desuspension), then, by definition,

$$\mathrm{Ch}(\alpha) = \Sigma^{-1}(\mathrm{Ch}(j(\alpha))) \in H^{\mathrm{odd}}(B), \alpha \in K^0((0,\pi) \times B).$$

If $\alpha \in K^1(B)$ is the index class of a continuous family A of self-adjoint Fredholm operators, $\alpha = \mathrm{Ind}(A)$, there is some additional structure. By definition $\mathrm{Ind}(A)$ equals the (regularized) virtual bundle $[\mathrm{null}(\mathrm{Sus}(A))] - [\mathrm{null}(\mathrm{Sus}(A)^*)]$ over $(0,\pi) \times B$. The Chern character of this virtual bundle is well defined as a closed differential form with compact support on $(0,\pi) \times B$. The Chern character of $j(\mathrm{Ind}(A))$ is obtained by extending this differential form to all of $\mathbb{S}^1 \times B$ so as to vanish near $\{0\} \times B$. Thus, in this case,

$$\begin{aligned}\mathrm{Ch}(\mathrm{Ind}(A)) &\equiv \frac{i}{2\pi} \int_{\mathbb{S}^1} \mathrm{Ch}(j(\mathrm{Ind}(A)) \\ &= \frac{i}{2\pi} \int_0^\pi \mathrm{Ch}([\mathrm{null}(\mathrm{Sus}(A))] - [\mathrm{null}(\mathrm{Sus}(A)^*)]) \\ &\in H^{\mathrm{odd}}(B).\end{aligned} \tag{3.3}$$

The choice of the normalizing factor will be explained after Proposition 7.

We first use a simple suspension argument to show that two $\mathrm{Cl}(1)$ spectral sections for a fixed family define an element in K^1 of the base.

Proposition 4. *If P_1 and P_2 are $\mathrm{Cl}(1)$ spectral sections for a fixed family of $\mathbb{Z}_2$-graded elliptic operators of positive order for a compact fibration with base B, then suspension defines a difference element*

$$[P_1 - P_2] \in K^1(B). \tag{3.4}$$

Proof. Let $A_i = A + A^0_{P_i}$ for $i = 1,2$ be deformations of the family A, as in Lemma 1, corresponding to the two $\mathrm{Cl}(1)$ spectral sections. In each case consider the family of elliptic operators:

$$A_i(t) = \sigma(1 + A_i^2)^{\frac{1}{2}} \cos t + A_i \sin t,\ t \in [0,\pi]. \tag{3.5}$$

These are also families of self-adjoint operators with $A_i(\frac{1}{2}\pi) = A_i$. Moreover $A_i(t)$ commutes with A_i for all t and on the two-dimensional space spanned by u and σu, where u is an eigenfunction of A_i with eigenvalue λ, $A_i(t)$ reduces to the matrix

$$\begin{pmatrix} (1+\lambda^2)^{\frac{1}{2}}\cos t & \lambda \sin t \\ \lambda \sin t & -(1+\lambda^2)^{\frac{1}{2}}\cos t \end{pmatrix}. \tag{3.6}$$

Thus $A_i(t)$ has eigenvalues $\pm(\cos^2 t(1+\lambda^2) + \sin^2 t\lambda^2)^{\frac{1}{2}}$ where the λ are the eigenvalues of A_i. In particular the $A_i(t)$ are invertible for all $t \in [0,\pi]$. Let $P_i(t)$ be the Atiyah-Patodi-Singer projection for $A_i(t)$. Then we define the class in (3.4) to be just

$$[P_1 - P_2] = [P_1(t) - P_2(t)] \in K^0((0,\pi)\times B). \tag{3.7}$$

Certainly the projections $P_1(t)$ and $P_2(t)$, for each point in B, differ by a finite rank operator. Thus, as discussed in [24] the difference is a well-defined virtual bundle over $[0,\pi]\times B$. We will briefly review the construction of this virtual bundle in §7. This difference bundle is trivial over $t=0$, and $t=\pi$ since the operators $A_i(0)$ and $A_i(\pi)$ are each independent of B and i. The class in (3.7) is therefore meaningful. Different choices of deformation give homotopic operators so the result is actually independent of all choices.

In particular it follows from (3.7) that the odd Chern character of the difference element in (3.4) is just the integral in t of the Chern character of the virtual bundle on the right in (3.7).

4. The odd index

Using Corallary 1 and the deformations given by Lemma 1 we can now associate a class in $K^1(B)$ to a choice of Cl(1) spectral section for the boundary family of the family of Dirac operators. As in [24], choose non-negative $\mathcal{C}^\infty$ functions ϕ_1 and $\phi_2 \in \mathcal{C}^\infty_c(\mathbb{R})$, where ϕ_1 is even and has integral 1, and ϕ_2 is identically equal to 1 near 0. For $\delta > 0$ small enough consider the operator

$$A_P = \delta^{-1}\phi_2(\delta x)\hat{\phi}_1(\delta x D_x)\phi_2(\delta x)A_P^0, \tag{4.1}$$

where A_P^0 is a perturbation of $\eth_0$ as in Lemma 1, and $\hat{\phi}_1$ is the Fourier transform of ϕ_1. Since A_P^0 is a finite rank smoothing operator, A_P is a b-pseudodifferential operator of order $-\infty$ on $[0,\infty)\times\partial X$ and its support is close to $x = 0$. Thus for $\delta > 0$ small enough A_P can be transferred to X by choice of a collar neighbourhood of the boundary.

Definition 2. If P is a $\mathrm{Cl}(1)$ spectral section for the boundary family of the family of Dirac operators associated to an Hermitian Clifford module with unitary Clifford connection for exact b-metrics on the odd-dimensional fibres of a fibration of a compact manifold with boundary, then $\mathrm{Ind}(\eth, P) = \mathrm{Ind}(\eth + A_P) \in K^1(B)$.

Since different choices of the perturbation A_P^0 produce homotopic self-adjoint Fredholm families, the index class in Definition 2 depends only on $\eth$ and P.

Although we take this definition of the index, involving Fredholm operators on the complete manifold, as the basic one, we shall also connect it to a more traditional definition of the index for an elliptic boundary problem in the usual, incomplete, sense. Thus, let $\tilde{M} \longrightarrow B$ be a fibration with compact odd-dimensional fibres being manifolds with boundary as before, but consider instead a smoothly varying family of metrics (incomplete, i.e., smooth and non-degenerate up to the boundary) with smoothly varying product decompositions near the boundary. If $\tilde{E}$ is an Hermitian Clifford module on the fibres, with unitary Clifford connection with product decomposition near the boundary, then the associated family of Dirac operators $\tilde{\eth}$ has a decomposition similar to (1.6),

$$\tilde{\eth} = \sigma(\frac{1}{i}\frac{\partial}{\partial \tilde{x}} + \frac{1}{i}\eth_0), \tag{4.2}$$

where we again write $\eth_0$ for the boundary Dirac operator which is $\mathbb{Z}_2$-graded odd for the same decomposition as in (1.5), except that $\sigma = \mathrm{cl}(d\tilde{x})$, where $\tilde{x}$ is the normal variable to the boundary.

Of course the connection between this incomplete case and the exact b-metric case discussed above is very close. Following [3] one can pass from the incomplete case to the (product) exact b-case by adding the semi-infinite cylinder $(-\infty, 0)_{\tilde{x}} \times \partial M$ and then compactifying this fibration by introducing the new boundary defining function $x = e^{\tilde{x}}$. Similarly one can pass from the product exact b-case to the incomplete case by introducing $\tilde{x} = \log(x/a)$ with $a > 0$ chosen so small that $x = a$ is in the product region of the metric. Then restricting to $\tilde{x} \geq 0$ gives a family of incomplete product metrics. The bundles can be similarly trivialized. To pass from the exact b-metric case to the product case, and hence to the incomplete product case, requires a small homotopy.

Now in the incomplete case a spectral section P for $\eth_0$, the existence of which follows as before, specifies a boundary condition of 'generalized Atiyah-Patodi-Singer' type. Namely the boundary problem

$$\widetilde{\eth}_z u = f, \ P_z(u_z \restriction \partial\tilde{M}_z) = 0 \tag{4.3}$$

is a (pseudodifferential) elliptic boundary problem varying smoothly with $z \in B$. Thus on the Sobolev spaces

$$\left\{u_z \in H^1(\tilde{M}_z; E_z); P_z(u_z \restriction \partial\tilde{M}_z) = 0\right\} \tag{4.4}$$

this gives a smooth family of Fredholm operators which are, by Green's formula (see for example [25]), self-adjoint in view of the requirement that P be a $\mathrm{Cl}(1)$ spectral section.

Proposition 5. *The index in $K^1(B)$ of the family of generalized APS boundary problems (3.4), for a family of Dirac operators with respect to incomplete metrics (as discussed above) is equal to the index of the Dirac operators, with perturbation fixed by the spectral section, as in Definition 2, for the (product) exact b-metrics obtained by adding a semi-infinite cylindrical end and extending the product structures.*

In case no perturbation is necessary (or for a single operator in general) this is the approach taken by Atiyah, Patodi and Singer to the proof of their index formula for a single even-dimensional manifold.

Proof. Using the notation above, let $\widetilde{\eth}$ be the Dirac operator for an incomplete metric structure on the fibres of a fibration $\tilde{M} \longrightarrow B$, and let $\eth$ be the corresponding Dirac operator arising from the extension to a (product type) exact b-metric structure. To show that the two index classes are the same we first deform the operator $\widetilde{\eth}$. With the boundary defining function $\tilde{x}$ consider the homotopy

$$\widetilde{\eth} + t\tilde{A}_P, \ \tilde{A}_P = \sigma\frac{1}{i}\rho(\tilde{x})A^0_P \tag{4.5}$$

for $t \in [0, 1]$. Here, $\rho \in \mathcal{C}^\infty_c(\mathbb{R})$ has $\rho(\tilde{x}) = 1$ near $\tilde{x} = 0$ and such small support that it vanishes outside the product neighbourhood. The perturbed operator is no longer a differential operator but it remains a self-adjoint Fredholm operator on the fixed domain (4.4). Moreover, for $t = 1$ the original boundary condition in (3.4) becomes the Atiyah-Patodi-Singer boundary condition for the perturbed boundary operator $\eth_0 + A^0_P$. Thus the index class of (3.4) is the same as that of (4.5) with the same boundary condition.

Consider the spectral family of $\widetilde{\eth} + \tilde{A}_P$, i.e., the operators

$$(\widetilde{\eth} + \tilde{A}_P)u - \lambda u = 0,\ P(u \restriction \partial\tilde{M}) = 0. \tag{4.6}$$

The operator is of product type and an eigenfunction u satisfies

$$\frac{\partial}{\partial\tilde{x}}u + (\eth_0 + A_P^0 - \sigma\lambda)u = 0 \tag{4.7}$$

near the boundary. Let μ_j be the positive eigenvalues of the self-adjoint operator $\eth_0 + A_P^0$, with e_j corresponding orthonormal eigenfunctions. The negative eigenvalues are $-\mu_j$ with eigenfunctions σe_j, and by construction 0 is never an eigenvalue. Let $\mu > 0$ be a lower bound for the eigenvalues (for all values of the parameter.) Then, for $\lambda \in \mathbb{C}$ small, $|\lambda| < \mu$, the eigenvalues of $\eth_0 + A_P^0 - \sigma\lambda$ are

$$\begin{aligned}&\delta_j = \sqrt{\mu_j^2 + \lambda^2} \text{ with eigenfunction } f_j^+ = e_j - \left(\tfrac{\delta_j - \mu_j}{\lambda}\right)\sigma e_j,\\ &-\delta_j \text{ with eigenfunction } f_j^- = \left(\tfrac{\lambda}{\delta_j + \mu_j}\right)e_j + \sigma e_j.\end{aligned} \tag{4.8}$$

For $t \in [0,1]$ consider the basis

$$\begin{aligned}f_j^+(t) &= e_j - t\left(\tfrac{\delta_j - \mu_j}{\lambda}\right)\sigma e_j,\\ f_j^-(t) &= t\left(\tfrac{\lambda}{\delta_j + \mu_j}\right)e_j + \sigma e_j,\end{aligned} \tag{4.9}$$

which is a homotopy from the e_j, σe_j basis to the f_j^+ and f_j^-. Let $P_\lambda(t)$ be the projection satisfying

$$P_\lambda(t)f_j^+(t) = f_j^+(t),\ P_\lambda(t)f_j^-(t) = 0. \tag{4.10}$$

This is a pseudodifferential operator of order 0 with the same principal symbol as P_0, the Atiyah-Patodi-Singer projection, for all $t \in [0,1]$ and $|\lambda| < \mu$.

