AN INDEX THEOREM FOR FIRST ORDER REGULAR SINGULAR OPERATORS

By Jochen Brüning and Robert Seeley

1. Introduction. In this paper we use the methods developed in [B+S1,2] to prove index theorems for certain first order elliptic operators. More precisely, let M be a Riemannian manifold of dimension $n+1$, E, F hermitian vector bundles over M, and $D: C_0^\infty(E) \to C_0^\infty(F)$ an elliptic first order differential operator. We think of M as a singular Riemannian manifold with singularities in an open subset U such that $M \setminus U$ is a smooth compact manifold with boundary. Our assumptions on the nature of the singularities and the behavior of D on U will be formulated abstractly in the following way.

($RS1$). There is a compact Riemannian manifold N of dimension n and a hermitian vector bundle G over N such that there are bijective linear maps

$$\Phi_E: C_0^\infty(E|U) \to C_0^\infty(I, C^\infty(G)),$$

$$\Phi_F: C_0^\infty(F|U) \to C_0^\infty(I, C^\infty(G)),$$

where $I := (0, \epsilon]$ for some ϵ with $0 < \epsilon \leq 1$.

($RS2$). Φ_E and Φ_F extend, respectively, to unitary maps $L^2(E|U) \to L^2(I, L^2(G))$ and $L^2(F|U) \to L^2(I, L^2(G))$.

($RS3$). For $\varphi \in C^\infty(I)$ with φ constant near 0 and ϵ let M_φ be the multiplication operator on $L^2(I, L^2(G))$. Then $\Phi_E^* M_\varphi \Phi_E = \Phi_F^* M_\varphi \Phi_F = \mathrm{M}_{\bar\varphi}$ for some $\bar\varphi \in C^\infty(M)$, and $\bar\varphi \in C_0^\infty(M)$ if φ vanishes in a neighborhood of 0.

Manuscript received 20 June 1986; revised 4 March 1987.

American Journal of Mathematics 110 (1988), 659–714.

(RS4). On $C_0^\infty(E|U)$ we have for some $\beta > -1/2$

$$T := \Phi_F D \Phi_E^* = \partial_x + x^{-1}S_0 + x^\beta S_1(x)$$

where

(a) S_0 is a self-adjoint first order elliptic differential operator on $C^\infty(G)$,
(b) $S_1(x)$ is a first order differential operator on $C^\infty(G)$ with smooth coefficients in $(0, \epsilon]$,
(c) $\|S_1(x)(|S_0| + 1)^{-1}\| + \|(|S_0| + 1)^{-1}S_1(x)\| \leq C$ uniformly in $(0, \epsilon]$.

If these assumptions are satisfied we refer to D as a *first order regular singular elliptic operator.* We will express this fact in writing

$$D \simeq \partial_x + x^{-1}S_0 + x^\beta S_1(x) \quad \text{on} \quad U,$$

and we will also identify φ and $\bar{\varphi}$ in (RS3) for simplicity. In addition, we use the notation listed in [B+S2] Section 1, which we recall for convenience of the reader at the end of this introduction.

Of course, the principal example of this situation is a manifold with conical singularities where certain index theorems for geometric operators are known [Che], [Cho]. It was our aim to unify and to generalize these results. The plan of the paper is as follows. In Section 2 we construct a closed extension D_δ (where δ refers to "Dirichlet") of D and show that it is Fredholm with index essentially independent of S_1. In Section 3 we impose slightly stronger conditions on β and S_1 and classify all closed extensions between the minimal $D_{\min}$ and the maximal $D_{\max}$ given by

$$\mathfrak{D}(D_{\max}) = \{u \in L^2(E) | Du \in L^2(F)\}.$$

It turns out that $D_{\min} = D_{\max}$ iff spec $S_0 \cap (-1/2, 1/2) = \emptyset$. The closed extensions are classified by the subspaces of

$$W := \bigoplus_{|s|<1/2} \ker(S_0 - s)$$

(Theorem 3.1), and their indices are related in a simple way (Theorem 3.2). In Section 4 we take up the calculation of the index of D_δ. This is done directly from the resolvent, using some results of [B+S2]. We obtain the index formula (Theorem 4.1)

$$\operatorname{ind} D_\delta = \int_M \omega_D - \frac{1}{2}(\eta_{S_0} + \dim \ker S_0) + R.$$

Here ω_D denotes the index form of D i.e. $\omega_D(p)$ is the constant term in the asymptotic expansion of

$$\operatorname{tr}_E e^{-tD_\delta^* D_\delta}(p,p) - \operatorname{tr}_F e^{-tD_\delta D_\delta^*}(p,p), \qquad p \in M,$$

as $t \searrow 0$, and the integral stands for a certain regularization of the possibly divergent integral; η_{S_0} is the usual η-invariant of S_0 as introduced in [A+P+S]; and R is a linear combination of residues of the η-function of S_0. We apply our results to the Gauß-Bonnet and the signature operator on manifolds with asymptotically cone-like singularities (see (5.1) for the definition), and recover the Gauß-Bonnet Theorem and the Signature Theorem of [Che] for suitable closed extensions in the conic case (Theorem 5.1, 5.2). Asymptotically cone-like singularities are still very close to conic ones, but they cannot be treated analytically by separation of variables. We hope, however, to extend the method given here to considerably more general situations.

We are indebted to Robert McOwen and Richard Melrose for helpful conversations. The first author also acknowledges the support of the Deutsche Forschungsgemeinschaft and the hospitality of MIT and Northeastern University, the second author acknowledges the hospitality of the University of Augsburg.

Notation.

$\mathbf{R}^*$ is the interval $(0, \infty)$, $\mathbf{R}_+$ is $[0, \infty)$.

$C_0^\infty(Y)$ is C^∞-functions with compact support in Y.

H is a fixed Hilbert space.

H_S is the common domain of the family of self-adjoint operators $S(x) = S_0 + x^{\beta+1}S_1(x)$, $x \in (0, \epsilon]$.

X denotes the operator $Xf(x) = xf(x)$ on $L^2(\mathbf{R}_+, H)$.

If $\psi \in L^\infty(\mathbf{R}_+)$, Ψ denotes the operator $\Psi f(x) = \psi(x)f(x)$ on $L^2(\mathbf{R}_+, H)$.

2. The construction of a boundary parametrix for the operator

$$T = \partial_x + X^{-1}S_0 + X^{\beta}S_1(x), \qquad \beta > -1/2, \tag{2.1}$$

acting in $L^2(\mathbf{R}_+, H)$ with domain $C_0^\infty(\mathbf{R}^*, H_S)$, amounts to the integration of first order ordinary differential equations. We assume as before that S_1 is smooth away from 0 and that for some constant C_0

$$\|S_1(x)(|S_0| + 1)^{-1}\|_H + \|(|S_0| + 1)^{-1}S_1(x)\|_H \le C_0 \tag{2.2}$$

uniformly in $x > 0$.

For $f \in L^2(0, \infty)$ we put

$$P_{0,s}f(x) := \int_0^x (y/x)^s f(y)\,dy, \qquad s > -1/2, \tag{2.3}$$

$$P_{1,s}f(x) := \int_1^x (y/x)^s f(y)\,dy, \qquad s < 1/2. \tag{2.4}$$

Note that

$$(\partial_x + X^{-1}s)P_{0,s} = (\partial_x + X^{-1}s)P_{1,s} = I. \tag{2.5}$$

Appropriate parametrices are constructed by combining $P_{0,s}$ and $P_{1,s}$.

LEMMA 2.1. *For f in $L^2(0, 1)$ and $x \to 0$ we have the following estimates.*

a) $|P_{0,s}f(x)| \le x^{1/2}|2s + 1|^{-1/2}\left(\int_0^x |f(y)|^2 dy\right)^{1/2}, s > -1/2.$

b) $|P_{1,s}f(x)| \le$

$$\begin{cases} x^{1/2}|2s + 1|^{-1/2}\|f\|_{L^2}, & s < -1/2, \\ x^{1/2}[|\log x|^{1/2}(\int_0^\delta |f(y)|^2 dy)^{1/2} & \\ \qquad + |\log \delta|^{1/2}\|f\|_{L^2}], & s = -1/2, \qquad 0 < \delta < 1. \end{cases}$$

c) *For* $-1/2 < s < 1/2$

$$\left|P_{1,s}f(x) + x^{-s}\int_0^1 y^s f(y)\,dy\right| \le x^{1/2}|2s + 1|^{-1/2}\|f\|_{L^2}.$$

Proof. We prove the second estimate in b); the other estimates in a) and b) are proved similarly, while c) follows from a). Let $0 < \delta < 1$ and $s = -1/2$; we find for $x \leq \delta$

$$|P_{1,-1/2}f(x)| = \left| x^{1/2} \int_1^x y^{-1/2} f(y)\, dy \right|$$

$$\leq x^{1/2} \left[\int_x^\delta + \int_\delta^1 \right] y^{-1/2} |f(y)|\, dy$$

$$\leq x^{1/2} \left[|\log x|^{1/2} \left(\int_0^\delta |f(y)|^2 dy \right)^{1/2} + |\log \delta|^{1/2} \|f\|_{L^2} \right]. \qquad \square$$

LEMMA 2.2. *Let $0 < \epsilon \leq 1$ and $-1 < \beta \leq 0$. Then in $L^2(0, \epsilon)$ we have*

$$\|X^\beta P_{0,s}\| + \|P_{1,-s} X^\beta\| \leq C^1(s, \epsilon)(|s| + 1)^{-1}, \qquad s > -1/2, \tag{2.6}$$

$$\|X^\beta P_{1,s}\| + \|P_{0,-s} X^\beta\| \leq C^2(s, \epsilon)(|s| + 1)^{-1}, \qquad s < 1/2 + \beta. \tag{2.7}$$

Here $\lim_{\epsilon \to 0} C^i(s, \epsilon) = 0$, $i = 1, 2$, *and uniformly for* $|s| \geq 2$.

Proof. We note first that

$$(X^\beta P_{0,s} X^\gamma)^* = -X^\gamma P_{1,-s} X^\beta \tag{2.8}$$

whenever $X^\beta P_{0,s} X^\gamma$ is bounded in $L^2(0, \epsilon)$. Thus it is sufficient to estimate the norm of the operators

$$u \mapsto x^{\beta - s} \int_0^x y^s u(y)\, dy, \qquad s > -1/2, \tag{2.9a}$$

and

$$u \mapsto x^s \int_0^x y^{-s+\beta} u(y)\, dy, \qquad s < \beta + 1/2. \tag{2.9b}$$

Now the assertion follows from standard estimates for integral operators, e.g. from Schur's test ([H+S], p. 22). $\square$

We introduce the "Dirichlet" boundary condition for the operator T at 0 by defining an operator T_δ as restriction of $T_{\max}$ to the domain

$$\mathfrak{D}(T_\delta) := \{u \in \mathfrak{D}(T_{\max}) \mid \|u(x)\|_H = o(1) \text{ as } x \to 0\}. \tag{2.10a}$$

This also gives rise to Dirichlet boundary conditions for D; we thus call D_δ the restriction of $D_{\max}$ to the domain

$$\mathfrak{D}(D_\delta) = \{u \in \mathfrak{D}(D_{\max}) \mid \|u(x)\|_H = o(1) \text{ as } x \to 0\}. \tag{2.10b}$$

The boundary parametrix P_δ is then defined by

$$P_\delta := \bigoplus_{\substack{s \in \operatorname{spec} S_0 \\ s \geq 0}} P_{0,s} \oplus \bigoplus_{\substack{s \in \operatorname{spec} S_0 \\ s < 0}} P_{1,s} \tag{2.11}$$

with each term $P_{0,s}$ or $P_{1,s}$ acting in the appropriate eigenspace. Since we assume that $\beta > -1/2$ in (2.1), Lemma 2.2 applies to P_δ. We will now establish that D_δ is a Fredholm operator.

LEMMA 2.3. *If* $\psi \in C_0^\infty(-1, 1)$ *then* ΨP_δ *maps* $L^2((0, 1), H)$ *into* $\mathfrak{D}(T_\delta)$.

Proof. By Lemma 2.1, setting $f(x) = \oplus_{s \in \operatorname{spec} S_0} f_s(x)$ we have

$$\Big\| \bigoplus_{s \geq 0} P_{0,s} f_s(x) \Big\|_H^2 = O\left(x \int_0^x \sum_{s \geq 0} |f_s(y)|^2 dy\right) = O\left(x \int_0^x \|f(y)\|_H^2 dy\right),$$

$$\begin{aligned} \Big\| \bigoplus_{s<0} P_{1,s} f_s(x) \Big\|_H^2 &= O\Big(\sum_{-1/2 < s < 0} x^{-2s} \|f_s\|^2 \Big) \\ &\quad + O(x |\log x| \, \|f_{-1/2}\|^2) + O\Big(x \sum_{s < -1/2} \|f_s\|^2\Big) \\ &= o(1) \|f\|^2, \end{aligned}$$

so $P_\delta f(x) = o(1)$ as $x \to 0$. Now

$$\begin{aligned} T\psi P_\delta f &= \psi T P_\delta f + \psi' P_\delta f \\ &= \psi f + \psi X^\beta S_1 P_\delta f + \psi' P_\delta f \end{aligned} \tag{2.12}$$

so in view of (2.2) it suffices to estimate

$$\|X^{\beta}(|S_o| + 1)P_{\delta}f\|^2 = \sum_{s\geq 0} (|s| + 1)^2\|X^{\beta}P_{0,s}f_s\|^2$$

$$+ \sum_{s<0} (|s| + 1)^2\|X^{\beta}P_{1,s}f_s\|^2 \leq o(\epsilon) \sum \|f_s\|^2 = o(\epsilon)\|f\|^2,$$

where we have used Lemma 2.2. □

LEMMA 2.4. *If $u \in \mathfrak{D}(T_\delta)$ and $u(x) \equiv 0$ for $x \geq 1$ then*

$$P_\delta Tu = u + (P_\delta X^\beta S_1)u. \tag{2.13}$$

Proof. Let $(e_s)_{s\in \mathrm{spec}\, S_0}$ be an orthonormal basis in H with $S_0 e_s = se_s$. For $x > 0$ we put

$$h(x) := (\partial_x + x^{-1}S_0)u(x) = Tu(x) - X^\beta S_1(x)u(x) \tag{2.14}$$

and

$$h_s(x) := \langle h(x), e_s\rangle = \langle Tu(x), e_s\rangle_H - \langle u(x), x^\beta S_1(x)^* e_s\rangle_H$$

$$= u_s'(x) + x^{-1}su_s(x), \qquad s \in \mathrm{spec}\, S_0.$$

In view of (2.2) and $\beta > -1/2$ we have $h_s \in L^1(0, 1)$, and since $u_s(1) = 0$ we obtain

$$u_s(x) = P_{1,s}h_s(x).$$

It remains to show that for $s \geq 0$, $P_{1,s}$ can be replaced by $P_{0,s}$. We write

$$u_s(x) = -x^{-s}\int_0^1 h_s(x)dx + P_{0,s}h_s(x) \tag{2.15}$$

$$=: c_s x^{-s} + P_{0,s}h_s(x).$$

For $s \geq 0$ and $h \in L^1$

$$|P_{0,s}h(x)| \leq \int_0^x |h(t)|dt = o(1),$$

and $u_s(x) = o(1)$ since $u \in \mathfrak{D}(T_\delta)$. So $c_s = 0$, and $u_s = P_{0,s}h_s$ if $s \geq 0$. The proof is complete. □

