AN INDUCTIVE JULIA-CARATHÉODORY THEOREM FOR PICK FUNCTIONS IN TWO VARIABLES

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Abstract. We study the asymptotic behavior of Pick functions, analytic functions which take the upper half plane to itself. We show that if a two variable Pick function f has real residues to order $2N-1$ at infinity and the imaginary part of the remainder between f and this expansion is of order $2N+1$, then f has real residues to order $2N$ and directional residues to order $2N+1$. Furthermore, f has real residues to order $2N+1$ if and only if the $2N+1$-th derivative is given by a polynomial, thus obtaining a two variable analogue of a higher order Julia-Carathéodory type theorem.

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1. Intoduction

The simplest form of classical Julia-Carathéodory theorem, given by Carathéodory[9] and Julia[11], follows.

Theorem 1.1 (Julia-Carathéodory theorem). *Let $f : \mathbb{D} \to \overline{\mathbb{D}}$ be an analytic function. The limit $\lim_{t\to 1} \frac{1-|f(t)|}{1-|t|}$ exists if and only if $\lim_{t\to 1} f(t)$ exists and the directional derivative at 1 exists for all directions pointing into the disk.*

The Julia-Carathéodory theorem was extended to higher derivatives by Bolotnikov and Kheifets in [7, 8] and earlier, on the upper half plane, by Nevanlinna in his solution of the Hamburger moment problem[12]. There has been some effort to prove an analogue of the Julia-Carathéodory theorem in several variables in the works of Abate [1, 2], Agler McCarthy, Young [4], Jafari [10], and Wlodarczyk [15]. We are interested in a fusion of the two approaches, that is, an analogue of the Julia-Carathéodory theorem in several variables concerning higher derivatives.

Let Π denote the upper half plane. On Π analogues of the above program exist since Π is conformally equivalent to $\mathbb{D}$. We will give an analogue of the Julia-Carathéodory theorem on the domain Π^2. We work in two variables since operator theoretic representation formulas exist for analytic functions $f : \Pi^2 \to \Pi$, but do not exist in general due to some classically notorious obstruction[13, 14].

On Π, we have the luxury of the Nevanlinna representation.

Theorem 1.2 (R. Nevanlinna [12])**.** *Let $h : \Pi \to \mathbb{C}$. There exists a finite positive Borel measure μ on $\mathbb{R}$ such that*

$$h(z) = \int \frac{1}{t-z}\, d\mu(t) \tag{1.3}$$

if and only if h is in the Pick class and

$$\liminf_{s\to\infty} s\,|h(is)| < \infty. \tag{1.4}$$

Moreover, for any Pick function h satisfying Equation (1.4) *the measure μ in Equation* (1.3) *is uniquely determined.*

Notably, the condition (1.4) is conformally equivalent to the limit in the classical Julia-Carathéodory theorem. The Nevanlinna representation can be used to develop a theory of higher order regularity, since essentially questions about regularity are equivalent to elementary questions in real analysis and measure theory. Namely, since

$$\frac{1}{t-z} = -\sum_{n=0}^{\infty} \frac{t^n}{z^{n+1}}$$

questions about regularity at ∞ can be reduced to questions about the existence of moments $\int t^n d\mu(t)$. The Nevanlinna representation in several variables is given in terms of operator theory, and so questions there can be reduced to questions some operator theoretic analogue of moments.

1.1. **The Löwner class.** We denote the two variable Pick class, the set of holomorphic functions from Π^2 to Π, as $\mathcal{P}_2$.

In [3], Agler and McCarthy defined the Löwner class at infinity.

Definition 1.5. *The Löwner class at ∞, denoted $\mathcal{L}^N$, is the set of functions $h \in \mathcal{P}_2$ such that $\lim_{s\to\infty} h(is, is) = 0$ and there exists a multi-indexed sequence of real numbers $(\rho_n)_{|n|\leq 2N-1}$ (here, each $n = (n_1, n_2)$ for some non-negative integers n_1 and n_2 and $|n| = n_1 + n_2$) such that*

$$h(z) = \sum_{|n|\leq 2N-1} \frac{\rho_n}{z^n} + o\left(\frac{1}{\|z\|^{2N-1}}\right) \text{ nontangentially.}$$

An asymptotic formula holds *nontangentially* at ∞ if for each $c \in \mathbb{R}$ the formula holds for all z large enough satisfying $\|z\| \leq c \min\{\mathrm{Im}(z_1), \mathrm{Im}(z_2)\}$.

A weaker notion of regularity is given by the intermediate Löwner class.

Definition 1.6. *The intermediate Löwner class at ∞, denoted $\mathcal{L}^{N-}$, is the set of functions $h \in \mathcal{P}_2$ such that $\lim_{s\to\infty} h(is, is) = 0$ and there exists a multi-indexed sequence of real numbers $(\rho_n)_{|n|\leq 2N-2}$ such that*

$$h(z) = \sum_{|n|\leq 2N-2} \frac{\rho_n}{z^n} + O\left(\frac{1}{\|z\|^{2N-1}}\right) \text{ nontangentially.}$$

We show that $\mathcal{L}^N \neq \mathcal{L}^{N-}$ in Section 4.

We examine an inductive relationship between $\mathcal{L}^{N-1}, \mathcal{L}^{N-}$, and $\mathcal{L}^N$, which is given in the following two theorems.