For $\lambda = is$, $s \in \mathbb{R}$ and with $|s| < \mu$, the δ_j are real and the $f_j^-(t)$ satisfy

$$\langle \sigma f_j^-(t), f_j^-(t)\rangle = \frac{ist}{\delta_j + \mu_j} + \frac{-ist}{\delta_j + \mu_j} = 0, \tag{4.11}$$

which means that the elliptic boundary problem

$$\begin{aligned}&Q_s u = i\left(\widetilde{\eth} + \tilde{A}_P - is\right), \quad u = 0,\\ &P_{is}(t)(u \restriction \partial\tilde{M}) = 0, \quad t \in [0,1],\ s \in [-\mu, \mu]\end{aligned} \tag{4.12}$$

satisfies the integral identity

$$\operatorname{Im}\langle(\widetilde{\eth}+\tilde{A}_P)u - isu, u\rangle = -s\|u\|^2 \tag{4.13}$$

on its domain and hence is invertible for $0 \neq s \in [-\frac{1}{2}\mu, \frac{1}{2}\mu]$ and $t \in [0,1]$. Moreover for $s = -\frac{1}{2}\mu$ the self-adjoint part of Q_s is strictly negative, so there is an obvious homotopy to a fixed invertible operator. Similarly for $s = \frac{1}{2}\mu$ the self-adjoint part is strictly positive. For $t = 0$ this family (as the parameter z, mostly suppressed in this discussion, varies) represents, by suspension, the K^1 index of the family of Dirac operators; *cf.* (3.1). By homotopy invariance the same is true for $t = 1$, i.e., the index is represented by the family of Fredholm operators

$$\begin{aligned} Q_s = i(\widetilde{\eth}+\tilde{A}_P) + s &: \operatorname{Dom}_s \longrightarrow L^2(\tilde{M};E),\\ \operatorname{Dom}_s = \Big\{u \in H^1(\tilde{M};E); P_{is}(1)&(u \restriction \tilde{M}) = 0\Big\},\\ &s \in [-\frac{1}{2}\mu, \frac{1}{2}\mu]. \end{aligned} \tag{4.14}$$

Now we extend the manifold, $\tilde{M}$, to a manifold, M, with cylindrical end and the boundary defining functions related by $x = \exp(\tilde{x})$. The domain Dom_s can be embedded as a subspace of $H^1_{\mathrm{b}}(M;E)$ by mapping each element u_s to the section u'_s over M where

$$\begin{aligned} &u'_s = u_s \qquad \text{in } x > 1,\\ &(\eth + \tilde{A}_P)u'_s - isu'_s = 0 \qquad \text{in } x \le 1. \end{aligned} \tag{4.15}$$

Notice that, by definition of $P_{is}(1)$, there is a unique square-integrable solution of the equation in $x < 1$ with boundary data at $x = 1$ in the null space of $P_{is}(1)$. Thus the map $u_s \longmapsto u'_s$ is an isomorphism onto a closed subspace of $H^1_{\mathrm{b}}(M;E)$; we can therefore write

$$\operatorname{Dom}_s \subset H^1_{\mathrm{b}}(M;E). \tag{4.16}$$

To find an appropriate complementary subspace choose $\chi \in \mathcal{C}^\infty(\mathbb{R})$ with $\chi(x) = 1$ in $x < 1$ but with $\chi(x) \in \mathcal{C}^\infty(M)$ having support in the collar neighbourhood of the boundary and set

$$\begin{aligned} G_s = \big\{u \in H^1_{\mathrm{b}}(M;E); u_0 = u \restriction (x=1)&\\ \text{satisfies } P_{is}(1)u_0 = u_0&\\ \text{and } u = \chi(x)u_0 \text{ in } x > 1\big\}&. \end{aligned} \tag{4.17}$$

Then

$$H^1_{\mathrm{b}}(M;E) = \operatorname{Dom}_s \oplus G_s, \tag{4.18}$$

which is a direct sum of closed subspaces, but not an orthogonal sum. That the intersection of these two closed subspaces is $\{0\}$ follows from the fact that if $u \in H^1_{\mathrm{b}}(M;E)$ is in both, then $u = 0$ on $x = 1$ and hence from (4.17) u vanishes in $x > 1$ and from (4.15) $u = 0$ in $x < 1$. Given $v \in H^1_{\mathrm{b}}(M;E)$ let $v_1 = v_+ + v_-$ be the decomposition of $v_1 = v \restriction (x=1)$ under $\mathrm{Id} = P_{is}(1) + (\mathrm{Id} - P_{is}(1))$. Then v_- can be continued into $x < 1$ as a solution, u'_s, of the equation (4.15). Let $v'' = v - u'_s$ in $x < 1$ and $v'' = \chi(x)v_+$ in $x > 1$. Then $v'' \in G_s$ and $v' = v - v'' \in \mathrm{Dom}_s$. This proves (4.18).

Consider the action of $\eth + \tilde{A}_P$ on G_s. In fact

$$(\eth + \tilde{A}_P - is)G_s = H_s \subset L^2_{\mathrm{b}}(M;E)$$

is closed and

$$(4.19) \qquad (\eth + \tilde{A}_P - is) : G_s \longrightarrow H_s \text{ is an isomorphism.}$$

To see this first note that

$$(4.20) \qquad \begin{aligned} (\eth + \tilde{A}_P - is) : \{u \in H^1_{\mathrm{b}}([0,1] \times \partial M; E);& \\ P_{is}(1)u(1) = u(1)\} \longrightarrow \tilde{H},& \\ \tilde{H} = \{f \in L^2_{\mathrm{loc}}((0,1) \times \partial M); \int_0^1 \int_{\partial M} |f(x,y)|^2 \frac{dx}{x} dy < \infty\}& \end{aligned}$$

is an isomorphism, always for $s \in \mathbb{R}$, $|s| \leq \frac{1}{2}\mu$, as can be seen using the Mellin transform (see [23]) or separation of variables. Thus the restriction map $H_s \longrightarrow \tilde{H}$ is an isomorphism, and this shows that H_s is closed in $L^2_{\mathrm{b}}(M;E)$ and that (4.19) holds. This in turn shows that the decomposition

$$(4.21) \qquad L^2_{\mathrm{b}}(M;E) = L^2(\tilde{M};E) \oplus H_s$$

is a topological (but again not an orthogonal) direct sum and hence that

$$i(\eth + \tilde{A}_P) + s = \begin{pmatrix} i(\tilde{\eth} + \tilde{A}_P) + s & 0 \\ 0 & i(\eth + \tilde{A}_P) + s \end{pmatrix} :$$

$$\mathrm{Dom}_s \oplus G_s \longrightarrow L^2(\tilde{M};E) \oplus H_s = L^2_{\mathrm{b}}(M;E)$$

with the second diagonal entry an isomorphism.

As in the initial part of the argument, $i\,(\eth + A_P - is)$ represents, by suspension, the K^1 class of the self-adjoint family $\eth + A_P$. Thus we have shown that the index classes, in K^1 of the base, of the elliptic boundary problem and of the complete problem, with perturbation, are the same.

5. Dirac suspension

To compute the Chern character of

$$\mathrm{Ind}(\eth, P) = \mathrm{Ind}(\eth + A_P) \in K^1(B)$$

we need to express the Atiyah-Singer suspended family (3.1) as a family of perturbed Dirac operators on even dimensional manifolds fibering over B'. It is only to such a family that the results of [24] apply.

We start by analyzing the structure of the external tensor product of two Clifford modules over the product of two Riemannian manifolds. Recall that by a Clifford module over a Riemannian manifold X we shall mean a complex vector bundle over X with a smooth non-trivial fibre action of the Clifford algebra, and in case the manifold is even-dimensional we demand that the module be $\mathbb{Z}_2$-graded, i.e., $L = L^+ \oplus L^-$ with $\mathrm{cl}(\alpha) : \mathcal{C}^\infty(X; L^\pm) \longrightarrow \mathcal{C}^\infty(X; L^\mp)$ for each $\alpha \in \mathcal{C}^\infty(X; T^*X)$. With these conventions we find:

Lemma 2. *If L_1 and L_2 are Clifford modules, with Clifford actions cl_1 and cl_2, over Riemannian manifolds X_1 and X_2, then for the product metric on $X = X_1 \times X_2$*

$$\begin{aligned} \mathrm{cl}(\alpha) &= \mathrm{cl}_1(\alpha) \otimes \mathrm{Id} \otimes \Gamma_1 \ \textit{for}\ \alpha \in \mathcal{C}^\infty(X_1; T^*X_1); \\ \mathrm{cl}(\beta) &= \mathrm{Id} \otimes \mathrm{cl}_2(\beta) \otimes \Gamma_2 \ \textit{for}\ \beta \in \mathcal{C}^\infty(X_2; T^*X_2) \end{aligned} \tag{5.1}$$

gives the bundle $L = L_1 \otimes L_2 \otimes \mathbb{C}^2$ a Clifford module structure provided Γ_1 and Γ_2 are anti-commuting involutions; for example

$$\Gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ \textit{and}\ \Gamma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \tag{5.2}$$

Proof. That (5.1) gives an action on L of the Clifford bundle for the product metric follows from the conditions

$$\Gamma_1^2 = \mathrm{Id},\ \Gamma_2^2 = \mathrm{Id} \ \text{and}\ \Gamma_1\Gamma_2 + \Gamma_2\Gamma_1 = 0. \tag{5.3}$$

If both X_1 and X_2 are even dimensional then, by assumption, both L_1 and L_2 are $\mathbb{Z}_2$-graded, and the $\mathbb{Z}_2$-grading on L can be taken to be

$$\begin{aligned} L^+ = & \ L_1^+ \otimes L_2^+ \otimes \mathbb{C}^2 + L_1^- \otimes L_2^- \otimes \mathbb{C}^2, \\ L^- = & \ L_1^+ \otimes L_2^- \otimes \mathbb{C}^2 + L_1^+ \otimes L_2^- \otimes \mathbb{C}^2. \end{aligned} \tag{5.4}$$

In case both X_1 and X_2 are odd dimensional, the $\mathbb{Z}_2$-grading can be taken as the trivial $\mathbb{Z}_2$-grading on $\mathbb{C}^2$:

$$\begin{aligned} L^+ &= L_1 \otimes L_2 \otimes (\mathbb{C} \oplus \{0\}), \\ L^- &= L_1 \otimes L_2 \otimes (\{0\} \oplus \mathbb{C}). \end{aligned} \tag{5.5}$$

If the Clifford actions are unitary for an Hermitian inner product on the bundles, then this product action is also unitary. Similarly if L_1 and L_2 carry unitary Clifford connections, then the tensor product connection on L, trivial on the factor $\mathbb{C}^2$, is also Clifford and unitary.

We are now ready to give a Dirac representative for the suspended family (see [14] for the closed case). Set $\mathbb{S}^1_\theta = \mathbb{R}/2\pi\mathbb{Z}$ and $\mathbb{S}^1_t = \mathbb{R}/\pi\mathbb{Z}$. Consider the fibration

$$\psi : \mathbb{S}^1_\theta \times \mathbb{S}^1_t \longrightarrow \mathbb{S}^1_t \tag{5.6}$$

and the Hermitian line bundle L, over the total space, obtained by identifying the points $(\theta, 0, v)$ and $(\theta, \pi, \exp(-i\theta)v)$. We endow L with the Hermitian connection

$$\nabla^L = d + i\frac{t-\frac{1}{2}}{\pi} d\theta. \tag{5.7}$$

These data restrict to each fibre to define a Hermitian Clifford module with Hermitian Clifford connection. The resulting family of Dirac operators

$$\eth^L_t = \frac{1}{i}\frac{\partial}{\partial\theta} + \frac{t-\frac{1}{2}}{\pi} \tag{5.8}$$

has spectral flow equal to one since

$$\text{the eigenvalues of } \eth^L_t \text{ are } \lambda_n(t) = n + \frac{t-\frac{1}{2}}{\pi}. \tag{5.9}$$

Notice that the family (5.8) is periodic precisely because of the definition of the line bundle L. The eigenfunctions of $\eth^L_t$ corresponding to the eigenvalues $\lambda_n(t) = n + (t - \frac{1}{2})/\pi$ are given explicitly by the smooth sections $e_n \in \mathcal{C}^\infty(\mathbb{S}^1_t \times \mathbb{S}^1_\theta; L)$ defined by

$$\begin{aligned} e_n(t,\theta) &= e^{in\theta} \quad \text{if } t \in [0,\pi), \\ e_n(t,\theta) &= e^{i(n-1)\theta} \quad \text{if } t = \pi. \end{aligned}$$

Let E be the original Clifford module for the fibration $\phi : M \longrightarrow B$. We consider the product fibre Clifford structure of Lemma 2 on the bundle $F = L \otimes E \otimes \mathbb{C}^2$ over the product fibration

$$\psi \times \phi : \mathbb{S}^1_\theta \times \mathbb{S}^1_t \times M \longrightarrow \mathbb{S}^1_t \times B. \tag{5.10}$$

The family of Dirac operators defined by these data is

(5.11)

$$\eth^F_{(t,z)} = \begin{pmatrix} 0 & \eth^L_t \otimes \mathrm{Id}(z) + \mathrm{Id}(t) \otimes (i\eth_z) \\ \eth^L_t \otimes \mathrm{Id}(z) - \mathrm{Id}(t) \otimes (i\eth_z) & 0 \end{pmatrix}$$

for each $(t,z) \in \mathbb{S}^1 \times B$.

Assume for the time being that

$$\text{the boundary family } \eth_0 \text{ is invertible.} \tag{5.12}$$

We shall deal with the technical difficulties of the general case in the second part of this section.

As explained in §1 the family $\eth$ fixes a smooth family of unbounded self-adjoint Fredholm operators and thus, according to §3, an index class $\mathrm{Ind}(\eth) = [\mathrm{Sus}(\eth/(1+\eth^2)^{\frac{1}{2}})] \in K^0((0,\pi) \times B)$. By applying the natural inclusion $j : K^0((0,\pi) \times B) \longrightarrow K^0(\mathbb{S}^1 \times B)$ we obtain an element in $K^0(\mathbb{S}^1 \times B)$.

Consider now the family $\eth^F$ defined by (5.11). Using Lemma 5 of §6 below we see that, under assumption (5.12), the family $\eth^F$ is Fredholm. Thus there is a well-defined index class $\mathrm{Ind}(\eth^F) \in K^0(\mathbb{S}^1 \times B)$.