LEMMA 2.5. *There is $0 < \epsilon \leq 1$ such that for $\varphi, \psi \in C_0^\infty(-\epsilon, \epsilon)$, with $\psi\varphi = \varphi$ and $u \in \mathfrak{D}(D_\delta)$*

$$\varphi u = \psi P_\delta V T_\delta \varphi u \tag{2.16}$$

for some bounded operator V in $L^2((0, 1), H)$. As a consequence,

$$\|\varphi X^\beta(|S_0| + 1)u\| \leq C\|T_\delta \varphi u\|. \tag{2.17}$$

Proof. Choose $\chi \in C_0^\infty(-\epsilon, \epsilon)$ with $\chi\psi = \psi$. Since $\varphi u \in \mathfrak{D}(T_\delta)$ we obtain from Lemma 2.4 with $f := T_\delta \varphi u$

$$\varphi u = \psi P_\delta \chi f - \psi P_\delta X^\beta S_1 \chi \varphi u.$$

Iterating,

$$\varphi u = \psi P_\delta \chi \sum_{j=0}^{n} (-X^\beta S_1 \psi P_\delta \chi)^j f + (-1)^{n+1}(\psi P_\delta X^\beta S_1 \chi)^{n+1} \varphi u.$$

For ϵ sufficiently small we have by Lemma 2.2 and (2.2) the operator norms

$$\|X^\beta S_1 \psi P_\delta \chi\|_{L^2((0,\epsilon),H)} + \|\psi P_\delta X^\beta S_1 \chi\|_{L^2((0,\epsilon),H)} < 1, \tag{2.18}$$

so we obtain (2.16) with

$$V := \sum_{j=0}^{\infty} (-X^\beta S_1 \psi P_\delta \chi)^j.$$

(2.17) follows from Lemma 2.2 and (2.2). □

LEMMA 2.6. *D_δ is a closed operator.*

Proof. If $(u_n) \subset \mathfrak{D}(D_\delta)$ with $u_n \to u$, $D_\delta u_n \to v$ in $L^2(E)$ then clearly $u \in \mathfrak{D}(D_{\max})$ and $v = Du$. So we have to show only that u satisfies the boundary condition (2.10b). If ϵ is chosen as in Lemma 2.5 and $\varphi \in C_0^\infty(-\epsilon, \epsilon)$ with $\varphi = 1$ near 0 then we derive from (2.16)

$$\varphi u_n = \psi P_\delta V(\varphi' u_n + \varphi D_\delta u_n)$$

hence

$$\varphi u = \psi P_\delta V(\varphi' u + \varphi v).$$

Thus it follows from Lemma 2.1 that $\|u(x)\|_H = o(1)$ as $x \to 0$. □

THEOREM 2.1. *$D_\delta : \mathfrak{D}(D_\delta) \to L^2(F)$ is a Fredholm operator.*

Proof. By Lemma 2.6, $\mathfrak{D}(D_\delta)$ is a Hilbert space under the graph norm, so we only have to prove that D_δ has finite kernel and cokernel; for this we construct right and left parametrices. Choose $\varphi, \tilde{\varphi} \in C_0^\infty(-\epsilon, \epsilon)$ such that $\varphi = 1$ near 0 and $\tilde{\varphi} = 1$ near supp φ, and choose $\psi, \tilde{\psi} \in C_0^\infty(M)$ such that $\varphi + \psi = 1$ and $\tilde{\psi} = 1$ in a neighborhood of supp ψ. Let $P_i : L^2(F) \to H^1_{\mathrm{loc}}(E)$ be an interior parametrix for D with

$$D\tilde{\psi} P_i \psi = \psi + R_i, \tag{2.19a}$$

$$\tilde{\psi} P_i \psi D = \psi + L_i, \tag{2.19b}$$

with R_i, L_i compact in $L^2(F)$ and $L^2(E)$, respectively. Define

$$Q_\delta := \tilde{\varphi} P_\delta \varphi + \tilde{\psi} P_i \psi.$$

By Lemma 2.3, Q_δ maps into $\mathfrak{D}(D_\delta)$ and

$$D_\delta Q_\delta = I + \tilde{\varphi}' P_\delta \varphi + \tilde{\varphi} X^\beta S_1 P_\delta \varphi + R_i.$$

Now if the support of φ is sufficiently small we have in view of Lemma 2.2 and (2.2)

$$\|\tilde{\varphi}' P_\delta \varphi + \tilde{\varphi} X^\beta S_1 P_\delta \varphi\| < 1/2$$

and we can write

$$D_\delta Q_\delta = I + R + R_i$$

where R_i is compact and $\|R\| < 1/2$. This implies

$$D_\delta Q_\delta (I + R)^{-1} = I + R_i (I + R)^{-1},$$

so D_δ has finite cokernel. Next we find with Lemma 2.4

$$Q_\delta D_\delta = \tilde{\varphi} P_\delta \varphi T_\delta + \psi + L_i$$

$$= I + \tilde{\varphi} P_\delta X^\beta S_1 \varphi - \tilde{\varphi} P_\delta \varphi' + L_i$$

and as before we obtain for small ϵ

$$Q_\delta D_\delta = I + L + L_i$$

where L_i is compact and $\|L\| < 1/2$. But then

$$(I + L)^{-1} Q_\delta D_\delta = I + (I + L)^{-1} L_i$$

so D_δ has finite kernel. □

To compute the index of D_δ it is convenient to have $S_1(x) \equiv 0$ near 0. This can always be achieved by a deformation of D_δ.

LEMMA 2.7. *Let $S(x) \in \mathfrak{L}(H_S, H)$ be a smooth function of x in* (0, 1] *and satisfy* (2.2). *Then for $\chi \in C_0^\infty(-\epsilon, \epsilon)$ with ϵ sufficiently small and $\beta > -1/2$*

$$D_\delta + \chi X^\beta S(x) =: \tilde{D}_\delta$$

is a Fredholm operator on $\mathfrak{D}(D_\delta)$ with

$$\operatorname{ind} D_\delta = \operatorname{ind} \tilde{D}_\delta .$$

Proof. By (2.17) and interior regularity

$$\chi X^\beta S(x) = S(x)(|S_0| + 1)^{-1} \chi X^\beta (|S_0| + 1)$$

is bounded from $\mathfrak{D}(D_\delta)$ to $L^2(F)$. Thus the family

$$D_\delta(\theta) := D_\delta + \theta \chi X^\beta S(x) \tag{2.20}$$

is a continuous function of $\theta \in [0, 1]$ with values in $\mathfrak{L}(\mathfrak{D}(D_\delta), L^2(F))$. Repeating the proof of Theorem 2.1 with $\tilde{\psi}$ such that $\chi\tilde{\psi} = 0$ we see that each $D_\delta(\theta)$ is a Fredholm operator, so the index must be constant. □

3. The closed realizations of D are all Fredholm operators; we show this by proving that $D_{\max}$ and $D'_{\max}$ are Fredholm. We then identify those realizations with the subspaces of $\oplus_{|s|<1/2} \ker(S_0 - s)$. Assume now that $\beta = 0$, that is for $x \le 1$

$$(3.1a) \qquad D \simeq \partial_x + X^{-1}S_0 + S_1(x) = T,$$

$$(3.1b) \qquad D' \simeq -\partial_x + X^{-1}S_0 + S_1(x) = T',$$

and maintain the hypothesis (2.2) on S_1. We have the following analog of Lemma 2.4.

LEMMA 3.1. *If $u \in \mathfrak{D}(T_{\max})$ and $u(x) \equiv 0$ for $x \ge 1$ then*

$$P_{\max} Tu = u + (P_{\max} S_1)u$$

where

$$P_{\max} = \bigoplus_{s<1/2} P_{1,s} \oplus \bigoplus_{s\ge 1/2} P_{0,s},$$

and $P_{\max} S_1$ is bounded in $L^2((0, 1), H)$.

Proof. Let $\{e_s\}_{s \in \operatorname{spec} S_0}$ be an orthonormal basis of eigensections for S_0. Let u and $Tu \in L^2$, and set

$$(3.3) \qquad h(x) := u'(x) + x^{-1}S_0 u(x) = Tu(x) - S_1(x)u(x).$$

By (2.2)

$$\|S_1(x)^* e_s\|_H = \|S_1(x)(|S_0| + 1)^{-1}(|S_0| + 1)e_s\|_H \le C_0(|s| + 1).$$

Hence for each s,

$$h_s(x) = \langle h(x), e_s\rangle_H = \langle (Tu)(x), e_s\rangle_H - \langle u(x), S_1(x)^* e_s\rangle_H \in L^2(0, 1)$$

since u and Tu are in L^2. For any s, $u_s(1) = 0$ implies that

$$(3.4) \qquad P_{1,s} h_s = P_{1,s}(u_s' + x^{-1} s u_s) = u_s \in L^2.$$

It follows that for $s \geq 1/2$

$$\int_0^1 y^s h_s(y)\,dy = 0 \tag{3.5}$$

since

$$x^{-s}\int_0^1 y^s h_s(y)\,dy = P_{0,s}h_s(x) - P_{1,s}h_s(x)$$

is in L^2, the last term by (3.4) and the other by Lemma 2.1. Now (3.4) and (3.5) give

$$P_{0,s}h_s = P_{1,s}h_s = u_s, \qquad s \geq 1/2.$$

Combining this with (3.4) for $s < 1/2$ gives $P_{\max}h = u$, and this proves the lemma, by (3.3). □

THEOREM 3.1. *$D_{\max}$ and $D_{\min}$ are Fredholm operators. The extensions of $D_{\min}$ are all Fredholm operators, and correspond to the subspaces of the finite-dimensional space*

$$\mathfrak{D}(D_{\max})/\mathfrak{D}(D_{\min}).$$

Proof. Choose φ, $\tilde{\varphi}$, ψ, $\tilde{\psi}$ as in Theorem 2.1 and define the parametrix

$$P = \tilde{\varphi}P_{\max}\varphi + \tilde{\psi}P_i\psi.$$

Then by Lemma 3.1 and (2.19*b*)

$$PD_{\max}u = u + [\tilde{\varphi}P_{\max}S_1\varphi - \tilde{\varphi}P_{\max}\varphi']u + L_i u.$$

As in the proof of Theorem 2.1 we see that, by an appropriate choice of φ and $\tilde{\varphi}$, the operator in brackets has small norm, while L_i is compact; hence $PD_{\max}$ is a Fredholm operator, and has finite nullity. Thus $D_{\max}$ has finite nullity. Since it is an extension of the Fredholm operator D_δ, it also has closed range with finite codimension; thus it is Fredholm. The same argument applies to $D'_{\max}$, hence its adjoint $D_{\min}$ is also Fredholm.

Now $\mathfrak{D}(D_{\min})$ and $\mathfrak{D}(D_{\max})$ are Hilbert spaces under the graph norm. Thus $\mathfrak{D}(D_{\min})$ is a closed subspace of $\mathfrak{D}(D_{\max})$, and it has finite codimension since both operators are Fredholm. Hence the inclusion map is Fredholm and

$$\operatorname{ind} D_{\max} = \operatorname{ind} D_{\min} + \dim \mathfrak{D}(D_{\max})/\mathfrak{D}(D_{\min}).$$

The conclusion of the theorem is now clear. □

We next show that $\mathfrak{D}(D_{\max})/\mathfrak{D}(D_{\min})$ is isomorphic to $\oplus_{|s|<1/2} \ker(S_0 - s)$, and relate the extensions of $D_{\min}$ to the asymptotic behavior of their elements at $x = 0$.

LEMMA 3.2. *For s in* $\operatorname{spec} S_0$, $|s| < 1/2$, *there are continuous linear functionals c_s on $\mathfrak{D}(D_{\max})$ such that for x in* $(0, 1)$ *and* $0 < \epsilon < 1$

$$\left\| u(x) - \sum_{|s|<1/2} c_s(u) x^{-s} e_s \right\|_H \leq \epsilon x^{1/2} |\log x|^{1/2} + C_{\epsilon,u} x^{1/2} \tag{3.6}$$

for u in $\mathfrak{D}(D_{\max})$. The same statement holds for D', mutatis mutandis. (If s has multiplicity $m > 1$, there are m corresponding functionals c_s.)

Proof. Just as Lemma 2.4 implies Lemma 2.5, Lemma 3.1 implies that if $u \in \mathfrak{D}(D_{\max})$ then $\|(|S_0| + 1)u(\cdot)\|_H \in L^2(0, 1)$, and

$$\int_0^1 \|(|S_0| + 1)u(x)\|^2 dx \leq C(\|Du\|^2 + \|u\|^2).$$

Hence in (3.3), $\|h(\cdot)\|_H \in L^2(0, 1)$. Since $h = u' + x^{-1}S_0 u$, we have for $s > -1/2$, for some constants $c_s(u)$,

$$u_s(x) = x^{-s}\left[c_s(u) + \int_0^x y^s h_s(y)dy\right]. \tag{3.7}$$

Since $x^{-s}\int_0^x y^s h_s(y)dy \in L^2$ by Lemma 2.2, we have

$$c_s(u) = 0, \qquad s \geq 1/2. \tag{3.8}$$

For $|s| < 1/2$, setting $x = 1$ gives

$$c_s(u) = u_s(1) - \int_0^1 y^s h_s(y)dy. \tag{3.9}$$

For $s < 1/2$,

$$u_s(x) = [x^{-s}u_s(1) + P_{1,s}h_s(x)]. \tag{3.10}$$

By interior regularity, $u' \in L^2((\frac{1}{2}, 1), H)$ and

$$\|u(1)\|_H^2 \leq C \int_{1/2}^1 (\|u(x)\|^2 + \|u'(x)\|^2)dx.$$

Hence the functionals in (3.9) are continuous on $\mathfrak{D}(D_{\max})$, and

$$\sum_{s \leq -1/2} x^{-2s}|u_s(1)|^2 \leq C_u x. \tag{3.11}$$

By Lemma 2.1*b*), for every positive $\delta < 1$,

(3.12)

$$\sum_{s \leq -1/2} |P_{1,s}h_s(x)|^2 \leq x\left[\|h\|^2 + 2\log\delta\|h_{-1/2}\|^2 + 2|\log x| \int_0^\delta |h_{-1/2}|^2\right].$$

By Lemma 2.1*a*) and (3.7), (3.8),

$$\sum_{s > -1/2} |u_s(x) - c_s(u)x^{-s}|^2 = \sum_{s > -1/2} |P_{0,s}h_s(x)|^2 \leq Cx\|h\|^2.$$

This together with (3.10)-(3.12) proves the Lemma. □

We can now define, for each subspace $W \subset \oplus_{|s|<1/2} \ker(S_0 - s)$, an extension D_W of $D_{\min}$ by restricting $D_{\max}$ to

$$\mathfrak{D}(D_W) = \{u \in \mathfrak{D}(D_{\max}) | \sum_{|s|<1/2} c_s(u)e_s \in W\}.$$

Note that D_W is automatically closed since the functionals c_s are continuous on $\mathfrak{D}(D_{\max})$.