Lemma 3. *Let $p_0 = [0] = [\pi] \in \mathbb{S}^1 \equiv \mathbb{R}/\pi\mathbb{Z}$ and let U_0 be a small neighborhood of p_0 in $\mathbb{S}^1$. For each $t \in U_0$ and each $z \in B$ the operator $\eth^F_{(t,z)}$ is invertible.*

Proof. It suffices to show that $(\eth^F_{(t,z)})^2$ is invertible $\forall\ (t,z) \in U_0 \times B$. Since

$$(\eth^F_{(t,z)})^2 = ((\eth^L_t)^2 \otimes \mathrm{Id}(z) + \mathrm{Id}(t) \otimes (\eth_z)^2) \otimes \mathrm{Id}_{\mathbb{C}^2}$$

and $(\eth^L_t)^2$ is a strictly positive operator for each $t \in U_0$, it follows that $\mathrm{spec}(\eth^F_{(t,z)})^2 \cap (-\infty, 0] = \emptyset$ for each $(t,z) \in U_0 \times B$ and the lemma is proved.

From this Lemma we obtain at once

Lemma 4. *The index class* $\mathrm{Ind}(\eth^F)$ *lies in the image of the natural inclusion* $j : K^0((0,\pi)\times B) \longrightarrow K^0(\mathbb{S}^1 \times B)$.

Proposition 6. *Under assumption (5.12) we have the following equality of K-classes:*

$$j([\mathrm{Sus}(\eth/(1+\eth^2)^{\frac{1}{2}})]) = \mathrm{Ind}(\eth^F) \ \textit{in}\ K^0(\mathbb{S}^1 \times B). \tag{5.13}$$

Proof. We certainly have $\mathrm{Ind}(\eth^F) = \mathrm{Ind}(\eth^F/(1+(\eth^F)^2)^{\frac{1}{2}})$. Let $\mathbb{S}^1(t)\times M_z$ be the fibre of (5.10) over the point $(t,z)\in\mathbb{S}^1\times B$. If $e_n(t)$ is the eigenfunction of $\eth^L_t$ corresponding to the eigenvalue $\lambda_n(t)$, then there is an orthogonal decomposition

$$L^2_b(\mathbb{S}^1(t)\times M_z;F) = \bigoplus_{n\in\mathbb{Z}} \left(\mathbb{C}e_n(t)\otimes L^2_b(M_z;E)\otimes\mathbb{C}^2\right),$$

which is valid globally on $(0,\pi)\times B$. By Lemma 4 it suffices to prove that

$$j^{-1}(\mathrm{Ind}(\frac{\eth^F}{(1+(\eth^F)^2)^{\frac{1}{2}}})) = [\mathrm{Sus}(\frac{\eth}{(1+\eth^2)^{\frac{1}{2}}})],$$

i.e., that $\eth^F/(1+(\eth^F)^2)^{\frac{1}{2}}$ and $\mathrm{Sus}(\eth/(1+\eth^2)^{\frac{1}{2}})$ are homotopic as continuous Fredholm families over $(0,\pi)\times B$.

The eigenvalues $\lambda_n(t)$ of the operators $\eth^L_t$ are given by (5.9), so are different from zero for each $n\neq 0$ and for each $t\in[0,\pi]$ whereas $\lambda_0(t) = (t-\frac{1}{2})/\pi$ vanishes only for $t=\frac{1}{2}$. On each summand of the decomposition the operator $\eth^F/(1+(\eth^F)^2)^{\frac{1}{2}}$ reduces to

$$\begin{pmatrix} 0 & \frac{\lambda_n(t)}{(1+\lambda_n(t)^2+\eth_z^2)^{\frac{1}{2}}} + \frac{(i\eth_z)}{(1+\lambda_n(t)^2+\eth_z^2)^{\frac{1}{2}}} \\ \frac{\lambda_n(t)}{(1+\lambda_n(t)^2+\eth_z^2)^{\frac{1}{2}}} - \frac{(i\eth_z)}{(1+\lambda_n(t)^2+\eth_z^2)^{\frac{1}{2}}} & 0 \end{pmatrix}. \tag{5.14}$$

This operator has spectrum equal to the range of

$$\pm(\lambda_n(t)^2+\alpha(z)^2)/1+\lambda_n(t)^2+\alpha(z)^2)^{\frac{1}{2}}$$

as $\alpha(z)$ runs over the spectrum of $\eth_z$. Thus if $n\neq 0$ it is invertible for each value of $(t,z)\in(0,\pi)\times B$, whereas if $n=0$ it is invertible for each $t\neq\frac{1}{2}$ and each $z\in B$. It follows that $\eth^F/(1+(\eth^F)^2)^{\frac{1}{2}}$ can be written as the direct sum of an invertible family and the family corresponding to (5.14) for $n=0$. Since the former does not contribute to the index, we conclude that $\operatorname{Ind}(\eth^F/(1+(\eth^F)^2)^{\frac{1}{2}})$ is equal in $K^0((0,\pi)\times B)$ to the index bundle associated to the continuous family

$$-\frac{\lambda_0(t)}{(1+\lambda_0(t)^2+\eth_z^2)^{\frac{1}{2}}}+\frac{(i\eth_z)}{(1+\lambda_0(t)^2+\eth_z^2)^{\frac{1}{2}}} \tag{5.15}$$

acting from $(\mathbb{C}e_0(t)\otimes L^2(M_z;E)\otimes\mathbb{C}^2)^+$ to $(\mathbb{C}e_0(t)\otimes L^2(M_z;E)\otimes\mathbb{C}^2)^-$, both spaces being isomorphic to $L^2(M_z;E)$. The family (5.15) is clearly homotopic to

$$-\lambda_0(t)+i\frac{\eth_z}{(1+\eth_z^2)^{\frac{1}{2}}},$$

which is in turn homotopic to the Atiyah-Singer suspension

$$\operatorname{Sus}(\frac{\eth}{(1+\eth^2)^{\frac{1}{2}}})_{(t,z)}=\cos t+i\frac{\eth_z}{(1+\eth_z^2)^{\frac{1}{2}}}\sin t.$$

The proposition is therefore proved.

According to the definition (3.4) of the odd Chern character we have

$$\operatorname{Ch}(\operatorname{Ind}(\eth))=\frac{i}{2\pi}\int_{\mathbb{S}^1}\operatorname{Ch}(\operatorname{Ind}(\eth^F)).$$

We now drop assumption (5.12) and consider the general case. Let P be a $\mathrm{Cl}(1)$ spectral section for $\eth$ and let $\operatorname{Ind}(\eth,P)=\operatorname{Ind}(\eth+A_P)$ be the associated index class in $K^1(B)$. It would be natural to consider the family of $\mathbb{Z}_2$−graded operators

(5.16)
$$\check{\eth}^F_{(t,z)}=\begin{pmatrix}0 & \eth^L_t\otimes\operatorname{Id}(z)+\operatorname{Id}(t)\otimes(i\eth+iA_P)_z\\ \eth^L_t\otimes\operatorname{Id}(z)-\operatorname{Id}(t)\otimes(i\eth+iA_P)_z & 0\end{pmatrix}$$

with $(t,z)\in\mathbb{S}^1\times B$. However to get a family for which the results of [24] apply we need to further deform the family $\check{\eth}^F$; the problem comes from the operator $\operatorname{Id}(t)\otimes(A_P)_z$ which is not b-pseudodifferential on $\mathbb{S}^1(t)\times M_z$. We consider instead the following family. Let $0<\epsilon<1,\epsilon$ small and let $f_\epsilon\in\mathcal{C}^\infty_c([0,\pi]\times\mathbb{R})$ be a smooth function equal to zero on $\{0\}\times\mathbb{R}$ and $\{\pi\}\times B$, and equal to one for $(t,x)\in[\delta,\pi-\delta]\times[-1/\epsilon,1/\epsilon],\delta$ small. Consider the smooth family of operators, R_ϵ, with $R_\epsilon(t)$ the smoothing finite rank operator with Schwartz kernel

$$K(R_\epsilon(t))(\theta,\theta')=\sum_n f_\epsilon(t,\lambda_n(t))e_n(t)\otimes\overline{e_n(t)}. \tag{5.17}$$

Let $\widetilde{\eth}^{F,\epsilon}$ be the family of b-pseudodifferential operators

$$\widetilde{\eth}^{F,\epsilon}_{(t,z)} = \begin{pmatrix} 0 & \eth^{F,-}_{(t,z)} + iR_\epsilon(t) \otimes (A_P)_z \\ \eth^{F,+}_{(t,z)} - iR_\epsilon(t) \otimes (A_P)_z & 0 \end{pmatrix}. \tag{5.18}$$

Notice that $\widetilde{\eth}^{F,\epsilon}$ is globally defined on $\mathbb{S}^1 \times B$; moreover for each value of the parameter it is the sum of a generalized Dirac operator and a b-pseudodifferential operator of order $-\infty$. Using the results of the next section we see that the indicial family of $\widetilde{\eth}^{F,\epsilon}$ is invertible for λ real; thus $\widetilde{\eth}^{F,\epsilon,+}$ defines a smooth family of Fredhom operators and so an index class $\mathrm{Ind}(\widetilde{\eth}^{F,\epsilon}) \in K^0(\mathbb{S}^1 \times B)$. The proofs of Lemma 3 and Proposition 6 can be easily modified to give:

Proposition 7. *As classes in* $K^0(\mathbb{S}^1 \times B)$

$$\mathrm{Ind}(\widetilde{\eth}^{F,\epsilon}) = j([\mathrm{Sus}(\eth + A_P)]) \equiv j(\mathrm{Ind}(\eth, P)).$$

Notice once again that it is really the restriction of $\widetilde{\eth}^{F,\epsilon}$ to $(0,\pi) \times B$, that determines the index class $\mathrm{Ind}(\eth, P)$.

In fact over $(0,\pi) \times B$ the family $\widetilde{\eth}^{F,\epsilon}$ is homotopic to the family obtained by considering instead of R_ϵ in (5.18) the family of finite rank smoothing operators $Q_\epsilon = (Q_\epsilon(t))_{t \in (0,\pi)}$ with Schwartz kernel

$$K(Q_\epsilon(t)) = \sum_n \phi_\epsilon(\lambda_n(t)) e_n(t) \otimes \overline{e_n(t)}, \tag{5.19}$$

with $\phi_\epsilon \in \mathcal{C}^\infty_c(\mathbb{R})$ a smooth function equal to one for $x \in [-1/\epsilon, 1/\epsilon]$ (thus, compared to f_ϵ in (5.13), with no dependence on the t variable). Hence if we denote by $\widehat{\eth}^{F,\epsilon}$ the family of b-pseudodifferential operators

$$\widehat{\eth}^{F,\epsilon}_{(t,z)} = \begin{pmatrix} 0 & \eth^{F,-}_{(t,z)} + iQ_\epsilon(t) \otimes (A_P)_z \\ \eth^{F,+}_{(t,z)} - iQ_\epsilon(t) \otimes (A_P)_z & 0 \end{pmatrix} \tag{5.20}$$

with $(t,z) \in (0,\pi) \times B$ and if we recall (3.4), we obtain

Proposition 8. *As classes in* $K^0((0,\pi) \times B)$

$$\mathrm{Ind}(\eth, P) \equiv [\mathrm{Sus}(\eth + A_P)] = j^{-1}(\mathrm{Ind}(\widetilde{\eth}^{F,\epsilon})) = \mathrm{Ind}(\widehat{\eth}^{F,\epsilon}). \tag{5.21}$$

Moreover

$$\mathrm{Ch}(\mathrm{Ind}(\eth, P)) = \frac{i}{2\pi} \int_0^\pi \mathrm{Ch}([\mathrm{null}(\widehat{\eth}^{F,\epsilon,+})] - [\mathrm{null}(\widehat{\eth}^{F,\epsilon,-})]). \tag{5.22}$$

We will obtain formula (4) for the Chern character of the index class associated to the original family $\eth$ and the choice of P by applying the family index theorem of [24] to $\widehat{\eth}^{F,\epsilon}$, computing the normalized t-integral in (5.22) and letting ϵ go to zero.

We end this section by observing that the analogue of Proposition 6 in the boundaryless case and a straightforward application of the Atiyah-Singer family index formula for the Dirac suspension $\eth^F$ of (5.8) imply that

$$\langle \mathrm{Ch}(\eth^L), [\mathbb{S}^1]\rangle = 1 = \text{ spectral flow of } \eth^L,$$

which is consistent with [5] and explains our choice of the normalizing factor in (3.4)

6. Boundary behaviour

In order to analyze the boundary behaviour of the Dirac suspension we shall follow the identifications explained in [24]. We consider the bundle $F \restriction \partial M' = L \otimes (E \restriction \partial M) \otimes \mathbb{C}^2$ over the manifold $\partial M \times \mathbb{S}^1 \times \mathbb{S}^1$. Let E_0 denote the restriction of E to ∂M; as in [24] we identify $F \restriction \partial M'$ with $(L \otimes E_0) \oplus (L \otimes E_0)$ through the isomorphism N given by the block-diagonal matrix

$$\begin{pmatrix} \mathrm{Id} & 0 \\ 0 & i\,\mathrm{Id} \otimes \mathrm{cl}_2(\frac{dx}{x}) \end{pmatrix}.$$

Clearly $N^{-1} : (L \otimes E_0) \oplus (L \otimes E_0) \longrightarrow F \restriction \partial M'$ is then

$$\begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \frac{1}{i}\,\mathrm{Id} \otimes \mathrm{cl}_2(\frac{dx}{x}) \end{pmatrix}.$$

Consider the additional matrices of (5.2). Writing the Dirac suspension in block form and using the definition of the isomorphism N we find

$$\begin{aligned} N \cdot I(\eth^F, \lambda) \cdot N^{-1} & \\ &= \Gamma_1 \lambda + \mathrm{Id} \otimes \eth_0 \otimes \Gamma_2 \\ &\quad - \eth^L \otimes \mathrm{cl}_2(\frac{dx}{x}) \otimes \Gamma_2 \end{aligned} \tag{6.1}$$

with $\eth_0$ equal to the boundary Dirac operator of the original structure with respect to the Clifford action

$$\mathrm{cl}_\partial(\eta) = i\,\mathrm{cl}(\frac{dx}{x})\,\mathrm{cl}(\eta).$$

Note that although the matrices Γ_i are different from the ones considered in [24], it is still true that if Γ_3 is the matrix

$$\Gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

then $i\Gamma_3\Gamma_1 = \Gamma_2$. This is the only relation among the 2×2 matrices appearing in the indicial family that was used in the proof of the family index theorem in [24].

There is a natural product boundary Clifford structure underlying formula (6.1). Namely, on the bundle $L \otimes E_0$ over the boundary fibration $\partial M' \longrightarrow \mathbb{S}^1 \times B$ consider the Clifford action:

$$\begin{aligned} \mathrm{cl}_\partial(\eta) &= \mathrm{Id} \otimes i\,\mathrm{cl}_2(\frac{dx}{x})\,\mathrm{cl}_2(\eta), \\ \mathrm{cl}_\partial(d\theta) &= i\,\mathrm{cl}_1(d\theta) \otimes i\,\mathrm{cl}_2(\frac{dx}{x}). \end{aligned} \tag{6.2}$$

The family of Dirac operators associated to this product structure (with respect to the connection induced on the boundary as in Lemma 2) is precisely the family

$$\mathrm{Id} \otimes \eth_0 - \eth^L \otimes \mathrm{cl}_2(\frac{dx}{x})$$

on the fibre-Clifford module $E_0 \otimes L$.

Proposition 9. *If the boundary operator* $\eth_{0,z}$ *is invertible, then so is the indicial family*

$$\Gamma_1\lambda + \mathrm{Id} \otimes \eth_{0,z} \otimes \Gamma_2 - \eth^L_t \otimes \mathrm{cl}_2(\frac{dx}{x}) \otimes \Gamma_2,$$

acting on $L \otimes E_0 \otimes \mathbb{C}^2$, *for each* $(t, z) \in \mathbb{S}^1 \times B$.

Proof. Since λ is real and $\mathrm{Id} \otimes \eth_{0,z} - \eth^L_t \otimes \mathrm{cl}_2(\frac{dx}{x})$ is self-adjoint, it suffices to check the result for $\lambda = 0$. Thus we need to show that $\mathrm{Id} \otimes \eth_{0,z} - \eth^L_t \otimes \mathrm{cl}_2(\frac{dx}{x})$ is invertible for each $(t, z) \in \mathbb{S}^1 \times B$. Let $\{e_n(t)\}$ and $\{f_k(z)\}$ be orthonormal bases of eigenfunctions for $\eth^L_t$ and $\eth_{0,z}$ respectively; let $\{\lambda_n(t)\}$ and $\{\mu_k(z)\}$ the corresponding eigenvalues. For each $(n, k) \in \mathbb{Z}^2$ consider the two dimensional subspace of $L^2(\mathbb{S}^1(t) \times \partial M_z; L \otimes E_0)$ spanned by the sections $v = e_n \otimes f_k$, $u = e_n \otimes \mathrm{cl}_2(\frac{dx}{x}) f_k$. We denote this subspace by $V(n, k)$. Clearly

$$L^2(\mathbb{S}^1(t) \times \partial M_z; L \otimes E_0) = \bigoplus_{(n,k)\in\mathbb{Z}^2} V(n, k), \tag{6.3}$$

and moreover each $V(n,k)$ is invariant under the action of

$$\mathrm{Id}(t) \otimes \eth_{0,z} - \eth^L_t \otimes \mathrm{cl}_2(\frac{dx}{x}),$$

the restriction being given by the self-adjoint matrix

$$\begin{pmatrix} \mu_k & -\lambda_n \\ -\lambda_n & -\mu_k \end{pmatrix} \tag{6.4}$$

with eigenvalues equal to $\pm(\lambda_n^2 + \mu_k^2)^{\frac{1}{2}}$. Since by assumption $\mu_k \neq 0$ for each $k \in \mathbb{Z}$, the proposition is proved.

Consider now the general case. Let P be a spectral section for the boundary family $\eth_0$ and let $\widehat{\eth}^{F,\epsilon}$ be the perturbed Dirac suspension considered in the previous section :

$$\widehat{\eth}^{F,\epsilon}_{(t,z)} = \begin{pmatrix} 0 & \eth^{F,-}_{(t,z)} + iQ_\epsilon(t) \otimes (A_P)_z \\ \eth^{F,+}_{(t,z)} - iQ_\epsilon(t) \otimes (A_P)_z & 0 \end{pmatrix}.$$

We can easily extend the formula for the indicial family (6.1); suppressing obvious tensor products we obtain

$$N \cdot I(\widehat{\eth}^{F,\epsilon}, \lambda) \cdot N^{-1} = \Gamma_1 \lambda + \eth_0 \Gamma_2 - \eth^L \otimes \mathrm{cl}_2(\frac{dx}{x})\Gamma_2 + Q_\epsilon \otimes A_P(\lambda)\Gamma_2.$$

Proposition 10. *For any* $(t,z) \in \mathbb{S}^1 \times B$ *the indicial family* $I(\widehat{\eth}^{F,\epsilon}_{(t,z)}, \lambda)$ *is invertible for* $\lambda \in \mathbb{R}$.

Proof. Using the explicit definition of A_P (see (4.1)) we obtain

$$I(\widehat{\eth}^{F,\epsilon}, \lambda) = \Gamma_1 \lambda + \eth_0 \Gamma_2 - \eth^L \otimes \mathrm{cl}_2(\frac{dx}{x})\Gamma_2 + Q_\epsilon \otimes \hat{\phi}_1(\delta\lambda) A^0_P \Gamma_2.$$

Since λ is real, it follows that $\widehat{\phi}_1$ is also real. Thus $I(\widehat{\eth}^{F,\epsilon}, \lambda)$ is invertible for each $\lambda \neq 0$ real. To check the result for $\lambda = 0$ i.e., for the boundary family

$$\eth_0 \Gamma_2 - \eth^L \otimes \mathrm{cl}_2(\frac{dx}{x})\Gamma_2 + Q_\epsilon \otimes A^0_P \Gamma_2, \tag{6.5}$$

we need to modify the argument given in Proposition 9. Let $\{e_n(t)\}$ be an orthonormal basis of eigenfunctions for $\eth^L_t$ and let $\{\lambda_n(t)\}$ be the corresponding eigeinvalues. Consider the function $\phi_\epsilon \in \mathcal{C}^\infty_c(\mathbb{R})$ used in the definition of the smoothing family Q_ϵ. Thus $\phi_\epsilon(x) = 1$ for $|x| \leq 1/\epsilon$, and there exists an $\epsilon' > 0, \epsilon' < \epsilon < 1$, such that $\phi_\epsilon(x) = 0$ for $|x| \geq 1/\epsilon'$. For each $(t, z) \in \mathbb{S}^1 \times B$ we consider the following decomposition of

$$L^2(\mathbb{S}^1(t) \times \partial M_z; L \otimes E_0) = U_1 \oplus U_2 \oplus U_3$$

with

$$U_1 = \bigoplus_{|\lambda_n(t)| \leq 1/\epsilon} (\mathbb{C}e_n(t) \otimes L^2(\partial M_z; E_0)),$$
$$U_2 = \bigoplus_{1/\epsilon < |\lambda_n(t)| < 1/\epsilon'} (\mathbb{C}e_n(t) \otimes L^2(\partial M_z; E_0)),$$
$$U_3 = \bigoplus_{|\lambda_n(t)| \geq 1/\epsilon'} (\mathbb{C}e_n(t) \otimes L^2(\partial M_z; E_0)).$$

Now decompose each U_i as in (6.3) but with $\{f_k(z)\}$ and $\{\mu_k(z)\}$ eigenfunctions and eigenvalues associated to the elliptic pseudodifferential operators $\eth_{0,z} + A^0_P(z)$ for U_1, to $\eth_{0,z} + \phi_\epsilon(\lambda_n(t))A^0_P(z)$ for U_2 and to the elliptic differential operator $\eth_{0,z}$ for U_3. On each two-dimensional invariant subspace of this decomposition the operator is equal to the matrix (6.4) with eigenvalues equal to $\pm(\lambda_n^2 + \mu_k^2)^{\frac{1}{2}}$. Since $\eth_{0,z} + A^0_P(z)$ is invertible we see that the restriction of $\eth_0 - \eth^L \otimes \mathrm{cl}_2(\frac{dx}{x}) + Q_\epsilon \otimes A^0_P$ to U_1 is invertible (the eigenvalues $\mu_k(z)$ are always different from zero); on the other hand on U_2 and U_3 it is always the case that $\lambda_n(t) \neq 0$. Thus $\eth_0 - \eth^L \otimes \mathrm{cl}_2(\frac{dx}{x}) + Q_\epsilon \otimes A^0_P$ is always invertible and the proposition follows.

It is important to point out that the boundary family (6.5) of $\widehat{\eth}^{F,\epsilon}$ is equal to a family of generalized Dirac operators,

$$\eth_0\Gamma_2 - \eth^L \otimes \mathrm{cl}_2(\frac{dx}{x})\Gamma_2, \tag{6.6}$$

plus a family of finite rank operators with values in a finite sum of eigenfunctions of the family (6.6). In other words the perturbation $Q_\epsilon \otimes A^0_P\Gamma_2$ corresponds to a spectral section for (6.6). It is for this reason that the index theorem in [24] can be applied to $\widehat{\eth}^{F,\epsilon}$.

7. Relative index theorem

As a first application of the Dirac suspension described above we will now prove the relative index theorem in the odd case using the corresponding relative index theorem of [24]. As in the even-dimensional setting the relative index theorem can be obtained as a corollary of the full index theorem; however it is much more elementary. We therefore give an independent proof which also serves to illustrate the reduction of odd- to even-dimensional cases by suspension.

Let P_1, P_2 be two Cl(1) spectral sections for the original boundary family $\eth_0$, and consider the difference of the two corresponding index classes $\mathrm{Ind}(\eth, P_1) - \mathrm{Ind}(\eth, P_2) \in K^1(B)$. By Proposition 7 this class corresponds under suspension to

$$\begin{aligned} \mathrm{Ind}(\widehat{\eth}_1^{F,\epsilon}) - \mathrm{Ind}(\widehat{\eth}_2^{F,\epsilon}) \in & \ K^0((0,\pi)\times B), \\ \text{with } \widehat{\eth}_i^{F,\epsilon} = & \ \eth^F + Q_\epsilon \otimes A_{P_i}\Gamma_2. \end{aligned} \tag{7.1}$$

To this difference we can apply the relative index theorem of [24]. Thus consider the boundary families corresponding to $\widehat{\eth}_i^{F,\epsilon,+}$ which are given by

$$\eth_0 - \eth^L \otimes \mathrm{cl}_2(\frac{dx}{x}) + Q_\epsilon \otimes A^0_{P_i}$$

respectively. Both these families are invertible by Proposition 10, and we can therefore consider the corresponding Atiyah-Patodi-Singer spectral projections $\widehat{P}_1$ and $\widehat{P}_2$. The relative index theorem of [24] then gives

$$\mathrm{Ind}(\widehat{\eth}_1^{F,\epsilon}) - \mathrm{Ind}(\widehat{\eth}_2^{F,\epsilon}) = [\widehat{P}_2 - \widehat{P}_1] \in K^0((0,\pi)\times B). \tag{7.2}$$

Recall how this difference class is defined. By construction there exists $r \in \mathbb{R}^+$ such that the range of $Q_\epsilon \otimes P_i$ is contained in the span of the eigenfunctions of $\eth_0 - \eth^L \otimes \mathrm{cl}_2(\frac{dx}{x})$ corresponding to the eigenvalues belonging to the interval $[-r, r]$. We can always choose an auxiliary spectral section Q for $\eth_0 - \eth^L \otimes \mathrm{cl}_2(\frac{dx}{x})$ such that Q annihilates all the eigenfunctions corresponding to eigenvalues less than or equal to r and with Q acting as the identity on the span of the eigenfunctions corresponding to eigenvalues greater than $R > r$, for some $R \in \mathbb{R}^+$. Consider the orthocomplements of $\operatorname{ran} Q$ in $\operatorname{ran} \widehat{P}_1$ and $\operatorname{ran} \widehat{P}_2$ respectively. These are smooth bundles over $\mathbb{S}^1 \times B$, denoted respectively $(\widehat{P}_1 - Q)$ and $(\widehat{P}_2 - Q)$. The K^0 class is given by the virtual bundle

$$[\widehat{P}_2 - \widehat{P}_1] = [\widehat{P}_2 - Q] - [\widehat{P}_1 - Q]. \tag{7.3}$$

By a further simple homotopy (see (7.5)), both Dirac suspensions in (7.1) can be reduced to $\eth^F$ near $t = 0$, and π. Thus $[\widehat{P}_2 - \widehat{P}_1]$ is an element of $K^0((0,\pi)\times B)$ as required. It is shown in [24] that this class is well defined, independent of the particular choice of Q.

Recall that in §3, Proposition 4, the difference class of two Cl(1) spectral sections, $[P_2 - P_1] \in K^1(B)$, was defined. After these preliminaries we can state and prove the relative index theorem in the odd case.