THEOREM 3.2. *The operators D_W give all closed extensions of $D_{\min}$, and $(D_W)^* = D'_{W^\perp}$. Moreover*

$$\operatorname{ind}(D_W) = \operatorname{ind}(D_{\min}) + \dim W.$$

Proof. We note first that for $u \in \mathfrak{D}(D_{\max})$ and $v \in \mathfrak{D}(D'_{\max})$

$$(Du, v) = (u, D'v) - \sum_{|s|<1/2} c_s(u)\overline{c'_{-s}(v)} \tag{3.13}$$

where c'_{-s} are the functionals for D'. For by Lemma 3.2, taking $\varphi \in C_0^\infty(-1, 1)$ with $\varphi(x) \equiv 1$ near $x = 0$, we have

$$u(x) = \varphi(x) \sum_{|s|<1/2} c_s(u)x^{-s}e_s + \tilde{u}(x),$$

$$v(x) = \varphi(x) \sum_{|s|<1/2} c'_{-s}(v)x^{s}e_s + \tilde{v}(x)$$

with $\|\tilde{u}(x)\| + \|\tilde{v}(x)\| \leq Cx^{1/2}|\log x|^{1/2}$ as $x \to 0$. Then

$$(Du, v) = \lim_{\epsilon\to 0} \int_\epsilon^1 \langle Du, \varphi v\rangle_H + (u, D'(1-\varphi)v)$$

$$= \lim_{\epsilon\to 0} [-\langle u(\epsilon), v(\epsilon)\rangle_H] + (u, D'v)$$

$$= -\sum_{|s|<1/2} c_s(u)c'_{-s}(v) + (u, D'v).$$

Note second that

$$\Big\{\sum_{|s|<1/2} c_s(u)e_s \,|\, u \in \mathfrak{D}(D_{\max})\Big\} = \bigoplus_{|s|<1/2} \ker(S_0 - s). \tag{3.14}$$

For, given any constants c_s, $|s| < 1/2$, we set

$$u(x) = \varphi(x) \sum_{|s|<1/2} c_s x^{-s}e_s$$

with φ as before, and find

$$Tu(x) = \varphi'(x) \sum_{|s|<1/2} c_s x^{-s}e_s$$

$$+ \varphi(x) \sum_{|s|<1/2} c_s x^{-s}[S_1(x)(|S_0| + 1)^{-1}](|s| + 1)e_s \in L^2.$$

We observe next that

$$(3.15) \qquad c_s(u) = 0 \quad \text{for all} \quad s \quad \text{iff} \quad u \in \mathfrak{D}(D_{\min}).$$

In fact, (3.13) implies that $u \in \mathfrak{D}((D'_{\max})^*)$ if u satisfies (3.15). But since $D'_{\max} = (D_{\min})^*$ we have $(D'_{\max})^* = D_{\min}$. The converse part of (3.15) is true since c_s is continuous on $\mathfrak{D}(D_{\max})$.

Now let D be any extension of $D_{\min}$ and define

$$W := \Big\{ \sum_{|s|<1/2} c_s(u) e_s \,\big|\, u \in \mathfrak{D}(D) \Big\}.$$

Then clearly $D \subset D_W$. Conversely, for $v \in \mathfrak{D}(D_W)$ there is $u \in \mathfrak{D}(D)$ with $c_s(v-u) = 0$ for all s by definition. But then $u - v \in \mathfrak{D}(D_{\min}) \subset \mathfrak{D}(D)$ by (3.15) proving $D = D_W$. The formula for ind D_W is clear from Theorem 3.1 and $\mathfrak{D}(D_W)/\mathfrak{D}(D_{\min}) \simeq W$. The relation $D_W^* = D'_{W^\perp}$ follows from (3.13). □

Example. For $u \in \mathfrak{D}(D_{\max})$ we have $u(x) = o(1)$ as $x \to 0$ iff $c_s(u) = 0$ for $s \geq 0$. Introducing

$$W_< := \bigoplus_{s<0} \ker(S_0 - s), \qquad W_\geq := \bigoplus_{s \geq 0} \ker(S_0 - s)$$

we see that

$$D_\delta = D_{W_<}, \qquad D_\delta^* = D'_{W_\geq}.$$

Thus we obtain for $W \subset \bigoplus_{|s|<1/2} \ker(S_0 - s)$ from Theorem 3.2

$$(3.16) \qquad \begin{aligned} \operatorname{ind} D_W &= \operatorname{ind} D_{\min} + \dim W \\ &= \operatorname{ind} D_\delta + \dim W - \dim W_<. \end{aligned}$$

4. The index of D_δ will be calculated in this section, using a variant of the approach in [B+S2]. We assume for small x the representation (2.1) with $\beta > -1/2$ and the regularity property (2.2). Moreover, at first we assume also that $S_1(x) \equiv 0$, that is

$$(4.1) \qquad S(x) \equiv S_0, \qquad 0 < x < \epsilon,$$

for some sufficiently small positive ϵ. We then pass to the general case by a limiting argument.

Since D_δ is closed, the operators

$$\Delta^+ = D_\delta^* D_\delta, \qquad \Delta^- = D_\delta D_\delta^* \tag{4.2}$$

are nonnegative and self-adjoint. We will show that the resolvent powers $(\Delta^\pm + \lambda)^{-m}$ are trace class for appropriate m, and $\operatorname{tr}(\Delta^\pm + \lambda)^{-m}$ has an expansion in powers of λ and $\log \lambda$ as $\lambda \to +\infty$. By a familiar argument, the nonzero eigenvalues of Δ^+ and Δ^- coincide, counting multiplicities; for the maps

$$\varphi \to D_\delta \varphi, \qquad \psi \to D_\delta^* \psi$$

are injective between the corresponding eigenspaces. Thus

$$\operatorname{tr}(\Delta^+ + \lambda)^{-m} - \operatorname{tr}(\Delta^- + \lambda)^{-m} = \lambda^{-m} \operatorname{ind} D_\delta. \tag{4.3}$$

For this difference, all terms in the expansion as $\lambda \to +\infty$ are zero, except for the term in λ^{-m}, and the one gives the index.

The expansion of $\operatorname{tr}(\Delta + \lambda)^{-m}$ comes from a parametrix. For $0 < x < \epsilon$, (4.1) implies that

$$\Delta^\pm \simeq -\partial_x^2 + X^{-2}(S_0^2 \pm S_0) \tag{4.4}$$

with $S_0^2 \pm S_0 + 1/4 = (S_0 \pm 1/2)^2 \geq 0$. Denote by $T^\pm$ the operators in $L^2(\mathbf{R}_+, H)$ defined by the right hand side of (4.4), with the appropriate boundary conditions:

$$\text{For} \quad T^+ : u(x) = o(1) \quad \text{and} \quad u' + x^{-1} S_0 u = O(1).$$

$$\text{For} \quad T^- : u(x) = O(1) \quad \text{and} \quad -u' + x^{-1} S_0 u = o(1).$$

The resolvent for $T^\pm$ is obtained as a direct sum over $s \in \operatorname{spec} S_0$,

$$(T^\pm + \lambda)^{-1} = \otimes_s (L_s^\pm + \lambda)^{-1} \otimes \pi_s \tag{4.5}$$

where $L_s^{\pm}$ is the appropriate realization of $-\partial_x^2 + X^{-2}(s^2 \pm s)$, and π_s is the projection on the s-eigenspace of S_0. Set

$$(4.6) \qquad \nu_{\pm} = \nu_{\pm}(s) := \sqrt{s^2 \pm s + 1/4} = |s \pm 1/2|.$$

We generally suppress the dependence of $\nu_{\pm}$ on s to simplify notation.

LEMMA 4.1. *Let* $\operatorname{Im} z^2 \neq 0$ *and* $x \leq y$. *Then* $(L_s^+ + z^2)^{-1}$ *has the kernel*

$$(4.7a) \qquad (xy)^{1/2} I_{\nu_+}(xz) K_{\nu_+}(yz) \quad \text{if} \quad s \leq -1/2 \quad \text{or} \quad s \geq 0$$

and

$$(4.7b) \qquad (xy)^{1/2} I_{-\nu_+}(xz) K_{\nu_+}(yz) \quad \text{if} \quad -1/2 < s < 0,$$

whereas $(L_s^- + z^2)^{-1}$ *has the kernel*

$$(4.8a) \qquad (xy)^{1/2} I_{\nu_-}(xz) K_{\nu_-}(yz) \quad \text{if} \quad s < 0 \quad \text{or} \quad s \geq 1/2$$

and

$$(4.8b) \qquad (xy)^{1/2} I_{-\nu_-}(xz) K_{\nu_-}(yz) \quad \text{if} \quad 0 \leq s < 1/2.$$

Proof. We consider L_s^+ only; L_s^- is treated similarly. To compute the resolvent kernel we may apply Theorem 16 in [D+S], XIII. 3, i.e. if $\varphi(x, z)$, $\psi(x, z)$ denote the (up to constants) unique solutions of $(L_s^+ + z^2)u(x) = 0$ satisfying the boundary conditions at 0 and ∞, respectively, then

$$(L_s^+ + z^2)^{-1}(x, y) = (\varphi'\psi - \varphi\psi')^{-1}(x, z)\varphi(x, z)\psi(y, z),$$
$$0 < x < y < \infty.$$

The equation

$$(4.9) \qquad [-\partial_x^2 + x^{-2}(s^2 + s) + z^2]u(x) = 0, \qquad x \in \mathbf{R}^*,$$

has the general solution

$$u(x) = x^{1/2}(\alpha I_{\nu+}(xz) + \beta K_{\nu+}(xz)) \tag{4.10a}$$

or, if ν_+ is not an integer,

$$u(x) = x^{1/2}(\gamma I_{\nu+}(xz) + \delta I_{-\nu_+}(xz)). \tag{4.10b}$$

The unique solution satisfying the boundary condition at ∞ is

$$\psi(x, z) := x^{1/2} K_{\nu+}(xz).$$

Unless $-1/2 < s < 0$, the function

$$\varphi(x, z) := x^{1/2} I_{\nu\pm(s)}(xz)$$

satisfies the relevant boundary condition $\varphi(x) = o(1)$ and $(\partial_x + x^{-1}s)\varphi(x) = O(1)$ as $x \to 0$. Since

$$I_\nu'(x)K_\nu(x) - I_\nu(x)K_\nu'(x) = \frac{1}{x},$$

the Wronskian of φ and ψ is 1, and (4.7a) follows. When $-1/2 < s < 0$ then

$$\varphi(x, z) := x^{1/2} I_{-\nu_+}(xz)$$

solves the above boundary conditions. Since $K_\nu = K_{-\nu}$, the Wronskian calculation is the same as above, and we obtain (4.7b). □

Now we construct the parametrix for $(\Delta + \lambda)^{-m}$. In the interior, away from $x = 0$, there is the standard pseudodifferential parametrix for $(\Delta + \lambda)^{-m}$ ([G], [S]), which we denote by P_i. If φ and ψ are C^∞, vanishing near $x = 0$, with $\psi \equiv 1$ near supp φ, then

$$(\Delta + \lambda)^m \psi P_i \varphi = \varphi - R_i \quad \text{with} \quad \|R_i\|_{\text{tr}} = O(\lambda^{-k}) \tag{4.12}$$

where k can be arbitrarily large. Moreover, where $M \simeq (0, x_0) \times N$, P_i has a kernel $P_i(x, x'; y, y'; \lambda)dy'\,dy$ with an expansion (when $(x, x') = (y, y')$)

$$\text{(4.13)} \qquad \operatorname{tr} P_i(x, x'; x, x'; \lambda)dx'dx = \sum_j p_j(x, x')\lambda^{-j/2}dx'dx.$$

The expansion is uniform for x' in the cross section N and $x > \epsilon$, with any $\epsilon > 0$. We patch this together with a boundary parametrix as in (4.5). To control the remainder arising from the patching, we use:

LEMMA 4.2. *If $\varphi \in C_0^\infty(-1, y_0)$ and $\psi \in C_0^\infty(x_0, \infty)$ with $y_0 < x_0$, then for all j, i, m, k and λ large,*

$$\|\varphi \partial_x^j S_0^i (T + \lambda)^{-m} \varphi\|_{\mathrm{tr}} \leq C_{jimk}\lambda^{-k}.$$

Proof. For high eigenvalues $s \in \operatorname{spec} S_0$ we use the a priori estimates (3.5) in [B+S2]; we identify the L_a in those estimates with $L_s^\pm$ in (4.5), taking

$$\text{(4.14)} \qquad a = s^2 \pm s.$$

We will prove inductively that for $|s|$ sufficiently large, and ψ, φ satisfying the conditions of Lemma 4.2,

$$\text{(4.15)} \quad \|\psi(L_s^\pm + \lambda)^{-m}\varphi\| + \|\psi\partial(L_s^\pm + \lambda)^{-m}\varphi\| \leq C_{mk}(s^2 + \lambda)^{-k-n}.$$

We abbreviate $L_s^\pm$ to L, and write $L = -\partial_x^2 + X^{-2}a$ with a in (4.14). Since ψ and φ have disjoint supports,

$$(L + \lambda)\psi(L + \lambda)^{-1}\varphi = -\psi''(L + \lambda)^{-1}\varphi - 2\psi'\partial(L + \lambda)^{-1}\varphi.$$

Thus if $\psi_1 \in C_0^\infty(0, \infty)$ and $\psi_1 \equiv 1$ near supp ψ,

(4.16*a*)

$$\psi(L + \lambda)^{-1}\varphi = -[\psi_1(L + \lambda)^{-1}\psi_1][\psi''(L + \lambda)^{-1}\varphi + 2\psi'\partial(L + \lambda)^{-1}\varphi].$$

Similarly, since

$$\partial^2(L + \lambda)^{-1} = -I + (aX^{-2} + \lambda)(L + \lambda)^{-1}$$

and

$$(L + \lambda)\partial = \partial(L + \lambda) + 2aX^{-3},$$

we have

$$(4.16b)\quad \psi\partial(L + \lambda)^{-1}\varphi = -[\psi_1(L + \lambda)^{-1}\psi_1][\psi''\partial(L + \lambda)^{-1}\varphi + 2\psi'(aX^{-2} + \lambda)(L + \lambda)^{-1}\varphi + 2\psi aX^{-3}(L + \lambda)^{-1}\varphi].$$

From (3.5) in [B+S2], the following have bounds independent of a and λ:

$$a\|X^{-2-j}(L + \lambda)^{-1}X^j\|, \qquad \|\lambda X^{-j}(L + \lambda)^{-1}X^j\|,$$

$$a^{1/2}\|X^{j-1}\partial(L + \lambda)^{-1}X^{-j}\|, \qquad \lambda^{1/2}\|X^{-j}\partial(L + \lambda)^{-1}X^j\|,$$

for any fixed integer j. Since

$$a^k(L + \lambda)^{-k} = X^{2k}[aX^{-2k}(L + \lambda)^{-1}X^{2k-2}] \cdots [aX^{-2}(L + \lambda)^{-1}]$$

we find for ψ with compact support that

$$(4.17)\quad \|\psi a^k(L + \lambda)^{-k}\| \le C_k \quad\text{and}\quad \|\lambda^k(L + \lambda)^{-k}\| \le 1$$

and hence

$$\|\psi(L + \lambda)^{-k}\| \le C(a + \lambda)^{-k}.$$

Likewise

$$a^{k-1/2}\partial(L + \lambda)^{-k} = X^{2k-1}[X^{1-2k}a^{1/2}\partial(L + \lambda)^{-1}X^{2k-2}] \cdots [aX^{-2}(L + \lambda)^{-1}],$$

so

$$(4.18)\quad \|\psi\partial(L + \lambda)^{-k}\| \le C(a + \lambda)^{1/2-k}.$$

Now differentiate (4.16) $k-1$ times with respect to λ and apply (4.17), (4.18) to obtain

$$\|\psi(L+\lambda)^{-k}\varphi\| \le C(a+\lambda)^{-k-1/2}, \tag{4.17'}$$

$$\|\psi\partial(L+\lambda)^{-k}\varphi\| \le C(a+\lambda)^{-k}, \tag{4.18'}$$

when ψ, φ have disjoint supports and ψ vanishes near 0. The proof of (4.15) is completed by induction; in (4.16), use (4.17) in the first factor on the right, and successive improvements of (4.17′) and (4.18′) in the other factors.