Proposition 11. *If P_1 and P_2 are two* Cl(1) *spectral sections for the boundary family $\eth_0$, then*

$$\operatorname{Ind}(\eth, P_1) - \operatorname{Ind}(\eth, P_2) = [P_2 - P_1] \in K^1(B). \tag{7.4}$$

Proof. Following the discussion above it only remains to show that the virtual bundle $[\widehat{P}_2 - \widehat{P}_1]$ associated to the difference of the index classes defined by the Dirac suspensions is equal to the right-hand side of (7.4).

We first perform a preliminary homotopy on the Dirac suspensions in (7.1). Let $\psi_\epsilon \in \mathcal{C}^\infty([0,\pi]_t \times \mathbb{R}_\lambda)$ be a function, which vanishes on the complement of the open rectangle $(0,\pi) \times (a, a+b\pi)$ and is equal to one on the rectangle $[\epsilon, \pi-\epsilon] \times [a+\epsilon, a+b\pi-\epsilon]$. Let R_ϵ be the family of operators defined by (5.17) but with ψ_ϵ in place of ϕ_ϵ, and consider the Dirac suspensions defined now as

$$\widehat{\eth}_i^{F,\epsilon} = \eth^F + R_\epsilon \otimes A_{P_i}\Gamma_2. \tag{7.5}$$

The behaviour of the eigenvalues of $\eth^L$ is described in the proof of Proposition 6. Since only the nearly zero modes of $\eth^L$ contribute to the index class of the Dirac suspension (see the proof of Proposition 6) the index class of (7.5) still represents, in $K^0((0,\pi)\times B)$, the Atiyah-Singer suspension of the index class $\operatorname{Ind}(\eth, P_i)$. Let $\eth_0 + R_\epsilon \otimes A^0_{P_i} - \eth^L \otimes \mathrm{cl}_2(\frac{dx}{x})$ be the boundary operator associated to $\widehat{\eth}_i^{F,\epsilon,+}$. For each $(t,z) \in \mathbb{S}^1 \times B$ we take the decomposition

$$L^2(\mathbb{S}^1(t) \times \partial M_z; L \otimes E_0) = V \oplus V_0$$

with

$$V(t,z) = V = \bigoplus_{n\neq 0} (\mathbb{C}e_n(t) \otimes L^2(\partial M_z; E_0))$$

and

$$V_0(t,z) = V_0 = \mathbb{C}e_0(t) \otimes L^2(\partial M_z; E_0).$$

These two subspaces are invariant under the action of the boundary operator

$$\eth_{0,z} + R_\epsilon(t) \otimes A^0_{P_i}(z) - \eth^L_t \otimes \mathrm{cl}_2(\frac{dx}{x}). \tag{7.6}$$

The Atiyah-Patodi-Singer spectral projection $\widehat{P}_i(t,z)$ with $i=1,2$ splits accordingly as a pair of projections $\widehat{P}_i^V(t,z)$ and $\widehat{P}_i^{V_0}(t,z)$ on V and V_0 respectively. As the point (t,z) in the base varies we obtain, for $i=1,2$, families of spectral projections which we still denote by $\widehat{P}_i^V$ and $\widehat{P}_i^{V_0}$. Using the definition of ψ_ϵ we conclude that the restrictions to V of the two boundary operators (7.6) are both equal to the unperturbed boundary operator and hence coincide. It follows that $[\widehat{P}_2-\widehat{P}_1]=[\widehat{P}_2^{V_0}-\widehat{P}_1^{V_0}]$.

Let $f_k^i(z)$ be a basis of eigenfunctions for the boundary operator (7.6) restricted to $V_0(t,z)$, which is just

$$\eth_{0,z}+\psi_\epsilon(t,\lambda_0(t))A^0_{P_i}-\lambda_0(t)\,\mathrm{cl}_2(\frac{dx}{x}), \tag{7.7}$$

and let $\{\mu_k^i(z)\}$ be the corresponding eigenvalues. We decompose V_0 as the direct sum of the two-dimensional invariant subspaces spanned by $e_0(t)\otimes f_k^i(z)$ and $e_0(t)\otimes \mathrm{cl}_2(\frac{dx}{x})f_k^i(z)$ as in the proof of Proposition 10. The action of the boundary operator on these invariant subspaces is given by the matrix

$$\begin{pmatrix}\mu_k^i(z) & -\lambda_0(t)\\ -\lambda_0(t) & -\mu_k^i(z)\end{pmatrix};$$

the eigenvalues are $\pm(\lambda_0(t)^2+(\mu_k^i(z))^2)^{\frac12}$. Consider now the families

$$(1+(\eth_0+A^0_{P_i})^2)^{\frac12}\cos(t)\,\mathrm{cl}_2(\frac{dx}{x})+(\eth_0+A^0_{P_i})\sin(t) \tag{7.8}$$

and

$$\begin{aligned}(1+(\eth_0+\psi_\epsilon(t,0)A^0_{P_i})^2)^{\frac12}\cos(t)\,\mathrm{cl}_2(\frac{dx}{x})\\ +(\eth_0+\psi_\epsilon(t,0)A^0_{P_i})\sin(t),\end{aligned} \tag{7.9}$$

and let $[P_2-P_1]$ and $[P_2^\epsilon-P_1^\epsilon]$ be the corresponding difference classes as in §3. Since the two families (7.8) and (7.9) are always invertible, from the assumed properties of ψ_ϵ it follows that $[P_2-P_1]=[P_2^\epsilon-P_1^\epsilon]$. Since $[P_2^\epsilon-P_1^\epsilon]=[\widehat{P}_2^{V_0}-\widehat{P}_1^{V_0}]$, the proposition is proved.

As mentioned in the Introduction any K^1 class can be represented by such a difference although we do not make use of this result here.

Proposition 12. *Provided an elliptic family of $\mathbb{Z}_2$-graded differential operators is trivial in K^0 of the base, so* Cl(1) *spectral sections exist, as P_1 ranges over all such spectral sections, for a fixed P_2, (3.4) exhausts*

$K^1(B)$. *Similarly, provided a given elliptic family of self-adjoint operators on the fibres is trivial in* $K^1(B)$, *the difference classes of spectral sections exhausts* $K^0(B)$

Proof. Consider the second case, of a self-adjoint family, A_z. The assumption of the vanishing of the index of A in $K^1(B)$ implies that A has a spectral section, P; see [24]. Also the results of [24] allow us to find a second spectral section, P'', commuting with P and such that $V'' = \text{null}(P'')/\,\text{null}(P)$ is a vector bundle with rank larger than any preassigned integer. Let V be a vector bundle over B. There is an integer, N, such that V is a subbundle of any vector bundle over B of rank at least N. Thus V can be embedded as a subbundle of V'' for an appropriate choice of P''. Let P' be the self-adjoint projection with null space $\text{null}(P) \oplus V$. Clearly the difference element $[P' - P]$ represents V in $K^0(B)$. It follows that the formal difference, $V \ominus W$, of any two vector bundles can be represented by a difference of spectral sections.

In order to prove the first case we consider the description of $K^1(B)$ and $K^0((0,\pi)\times B)$ as equivalences classes of $\text{Cl}^{0,1}$ and $\text{Cl}^{0,0}$ bundles with involutions as in Karoubi's book [20]. Following the notation given there we can represent a class in $K^1(B)$ as $d(V, \eta_1, \eta_2)$ with V a $\text{Cl}^{0,1}$-module, i.e., a $\mathbb{Z}_2$-graded vector bundle $V = V_0 \oplus V_1$, and η_i odd involutions on V. Similarly a class in $K^0((0,\pi)\times B)$ is given by a triple $d(W, \zeta_1, \zeta_2)$ with W a vector bundle on $[0,\pi]\times B$ and ζ_i involutions on W with $\zeta_1 = \zeta_2$ on $\partial[0,\pi]$. The suspension isomorphism in this context is given by the map ([20, Theorem III.5.10])

$$
\begin{aligned}
\text{Sus} : K^1(B) &\longrightarrow K^0([0,\pi]\times B, \partial[0,\pi]\times B),\\
d(V, \eta_1, \eta_2) &\longrightarrow d(V', \zeta_1(t), \zeta_2(t)),
\end{aligned}
\tag{7.10}
$$

where $V' = \pi^* V$, $\pi : [0,\pi]\times B \longrightarrow B$, and

$$\zeta_i(t) = \sigma \cos t + \eta_i \sin t.$$

Fix a class $d(V_0 \oplus V_1, \eta_1, \eta_2) \in K^1(B)$ as above. Up to the addition of a trivial element we can assume that V_0 and V_1 have the same dimension ([20, Proposition III.4.26]). Using any one $\text{Cl}(1)$ spectral section, P', to deform the operator, as in Lemma 1, it suffices to consider the case that the self-adjoint family is everywhere invertible. The bundle V_0 can then be embedded in the image of the span of the positive eigenspaces, i.e., the range of the Atiyah-Patodi-Singer projection, to be orthogonal to all eigenfunctions with sufficiently large eigenvalues. Similarly V_1 can be taken to be σV_0, thus a subbundle in the span of the negative eigenspaces. Consider the spectral section P_i equal to P' on the orthocomplement of $V_0 \oplus V_1$ and to the projection onto the $+1$ eigenspace of η_i on $V_0 \oplus V_1$. Since η_i is odd with respect to σ, it follows that P_i is a $\mathrm{Cl}(1)$ spectral section. The proposition then follows by observing that, by definition of the difference class in (3.5), $\mathrm{Sus}(d(V, \eta_1, \eta_2)) = [P_1(t) - P'(t)] - [P_2(t) - P'(t)]$.

8. Suspended superconnection

As a first step towards the application of the index formula of [24] we shall find the explicit form of the Bismut superconnection adapted to the operator $\eth^F$ for the fibration (5.10), we call this simply the suspended Bismut superconnection and we denote it by $\mathbb{A}'$.

Set $M' = \mathbb{S}^1_\theta \times \mathbb{S}^1_t \times M$ and $B' = \mathbb{S}^1_t \times B$. Let π_2 and π_3 be the obvious projections from M' onto $\mathbb{S}^1_t$ and M respectively. Consider a choice of connection for the original fibration:

$$TM = T_H M \oplus T_V M. \tag{8.1}$$

On the fibration $\pi' : M' \longrightarrow B'$ we take the product connection:

$$T_H M' = (\pi_2^* T\mathbb{S}^1_t) \oplus (\pi_3^* T_H M). \tag{8.2}$$

Let π'_V, π'_H denote the projections onto the vertical and horizontal subbundles. Then denote by $\langle \cdot, \cdot \rangle$ the inner product induced on the vertical tangent bundle of M' by the family of exact b-metrics

$$g'_{M'/B'} = d\theta^2 + g_{M/B},$$

and by $\langle \cdot, \cdot \rangle_0$ the degenerate inner product on TM' which extends $\langle \cdot, \cdot \rangle$ to be identically zero on the horizontal bundle (8.2).

The curvature tensor, Ω', of the connection (8.2)

$$\Omega'(X', Y')(Z') = -\langle [\pi'_H X', \pi'_H Z'], Z' \rangle_0$$

is determined by the curvature tensor, Ω, of the original fibration through

$$\begin{aligned} \Omega'(X, Y)(Z) &= \ \Omega(X, Y)(Z), \\ \Omega'(X, \tfrac{\partial}{\partial t})(Z') &= \ 0 \text{ and } \Omega'(X', Y')(\tfrac{\partial}{\partial \theta}) = 0 \end{aligned}$$

for each $X, Y, Z \in \mathcal{C}^\infty(M; TM)$ and $X', Y', Z' \in \mathcal{C}^\infty(M'; TM')$. Let $\mathbb{A}$ be the Bismut superconnection for the original fibration. It follows from the above remarks that the 2-form piece of the suspended Bismut superconnection is

$$\mathbb{A}'_{[2]} = \pi_3^*(\mathrm{Id} \otimes \mathbb{A}_{[2]} \otimes \Gamma_2) \in \mathcal{C}^\infty(M'; (\hom F) \otimes \ (\phi \circ \pi_3)^* \Lambda^* B).$$

As far as the 1-form piece is concerned we simply observe that the mean curvature of the product fibration is just

$$k' = \pi_3^* k \in \mathcal{C}^\infty(M'; (\phi \circ \pi_3)^* \Lambda^* B).$$

Suppressing all the pullbacks and all the obvious tensor products involved, the suspended Bismut superconnection is therefore

$$\mathbb{A}' = \eth \otimes \Gamma_2 + \eth^L \otimes \Gamma_1 + \mathbb{A}_{[1]} + \frac{1}{i} dt \frac{\partial}{\partial t} + \mathbb{A}_{[2]} \otimes \Gamma_2. \tag{8.3}$$

Direct computation then shows that

$$(\mathbb{A}')^2 = \mathbb{A}^2 + (\eth^L)^2 + \frac{1}{i\pi} dt (\mathrm{cl}_1(d\theta) \otimes \Gamma_1), \tag{8.4}$$

where cl_1 is the Clifford action on L.

To find the boundary behaviour of $\mathbb{A}'$ we write the suspended Bismut superconnection in block form and use the definition of the isomorphism N, obtaining

$$\begin{aligned} N \cdot I(\mathbb{A}', \lambda) \cdot N^{-1} & \\ &= \Gamma_1 \lambda + \mathrm{Id} \otimes \eth_0 \otimes \Gamma_2 \\ &\quad - \eth^L \otimes \mathrm{cl}_2(\frac{dx}{x}) \otimes \Gamma_2 \\ &\quad + \mathbb{B}_{[1]} + \frac{1}{i} dt \frac{\partial}{\partial t} + \mathrm{Id} \otimes \mathbb{B}_{[2]} \otimes \Gamma_2, \end{aligned} \tag{8.5}$$

where $\eth_0 + \mathbb{B}_{[1]} + \mathbb{B}_{[2]}$ is the boundary Bismut superconnection of the original structure with respect to the Clifford action

$$\mathrm{cl}_\partial(\xi) = i \, \mathrm{cl}(\frac{dx}{x}) \, \mathrm{cl}(\xi).$$

The boundary Bismut superconnection on $(L \otimes E_0) \oplus (L \otimes E_0)$,

$$\begin{aligned} \mathbb{B}' = \ & (\eth_0 \Gamma_2 - \eth^L \otimes \mathrm{cl}_2(\tfrac{dx}{x}) \Gamma_2) + \mathbb{B}_{[1]} \\ & + \frac{1}{i} dt \frac{\partial}{\partial t} + \mathbb{B}_{[2]} \Gamma_2, \end{aligned} \tag{8.6}$$

is a $\mathrm{Cl}(1)$ superconnection with respect to the Clifford action (6.2) with square given by

$$(\mathbb{B}')^2 = \mathbb{B}^2 + (\eth^L)^2 - \frac{1}{i\pi} dt (\mathrm{cl}_1(d\theta) \otimes \mathrm{cl}_2(\frac{dx}{x}) \Gamma_2).$$

9. Odd eta forms

In this section, to show how the computation of the odd eta forms proceeds, we shall at first assume that the boundary family $\eth_0$ is invertible. The eta form associated to the boundary superconnection

$$\mathbb{B}' = (\eth_0\Gamma_2 - \eth^L \otimes \mathrm{cl}_2(\frac{dx}{x})\Gamma_2) + \mathbb{B}_{[1]} + \frac{1}{i}dt\frac{\partial}{\partial t} + \mathbb{B}_{[2]}\Gamma_2$$

is

$$\widehat{\eta} = \int_0^\infty \widehat{\eta}(u)du = \int_0^\infty \frac{1}{\sqrt{\pi}}\mathrm{STr}_{\mathrm{Cl}(1)}\left(\frac{d\mathbb{B}'_u}{du}e^{-(\mathbb{B}'_u)^2}\right)du. \tag{9.1}$$

Recall here that $\mathrm{str}_{\mathrm{Cl}(1)}(A + B\Gamma_2) = \mathrm{tr}(B)$ for each endomorphism $A + B\Gamma_2$ of the bundle $(L \otimes E_0 \otimes \mathbb{C}^2)$ and that the rescaled superconnection is

$$\mathbb{B}'_u = u^{\frac{1}{2}}(\eth_0\Gamma_2 - \eth^L \otimes \mathrm{cl}_2(\frac{dx}{x})\Gamma_2) + \mathbb{B}_{[1]} + u^{-\frac{1}{2}}\left(\frac{1}{i}dt\frac{\partial}{\partial t} + \mathbb{B}_{[2]}\Gamma_2\right).$$

Proposition 9 is needed here to ensure convergence at infinity.

To relate $\widehat{\eta}$ to the odd eta form as described in the Introduction we need to compute the normalized integral over $\mathbb{S}^1_t$:

$$\frac{i}{2\pi}\int_{\mathbb{S}^1_t}\left(\frac{1}{\sqrt{\pi}}\int_0^\infty \mathrm{STr}_{\mathrm{Cl}(1)}\left(\frac{d\mathbb{B}'_u}{du}e^{-(\mathbb{B}'_u)^2}\right)du\right). \tag{9.2}$$

Lemma 5. *Assuming that the boundary family $\eth_0$, acting on sections of E_0, is invertible, the form in (9.2) is equal to the eta form of (7) in the introduction, where*

$$\mathbb{B}_u = u^{\frac{1}{2}}\eth_0 + \mathbb{B}_{[1]} + u^{-\frac{1}{2}}\mathbb{B}_{[2]} \tag{9.3}$$

is the rescaled boundary superconnection.

Notice that, according to the discussion in §1,

$$\mathrm{str}_{\partial M}(\cdot) = \mathrm{tr}(\mathrm{cl}(\frac{dx}{x})\cdot).$$

Proof. Consider in detail the u-dependence of the suspended eta form (where we suppress all the obvious tensor products involved):

$$\begin{aligned}
\sqrt{\pi}\widehat{\eta}(u) &= \mathrm{STr}_{\mathrm{Cl}(1)}\left(\frac{d\mathbb{B}'_u}{du}e^{-(\mathbb{B}')^2_u}\right)\\
&= \mathrm{STr}_{\mathrm{Cl}(1)}\left(\frac{u^{-\frac{1}{2}}}{2}(\eth_0\Gamma_2 - \eth^L\otimes \mathrm{cl}_2(\frac{dx}{x})\Gamma_2) - \frac{u^{-\frac{3}{2}}}{2}\mathbb{B}_{[2]}\Gamma_2\right)\\
&\qquad \times \exp(-\mathbb{B}^2_u - u(\eth^L)^2 + \frac{u^{\frac{1}{2}}}{i\pi}dt(\mathrm{cl}_1(d\theta)\otimes \mathrm{cl}_2(\frac{dx}{x})\Gamma_2)).
\end{aligned}$$

Applying Duhamel's principle to the exponential and noting the idempotence of dt shows that

$$\begin{aligned}
&\exp\left(-\mathbb{B}^2_u - u(\eth^L)^2 + \frac{u^{\frac{1}{2}}}{i\pi}dt(\mathrm{cl}_1(d\theta)\otimes \mathrm{cl}_2(\frac{dx}{x})\Gamma_2\right)\\
&= \exp\left(-\mathbb{B}^2_u - u(\eth^L)^2\right)\\
&\quad + \int_0^1 \exp(-s(\mathbb{B}^2_u + u(\eth^L)^2))(\frac{u^{\frac{1}{2}}}{i\pi}dt(\mathrm{cl}_1(d\theta)\otimes \mathrm{cl}_2(\frac{dx}{x})\Gamma_2)\\
&\quad \times \exp(-(1-s)(\mathbb{B}^2_u + u(\eth^L)^2)))ds.
\end{aligned} \tag{9.4}$$

This allows $\sqrt{\pi}\widehat{\eta}(u)$ to be written as the sum of two terms:

$$\begin{aligned}
&\mathrm{STr}_{\mathrm{Cl}(1)}\Big(\frac{u^{-\frac{1}{2}}}{2}(\eth_0\Gamma_2 - \eth^L\otimes \mathrm{cl}_2(\frac{dx}{x})\Gamma_2)\\
&\quad -\frac{u^{-\frac{3}{2}}}{2}\mathbb{B}_{[2]}\Gamma_2\Big)\exp(-\mathbb{B}^2_u - u(\eth^L)^2)
\end{aligned} \tag{9.5}$$

and

$$\begin{aligned}
&\mathrm{STr}_{\mathrm{Cl}(1)}\big((\frac{u^{-\frac{1}{2}}}{2}(\eth_0\Gamma_2 - \eth^L\otimes \mathrm{cl}_2(\frac{dx}{x})\Gamma_2) - \frac{u^{-\frac{3}{2}}}{2}\mathbb{B}_{[2]}\Gamma_2)\\
&\quad \times (\frac{u^{\frac{1}{2}}}{i\pi}dt(\mathrm{cl}_1(d\theta)\otimes \mathrm{cl}_2(\frac{dx}{x})\Gamma_2))\exp(-\mathbb{B}^2_u - u(\eth^L)^2)\big).
\end{aligned} \tag{9.6}$$

The first of these, (9.5), does not involve dt and so makes no contribution to the $\mathbb{S}^1_t$-integral. Hence we only need to examine (9.6).

Carrying out the decomposition

$$\begin{aligned}
&\exp(-\mathbb{B}^2_u - u(\eth^L)^2)\\
&\quad = (\exp(-\mathbb{B}^2_u - u(\eth^L)^2))_{\mathrm{even}} + (\exp(-\mathbb{B}^2_u - u(\eth^L)^2))_{\mathrm{odd}}\Gamma_2
\end{aligned}$$

allows us to rewrite the second contribution as

$$\begin{aligned}(9.7)\qquad &\operatorname{Tr}\big((\frac{u^{-\frac{1}{2}}}{2}(\eth_0 - \eth^L \otimes \mathrm{cl}_2(\frac{dx}{x})) - \frac{u^{-\frac{3}{2}}}{2}\mathbb{B}_{[2]})\\ &\times(\frac{u^{\frac{1}{2}}}{i\pi}dt(\mathrm{cl}_1(d\theta)\otimes \mathrm{cl}_2(\frac{dx}{x})))(\exp(-\mathbb{B}_u^2 - u(\eth^L)^2))_{\mathrm{odd}}\big),\end{aligned}$$

which is an even differential form on $\mathbb{S}^1 \times B$ as it should be. Consider, for a fixed u in this interval, the normalized $\mathbb{S}^1$-integral of (9.7). This is the sum of the two terms

$$\begin{aligned}(9.8)\qquad &\frac{i}{2\pi}\int_{\mathbb{S}^1_t} \operatorname{Tr}\big((\frac{u^{-\frac{1}{2}}}{2}\eth_0 - \frac{u^{-\frac{3}{2}}}{2}\mathbb{B}_{[2]})(\frac{u^{\frac{1}{2}}}{i\pi}dt(\mathrm{cl}_1(d\theta)\otimes \mathrm{cl}_2(\frac{dx}{x})))\\ &\qquad\times(\exp(-\mathbb{B}_u^2 - u(\eth^L)^2))_{\mathrm{odd}}\big)\end{aligned}$$

and

$$(9.9)\qquad -\frac{i}{2\pi}\int_{\mathbb{S}^1_t} \operatorname{Tr}\left(\frac{1}{2i\pi}dt\, \mathrm{cl}_1(d\theta)\eth^L(\exp(-\mathbb{B}_u^2 - u(\eth^L)^2))_{\mathrm{odd}}\right).$$

The fibres of the suspended fibration are simply the products of the original fibres with circles, L is a line bundle over the circles, and the actions of the operators $\mathbb{B}_u$ and $\eth^L$ are in these respective factors. The heat kernel in (9.9) therefore splits into the tensor product of the heat kernels. Similarly the other factors split as products so (9.8) reduces to

$$\begin{aligned}(9.10)\qquad &\operatorname{Tr}\left(\mathrm{cl}(\frac{dx}{x})(\frac{d\mathbb{B}_u}{du}(e^{-\mathbb{B}_u^2})_{\mathrm{odd}})\right)\\ &\qquad\cdot\left(\frac{i}{2\pi}\int_{\mathbb{S}^1_t}\operatorname{Tr}\left(\frac{u^{\frac{1}{2}}}{i\pi}\mathrm{cl}_1(d\theta)e^{-u(\eth^L)^2}\right)dt\right).\end{aligned}$$

The first of these two factors is just

$$\mathrm{STr}_{\partial M}\left(\frac{d\mathbb{B}_u}{du}e^{-\mathbb{B}_u^2}\right).$$

Notice that $d\mathbb{B}_u/du$ is odd with respect to the grading of E_0 induced by the involution $\mathrm{cl}(dx/x)$. Since it is composed with $\exp(-\mathbb{B}_u^2)_{\mathrm{odd}}$, which is also odd with respect to this $\mathbb{Z}_2$-grading, this is an even family of smoothing operators with odd-degree differential form coefficients. Applying the supertrace gives an odd differential form on the base B as it should. The second factor in (9.10) is equal to

$$\frac{1}{2\pi i}\int\limits_{\mathbb{S}^1_t} \mathrm{Tr}^{\mathrm{odd}}\, e^{-(u^{\frac12}\eth^L+\frac{1}{i}dt\frac{\partial}{\partial t})^2}$$

by Duhamel's principle; using the results of Bismut-Freed on the odd-family index theorem this is precisely the spectral flow of the family (5.8) on the torus fibration. By construction it is equal to one.

Next consider the term in (9.9). This can be rewritten as

$$-\operatorname{Tr}(\exp(-\mathbb{B}_u^2)_{\mathrm{odd}})\cdot\left(\frac{i}{2\pi}\int_{\mathbb{S}^1_t}\operatorname{Tr}(\frac{i}{2\pi}\operatorname{cl}_1(d\theta)\eth^L e^{-u(\eth^L)^2}dt)\right) \tag{9.11}$$

and hence seen to vanish, as the trace of an odd-term with respect to the $\mathbb{Z}_2$-grading of E_0 considered above.

Thus we have shown that

$$\frac{i}{2\pi}\int_{\mathbb{S}^1_t}\widehat{\eta}(u)=\frac{1}{\sqrt{\pi}}\operatorname{STr}_{E_0}\left(\frac{d\mathbb{B}_u}{du}e^{-(\mathbb{B})^2_u}\right). \tag{9.12}$$

Since the integral in u defining the eta form $\widehat{\eta}$ is absolutely convergent, we can interchange the $\mathbb{S}^1$-integral and the u-integral on the right-hand side of (9.2). Together with (9.12) this proves the Lemma.

10. Index theorem in the invertible case

Although the odd family index theorem is derived in full generality in the next section, we pause to show how the results in §3 and §9 combine to give the odd family index theorem in case the boundary family is invertible. The formula we give was conjectured by Bismut and Cheeger in [12].