It remains to obtain estimates like (4.15) for low eigenvalues s. There we use the kernels (4.7)-(4.8). From the asymptotics of the Bessel functions, and noting that $\psi \in C^\infty(x_0, \infty)$ and $\varphi \in C_0^\infty(-1, y_0)$ with $y_0 < x_0$, we can estimate the kernel of $\psi(L_s + z^2)^{-1}$ by

$$\begin{aligned} |\psi(x)(xy)^{1/2}K_\nu(xz)I_{\pm\nu}(yz)\varphi(y)| &\le C_\nu e^{-z(x-y)}\left(\frac{yz}{1+yz}\right)^{1/2\pm\nu} \\ &\le C_\nu e^{-z(x_0-y_0)} \end{aligned}$$

when $\psi(x)\varphi(y) \neq 0$; note that $-\nu$ occurs only when $\nu \le 1/2$. Similar estimates for the derivatives of the Bessel functions (see e.g. (3.11) in [B+S2]) yield the necessary inequalities for low eigenvalues, with exponential decay in $z = \sqrt{\lambda}$, proving (4.15) for all s, and λ large.

To complete the proof of Lemma 4.2 we need trace estimates. The operators $\psi_1(L+\lambda)^{-k}\psi_1$ are positive, with trace norm equal the trace. The estimate of these trace norms (and indeed the index calculation below) uses the Mellin transforms from [O, p. 123]:

$$\begin{aligned} &\int_0^\infty \zeta^w\left(-\frac{1}{2\zeta}\frac{\partial}{\partial\zeta}\right)^{m-1} I_\nu K_\nu(\zeta)\,d\zeta \\ &\qquad = \frac{\Gamma\left(\frac{w+1}{2}\right)\Gamma\left(m-1-\frac{w}{2}\right)\Gamma\left(\nu-m+\frac{w+3}{2}\right)}{4\sqrt{\pi}\,\Gamma\left(\nu+1+m-\frac{w+3}{2}\right)} \end{aligned} \tag{4.19a}$$

if $\nu \geq 0$ and $\max\{-1, 2m - 2\nu - 3\} < \operatorname{Re}(w) < 2m - 2$; and

$$(4.19b)\quad \int_0^\infty \zeta^w \left(-\frac{1}{2\zeta}\frac{\partial}{\partial\zeta}\right)^{m-1} I_{-\nu}K_\nu(\zeta)\,d\zeta = 2^{w-1}\frac{\Gamma\left(\frac{w+1}{2}\right)\Gamma(-w)\Gamma\left(\frac{w+1}{2}-\nu\right)}{\Gamma\left(\frac{1-w}{2}\right)\Gamma\left(\frac{1-w}{2}-\nu\right)}$$

if $0 \leq \nu < 1/2$ and $2m + 2\nu - 3 < 2m - 2$. So the trace norm of $\psi_1(L + \lambda)^{-k}\psi_1$ is, for the kernels with $I_\nu K_\nu$,

$$(4.20)\quad \frac{1}{(k-1)!}\int_0^\infty \psi_1(x)^2 x\left(-\frac{1}{2z}\frac{\partial}{\partial z}\right)^{k-1} I_\nu K_\nu(xz)\,dx$$

$$= \frac{z^{-2k}}{(k-1)!}\int_0^\infty \psi_1\left(\frac{\zeta}{z}\right)^2 \zeta^{2k-1}\left(-\frac{1}{2\zeta}\frac{\partial}{\partial\zeta}\right)^{k-1} I_\nu K_\nu(\zeta)\,d\zeta$$

$$\leq Cz^{-2k}\int_0^\infty \left(1+\frac{\zeta}{z}\right)^{-2} \zeta^{2k-1}\left(-\frac{1}{2\zeta}\frac{\partial}{\partial\zeta}\right)^{k-1} I_\nu K_\nu(\zeta)\,d\zeta$$

$$\leq Cz^{2-2k}\int_0^\infty \zeta^{2k-3}\left(-\frac{1}{2\zeta}\frac{\partial}{\partial\zeta}\right)^{k-1} I_\nu K_\nu(\zeta)\,d\zeta$$

$$= Cz^{2-2k}\frac{\Gamma(k-1)\Gamma(-5/2)}{4\sqrt{\pi}\nu}, \qquad \nu \neq 0.$$

Similar estimates hold for $\nu = 0$ (use a different power to estimate $\psi_1(\zeta/z)^2$), and for the kernels involving $I_{-\nu}$. Thus

$$\|\psi_1(L+\lambda)^{-k}\psi_1\|_{\mathrm{tr}} \leq C_k\lambda^{1-k}(1+|s|)^{-1}.$$

This with (4.16) gives (4.15) again, but now with trace norms:

$$\|\psi(L_s^\pm + \lambda)^{-m}\|_{\mathrm{tr}} + \|\psi\partial(L_s^\pm + \lambda)^{-m}\varphi\|_{\mathrm{tr}} \leq C_{mk}(s^2+\lambda)^{-k-n}.$$

Since the power k is arbitrary, we get the same inequalities with arbitrary powers of s on the left. Moreover, using $\partial_x^2(T+\lambda)^{-m} = -(T+\lambda)^{1-m} + (X^{-2}A+\lambda)(T+\lambda)^{-m}$, we get

$$\|\psi s^i \partial_x^j (L_s^{\pm} + \lambda)^{-m}\varphi\|_{\mathrm{tr}} \leq C_{ijkm}(s^2+\lambda)^{-k-n}. \tag{4.21}$$

Since S_0 is a first-order elliptic operator on the compact n-dimensional manifold N, then

$$\Sigma(1+|s|)^{-n-\delta} < \infty \tag{4.22}$$

for all $\delta > 0$. Thus Lemma 4.2 follows by summing (4.21) over $s \in \operatorname{spec} S_0$. □

To construct the parametrix for $(\Delta+\lambda)^{-m}$, choose φ in $C_0^\infty(-1, y_0)$ with $y_0 < \epsilon$ and $\varphi \equiv 1$ near $x = 0$; and ψ in $C_0^\infty(-1,\epsilon)$ with $\psi(x) \equiv 1$ for $0 \leq x \leq x_0$, where $x_0 > y_0$. Let $\varphi_i = 1 - \varphi$, and choose ψ_i in $C_0^\infty(M)$, vanishing near $x = 0$, with $\psi_i \equiv 1$ near supp φ_i. Let $P_i^{\pm}$ be an interior parametrix for $(\Delta^{\pm}+\lambda)^{-m}$ as in (4.12) above. With slight abuse of notation, and suppressing the superscript $\pm$, define

$$P = \psi(T+\lambda)^{-m}\varphi + \psi_i P_i \varphi_i. \tag{4.23}$$

Then $(\Delta+\lambda)^m P = I - R$, where

$$R = R_i + \sum_{j>0} c_{\alpha\beta\gamma jk}\psi^{(j)} X^{\alpha}\partial_x^{\beta} S_o^{\gamma}(T+\lambda)^{-k}\varphi.$$

By (4.12) and Lemma 4.2, $\|R\|_{\mathrm{tr}} \leq C\lambda^{-k}$. Hence for large λ

$$(\Delta+\lambda)^{-m} = P + P\sum_{j=1}^{\infty} R^j$$

and $\|P\,\Sigma_1^\infty R^j\|_{\mathrm{tr}} \leq C\lambda^{-k}$. So we may compute the asymptotics of $\mathrm{tr}(\Delta+\lambda)^{-m}$ from P. The interior term gives, by (4.13),

$$\sum_j \int_M \varphi_i p_j \lambda^{-j/2}.$$

We will show that the boundary contribution to $\operatorname{tr}(\psi(T+\lambda)^{-m}\varphi)$ has the form

$$\int_0^\infty \sigma(x, xz)\,dx,$$

with σ satisfying the conditions for the expansion theorem in [B+S1].

The operators $(T^\pm + \lambda)^{-1}$ have, on the diagonal $x = y$, the kernels given in Lemma 4.1: for $(T^+ + \lambda)^{-1}$

$$x\left[\bigoplus_{\substack{s<-1/2\\ s\geq 0}} I_{\nu_+}(xz)K_{\nu_+}(xz) + \bigoplus_{-1/2<s<0} I_{-\nu_+}(xz)K_{\nu_+}(xz)\right]$$

and for $(T^- + \lambda)^{-1}$

$$x\left[\bigoplus_{\substack{s\geq 1/2\\ s<0}} I_{\nu_-}(xz)K_{\nu_-}(xz) + \bigoplus_{0\leq s<1/2} I_{\nu_-}(xz)K_{-\nu_-}(xz)\right]$$

where $z^2 = \lambda$ and $\nu_\pm(s) = |s \pm 1/2|$ as in (4.6). Noting that

$$(T+z^2)^{-m} = \frac{1}{(m-1)!}\left(-\frac{1}{2z}\frac{\partial}{\partial z}\right)^{m-1}(T+z^2)^{-1},$$

and setting $xz = \zeta$, we are led to define formally

$$\text{(4.24}a\text{)}\quad \sigma_+(x,\zeta) = \frac{x^{2m-1}}{(m-1)!}\left(-\frac{1}{2\zeta}\frac{\partial}{\partial\zeta}\right)^{m-1} \times \left[\sum_{\substack{s\leq -1/2\\ s\geq 0}} I_{\nu_+}(\zeta)K_{\nu_+}(\zeta) + \sum_{-1/2<s<0} I_{-\nu_+}(\zeta)K_{\nu_+}(\zeta)\right]$$

$$\text{(4.24}b\text{)}\quad \sigma_-(x,\zeta) = \frac{x^{2m-1}}{(m-1)!}\left(-\frac{1}{2\zeta}\frac{\partial}{\partial\zeta}\right)^{m-1} \times \left[\sum_{\substack{s\geq 1/2\\ s<0}} I_{\nu_-}(\zeta)K_{\nu_-}(\zeta) + \sum_{0\leq s<1/2} I_{-\nu_-}(\zeta)K_{\nu_-}(\zeta)\right].$$

LEMMA 4.3. *If* $2m > n + 1$ *then each series* (4.24a, b) *converges to a* C^∞ *function for* $\zeta > 0$, *and*

$$\operatorname{tr}[\psi(T^\pm + z^2)^{-m}\varphi] = \int_0^\infty \varphi(x)\sigma_\pm(x, xz)\,dx. \tag{4.25}$$

Proof. Calculating as in (4.20), for $|s|$ so large that $\nu = |s \pm 1/2| \geq m - 1$, the positive operator $\psi(L_s^\pm + \lambda)^{-m}\psi$ has trace norm

$$\frac{1}{(m-1)!}\int_0^\infty \psi^2(x)x\left(-\frac{1}{2z}\frac{\partial}{\partial z}\right)^{m-1} I_\nu K_\nu(xz)\,dx$$

$$= \frac{z^{-2m}}{(m-1)!}\int_0^\infty \psi\left(\frac{\zeta}{z}\right)^2 \zeta^{2m-1}\left(-\frac{1}{2\zeta}\frac{\partial}{\partial \zeta}\right)^{m-1} I_\nu K_\nu(\zeta)\,d\zeta$$

$$\leq C_\theta z^{\theta-2m}\int_0^\infty \zeta^{2m-1-\theta}\left(-\frac{1}{2\zeta}\frac{\partial}{\partial \zeta}\right)^{m-1} I_\nu K_\nu(\zeta)\,d\zeta$$

$$\leq C_\theta z^{\theta-2m}\frac{\Gamma(\nu + 1 - \theta/2)}{\Gamma(\nu + \theta/2)} \leq C_\theta z^{\theta-2m}\nu^{1-\theta}$$

for $1 < \theta < 2m$. If $2m > n + 1$, we can choose $2m > \theta > n + 1$ and deduce from (4.22) that the sum over s of the terms in (4.26) is convergent. Further, each integrand in (4.26) is the restriction to the diagonal of the kernel of a positive operator, so the integrand is positive, hence the sum of the integrands in (4.26) is still positive with m replaced by $m + 1$, so

$$\left(-\frac{1}{2z}\frac{\partial}{\partial z}\right)^{m-1} I_\nu K_\nu(xz)$$

is a decreasing function of z, and it follows that (4.24a, b) converge uniformly; so do their derivatives. This proves Lemma 4.3. □

In order to expand $\int_0^\infty \sigma_\pm(x, xz)\,dx$ as $z \to +\infty$, we must verify the conditions (1.2a, b) in [B+S1]. The main point is an asymptotic expansion

$$\sigma_\pm(x, \zeta) \sim \sum_{j=1}^\infty \sigma_j^\pm(x)\zeta^{-j}, \qquad \zeta \to +\infty. \tag{4.27}$$

Now $\sigma_{\pm}(x, xz)$ gives the trace of the kernel of $(T^{\pm} + z^2)^{-m}$ on the diagonal. This kernel can be approximated by pseudodifferential methods. Denote by $p_{0,j}^{\pm}(x, x')dx'dx$ the forms (4.13) computed for

$$(\partial + X^{-1}S_0)^*(\partial + X^{-1}S_0) \quad \text{and} \quad (\partial + X^{-1}S_0)(\partial + X^{-1}S_0)^*.$$

Then

$$\sigma^{\pm}(1, \zeta) \sim \sum_j \int_N p_{0,j}^{\pm}(1, x')dx'\zeta^{-j}.$$

Since $\sigma_{\pm}(x, \zeta) = x^{2m-1}\sigma_{\pm}(1, \zeta)$, from (4.24) we get (4.27) with

$$\sigma_j^{\pm}(x) = x^{2m-1} \int_N p_{0,j}^{\pm}(1, x')dx'. \tag{4.28}$$

Now apply the expansion theorem of [B+S1]; note that $\varphi(x) \equiv 1$ near 0, and drop the "$\pm$":

$$\operatorname{tr} \varphi(\Delta + z^2)^{-m} \sim \int_0^\infty \varphi(x)\sigma(x, xz)dx + \sum_j \left(\int_M \varphi_i p_j\right) z^{-j}$$

$$\sim \sum_j \left(\int_M \varphi_i p_j\right) z^{-j}$$

$$+ \sum_j \int_0^\infty \varphi(x)\sigma_j(x)(xz)^{-j}dx \tag{4.29a}$$

$$+ \sum_{k=0}^{\infty} z^{-k-1} \int_0^\infty \frac{1}{k!} \zeta^k \sigma^{(k)}(0, \zeta)d\zeta \tag{4.29b}$$

$$+ \sum_{k=0}^{\infty} z^{-k-1} \log z\sigma_{k+1}^{(k)}(0)/k! \tag{4.29c}$$

with $\sigma^{(k)}(x, \zeta) = (\partial_x)^k\sigma(x, \zeta)$. In particular, there is precisely one logarithmic term in (4.29c) namely $z^{-2m} \log z$, and the coefficient of that term is

$$\sigma^{(2m-1)}(0)/(2m - 1)! = \int_N p_{0,2m}(1, x')dx'.$$

From (4.3),

$$\text{(4.30)}\qquad \operatorname{tr}[(\Delta^+ + z^2)^{-m} - (\Delta^- + z^2)^{-m}] = z^{-2m} \operatorname{ind} D_\delta$$

so the terms in $z^{-2m} \log z$ coming from Δ^+ and Δ^- must cancel:

$$\text{(4.31)}\qquad \int_N p_{0,2m}^+(1, x')dx' = \int_N p_{0,2m}^-(1, x')dx',$$

and so $\sigma_{2m}^+(x) = \sigma_{2m}^-(x)$. Hence in (4.29$a$)

$$\int_0^\infty \varphi(x)\sigma_{2m}^+(x)x^{2m-1}dx = \int_0^\infty \varphi(x)\sigma_{2m}^-(x)x^{2m-1}dx$$

and these two terms cancel from the expansion of (4.30), leaving

$$\text{(4.32)}\quad \operatorname{ind} D_\delta = \int_M \varphi_i(p_{2m}^+ - p_{2m}^-)$$

$$+ \int_0^\infty \frac{\zeta^{2m-1}}{(2m-1)!}[\sigma_+^{(2m-1)}(0, \zeta) - \sigma_-^{(2m-1)}(0, \zeta)]d\zeta.$$

In the integral over M, we have the usual locally computed "index form"

$$\omega_D := p_{2m}^+ - p_{2m}^-.$$

Near $x = 0$, $D_\delta \simeq \partial_x + X^{-1}S_0$, so $p_{2m}^\pm = p_{0,2m}^\pm$; hence from (4.31),

$$\int_N \omega_D(x, x')dx' = 0 \quad \text{for} \quad x \text{ near } 0.$$

It remains to compute the second integral in (4.32), which is defined by analytic continuation in the power of ζ. Define

$$h_\pm(w) = \int_0^\infty \frac{\zeta^w}{(2m-1)!}\sigma_\pm^{(2m-1)}(0, \zeta)d\zeta.$$

For a meromorphic function $h(w)$, denote by $\mathrm{Res}_k\, h(w_0)$ the coefficient of $(w - w_0)^{-k}$ in the Laurent expansion of h; we want

$$\mathrm{Res}_0[h_+ - h_-](2m - 1).$$

In view of (4.24), we decompose

$$h_\pm = h^1_\pm + h^2_\pm \tag{4.33}$$

where

$$h^1_+(w) = \sum_s \int_0^\infty \zeta^w \left(-\frac{1}{2\zeta}\frac{\partial}{\partial \zeta}\right)^{m-1} I_{\nu_+} K_{\nu_+}(\zeta)\, d\zeta/(m-1)!,$$

$$h^2_+(w) = \sum_{-1/2<s<0} \int_0^\infty \zeta^w \left(-\frac{1}{2\zeta}\frac{\partial}{\partial \zeta}\right)^{m-1} \times [I_{-\nu_+}(\zeta) - I_{\nu_+}(\zeta)] K_{\nu_+}(\zeta)\, d\zeta/(m-1)!$$

and h_- is similarly decomposed on the basis of (4.24*b*). From (4.19)

(4.34)

$$h^1_+(w) = \frac{\Gamma\left(\frac{w+1}{2}\right)\Gamma\left(m-1-\frac{w}{2}\right)}{4\sqrt{\pi}\,\Gamma(m)} \sum_s \frac{\Gamma\left(\nu_+ + \frac{w+3}{2} - m\right)}{\Gamma\left(1 + \nu_+ - \frac{w+3}{2} + m\right)}.$$

The sum (4.34) is analyzed in [B+S2], equations (7.12)-(7.16), where $z = (w + 1 - 2m)/2$. The analytic continuation is expressed in terms of the zeta function of a complex variable t,

$$\zeta_+(t) = \sum_{\nu_+ \neq 0} (\nu_+)^{-t} = \sum_{s \neq -1/2} \left| s + \frac{1}{2} \right|^{-t}, \tag{4.35}$$

as follows:

(4.36) $\mathrm{Res}_0 h^1_+(2m-1)$

$$= -\frac{1}{2}\,\mathrm{Res}_0\zeta+(-1) - \frac{1}{4}\sum_{k\geq 1}(-1)^k k^{-1} B_k\,\mathrm{Res}_1\zeta+(2k-1)$$

$$+\, c_m\,\mathrm{Res}_1\zeta+(-1)$$

where the B_k are Bernoulli numbers and

$$c_m = \frac{\Gamma'(-1/2)}{8\sqrt{\pi}} - \frac{\gamma}{4} + \frac{1}{4}\sum_1^{m-1}\frac{1}{j}, \tag{4.37}$$

with γ the Euler constant. For the correction term h^2_+ we have from (4.19a), continued analytically to $\nu_+ > -1/2$,

$$\frac{-1}{(m-1)!}\,\mathrm{Res}_0\left[\int_0^\infty \zeta^w\left(-\frac{1}{2\zeta}\frac{\partial}{\partial\zeta}\right)^{m-1} I_{\nu_+}K_{\nu_+}(\zeta)\,d\zeta\right]_{w=2m-1} = \frac{1}{2}\nu_+$$

and from (4.19b), for $\nu_+ < 1/2$

$$\frac{1}{(m-1)!}\,\mathrm{Res}_0\left[\int_0^\infty I_{-\nu_+}K_{\nu_+}(\zeta)\,d\zeta\right]_{w=2m-1} = \frac{1}{2}\nu_+.$$

Hence

$$\mathrm{Res}_0 h^2_+(2m-1) = \sum_{-1/2<s<0}\left(s+\frac{1}{2}\right). \tag{4.38}$$

The computation for h_- is the same, except for the possible occurrence of $\nu_-(s) = 1/2$ when $s = 0$. But then

$$I_{-1/2}K_{-1/2}(\zeta) = \frac{1}{2\zeta} + \frac{e^{-2\zeta}}{2\zeta}$$

and by the prescription of the Singular Asymptotics Lemma in [B+S1]

$$\int_0^\infty \zeta^w\left(-\frac{1}{2\zeta}\frac{\partial}{\partial\zeta}\right)^{m-1}\frac{1}{2\zeta}\,d\zeta = 0$$

while

$$\frac{1}{(m-1)!}\operatorname{Res}_0\left[\int_0^\infty \zeta^w\left(-\frac{1}{2\zeta}\frac{\partial}{\partial\zeta}\right)^{m-1}\frac{e^{-2\zeta}}{2\zeta}\,d\zeta\right]_{w=2m-1}=\frac{1}{4}.$$

Thus

$$\operatorname{Res}_0 h_-^2(2m-1)=\sum_{0\le s<1/2}\left(\frac{1}{2}-s\right).$$

Thus the second integral in the index formula (4.32) is

$$(4.39)\qquad \int_0^\infty \frac{\zeta^{2m-1}}{(2m-1)!}\left[\sigma_+^{(2m-1)}(0,\zeta)-\sigma_-^{(2m-1)}(0,\zeta)\right]d\zeta$$

$$=\operatorname{Res}_0(h_+^1+h_+^2-h_-^1-h_-^2)(2m-1)$$

$$=-\frac{1}{2}\operatorname{Res}_0(\zeta_+-\zeta_-)(1)-\frac{1}{4}\sum_{k\ge 1}(-1)^k k^{-1}B_k\operatorname{Res}_1(\zeta_+-\zeta_-)(2k-1)$$

$$+\,c_m\operatorname{Res}_1(\zeta_+-\zeta_-)(-1)+\sum_{-1/2<s<0}\left(s+\frac{1}{2}\right)-\sum_{0\le s<1/2}\left(\frac{1}{2}-s\right)$$

with

$$(4.40)\qquad \zeta_\pm(t)=\sum_{|s\pm 1/2|\neq 0}\left|s\pm\frac{1}{2}\right|^{-t}.$$

We shall see from (4.41) below that $\operatorname{Res}_1(\zeta_+-\zeta_-)(-1)$ is the residue of the eta function of S_0 at the origin, which is known to be zero for a differential operator S_0; but this vanishing can be deduced from (4.39) and (4.32). For, the first integral in (4.32) gives the coefficient of λ^{-m} in the expansion of $\operatorname{tr}\varphi_i[(\Delta^+ +\lambda)^{-m}-(\Delta^- +\lambda)^{-m}]\psi_i$, which is independent of m as one sees by differentiating the expansion with respect to λ. (Note that this expansion has no term in $\lambda^{-m}\log\lambda$.) The second integral is given by the right hand side of (4.39), where the only term depending on m is the one with c_m: since c_m in (4.37) varies with m its coefficient in (4.39) must be zero.

We next relate the difference $\zeta_+ - \zeta_-$ in (4.39) to the eta function of S_0:

$$\eta(z) = \sum_{\substack{s \in \operatorname{spec} S_0 \\ s \neq 0}} |s|^{-z} \operatorname{sgn} s.$$

Introduce

$$\tilde{\zeta}_\pm(z) := \sum_{\substack{s \in \operatorname{spec} S_0 \\ |s| > 1/2}} |s \pm 1/2|^{-z}.$$

For Re z large,

$$(4.41)\quad \tilde{\zeta}_+(z) - \tilde{\zeta}_-(z) = \sum_{\substack{s \in \operatorname{spec} S_0 \\ |s| > 1/2}} (|s + 1/2|^{-z} - |s - 1/2|^{-z})$$

$$= \sum_{\substack{s \in \operatorname{spec} S_0 \\ |s| > 1/2}} |s|^{-z}((1 + 1/2s)^{-z} - (1 - 1/2s)^{-z})$$

$$= 2 \sum_{\substack{s \in \operatorname{spec} S_0 \\ |s| > 1/2}} |s|^{-z} \sum_{k \geq 0} \binom{-z}{2k+1} (2s)^{-2k-1}$$

$$= \sum_{k \geq 0} 2^{-2k} \binom{-z}{2k+1} \sum_{\substack{s \in \operatorname{spec} S_0 \\ |s| > 1/2}} |s|^{-z-2k-1} \operatorname{sgn} s$$

$$= \sum_{k \geq 0} 2^{-2k} \frac{(-z)(-z-1)\cdots(-z-2k)}{(2k+1)!} \eta_{S_0}(z + 2k + 1)$$

$$- \sum_{k \geq 0} 2^{-2k} \frac{(-z)(-z-1)\cdots(-z-2k)}{(2k+1)!} \sum_{0 < |s| \leq 1/2} |s|^{-z-2k-1} \operatorname{sgn} s.$$

Since S_0 is a first order elliptic differential operator we know e.g. from [G], Lemma 1.10.3 that η_{S_0} is meromorphic in the whole complex plane with possibly simple poles at $n, n-1, \cdots, n = \dim N$, and holomorphic at $z = 0$ and in Re $z > n$. In particular, the η-invariant of S_0,

$$\eta_{S_0} := \eta_{S_0}(0),$$

is well defined. The right hand side of (4.19) can then be written as a finite sum plus a remainder holomorphic in $\operatorname{Re} z > -2$ and vanishing at $z = -1$. This gives

$$\text{(4.42}a\text{)} \quad \operatorname{Res}_0(\zeta_+ - \zeta_-)(-1) = \sum_{|s|<1/2} [s + 1/2 - (1/2 - s)] + \dim\ker(S_0 - 1/2) - \dim\ker(S_0 + 1/2) + \eta_{S_0} - \sum_{0<|s|\le 1/2} \operatorname{sgn} s + \sum_{k\ge 1} \frac{2^{-2k}}{2k(2k+1)} \operatorname{Res}_1 \eta_{S_0}(2k)$$

and for $j \geq 1$

$$\text{(4.42}b\text{)} \quad \operatorname{Res}_1(\zeta_+ - \zeta_-)(2j-1) = \sum_{k\ge 0} 2^{-2k} \operatorname{Res}_1\left(\binom{-z}{2k+1}\eta_{S_0}(z + 2k + 1)\right)_{z=2j-1} = -\sum_{k\ge 0} 2^{-2k}\binom{2k+2j-1}{2k+1} \operatorname{Res}_1 \eta_{S_0}(2j + 2k).$$

Also,

$$\text{(4.42}c\text{)} \qquad \operatorname{Res}_1(\zeta_+ - \zeta_-)(-1) = \operatorname{Res}_1 \eta_{S_0}(0).$$

This gives, using (4.39)

$$\text{(4.42}c\text{)} \quad \int_0^\infty \frac{\zeta^{2m-1}}{(2m-1)!} [\sigma_+^{(2m-1)}(0,\zeta) - \sigma_-^{(2m-1)}(0,\zeta)]\, d\zeta = -\frac{1}{2}(\eta_{S_0} + \dim\ker S_0) + \sum_{k\ge 1} \alpha_k \operatorname{Res}_1 \eta_{S_0}(2k)$$

as the second contribution in the index formula (4.32). The coefficients α_k can be computed from (4.39) and (4.42a, b); they are independent of S_0. The residues are given by "local" formulae ([G], Lemma 1.10.3) unlike η_{S_0} and dim ker S_0. For the classical geometric operators, they vanish, according to [A+P+S].

Summing up, we have proved:

LEMMA 4.4. *Suppose that $S_1(x) \equiv 0$ for $x \leq \epsilon$. Then*

(4.43)

$$\operatorname{ind} D_\delta = -\frac{1}{2}(\eta_{S_0} + \dim\ker S_0) + \sum_{k\geq 1} \alpha_k \operatorname{Res}_1 \eta_{S_0}(2k) + \int_{M_\epsilon} \omega_D$$

where ω_D is the usual locally computable index form for D, and $M_\epsilon = M\backslash\{x \leq \epsilon\}$.

Suppose now that we are in the general case, where $S_1(x)$ need not be zero for small x; we then obtain the index formula by a limiting process. Choose $\psi \in C^\infty(\mathbf{R})$ with $\psi(x) = 1$ if $x \geq -1$ and $\psi(x) = 0$ if $x \leq -2$. Put

$$\psi(x) := \psi(nx - 3),$$

so that $\psi_n(x) = 1$ if $x \geq 2/n$ and $\psi_n(x) = 0$ if $x \leq 1/n$. The operators D_n defined by

$$D_n = D \quad \text{on} \quad M\backslash U,$$

$$D_n \simeq \partial_x + X^{-1}S_0 + X^\beta \psi_n(x) S_1(x)$$

satisfy the same assumptions as D, and Lemmas 2.7 and 4.5 give

(4.44)

$$\operatorname{ind} D_\delta = \operatorname{ind} D_{n,\delta}$$

$$= -\frac{1}{2}(\eta_{S_0} + \dim\ker S_0) + \sum_{k\geq 1} \alpha_k \operatorname{Res}_1 \eta_{S_0}(2k) + \int_{M_{1/n}} \omega_n$$

where ω_n is the index form for D_n. Denote by ω_0 the index form for $\partial_x + X^{-1}S_0$. Then $\omega_n = \omega_0$ for $x < 1/n$, and as we noted after (4.32),

$$\int_N \omega_0(x, x')dx' = 0.$$

Thus the integral of ω_n in (4.44) can be written as

$$\int_{M\setminus U} \omega_n + \int_0^1 \int_N \omega_n(x, x')dx'dx. \tag{4.45}$$

Moreover, since the coefficients of D_n converge to those of D in C^∞ on each compact subset, $\omega_n \to \omega_D$ pointwise, and uniformly on compact subsets. Thus to pass to the limit as $n \to \infty$ in (4.45) we need only:

LEMMA 4.5. *Suppose that each coefficient $a(x, x')$ in the differential operator $S_1(x)$ satisfies*

$$|x^k \partial_x^k \partial_{x'}^\ell a(x, x')| = O(1) \tag{4.46}$$

uniformly for x in I and x' in compact subsets of the local coordinate patch on N. Then, with the same uniformity,

$$|\omega_n(x, x') - \omega_0(x, x')| \le Cx^\beta \tag{4.47}$$

with C independent of n.