Thus let $\eth_{0,z}$ be invertible for each $z\in B$. Proposition 6 shows that

$$\operatorname{Ind}(\eth^F)=j(\operatorname{Sus}(\operatorname{Ind}(\eth)))\text{ in }\widetilde{K}^0(\mathbb{S}^1\times B).$$

Thus, from the definition (3.4) of the odd Chern character, we obtain

$$\operatorname{Ch}(\operatorname{Ind}(\eth))=\frac{i}{2\pi}\int_{\mathbb{S}^1}\operatorname{Ch}(\operatorname{Ind}(\eth^F)).$$

By Proposition 9, $\eth^F$ is a $\mathbb{Z}_2$-graded family of Dirac operators with invertible boundary family. Applying the results of [10], [24] yields

$$\mathrm{Ch}(\mathrm{Ind}(\eth^F)) = (2\pi i)^{-\frac{n+1}{2}} \int\limits_{M'/B'} \widehat{A}(M'/B')\,\mathrm{Ch}'(F) - \frac{1}{2}\widehat{\eta} \tag{10.1}$$

in $H^*(\mathbb{S}^1 \times B)$. Here n is the dimension of M_z, and $\widehat{\eta}$ is the eta form (9.1). Consider the first term in (10.1). Since the Riemannian fibration (5.10) splits as a product over $\mathbb{S}^1_t$ and B, we conclude that

$$\widehat{A}(M'/B') = \widehat{A}(M/B).$$

Similarly, using the multiplicative properties of the Chern character and suppressing obvious pull-backs

$$\mathrm{Ch}'(F) = \mathrm{Ch}(L) \wedge \mathrm{Ch}'(E) = (1 - \frac{i}{\pi}(dt \wedge d\theta)) \wedge \mathrm{Ch}'(E).$$

Since $\mathbb{S}^1_\theta = \mathbb{R}/2\pi\mathbb{Z}$ and $\mathbb{S}^1_t = \mathbb{R}/\pi\mathbb{Z}$, it follows that

$$\begin{aligned} &\frac{i}{2\pi}\int_{\mathbb{S}^1_t}\left((2\pi i)^{-\frac{n+1}{2}} \int\limits_{M'/B'} \widehat{A}(M'/B')\,\mathrm{Ch}'(F)\right) dt \\ &\qquad = (2\pi i)^{-\frac{n+1}{2}} \int\limits_{M/B} \widehat{A}(M/B)\,\mathrm{Ch}'(E), \end{aligned} \tag{10.2}$$

which is precisely the first term in formula (6) in the introduction when we consider a general Clifford module E instead of the spinor bundle S. By applying Lemma 5 we finally obtain the odd family index formula in the invertible case:

$$\mathrm{Ch}(\mathrm{Ind}(\eth)) = (2\pi i)^{-\frac{n+1}{2}} \int\limits_{M/B} \widehat{A}(M/B)\,\mathrm{Ch}'(E) - \frac{1}{2}\eta_{\mathrm{odd}}.$$

11. Index theorem in the general case

Let $\eth$ be a family of Dirac operators on odd dimensional manifolds with boundary as in the Introduction. By Corollary 1 the boundary family $\eth_0$ admits a spectral section P, and we can thus associate to $\eth$

and P an index class $\operatorname{Ind}(\eth, P) \in K^1(B)$ as in §4. From Proposition 7 we know that

$$\operatorname{Ch}(\operatorname{Ind}(\eth, P)) = \frac{i}{2\pi}\int_{\mathbb{S}^1} \operatorname{Ch}(\operatorname{Ind}(\widehat{\eth}^{F,\epsilon})) \tag{11.1}$$

with

$$\widehat{\eth}^{F,\epsilon}_{(t,z)} = \begin{pmatrix} 0 & \eth^{F,-}_{(t,z)} + iQ_\epsilon(t)\otimes (A_P)_z \\ \eth^{F,+}_{(t,z)} - iQ_\epsilon(t)\otimes (A_P)_z & 0 \end{pmatrix}$$

and A_P and Q_ϵ defined respectively by (4.1) and (5.17).

To compute $\operatorname{Ch}(\operatorname{Ind}(\widehat{\eth}^F))$ we consider the (perturbed suspended rescaled) Bismut superconnection

$$\begin{aligned} \mathbb{A}'_u(\epsilon) = \quad & u^{\frac12}(\operatorname{Id}\otimes\eth\otimes\Gamma_2 + \eth^L\otimes\operatorname{Id}\otimes\Gamma_1 \\ & +\chi(u)Q_\epsilon\otimes A_P\otimes\Gamma_2) \\ & +\mathbb{A}_{[1]} + \frac{1}{i}dt\frac{\partial}{\partial t} + u^{-\frac12}\operatorname{Id}\otimes\mathbb{A}_{[2]}\otimes\Gamma_2 \end{aligned} \tag{11.2}$$

with $\chi \in \mathcal{C}^\infty(\mathbb{R})$, $\chi(u) = 0$ for $u < 1$ and $\chi(u) = 1$ for $u > 2$. Suppressing obvious tensor products, the boundary superconnection is then equal to

$$\begin{aligned} \mathbb{B}'_u(\epsilon) = \quad & u^{\frac12}(\eth_0\Gamma_2 - \eth^L\otimes\operatorname{cl}_2(\frac{dx}{x})\Gamma_2 \\ & +\chi(u)Q_\epsilon\otimes A^0_P\Gamma_2) + \mathbb{B}_{[1]} \\ & +\frac{1}{i}dt\frac{\partial}{\partial t} + u^{-\frac12}\mathbb{B}_{[2]}\Gamma_2 \end{aligned} \tag{11.3}$$

with square equal to

$$\begin{aligned} (\mathbb{B}'_u(\epsilon))^2 = & (\mathbb{B}_u\Gamma_2 + u^{\frac12}\chi(u)Q_\epsilon\otimes A^0_P\Gamma_2)^2 \\ & + u(\eth^L)^2 - \frac{u^{\frac12}}{i\pi}dt(\operatorname{cl}_1(d\theta)\otimes\operatorname{cl}_2(\frac{dx}{x})\Gamma_2) \\ & + \frac{u^{\frac12}}{i}dt\chi(u)(\frac{\partial}{\partial t}Q_\epsilon)\otimes A^0_P\Gamma_2. \end{aligned} \tag{11.4}$$

Finally consider the following superconnection on the original boundary fibration

$$\widetilde{\mathbb{B}}_u = u^{\frac12}(\eth_0 + \chi(u)A^0_P) + \mathbb{B}_{[1]} + u^{-\frac12}\mathbb{B}_{[2]}. \tag{11.5}$$

Applying the main result of [24] we obtain the following expression for the Chern character of the suspended family

$$\begin{aligned}\operatorname{Ch}(\operatorname{Ind}(\widehat{\eth}^{F,\epsilon})) = \; & (2\pi i)^{-\frac{n+1}{2}} \int\limits_{M'/B'} \widehat{A}(M'/B')\operatorname{Ch}'(F) \\ & -\frac{1}{2}\widehat{\eta}_P(\epsilon)\end{aligned} \tag{11.6}$$

in $H^*(\mathbb{S}^1 \times B)$ with

$$\begin{aligned}\widehat{\eta}_P(\epsilon) &= \int\limits_0^\infty \widehat{\eta}_P(\epsilon, u)du \\ &= \frac{1}{\sqrt{\pi}}\int\limits_0^\infty \operatorname{STr}_{\operatorname{Cl}(1)}\left(\frac{d\mathbb{B}'_u(\epsilon)}{du}e^{-(\mathbb{B}'_u(\epsilon))^2}\right)du.\end{aligned} \tag{11.7}$$

Let $\eta_{\mathrm{odd},P}$ be the differential form on B given by

$$\eta_{\mathrm{odd},P} = \frac{1}{\sqrt{\pi}}\int\limits_0^\infty \operatorname{STr}_{E_0}\left(\frac{d\widetilde{\mathbb{B}}_u}{du}e^{-(\widetilde{\mathbb{B}}_u)^2}\right)du. \tag{11.8}$$

Slight modification to the proof given in [24], along the lines of Lemma 6 below, shows that, up to an exact form on B, $\eta_{\mathrm{odd},P}$ depends only on P and not on the particular choice of the deformation A_P^0.

The main step remaining in the proof of the odd family index theorem is

Lemma 6. *Up to an exact form on B the normalized $\mathbb{S}^1$-integral of the suspended eta form (11.7) is equal to the eta form (11.8).*

Proof. Fix $\epsilon_0 > 0$. We wish to show that

$$\frac{i}{2\pi}\frac{1}{\sqrt{\pi}}\int\limits_0^\infty \int_{\mathbb{S}^1}\operatorname{STr}_{\operatorname{Cl}(1)}\left(\frac{d\mathbb{B}'_u(\epsilon_0)}{du}e^{-(\mathbb{B}'_u(\epsilon_0))^2}\right)du$$

is equal to $\eta_{\mathrm{odd},P}$ plus an exact form $d\alpha(\epsilon_0)$, with $\alpha(\epsilon_0) \in \mathcal{C}^\infty(B, \Lambda^* B)$.

Consider the derivative with respect to ϵ of $\widehat{\eta}_P(\epsilon, u)$. By Proposition 14 in [24] this is given by

$$\begin{aligned}\frac{d}{d\epsilon}\widehat{\eta}_P(\epsilon, u) = \; & \frac{d}{du}\left(\frac{u^{\frac{1}{2}}}{\sqrt{\pi}}\operatorname{STr}_{\operatorname{Cl}(1)}(\Gamma_2\chi(u)\frac{d}{d\epsilon}(Q_\epsilon \otimes A_P^0)e^{-(\mathbb{B}'_u(\epsilon))^2})\right) \\ & + d_{B\times\mathbb{S}^1}\alpha(u,\epsilon)\end{aligned}$$

with

$$\alpha(u,\epsilon) = -\operatorname{STr}_{\mathrm{Cl}(1)}\Big(\int_0^1 \frac{u^{\frac12}}{\sqrt{\pi}} e^{-v(\mathbb{B}'_u(\epsilon))^2}\Gamma_2\chi\frac{d}{d\epsilon}(Q_\epsilon\otimes A_P^0) \times e^{-(1-v)(\mathbb{B}'_u(\epsilon))^2}\frac{d\mathbb{B}'_u(\epsilon)}{du}dv\Big). \tag{11.9}$$

Thus if $0<\epsilon<\epsilon_0$ we have ([24]) :

$$\frac{i}{2\pi}\int_{\mathbb{S}^1}\widehat{\eta}_P(\epsilon_0) - \frac{i}{2\pi}\int_{\mathbb{S}^1}\widehat{\eta}_P(\epsilon) = d_B\frac{i}{2\pi}\int_{\mathbb{S}^1}\int_\epsilon^{\epsilon_0}\int_0^\infty \alpha(u,\delta)dud\delta. \tag{11.10}$$

The lemma will be proved by taking the limit as $\epsilon\downarrow 0$ in formula (11.10). Using Duhamel's principle and the idempotence of dt we obtain for $\sqrt{\pi}\widehat{\eta}_P(\epsilon,u)$ the analogues of (9.5) and (9.6). Thus $\sqrt{\pi}\widehat{\eta}_P(\epsilon,u)$ is equal to a differential form, not involving dt, plus the following expression

$$\begin{aligned}\operatorname{Tr}\Big(&\big(\frac{d\overline{\mathbb{B}}_u(\epsilon)}{du} - \frac{u^{-\frac12}}{2}\eth^L\otimes \mathrm{cl}_2(\frac{dx}{x})\big)\\ &\times\Big(\int_0^1 e^{-s(\overline{\mathbb{B}}_u(\epsilon)^2+u(\eth^L)^2)}\\ &\quad\times\Big(\frac{u^{\frac12}}{i\pi}dt\big((\mathrm{cl}_1(d\theta)\otimes\mathrm{cl}_2(\frac{dx}{x})\big) - \pi\chi\frac{d}{dt}Q_\epsilon\otimes A_P^0)\Big)\\ &\quad\times e^{-(1-s)(\overline{\mathbb{B}}_u(\epsilon)^2+u(\eth^L)^2)}\Big)_{\mathrm{odd}}ds\Big)\end{aligned} \tag{11.11}$$

with

$$\overline{\mathbb{B}}_u(\epsilon) = \mathrm{Id}\otimes\mathbb{B}_u + u^{\frac12}\chi(u)Q_\epsilon\otimes A_P^0.$$

The new technical problem in computing the normalized $\mathbb{S}^1$-integral of this expression comes from the mixed terms of the type $Q_\epsilon\otimes A_P^0$ appearing in the heat kernel. These terms prevent us from directly expressing the trace on the product manifold as the product of the traces as we did in the proof of Lemma 5, when we passed from (9.8) and (9.9) to (9.10) and (9.11) respectively. To get around this point we consider the usual orthonormal basis $\{e_k(t)\}$ of eigenfunctions of $\eth^L$, with eigenvalues $\{\lambda_k(t)\}$, and the decomposition

$$L^2(\mathbb{S}^1(t)\times\partial M_z;L\otimes E_0\otimes\mathbb{C}^2) = \bigoplus_{k\in\mathbb{Z}}\big(\mathbb{C}e_k(t)\otimes L^2(\partial M_z;E_0)\otimes\mathbb{C}^2\big)\,. \tag{11.12}$$

With respect to (11.12), we can express the trace appearing in (11.11) as the following absolutely convergent series :

$$\sum_{k\in\mathbb{Z}}\mathrm{Tr}\,\Big(\big(\frac{d\overline{\mathbb{B}}_u(k,\epsilon)}{du}-\frac{u^{-\frac12}}{2}\lambda_k(t)\,\mathrm{cl}_2(\frac{dx}{x})\big)\big(\int_0^1 e^{-s(\overline{\mathbb{B}}_u(k,\epsilon)^2+u\lambda_k(t)^2)}\times\big(\frac{u^{\frac12}}{i\pi}dt((\mathrm{cl}_1(d\theta)\otimes\mathrm{cl}_2(\frac{dx}{x}))-\chi\phi'_\epsilon(\lambda_k(t))A^0_P)\big)\times e^{-(1-s)(\overline{\mathbb{B}}_u(k,\epsilon)^2+u(\eth^L)^2)}\big)_{\mathrm{odd}}ds\Big),$$

where $\overline{\mathbb{B}}_u(k,\epsilon)=\mathbb{B}_u+u^{\frac12}\chi(u)\phi_\epsilon(\lambda_k(t))A^0_P$. We rewrite this expression as

$$\sum_{k\in\mathbb{Z}}\mathrm{Tr}\,\big(\mathrm{cl}_2(\frac{dx}{x})\frac{d\overline{\mathbb{B}}_u(k,\epsilon)}{du}(e^{-\overline{\mathbb{B}}_u(k,\epsilon)^2})_{\mathrm{odd}}\big)\big(\frac{u^{\frac12}}{i\pi}\,\mathrm{cl}_1(d\theta)e^{-u\lambda_k(t)^2}\big)dt - \sum_{k\in\mathbb{Z}}\mathrm{Tr}\,\big((e^{-\overline{\mathbb{B}}_u(k,\epsilon)^2})_{\mathrm{odd}}\big)\big(\frac{i}{2\pi}\lambda_k(t)\,\mathrm{cl}_1(d\theta)e^{-u\lambda_k(t)^2}\big)dt \tag{11.13}$$

plus a remainder given by

$$\sum_{k\in\mathbb{Z}}\ \mathrm{Tr}\,\Big(\big(\frac{d\overline{\mathbb{B}}_u(k,\epsilon)}{du}-\frac{u^{-\frac12}}{2}\lambda_k(t)\,\mathrm{cl}_2(\frac{dx}{x})\big)\big(\int_0^1 e^{-s(\overline{\mathbb{B}}_u(k,\epsilon)^2+u\lambda_k(t)^2)}\times\big(\frac{u^{\frac12}}{i\pi}dt(\chi\phi'_\epsilon(\lambda_k(t))A^0_P)\big)e^{-(1-s)(\overline{\mathbb{B}}_u(k,\epsilon)^2+u(\eth^L)^2)}ds\big)_{\mathrm{odd}}\Big). \tag{11.14}$$

The last term, involving the derivative of ϕ_ϵ, vanishes in the limit as $\epsilon\to 0$.