Proof. The cut-off functions $\psi_n(x) = \psi(nx - 3)$ satisfy

$$|x^k \partial_x^k \psi_n(x)| \le C_k$$

uniformly in n, so if we replace $S_1(x)$ by $\psi_n(x)S_1(x)$ then (4.46) remains valid uniformly in n. Thus it is enough to show the constant C in (4.47) depends only on the constants implied in (4.46); so our notation ignores the dependence on n. We obtain uniformity in x by rescaling to $x = 1$. Near $x = 0$,

$$D \simeq \partial_x + X^{-1}S(x) \quad \text{with} \quad S(x) = S_0 + X^{\beta+1}S_1(x).$$

For $c < 1$, set $D_c \simeq \partial_x + x^{-1}S(cx)$, and let $\Delta_c^\pm$ be the corresponding Laplaceans. We have locally computable forms $\omega_c^\pm(x, x')dx'dx$ such that for φ in $C_0^\infty(0, 1)$

$$\operatorname{tr} \varphi(\Delta_c^{\pm} + \lambda)^{-m} \sim \lambda^{-m} \int_0^\infty \int_N \omega_c^{\pm}(x, x')dx'\varphi(x)dx$$

$$+ \text{ other powers of } \lambda.$$

The change of variable $x = cy$ converts $\partial_x + X^{-1}S(x)$ to $c^{-1}[\partial_y + Y^{-1}S(cy)]$, hence

$$\operatorname{tr} \varphi(c^{-2}\Delta_c^{\pm} + \lambda)^{-m} \sim \lambda^{-m} \int_0^\infty \int_N \omega_1^{\pm}(cy, x')dx'\varphi(y)dy + \text{other powers.}$$

Replacing λ by $c^{-2}\lambda$ and comparing these two expansions, we find

$$\omega_c^{\pm}(y, x') = c\omega_1^{\pm}(cy, x') =: c\omega^{\pm}(cy, x').$$

Set $y = 1$, and find that the index form $\omega = \omega^+ - \omega^-$ for D satisfies

$$\omega(0, x') = c^{-1}\omega_c(1, x') \tag{4.48}$$

where ω_0 is the form for D_0. Thus

$$\omega(c, x') - \omega_0(c, x') = c^{-1}[\omega_c(1, x') - \omega_0(1, x')]$$

and it suffices to show that

$$\frac{\partial}{\partial c}\, \omega_c(1, x') = O(c^{\beta}). \tag{4.49}$$

Now let $\sigma(S(cx)) = \sigma(S_0) + (cx)^{\beta+1}\sigma(S_1(cx)) =: \sigma_{c1} \cdot \xi' + \sigma_{c0}$, where ξ' denotes the cotangent variables dual to x', and $\cdot$ denotes the scalar product. The usual pseudodifferential parametrices for $(\Delta_c^{\pm} + \lambda)^{-1}$, differentiated $m - 1$ times with respect to λ, give

$$\omega_c(x, x') = \int\int Q[\xi, p_2, \partial^{\alpha}x^{-1}\sigma_{c1} \cdot \xi', \partial^{\gamma}x^{-1}\sigma_{c0}]d\xi d\xi' \tag{4.50}$$

where $p_2 = [\xi^2 + 1 + x^{-2}(\sigma_{c1} \cdot \xi')^2]^{-1}$, $\partial = \partial_{x,x'}$, and Q is a polynomial such that the integral (4.50) converges. The conditions (4.46) give

$$\frac{\partial}{\partial c}(\partial_x)^k(\partial_{x'})^\alpha x^{-1}\sigma_{cj} = O(c^\beta)$$

for x near 1. This with (4.50) proves (4.49), hence the lemma. □

Thus we may pass to the limit in (4.45) and (4.44) to obtain the following index theorem:

THEOREM 4.1. *Assume that D satisfies* (RS1) *to* (RS4) *and that the assumption* (4.46) *is satisfied. Then D_δ is a Fredholm operator with index*

(4.51)

$$\operatorname{ind} D_\delta = \int_M \omega_D - \frac{1}{2}(\eta_{S_0} + \dim\ker S_0) + \sum_{k\geq 1}\alpha_k \operatorname{Res}_1 \eta_{S_0}(2k)$$

where

$$\int_M \omega_D = \lim_{\epsilon\to 0}\int_{x>\epsilon}\omega_D.$$

If D_V denotes the closed extension of D corresponding to V as in Theorem 3.1, *then D_V is also Fredholm and*

(4.52)

$$\operatorname{ind} D_V = \int_M \omega_D - \frac{1}{2}(\eta_{S_0} + \dim\ker S_0) + \sum_{k\geq 1}\alpha_k \operatorname{Res}_1 \eta_{S_0}(2k)$$
$$+ \dim V - \sum_{-1/2<s<0} \dim\ker(S_0 - s).$$

5. Applications of Theorem 4.1 will be given to the Gauß-Bonnet and the Signature operators on manifolds with asymptotically cone-like singularities. By this we mean Riemannian manifolds M which possess an open subset U such that $M\backslash U$ is a smooth compact manifold with boundary and U is isometric to $(0, \epsilon) \times N$, where N is a smooth compact manifold of dimension n, with metric

(5.1) $$g_M = dx^2 + x^2 g_N(x), \qquad x \in (0, \epsilon),$$

where $g_N(x)$ is a family of Riemannian metrics on N and smooth on $[0, \epsilon)$. We denote by Ω^p the smooth p-forms and by Ω_0^p those with compact support. With $I := (0, \epsilon)$ we define a bijective map

$$\begin{aligned} &\psi_p : C_0^\infty(I, \Omega^{p-1}(N) \oplus \Omega^p(N)) \to \Omega_0^p(U), \\ (5.2)\quad &(\phi_{p-1}, \phi_p) \mapsto x^{p-1-n/2}\pi^*(\phi_{p-1}(x)) \wedge dx + x^{p-n/2}\pi^*(\phi_p(x)), \end{aligned}$$

where $\pi : I \times N \to N$ is the projection on the second factor and x is the canonical coordinate on I. Denoting by $*$ and $*_x$ the Hodge operator on U and on N (with respect to the metric $g_N(x)$), respectively, one computes that

$$(5.3)\qquad *\psi_p(\phi_{p-1}, \phi_p) = \psi_{n+1-p}(*_x\phi_p, (-1)^{n+1-p}*_x\phi_{p-1})$$

and

(5.4)

$$\|\psi_p(\phi_{p-1}, \phi_p)\|^2_{L^{2,p}(U)} = \int_0^1 [\|\phi_{p-1}(x)\|^2_{L^{2,p-1}(N_x)} + \|\phi_p(x)\|^2_{L^{2,p}(N_x)}]dx,$$

where $L^{2,p}$ denotes the completion of Ω_0^p with respect to the scalar product defined by the metric.

Next we find that with d, d_N the exterior derivative on U, N

$$\begin{aligned} (5.5)\quad &d\psi_p(\phi_{p-1}, \phi_p) \\ &= \psi_{p+1}((-1)^p[\partial_x + (p - n/2)x^{-1}]\phi_p + x^{-1}d_N\phi_{p-1}, x^{-1}d_N\phi_p) \end{aligned}$$

and with similar notation

$$\begin{aligned} (5.6)\quad &\delta\psi_p(\phi_{p-1}, \phi_p) \\ &= \psi_{p-1}(x^{-1}\delta_{N,x}\phi_{p-1}, (-1)^p[\partial_x + (n/2 - p + 1)x^{-1}]\phi_{p-1} + x^{-1}\delta_{N,x}\phi_p) \\ &\qquad + \psi_{p-1}(0, (-1)^{n+1+np}[*_x, \partial_x]*_x\phi_{p-1}). \end{aligned}$$

Here $[*_x, \partial_x]$ denotes the commutator of operators on $C_0^\infty(I, \Omega(N))$, where $\Omega(N) := \bigoplus_{p\geq 0} \Omega^p(N)$. Note that

$$(5.7) \qquad b_p(x) := [*_x, \partial_x]*_x : C_0^\infty(I, \Omega^p(N)) \to C_0^\infty(I, \Omega^p(N))$$

is a differential operator of order 0 with coefficients depending smoothly on $x \in [0, \epsilon)$.

Now assume $n + 1$ *even*. The Gauß-Bonnet operator on M is

$$(5.8) \qquad D_{GB} := d + \delta : \Omega^{\mathrm{ev}}(M) \to \Omega^{\mathrm{odd}}(M),$$

where Ω^{ev}, Ω^{odd} denotes even and odd forms, respectively. Introducing

$$(5.9a) \quad \psi_{\mathrm{ev}} : C_0^\infty(I, \Omega(N)) \to \Omega_0^{\mathrm{ev}}(U),$$

$$(\phi_0, \ldots, \phi_n) \mapsto (\psi_0(0, \phi_0), \psi_2(\phi_1, \phi_2), \ldots, \psi_{n+1}(\phi_n, 0)),$$

$$(5.9b) \quad \psi_{\mathrm{odd}} : C_0^\infty(I, \Omega(N)) \to \Omega_0^{\mathrm{odd}}(U),$$

$$(\phi_0, \ldots, \phi_n) \mapsto (\psi_1(\phi_0, \phi_1), \psi_3(\phi_2, \phi_3), \ldots, \psi_n(\phi_{n-1}, \phi_n)),$$

a straightforward computation using (5.5) and (5.6) shows that on $C_0^\infty(I, \Omega(N))$

$$\tilde{T}_{GB} := \psi_{\mathrm{odd}}^{-1} D_{GB} \psi_{\mathrm{ev}} = \partial_x + x^{-1}\tilde{S}(x),$$

where $\tilde{S}(x)$ is the operator

(5.10)

$$\begin{pmatrix} c_0 & \delta_{N,x} & 0 & \cdots & 0 \\ d_N & c_1 & & & \vdots \\ 0 & & & & 0 \\ \vdots & & & c_{n-1} & \delta_{N,x} \\ 0 & \cdots & 0 & d_N & c_n \end{pmatrix} + x \begin{pmatrix} 0 & & & 0 \\ & b_1(x) & & \\ & & 0 & \\ & & & \ddots \\ 0 & & & & 0 \\ & & & & & b_n(x) \end{pmatrix}$$

with

$$c_p := (-1)^p(p - n/2). \tag{5.11}$$

Now $\tilde{T}_{GB}$ does not yet satisfy the assumptions of Section 1 since we do not have a fixed Hilbert space fiber in (5.4). To achieve this we denote by $\langle \cdot | \cdot \rangle_{p,x}$ the scalar product defined by $g_N(x)$ on $\Omega^p(N)$. Then we can write

$$\langle v | w \rangle_{p,x} = \langle A_p(x) v | w \rangle_{p,0}$$

where $A_p(x)$ is a family of bounded positive definite operators with respect to $\langle \cdot | \cdot \rangle_{p,0}$. Moreover, $A_p(x)$ is smooth in $[0, \epsilon)$ and satisfies $A_p(0) = \mathrm{Id}$. We put

$$\begin{gathered} R : C_0^\infty(I, \Omega(N)) \to C_0^\infty(I, \Omega(N)), \\ R(\phi_0, \ldots, \phi_n)(x) := (A_0(x)^{-1/2}\phi_0(x), \ldots, A_n(x)^{-1/2}\phi_n(x)). \end{gathered} \tag{5.12}$$

Then

$$T_{GB} := R^{-1}\tilde{T}_{GB}R$$

is defined in $L^2(I, \oplus_{p\geq 0} L^{2,p}(N))$ with domain $C_0^\infty(I, \Omega(N))$ where $L^{2,p}(N)$ now denotes the L^2 structure on $\Omega^p(N)$ defined by $g_N(0)$. Unless otherwise stated all geometric quantities on N will now be computed with respect to the metric $g_N(0)$. From (5.10) we obtain

$$T_{GB} = \partial_x + x^{-1}(S_0 + xS_1(x))$$

where

$$S_0 = \begin{pmatrix} c_0 & \delta_N & 0 & & \cdots & 0 \\ d_N & c_1 & & & & \vdots \\ 0 & & & & & 0 \\ \vdots & & & & c_{n-1} & \delta_N \\ 0 & \cdots & & 0 & d_N & c_n \end{pmatrix} \tag{5.13}$$

is clearly a symmetric first order elliptic differential operator on $\Omega(N)$ and $S_1(x)$ is a smooth family of first order differential operators on $\Omega(N)$ with smooth coefficients in $[0, \epsilon)$. So T_{GB} is well defined with domain $C_0^\infty(I, \oplus_{p\geq 0} H^{1,p}(N))$ where $H^{1,p}(N)$ is the space of p forms with square integrable derivatives of order ≤ 1.

To determine the closed extensions of D_{GB} in $\oplus_{p\geq 0} L^{2,2p}(M)$ and their indices we have to investigate the spectrum of S_0. We denote by Δ_p the (positive) Laplacian on p-forms, and by

$$H^p_{\lambda,\mathrm{ccl}}(N) := \{\omega \in \Omega^p(N) \mid \Delta_p \omega = \lambda\omega,\ \delta_N \omega = 0\}$$

the space of coclosed eigenfunctions of Δ_p with eigenvalue λ.

LEMMA 5.1. *$\mu \in \operatorname{spec} S_0$ iff*

$$(\mu - c_p)(\mu - c_{p+1}) =: \lambda_p(\mu) \tag{5.14}$$

is an eigenvalue of Δ_p such that

$$H^p_{\lambda_p(\mu),\mathrm{ccl}}(N) \neq \{0\}. \tag{5.15}$$

1. *If $\mu \in \operatorname{spec} S_0$ and $\mu \notin \{c_o, \ldots, c_n\}$ then the multiplicity of μ is*

$$\sum_{p\geq 0} \dim H^p_{\lambda_p(\mu),\mathrm{ccl}}(N). \tag{5.16a}$$

2. *If $\mu = c_p$ for some p and $\mu \neq c_i$, $0 \leq i \leq p-1$, then the multiplicity of μ is*

$$\sum_{\substack{j\geq 0\\ j\neq p-1}} \dim H^j_{\lambda_j(\mu),\mathrm{ccl}}(N). \tag{5.16b}$$

Proof. 1. Let $\mu \in \operatorname{spec} S_0$, $\mu \notin \{c_o, \ldots, c_n\}$, and put $S_{0\mu} := \ker(S_0 - \mu)$. By elliptic regularity we have $S_{0\mu} \subset \Omega(N)$. We define a map

$$\psi : S_{0\mu} \to \Omega(N), \qquad \phi = \begin{pmatrix} \phi_0 \\ \vdots \\ \phi_n \end{pmatrix} \mapsto \begin{pmatrix} \psi(\phi)_o \\ \vdots \\ \psi(\phi)_n \end{pmatrix} = \psi(\phi), \tag{5.17a}$$

as follows:

(5.17b)

$$\psi(\phi)_0 := \phi_0, \qquad \psi(\phi)_p := \phi_p - (\mu - c_p)^{-1} d(\phi)_{p-1} \quad \text{if} \quad p > 0.$$

Then we claim that ψ is a bijection of $S_{0\mu}$ onto $\bigoplus_{p\geq 0} H^p_{\lambda_p(\mu),\mathrm{ccl}}(N)$. First we show that

$$\psi(\phi)_p \in H^p_{\lambda_p(\mu),\mathrm{ccl}}(N). \tag{5.18}$$

The proof of (5.18) is based on two observations. First suppose that $\phi \in S_{0\mu}$ and for some $p \geq 0$ $\phi_i = 0$ if $0 \leq i < p$. Then $S_0\phi = \mu\phi$ implies the equations

$$\begin{aligned} \delta\phi_p &= 0, \\ c_p\phi_p + \delta\phi_{p+1} &= \mu\phi_p, \\ d\phi_p + c_{p+1}\phi_{p+1} + \delta\phi_{p+2} &= \mu\phi_{p+1}. \end{aligned} \tag{5.19}$$

Hence ϕ_p is coclosed and

$$\Delta_p\phi_p = \delta d\phi_p = (\mu - c_{p+1})\delta\phi_{p+1} = \lambda_p(\mu)\phi_p.$$

Thus $\phi_p = \psi(\phi)_p \in H^p_{\lambda_p(\mu),\mathrm{ccl}}(N)$; in particular, this is always true if $p = 0$. Next let $\phi_p \in H^p_{\lambda_p(\mu),\mathrm{ccl}}(N)$ and define $\tilde{\phi}$ by

$$\tilde{\phi}_j := \begin{cases} \phi_p, & j = p, \\ (\mu - c_{p+1})^{-1} d\phi_p, & j = p+1, \\ 0 & \text{otherwise.} \end{cases}$$

Then it follows from the equations (5.19) that $\tilde{\phi} \in S_{0\mu}$; in fact

$$\delta\tilde{\phi}_p = \delta\phi_p = 0,$$

$$\begin{aligned} c_p\tilde{\phi}_p + \delta\tilde{\phi}_{p+1} &= c_p\phi_p + \delta(\mu - c_{p+1})^{-1} d\phi_p \\ &= c_p\phi_p + (\mu - c_{p+1})^{-1}\Delta_p\phi_p = \mu\phi_p = \mu\tilde{\phi}_p, \end{aligned}$$

$$d\tilde{\phi}_p + c_{p+1}\tilde{\phi}_{p+1} = d\phi_p + c_{p+1}(\mu - c_{p+1})^{-1}d\phi_p$$

$$= \mu(\mu - c_{p+1})^{-1}d\phi_p = \mu\tilde{\phi}_{p+1},$$

$$d\tilde{\phi}_{p+1} = 0.$$

Using these facts it follows easily by induction that for $\phi \in S_{0\mu}$ and $p \geq 0$

$$\psi(\phi)_p \in H^p_{\lambda_p(\mu),\mathrm{ccl}}(N)$$

and

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \psi(\phi)_{p+1} \\ \phi_{p+2} \\ \vdots \\ \phi_n \end{pmatrix} \in S_{0\mu}.$$

Hence (5.18) is proved. Moreover, the same arguments show that the map

$$\psi : S_{0\mu} \to \bigoplus_{p \geq 0} H^p_{\lambda_p(\mu),\mathrm{ccl}}(N)$$

is bijective, proving the assertion on the multiplicity of μ.

2. Now assume that $\mu = c_p$ but $\mu \neq c_i$, $0 \leq i \leq p - 1$. From the arguments above we conclude that the map

$$S_{0\mu} \ni \phi \mapsto \begin{pmatrix} \psi(\phi)_0 \\ \vdots \\ \psi(\phi)_{p-2} \end{pmatrix} \in \bigoplus_{i=0}^{p-2} H^i_{\lambda_i(\mu),\mathrm{ccl}}(N)$$

is surjective and that

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \psi(\phi)_{p-1} \\ \phi_p \\ \vdots \\ \phi_n \end{pmatrix} \in S_{0\mu}.$$

Writing out the eigenvalue equation as before we find that $\psi(\phi)_{p-1}$ is in the range of δ, while $\delta d\psi(\phi)_{p-1} = 0$, so

$$\psi(\phi)_{p-1} = 0.$$

Thus

$$\begin{pmatrix} 0 \\ \vdots \\ \phi_p \\ \vdots \\ \phi_n \end{pmatrix} \in S_{0\mu}$$

and we conclude as before that the map

$$S_{0\mu} \ni \phi \mapsto \begin{pmatrix} \psi(\phi)_0 \\ \vdots \\ \psi(\phi)_{p-2} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \psi \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \phi_p \\ \vdots \\ \phi_n \end{pmatrix} \in \bigoplus_{\substack{i \geq 0 \\ i \neq p-1}} H^i_{\lambda_i(\mu),\mathrm{ccl}}(N)$$

is bijective. The proof is complete. □

We can now investigate the small eigenvalues of S_0. Denote by λ_{pj}, $0 \leq p \leq n, j \geq 0$, the different eigenvalues of Δ_p with nontrivial coclosed eigenforms, where $\lambda_{pj} < \lambda_{p,j+1}$. Then $\mu \in \operatorname{spec} S_0$ iff for some p and j

$$\lambda_{pj} = (\mu - c_p)(\mu - c_{p+1})$$

or

$$(\mu - c_p)^2 + (c_p - c_{p+1})(\mu - c_p) = \lambda_{pj}.$$

Thus λ_{pj} generates two eigenvalues namely

$$\begin{aligned} \mu_{pj}^{\pm} &:= \frac{c_{p+1} + c_p}{2} \pm \sqrt{\lambda_{pj} + \left(\frac{c_p - c_{p+1}}{2}\right)^2} \\ &= \frac{(-1)^{p+1}}{2} \pm \sqrt{\lambda_{pj} + \left(p - \frac{n-1}{2}\right)^2}. \end{aligned} \tag{5.20}$$

If $\lambda_{pj} > 0$ it follows from Lemma 5.1 that assigning to $\mu_{pj}^{\pm}$ the multiplicity

$$m_{pj} := \dim H^p_{\lambda_{pj},\mathrm{ccl}}(N), \tag{5.21}$$

we obtain that for each eigenvalue $\mu \notin \{c_1, \ldots, c_n\}$

$$\text{multiplicity of } \mu = \sum_{\substack{p,j \\ \mu = \mu_{p,j}^{\pm}}} m_{pj}.$$

Now consider the eigenvalues $\lambda_{p0} = 0$ with nontrivial coclosed eigenspace $H^p(N)$. By Lemma 5.1 again they contribute to the eigenvalue c_p only, and with multiplicity

$$b_p := \dim H^p(N).$$

Moreover, if $p \neq (n-1)/2$ and $j \geq 0$ then

$$\mu_{pj}^{+} \quad \text{and} \quad \mu_{pj}^{-} \quad \text{have different signs} \tag{5.22a}$$

and

$$|\mu_{pj}^{\pm}| \geq 1/2. \tag{5.22b}$$

Therefore, eigenvalues with absolute value $<1/2$ can occur only if $p = (n-1)/2$. In that case we find that eigenvalues μ with $|\mu| < 1/2$ arise precisely from eigenvalues $\lambda_{(n-1)/2,j}$ with $0 < \lambda_{(n-1)/2,j} < 1$. For these j

(5.23a) $\mu^{\pm}_{(n-1)/2,j}$ have the same sign if $\lambda_{(n-1)/2,j} < 1/4$,

(5.23b) $\mu^{\pm}_{(n-1)/2,j}$ have different signs if $1/4 < \lambda_{(n-1)/2,j} < 1$,

(5.23c) the multiplicity of the eigenvalue 0 of S_0 is $\dim H^{(n-1)/2}_{1/4,\mathrm{ccl}}(N)$.

Thus we obtain from Theorem 3.2.

LEMMA 5.2. *A choice of boundary conditions for D_{GB} is necessary iff*

$$\operatorname{spec} \Delta_{(n-1)/2,\mathrm{ccl}} \cap (0, 1) \neq \emptyset.$$

Our next goal is the computation of the η-invariant η_{S_0}. We know from [G], Lemma 1.10.3 that the η-function $\eta_{S_0}(z)$ of S_0 is meromorphic in $\mathbf{C}$ and regular at $z = 0$, and by the previous discussion it is given for Re z large by

$$(5.24)\quad \eta_{S_0}(z) = \sum_{s \in \operatorname{spec} S_0 \setminus \{0\}} \operatorname{sgn} s\, |s|^{-z} = \sum_{p=0}^{n} \operatorname{sgn} c_p |c_p|^{-z} b_p$$
$$+ \Big[\sum_{\substack{0<\lambda_{(n-1)/2,j} \leq 1/4 \\ \mu_{(n-1)/2,j} \neq 0}} + \sum_{\lambda_{p,j} + (2p+1-n)^2/4 > 1/4} \Big]$$
$$\times\, m_{pj} (\operatorname{sgn} \mu^+_{pj} |\mu^+_{pj}|^{-z} + \operatorname{sgn} \mu^-_{p,j} |\mu^-_{p,j}|^{-z})$$
$$=: \eta_1(s) + \eta_2(s) + \eta_3(s).$$

Clearly, η_1 and η_2 are entire functions. The contribution of η_1 to η_{S_0} is given by (setting $n =: 2k+1$)

$$(5.25)\quad \begin{aligned} \eta_1(0) &= \sum_{p=0}^{n} \operatorname{sgn} c_p \cdot b_p \\ &= \sum_{p=0}^{k} (-1)^{p+1} b_p + \sum_{p=k+1}^{2k+1} (-1)^p b_p. \end{aligned}$$

To determine the contribution of η_2 we have to distinguish two cases.

Case 1. $(n-1)/2 = k$ is odd. Then if $0 < \lambda_{k,j} < 1/4$ the eigenvalues of S_0 between 0 and 1/2 are precisely

$$\mu_{k,j}^- = \frac{1}{2} - \sqrt{\lambda_{k,j}}.$$

If $\lambda_{k,j} = 1/4$ then $\mu_{k,j}^-$ is the zero eigenvalue and $\mu_{k,j}^+ > 0$ has the same multiplicity. Thus we obtain in this case

$$(5.26a) \qquad \eta_2(0) = \dim\ker S_0 + 2 \sum_{0<s<1/2} \dim\ker(S_0 - s).$$

Case 2. k is even. A similar discussion shows that in this case

$$(5.26b) \quad \eta_2(0) = -\dim\ker S_0 - 2 \sum_{-1/2<s<0} \dim\ker(S_0 - s).$$

We turn to the contribution of η_3. By the above, η_3 is meromorphic, and regular at $z = 0$. Writing $d_p := (p - (n-1)/2)^2$ and recalling $H^n_{\lambda,\mathrm{ccl}}(N) = \{0\}$ if $\lambda > 0$ we have for Re z large

$$(5.27) \quad \eta_3(z) = \sum_{\substack{0\le p\le n-1\\ \lambda_{p,j}+d_p^2>1/4}} (-1)^{p+1} m_{pj}[|1/2 + (\lambda_{p,j} + d_p^2)^{1/2}|^{-z} - |1/2 - (\lambda_{p,j} + d_p^2)^{1/2}|^{-z}] = \sum_{\substack{0\le p\le n-1\\ \lambda_{p,j}+d_p^2>1/4}} (-1)^{p+1} m_{pj} \times \sum_{k\ge 0} \binom{-z}{2k+1} 2^{-2k}(\lambda_{p,j} + d_p^2)^{1/2(-z-2k-1)}.$$

Denote by Q_p the orthogonal projection in $L^{2,p}(N)$ onto the space of co-closed forms and put

$$(5.28) \qquad \zeta_p(z) := \operatorname{tr} Q_p(\Delta_p + d_p^2)^{-z/2}.$$

It follows from standard arguments that ζ_p is meromorphic and holomorphic for Re z sufficiently large. From (5.27) it follows that

(5.29)

$$\eta_3(z) = \sum_{0 \le p \le n-1} (-1)^{p+1} \sum_{k=0}^{N} 2^{-2k} \binom{-z}{2k+1} \zeta_p(z+2k+1) + zR_N(z)$$

where R_N is holomorphic in Re $z > \alpha_N \to -\infty$. Thus we obtain

$$\begin{aligned}\eta_3(0) &= \sum_{p=0}^{n-1} (-1)^p \operatorname{Res}_1 \zeta_p(1) + \sum_{p=0}^{n-1} (-1)^p \sum_{k \ge 1} \beta_k \operatorname{Res}_1 \zeta_p(2k+1) \\ &=: \sum_{p=0}^{n-1} (-1)^p \operatorname{Res}_1 \zeta_p(1) + R.\end{aligned} \tag{5.30}$$

If we know that η_{S_0} is regular in Re $z > -1/2$ then (5.29) clearly implies that

$$z \sum_{p=0}^{n-1} (-1)^{p+1} \zeta_p(z+1)$$

is holomorphic in Re $z > -1/2$. Thus $R = 0$ in this case. Since the coefficients of $S_1(x)$ in T_{GB} are smooth in $[0, \epsilon)$ Theorem 4.1 applies and we can state the Gauß-Bonnet Theorem for manifolds with asymptotically cone-like singularities.

THEOREM 5.1. *Let M be a Riemannian manifold of dimension $n + 1 = 2k$ with asymptotically cone-like singularities. If k is odd then the maximal closed extension $D_{GB,\max}$ of the Gauß-Bonnet operator D_{GB} is a Fredholm operator with index*

$$\begin{aligned}\operatorname{ind} D_{GB,\max} &= \int_M \omega_{GB} + \sum_{k \ge 1} \left[\alpha_k \operatorname{Res}_1 \eta_{S_0}(2k) + \beta_k \sum_{p=0}^{n-1} (-1)^p \operatorname{Res}_1 \zeta_p(2k+1) \right] \\ &+ \frac{1}{2} \sum_{p=0}^{k} (-1)^p b_p + \frac{1}{2} \sum_{p=k+1}^{2k+1} (-1)^{p+1} b_p \\ &+ \frac{1}{2} \sum_{p=0}^{2k+1} (-1)^{p+1} \operatorname{Res}_1 \zeta_p(1).\end{aligned}$$

Here b_p is the p^{th} Betti number of N, ζ_p is defined in (5.28), *and the constants β_k are determined from* (5.29). ω_{GB} *denotes the Chern-Gauß-Bonnet form on M, and the integral exists in the sense of Theorem* 4.1. *If k is even the same formula holds for the index of $D_{GB,\min}$, the closure of D_{GB}.*

Proof. The proof follows from Theorem 4.1 and (5.25), (5.26*a*), (5.26*b*), and (5.30). □

The index of D_{GB} is also equal to the L^2-Euler characteristic of M. Assuming that η_{S_0} is regular in Re $z > -1/2$ the second sum on the right vanishes and the expression (5.31) thus gives essentially Cheeger's formula ([Che] Theorem 5.1). As a corollary it gives the Gauß-Bonnet theorem for manifolds with boundary and identifies the boundary contribution as a spectral invariant of the boundary; this is explained in [Che] Section 5. Note also that our approach expresses the boundary contribution by means of an η-invariant.

We now turn to the signature operator D_S. Assume that $n + 1 = \dim M =: 4k$ and denote by τ the involution of $\Omega(M) = \oplus_{p\geq 0}\Omega^p(M)$ which equals

$$(\sqrt{-1})^{2k+p(p-1)} * \quad \text{on} \quad \Omega^p(M).$$

Denoting by $\Omega^{\pm}(M)$ the eigenspace of τ with eigenvalue ± 1 we obtain the decomposition

$$\Omega(M) = \Omega^+(M) \oplus \Omega^-(M).$$

Now $d + \delta$ anticommutes with τ so

$$D_S := d + \delta : \Omega^+(M) \to \Omega^-(M)$$

defines a first order elliptic differential operator, the signature operator. With ψ_j as in (5.2) we introduce the bijections

$$\psi^{\pm} : C_0^\infty(I, \Omega(N)) \to \Omega_0^{\pm}(U),$$

$$(\psi^{\pm}(\phi_0, \cdots, \phi_{4k-1}))_j := \psi_j(\pm(-1)^{k+(1/2)j(j+1)} *_x \phi_{4k-j}, \phi_j),$$

$$0 \leq j \leq 4k,$$

and a map $\sigma : \Omega(N) \to \Omega(N)$,

$$(\sigma\phi)_j := (-1)^{k+(1/2)(4k-j)(4k-j-1)}\phi_j, \qquad 0 \leq j \leq 4k - 1.$$

Then a straightforward computation using (5.3), (5.5), and (5.6) shows that

$$(\psi^-)^{-1} D_S \psi^+ \sigma * =: \partial_x + x^{-1}(\tilde{S}_0(x) + x\tilde{S}_1(x))$$

where $\tilde{S}_i(x)$ is a first order differential operator on $\Omega(N)$ with smooth coefficients in $[0, \epsilon)$, $i = 0, 1$, and in particular for $\phi \in \Omega^j(N)$

(5.32)

$$\tilde{S}_0(x)\phi = \left(\frac{4k-1}{2} - j\right)\phi + (-1)^{k+1+[(j+1)/2]}((-1)^j *_x d_N - d_N *_x)\phi,$$

where $[(j+1)/2]$ denotes the greatest integer $\leq (j+1)/2$. Modifying ψ^- and $\psi^+\sigma*$ by R in (5.12) we obtain that

$$T := R^{-1}(\psi^-)^{-1} D_S \psi^+ \sigma * R$$

$$=: \partial_x + x^{-1}(S_0 + xS_1(x))$$

with domain $C_0^\infty(I, \Omega(N))$ in $L^2(I, H)$, where again $H = \oplus_{p\geq 0} L^{2,p}(N)$. Then T is unitarily equivalent to D_S on $\Omega_0^+(U)$ with respect to the L^2 structure defined by the metric of M. Here

$$S_0 = \tilde{S}_0(0)$$

and $S_1(x)$ is again a first order differential operator on $\Omega(N)$ with smooth coefficients in $[0, \epsilon)$. Note that S_0 differs only by a diagonal operator with constant coefficients from the operator introduced in [A+P+S], p. 63. Also it is easy to see that S_0 is a self-adjoint first order elliptic operator on $\Omega(N)$. To apply our Index Theorem in this case we have to investigate spec S_0. This analysis is very similar to the arguments given in Lemma 5.1 so we only sketch the proof of the following result. We denote by $H^p_{\lambda,\mathrm{cl}}(N)$ and $H^p_{\lambda,\mathrm{ccl}}(N)$ the spaces of closed and coclosed eigenforms of Δ_p on N with eigenvalue λ.