As far as the right-hand side of (11.10) is concerned we apply the same steps as above, namely Duhamel's principle, the idempotence of dt and the expansion (11.12) to express it as the sum of six terms:

$$\frac{i}{2\pi}\int_{\mathbb{S}^1}\int_\epsilon^{\epsilon_0}\int_0^\infty d_{B\times\mathbb{S}^1}\alpha(u,\delta)dud\delta = (I+II+III)+(I'+II'+III') \tag{11.15}$$

with

$$\begin{aligned}
I = \quad & d_B\Big(\frac{i}{2\pi}\int_{\mathbb{S}^1}\int_{\epsilon_0}^{\epsilon}\int_0^{\infty}\sum_{k\in\mathbb{Z}}\big(\frac{u^{\frac12}}{i\pi}\operatorname{cl}_1(d\theta)e^{-u\lambda_k(t)^2}\big)\\
& \times \operatorname{Tr}\Big(\int_0^1 \operatorname{cl}_2(\frac{dx}{x})\frac{d\overline{\mathbb{B}}_u(k,\epsilon)}{du}\frac{u^{\frac12}}{\sqrt{\pi}}e^{-v(\overline{\mathbb{B}}_u(k,\epsilon)^2)}\\
& \qquad \times\chi\frac{d}{d\delta}\phi_\delta(\lambda_k(t))A_P^0e^{-(1-v)(\overline{\mathbb{B}}_u(k,\epsilon)^2)}dv\Big)dud\delta dt\Big),\\
II = \quad & d_B\Big(\frac{1}{2\pi i}\int_{\mathbb{S}^1}\int_{\epsilon_0}^{\epsilon}\int_0^{\infty}\sum_{k\in\mathbb{Z}}\big(\frac{i}{2\pi}\lambda_k(t)\operatorname{cl}_1(d\theta)e^{-u\lambda_k(t)^2}\big)\\
& \times \operatorname{Tr}\Big(\int_0^1 \frac{u^{\frac12}}{\sqrt{\pi}}e^{-v(\overline{\mathbb{B}}_u(k,\epsilon)^2)}\chi\frac{d}{d\delta}\phi_\delta(\lambda_k(t))A_P^0\\
& \qquad \times e^{-(1-v)(\overline{\mathbb{B}}_u(k,\epsilon)^2)}dv\Big)dud\delta dt\Big),\\
III = \quad & d_B\Big(\frac{1}{2\pi i}\int_{\mathbb{S}^1}\int_{\epsilon_0}^{\epsilon}\int_0^{\infty}\sum_{k\in\mathbb{Z}}\big(\frac{u^{\frac12}}{\sqrt{\pi}}e^{-u\lambda_k(t)^2}\big)\\
& \times \operatorname{Tr}\Big(\int_0^1\int_0^1\big(\frac{d\overline{\mathbb{B}}_u(k,\epsilon)}{du}-\frac{u^{-\frac12}}{2}\lambda_k(t)\operatorname{cl}_2(\frac{dx}{x})\big)\\
& \qquad \times\frac{u^{\frac12}}{\sqrt{\pi}}e^{-vs(\overline{\mathbb{B}}_u(k,\epsilon)^2)}\chi\phi_\delta'(\lambda_k(t))A_P^0\\
& \qquad \times e^{-v(1-s)(\overline{\mathbb{B}}_u(k,\epsilon)^2)}\chi\frac{d}{d\delta}\phi_\delta(\lambda_k(t))A_P^0\\
& \qquad \times e^{-(1-v)(\overline{\mathbb{B}}_u(k,\epsilon)^2)}\Big)dud\delta dt\Big)
\end{aligned}$$

and where I', II' and III' are the corresponding terms coming from Duhamel's expansion, in powers of dt, of the second heat kernel in (11.9).

We now take the limit as $\epsilon \downarrow 0$ in formula (11.10). We first concentrate on

$$\lim_{\epsilon\downarrow 0}\frac{i}{2\pi}\int_{\mathbb{S}^1}\int_0^1\widehat{\eta}_P(\epsilon,u)du. \tag{11.16}$$

Using (11.13) (11.14) and the absolute convergence of the u-integral and of the series we can write (11.16) as

$$\begin{aligned}(11.17)\qquad &\int_0^\infty \frac{i}{2\pi}\int_{\mathbb{S}^1}\sum_{k\in\mathbb{Z}}\lim_{\epsilon\downarrow 0}\frac{1}{\sqrt{\pi}}\operatorname{Tr}\big(\operatorname{cl}_2(\frac{dx}{x})\frac{d\overline{\mathbb{B}}_u(k,\epsilon)}{du}(e^{-\overline{\mathbb{B}}_u(k,\epsilon)^2})_{\mathrm{odd}}\big)\\ &\qquad\times(\frac{u^{\frac12}}{i\pi}\operatorname{cl}_1(d\theta)e^{-u\lambda_k(t)^2})dtdu\\ &-\int_0^\infty \frac{i}{2\pi}\int_{\mathbb{S}^1}\sum_{k\in\mathbb{Z}}\lim_{\epsilon\downarrow 0}\frac{1}{\sqrt{\pi}}\operatorname{Tr}\big((e^{-\overline{\mathbb{B}}_u(k,\epsilon)^2})_{\mathrm{odd}}\big)\\ &\qquad\times(\frac{i}{2\pi}\lambda_k(t)\operatorname{cl}_1(d\theta)e^{-u\lambda_k(t)^2})dtdu\end{aligned}$$

plus

(11.18)

$$\begin{aligned}&\int_0^\infty \frac{i}{2\pi}\int_{\mathbb{S}^1}\lim_{\epsilon\downarrow 0}\frac{1}{\sqrt{\pi}}\sum_{k\in\mathbb{Z}}\operatorname{Tr}\Big((\frac{d\overline{\mathbb{B}}_u(k,\epsilon)}{du}-\frac{u^{-\frac12}}{2}\lambda_k(t)\operatorname{cl}_2(\frac{dx}{x}))\\ &\quad\times\Big(\int_0^1 e^{-s(\overline{\mathbb{B}}_u(k,\epsilon)^2+u\lambda_k(t)^2)}\\ &\qquad\times(\frac{u^{\frac12}}{i\pi}dt(\chi\phi'_\epsilon(\lambda_k(t))A_P^0))e^{-(1-s)(\overline{\mathbb{B}}_u(k,\epsilon)^2+u(\eth^L)^2)}ds\Big)_{\mathrm{odd}}\Big).\end{aligned}$$

To compute the limit of each individual term we use Duhamel's principle to express the heat kernel

$$e^{-(\mathbb{B}_u)^2-u^{\frac12}\chi(u)\phi_\epsilon(\lambda_k(t))[A_P^0,\mathbb{B}_u]-(u^{\frac12}\chi(u)\phi_\epsilon(\lambda_k(t))A_P^0)^2}$$

on the fibres of ∂M as

$$(11.19)\qquad e^{-(\mathbb{B}_u)^2}+\sum_{n>0}I_n(u,\epsilon),$$

where

$$\begin{aligned}I_n(u,\epsilon)=(-u)^n\int_{\Delta^n}&e^{-\sigma_0\mathbb{B}_u^2}(u^{\frac12}\chi\phi_\epsilon(\lambda_k(t))[A_P^0,\mathbb{B}_u]\\ &+(u^{\frac12}\chi\phi_\epsilon(\lambda_k(t))A_P^0)^2)e^{-\sigma_1\mathbb{B}_u^2}\dots e^{-\sigma_n\mathbb{B}_u^2}d\sigma,\end{aligned}$$

and the series converges rapidly in each C^k norm. Thus we can again bring the limit as $\epsilon \downarrow 0$ under the sum in (11.19). Bearing in mind that ϕ_ϵ approaches 1 as $\epsilon \downarrow 0$ we can finally compute (11.17) as

$$\begin{aligned}\int_0^\infty \frac{1}{\sqrt{\pi}} \operatorname{Tr}\big(\operatorname{cl}(\frac{dx}{x})(\frac{d\widetilde{\mathbb{B}}_u}{du}(e^{-\widetilde{\mathbb{B}}_u^2})_{\mathrm{odd}})\big)du \\ \cdot\big(\frac{i}{2\pi}\int_{\mathbb{S}^1_t}\sum_{k\in\mathbb{Z}}\frac{u^{\frac12}}{i\pi}\operatorname{cl}_1(d\theta)e^{-u\lambda_k(t)^2}\big) \\ -\int_0^\infty \frac{1}{\sqrt{\pi}}\operatorname{Tr}(e^{-\widetilde{\mathbb{B}}_u^2})_{\mathrm{odd}})du \\ \cdot\big(\frac{i}{2\pi}\int_{\mathbb{S}^1_t}(\frac{i}{2\pi}\operatorname{cl}_1(d\theta)\lambda_k(t)e^{-u\lambda_k(t)^2}dt)\big),\end{aligned} \tag{11.20}$$

which in turn is equal to

$$\int_0^\infty \frac{1}{\sqrt{\pi}} \operatorname{Tr}\big(\operatorname{cl}(\frac{dx}{x})(\frac{d\widetilde{\mathbb{B}}_u}{du}(e^{-\widetilde{\mathbb{B}}_u^2})_{\mathrm{odd}})\big)du\cdot\big(\frac{1}{2\pi i}\int_{\mathbb{S}^1_t}\operatorname{Tr}^{\mathrm{odd}} e^{-(u^{\frac12}\eth^L+\frac1i dt\frac{d}{dt})^2}\big)$$

with the second factor equal to the spectral flow of $\eth^L$, which is equal to one. Here, and in (11.20), definition (11.5) has been used. Notice that in (11.20) we have again used Duhamel's principle in order to sum up the series of the limits of the traces resulting from (11.19). Since ϕ'_ϵ approaches 0 as $\epsilon \downarrow 0$, it is straightforward to check, using again (11.19) (11.20), that the limit as $\epsilon \downarrow 0$ of the remainder (11.18) is equal to 0.

Thus we have shown that

$$\lim_{\epsilon\downarrow 0}\frac{i}{2\pi}\int_{\mathbb{S}^1}\int_0^1 \widehat{\eta}_P(\epsilon,u)du = \int_0^\infty \frac{1}{\sqrt{\pi}}\operatorname{STr}_{\mathrm{Cl}(1)}\big(\frac{d\widetilde{\mathbb{B}}_u}{du}e^{-\widetilde{\mathbb{B}}_u^2}\big)du = \eta_{\mathrm{odd},P}.$$

On the other hand, direct inspection of formula (11.15) shows that

$$\lim_{\epsilon\downarrow 0}\frac{i}{2\pi}\int_{\mathbb{S}^1}\int_\epsilon^{\epsilon_0}\int_0^\infty d_{B\times\mathbb{S}^1}\alpha(u,\delta)du\,d\delta$$

exists and is equal to an exact form on B. The lemma is therefore proved.

By applying (11.1), (11.6), Lemma 6 and (10.2) we obtain the main result of this paper

Theorem. *Let* $\eth$ *be a family of generalized Dirac operators on odd dimensional manifolds with boundary as in §1, and let* P *be a* $\mathrm{Cl}(1)$ *spectral section for the boundary family* $\eth_0$. *If* $\mathrm{Ind}(\eth, P) \in K^1(B)$ *is the index class associated to* $\eth$ *and* P *as in §4, then the following formula holds*

$$\mathrm{Ch}(\mathrm{Ind}(\eth, P)) = (2\pi i)^{-\frac{n+1}{2}} \int_{M/B} \widehat{A}(M/B)\,\mathrm{Ch}'(E) - \frac{1}{2}\eta_{\mathrm{odd},P} \quad \text{in } H^{\mathrm{odd}}(B),$$

where $\eta_{\mathrm{odd},P}$ *is the eta form defined in (11.8).*

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