LEMMA 5.3. *Let* $b_{2j} := ((4k-1)/2 - 2j)$, $0 \leq j \leq 2k-1$, *and*

$$\alpha_i := \frac{1}{2}(b_{2k-2i-2} + b_{2k+2i}),$$

$$\gamma_i := \frac{1}{4}(b_{2k-2i-2} - b_{2k+2i})^2, \qquad 0 \leq i \leq k-1,$$

$$\beta_i := \frac{1}{2}(b_{2k-2i-2} + b_{2k+2i+2}),$$

$$\delta_i := \frac{1}{4}(b_{2k-2i-2} - b_{2k+2i+2})^2, \qquad 0 \leq i \leq k-2.$$

Then the spectrum of S_0 consists precisely of the following series:

(a) $\pm((\alpha_i/2) \pm \sqrt{\lambda + \gamma_i})$, *all with multiplicity* $\dim H^{2k-2i-2}_{\lambda,\mathrm{ccl}}(N)$ *for* $\lambda > 0$ *and* $0 \leq i \leq k-1$;

(b) $\pm((\beta_i/2) \pm \sqrt{\lambda + \gamma_i})$, *all with multiplicity* $\dim H^{2k+2i+2}_{\lambda,\mathrm{cl}}(N)$ *for* $\lambda > 0$ *and* $0 \leq i \leq k-2$;

(c) $\pm b_{2j}$, *both with multiplicity* $\dim H^{2j}(N)$, $0 \leq j \leq 2k-1$;

(d)

$$\pm\, b_{2k} + \sqrt{\lambda}, \qquad \textit{both with multiplicity } m_\lambda^+,$$

$$\pm\, b_{2k} - \sqrt{\lambda}, \qquad \textit{both with multiplicity } m_\lambda^-,$$

where $m_\lambda^\pm$ denotes the dimension of the ± 1 eigenspace of the involution $\lambda^{-1/2} d*$ on $H^{2k}_{\lambda,\mathrm{cl}}(N)$.

Proof. Observe first that $\Omega^{\mathrm{ev}}(N)$ and $\Omega^{\mathrm{odd}}(N)$ are invariant under S_0, inducing the decomposition $S_0 = S_0^{\mathrm{ev}} \oplus S_0^{\mathrm{odd}}$. Denoting by $\hat{S}_0^{\mathrm{ev}}$ the operator arising from S_0^{ev} by changing all b_{2j} to $-b_{2j}$ one checks that

$$S_0^{\mathrm{odd}} \epsilon * = \epsilon * \hat{S}_0^{\mathrm{ev}}$$

where $\epsilon : \Omega(N) \to \Omega(N)$ is given by $(\epsilon\phi)_j = (-1)^{[(j+1)/2]}\phi_j$. Hence it is sufficient to treat S_0^{ev}. Next we note that

$$\Omega_i^{\mathrm{ev}}(N) := \Omega^{2k-2i-2}_{\mathrm{ccl}}(N) \oplus \Omega^{2k-2i}(N) \oplus \cdots \oplus \Omega^{2k+2i}(N),$$

$0 \leq i \leq k-1$, is also invariant under S_0^{ev}. Denoting the restriction by $S_0^{\mathrm{ev},i}$ we prove the assertion by induction on i.

<u>$i = 0$</u> Let $\mu \neq b_{2k}$ be an eigenvalue of $S_0^{\mathrm{ev},0}$. This means that

$$(5.33a) \qquad (b_{2k-2} - \mu)\phi_{2k-2} - *d\phi_{2k} = 0,$$

$$(5.33b) \qquad (b_{2k} - \mu)\phi_{2k} + *d\phi_{2k-2} + d * \phi_{2k} = 0.$$

Eliminating $*d\phi_{2k}$ from (5.33a) by (5.33b) we see that

$$\phi_{2k-2} \in H^{2k-2}_{\lambda,\mathrm{ccl}}(N) \tag{5.34}$$

with

$$\lambda = \lambda(\mu) = (b_{2k-2} - \mu)(b_{2k} - \mu).$$

On the other hand, if ϕ_{2k-2} satisfies (5.34) then it is easy to see that

$$\phi_1 := \begin{pmatrix} \phi_{2k-2} \\ (\mu - b_{2k})^{-1} * d\phi_{2k-2} \end{pmatrix} \in S^{\mathrm{ev},0}_{0,\mu}$$

for $\mu = (\alpha_0/2) \pm \sqrt{\lambda + \gamma_0}$. Hence we may assume $\phi_{2k-2} = 0$ in (5.33). Then we must have

$$\phi_{2k} \in H^{2k}_{\lambda,\mathrm{cl}}(N) \tag{5.35}$$

where

$$\lambda = \lambda(\mu) = (b_{2k} - \mu)^2$$

and

$$\pm\phi_{2k} + \lambda^{-1/2} d * \phi_{2k} = 0 \tag{5.36}$$

with $\pm$ according to $b_{2k} - \mu = \pm\sqrt{\lambda}$. On the other hand, if ϕ_{2k} satisfies (5.35) and (5.36) then

$$\phi_2 := \begin{pmatrix} 0 \\ \phi_{2k} \end{pmatrix} \in S^{\mathrm{ev},0}_{0,\mu}$$

where

$$\mu = b_{2k} \mp \sqrt{\lambda}.$$

The eigenvalues $\mu \neq b_{2k}$ of $S_0^{\mathrm{ev},0}$ are therefore precisely the following:

(5.37a)
$$\frac{\alpha_0}{2} \pm \sqrt{\lambda + \gamma_0}, \qquad \text{both with multiplicity} \quad \dim H^{2k-2}_{\lambda,\mathrm{ccl}}(N), \qquad \lambda > 0;$$

(5.37b)
$$b_{2k-2} \quad \text{with multiplicity} \quad \dim H^{2k-2}(N),$$

(5.37c)
$$b_{2k} \pm \sqrt{\lambda} \quad \text{with multiplicity} \quad m_\lambda^{\pm}, \qquad \lambda > 0.$$

If $\mu = b_{2k}$ then we obtain from (5.33b) that ϕ_{2k-2} is closed, hence harmonic, and ϕ_{2k} is coclosed. But then we get from (5.33a) that ϕ_{2k} is also closed and $\phi_{2k-2} = 0$ since $b_{2k-2} - b_{2k} \neq 0$. Thus we find an additional eigenvalue

(5.37d)
$$b_{2k} \quad \text{with multiplicity} \quad \dim H^{2k}(N).$$

<u>$i \mapsto i + 1$</u> Using completely analogous arguments we find the following description of the spectrum of $S_0^{\mathrm{ev},i+1}$: it consists of

(5.38a)
$$\frac{\alpha_{i+1}}{2} \pm \sqrt{\lambda + \gamma_{i+1}}, \qquad \text{both with multiplicity} \quad \dim H^{2k-2i-4}_{\lambda,\mathrm{ccl}}(N) \quad \text{for } \lambda > 0;$$

(5.38b)
$$\frac{\beta_i}{2} \pm \sqrt{\lambda + \delta_i}, \qquad \text{both with multiplicity} \quad \dim H^{2k+2i+2}_{\lambda,\mathrm{cl}}(N) \quad \text{for } \lambda > 0;$$

(5.38c)
$$b_{2k-2i-4} \quad \text{with multiplicity} \quad \dim H^{2k-2i-4}(N),$$
$$b_{2k+2i+2} \quad \text{with multiplicity} \quad \dim H^{2k+2i+2}(N);$$

(5.38d)
$$\mu \quad \text{with multiplicity} \quad \dim S^{\mathrm{ev},i}_{0,\mu}.$$

The assertion of the lemma now follows inductively from (5.37) and (5.38). □

As an immediate consequence of Lemma 5.3 we see that eigenvalues μ of S_0 with $|\mu| < 1/2$ are of the form

$$(5.39a)\qquad -1/2 + \sqrt{\lambda} \quad \text{with multiplicity} \quad m_\lambda^+$$

and

$$(5.39b)\qquad 1/2 - \sqrt{\lambda} \quad \text{with multiplicity} \quad m_\lambda^-$$

for $0 < \lambda < 1$. This implies

LEMMA 5.4. *A choice of boundary conditions for D_S is necessary iff*

$$\bigoplus_{0<\lambda<1} H^{2k}_{\lambda,\mathrm{cl}}(N) \neq \{0\}.$$

It remains to study the η-function of S_0. Since the eigenvalues in (a), (b), (c) of Lemma 5.3 occur in pairs with opposite sign we have for Re z large ($b_{2k} = -1/2$)

$$(5.40)\quad \eta_{S_0}(z) = \sum_{0<\lambda<1/4} [m_\lambda^+(-|-1/2 + \sqrt{\lambda}|^{-z} + |1/2 + \sqrt{\lambda}|^{-z})$$

$$+ m_\lambda^-(-|-1/2 - \sqrt{\lambda}|^{-z} + |1/2 - \sqrt{\lambda}|^{-z})]$$

$$+ m_{1/4}^+ - m_{1/4}^- + \sum_{\lambda>1/4} (m_\lambda^+ - m_\lambda^-)$$

$$\times (|1/2 + \sqrt{\lambda}|^{-z} + |-1/2 + \sqrt{\lambda}|^{-z})$$

$$=: \eta_1(z) + m_{1/4}^+ - m_{1/4}^- + \eta_2(z).$$

Clearly, η_1 is entire and satisfies

$$(5.41)\qquad \eta_1(0) = 0.$$

The study of η_2 is analogous to that of η_3 in (5.23); we obtain the representation

$$\eta_2(z) = 2\bar{\eta}(z) - 2 \sum_{0<\lambda\leq 1/4} (m_\lambda^+ - m_\lambda^-)\lambda^{-z\,2} \tag{5.42}$$
$$+ \sum_{j=1}^{N} 2^{1-2j}\binom{-z}{2j}\bar{\eta}(z+2j) + zR_N(z)$$

where

$$\bar{\eta}(z) := \sum_{\lambda>0} (m_\lambda^+ - m_\lambda^-)\lambda^{-z/2} \tag{5.43}$$

and R_N is holomorphic in $|z| \leq c_N$ with $\lim_{N\to\infty} c_N = \infty$. The arguments given in Lemma 5.3 can be applied to the case that all $b_j = 0$ also. This gives

LEMMA 5.5. *$2\bar{\eta}$ is the η-function of N in the sense of* [A+P+S].

In particular, $\bar{\eta}$ is holomorphic in $\operatorname{Re} z > -1/2$ and we obtain from (5.39), (5.40), and (5.41)

$$\eta_{S_0}(0) = m_{1/4}^+ - m_{1/4}^- + \eta(N) - 2 \sum_{0<\lambda\leq 1/4} (m_\lambda^+ - m_\lambda^-).$$

As before, (4.31) is satisfied and Theorem 4.1 applies. If $D_{S,V}$ is the closed extension corresponding to the subspace V of $\oplus_{|\mu|<1/2} S_{0,\mu}$ then the singular contribution to the index formula is according to Theorem 4.2

$$-\frac{1}{2}(\eta_{S_0}(0) + \dim\ker S_0) + \dim V - \sum_{-1/2<\mu<0} \dim S_{0,\mu}$$
$$= -\frac{1}{2}\eta(N) - m_{1/4}^+ + \sum_{0<\lambda\leq 1/4} (m_\lambda^+ - m_\lambda^-)$$
$$+ \dim V - \sum_{-1/2<\mu<0} \dim S_{0,\mu}.$$

By (5.39) we have

$$\sum_{-1/2<\mu<0} \dim S_{0,\mu} = \sum_{0<\lambda<1/4} m_\lambda^+ + \sum_{1/4<\lambda<1} m_\lambda^-$$

hence the singular contribution becomes

$$-\frac{1}{2}\eta(N) + \dim V - \sum_{0<\lambda<1} m_\lambda^- .$$

Thus we can state the Signature Theorem for our case.

THEOREM 5.2. *Let M be a Riemannian manifold of dimension $n = 4k$ with asymptotically cone-like singularities. Then the closed extension $D_{S,V}$ of the signature operator corresponding to the eigenvalues* (5.39b) *is a Fredholm operator with index*

$$\operatorname{ind} D_{S,V} = \int_M \omega_S - \frac{1}{2}\eta(N).$$

Here ω_S is the Hirzebruch L_k-polynomial in the Pontrjagin classes of M, and the integral exists in the sense of Theorem 4.2.

INST. FÜR MATHEMATIK DER UNIVERSITÄT AUGSBURG
UNIVERSITY OF MASSACHUSETTS AT BOSTON

REFERENCES

[A+P+S] M. F. Atiyah, V. K. Patodi, and I. M. Singer, Spectral asymmetry and Riemannian geometry I. *Math. Proc. Camb. Philos. Soc.*, **77** (1975), 43-69.

[B+S1] J. Brüning and R. Seeley, Regular singular asymptotics. *Adv. Math.*, **58** (1985), 133-148.

[B+S2] ______ and ______, The resolvent expansion for second order regular singular operators, *J. Funct. Anal.*, **73** (1987), 369-429.

[Che] J. Cheeger, Spectral geometry of singular Riemannian spaces. *J. Differ. Geom.*, **18** (1983), 575-657.

[Cho] A. Chou, The Dirac operator on spaces with conical singularities and positive scalar curvature. *Trans. Am. Math. Soc.*, **289** (1985), 1-40.

[D+S] N. Dunford and J. Schwartz, *Linear Operators.* **I, II, III**. Interscience, New York 1963.

[G] P. Gilkey, *Invariance Theory, the Heat Equation, and the Atiyah-Singer Index Theorem*. Wilmington: Publish or Perish 1984.

[H+S] P. Halmos and V. Sunder, *Bounded Integral Operators on L^2 spaces*. Berlin, Springer 1978.

[O] F. Oberhettinger, *Tables of Mellin Transforms.* Berlin, *Springer 1974.*

[S] R. Seeley, Complex powers of an elliptic operator. *AMS Proceedings of Symposia in Pure Math*, **10**.

LINKED CITATIONS

You have printed the following article:

An Index Theorem for First Order Regular Singular Operators
Jochen Brüning; Robert Seeley
American Journal of Mathematics, Vol. 110, No. 4. (Aug., 1988), pp. 659-714.
Stable URL:
http://links.jstor.org/sici?sici=0002-9327%28198808%29110%3A4%3C659%3AAITFFO%3E2.0.CO%3B2-6

References

[Cho] **The Dirac Operator on Spaces with Conical Singularities and Positive Scalar Curvatures**
Arthur Weichung Chou
Transactions of the American Mathematical Society, Vol. 289, No. 1. (May, 1985), pp. 1-40.
Stable URL:
http://links.jstor.org/sici?sici=0002-9947%28198505%29289%3A1%3C1%3ATDOOSW%3E2.0.CO%3B2-4

NOTE: *The reference numbering from the original has been maintained in this citation list.*