Our first main result describes when a function in $\mathcal{L}^{N-1}$ is in $\mathcal{L}^{N-}$.

Theorem 1.7. *Let $h \in \mathcal{P}_2$. The following are equivalent:*
(1) $h \in \mathcal{L}^{N-}$.
(2) $h \in \mathcal{L}^{N-1}$ *and for each* $b \in (\mathbb{R}^+)^2$,
$$s^{2N-1} Im \left[h(isb) - \sum_{|n| \leq 2N-3} \frac{\rho_n}{(isb)^n} \right]$$
is bounded for large s.

We prove Theorem 1.7 as Theorem 2.3 in terms of the language of the Agler-McCarthy vector moment theory, which we will discuss later.

Our second main result describes when a function in $\mathcal{L}^{N-}$ is in $\mathcal{L}^N$.

Theorem 1.8. *Let $h \in \mathcal{P}_2$. The following are equivalent:*
(1) $h \in \mathcal{L}^N$.
(2) $h \in \mathcal{L}^{N-}$ *and there are residues, not necessarily real,* $\{\rho_n\}_{n \leq 2N-1}$ *such that*
$$h(z) = \sum_{|n| \leq 2N-1} \frac{\rho_n}{z^n} + o\left(\frac{1}{\|z\|^{2N-1}}\right)$$
nontangentially.

We prove Theorem 1.8 as Theorem 2.4 in terms of the language of the Agler-McCarthy vector moment theory.

For $N = 1$ our theorems are conformally equivalent the two-variable Julia-Carathéodory theorem proven on $\mathbb{D}^2$ in Agler McCarthy, Young [4] where the conformal analogues of $\mathcal{L}^1$ and $\mathcal{L}^{1-}$ were called C-points and B-points. An analysis for $N = 1$ was given on Π^2 in Agler, Tully-Doyle, Young[6, 5].

2. The Agler-McCarthy vector moment theory

A calculus was developed to calculate the residues of functions in $\mathcal{P}_2$ at ∞ in [6, 3].

Theorem 2.1 (Type I two variable Nevanlinna representation [6])**.** *Let $h \in \mathcal{P}_2$ and suppose that $sh(is, is)$ is bounded for real s large enough. Then, there is a separable Hilbert space $\mathcal{H}$, an unbounded self-adjoint operator A on $\mathcal{H}$, a positive contraction Y and a vector $\alpha \in \mathcal{H}$ such that*
$$\langle (A - z_Y)^{-1} \alpha, \alpha \rangle$$
where $z_Y = Y z_1 + (1 - Y) z_2$.

In terms of the above representation, Agler and McCarthy defined *vector moments,* which occur in a way algebraically analogous to the way classical moments occur in the theory of the Nevanlinna representation in one variable.[12].

Definition 2.2. *Given a separable Hilbert space $\mathcal{H}$, an unbounded self-adjoint operator A on $\mathcal{H}$, a positive contraction Y and a vector $\alpha \in \mathcal{H}$, we say A has* vector moments *to order N denoted $(R_k)_{i=1}^N$ if*

$$R_k(b) = (b_Y^{-1}A)^{k-1}b_Y^{-1}\alpha$$

exists for every $b \in (\mathbb{R}^+)^2$.

If R_k is a vector-valued polynomial in $\frac{1}{b_1}$ and $\frac{1}{b_2}$, that is, there are vectors $(\alpha_n)_{|n|=k}$ such that

$$R_k(b) = \sum_{|n|=k} \frac{1}{b^n}\alpha_n,$$

we extend R_k to all of $\mathbb{C}^2$ via its formula.

To prove Theorem 1.7 we prove the following equivalence in terms of the Agler-McCarthy vector moment theory.

Theorem 2.3. *Let $h \in \mathcal{P}_2$. The following are equivalent:*

(1) *$h \in \mathcal{L}^{N-}$.*
(2) *$h \in \mathcal{L}^{N-1}$ and for any type I representation of h,*

$$h(z) = \langle (A - z_Y)^{-1}\alpha, \alpha\rangle,$$

A has real vector (Y, α)-moments to order $N - 1$.

(3) *For each $b \in (\mathbb{R}^+)^2$,*

$$s^{2N-1} Im \left[h(isb) - \sum_{|n|\leq 2N-3} \frac{\rho_n}{(isb)^n} \right]$$

is bounded for large s.

We prove Theorem 2.3 in several parts. The implication (1) $\Rightarrow$ (2) is given in Proposition 3.1. The implication (2) $\Rightarrow$ (1) is given in Proposition 3.2. The implication (2) $\Leftrightarrow$ (3) is given in Proposition 3.4.

Our second result, Theorem 1.8, becomes the following in the language of the Agler-McCarthy moment theory.

Theorem 2.4. *Let $h \in \mathcal{P}_2$. The following are equivalent:*

(1) *$h \in \mathcal{L}^{N}$.*
(2) *$h \in \mathcal{L}^{N-}$ and for any type I representation of h,*

$$h(z) = \langle (A - z_Y)^{-1}\alpha, \alpha\rangle,$$

A has vector (Y, α)-moments to order $N - 1$ and R_{N-1} is a vector valued polynomial.

(3) *$h \in \mathcal{L}^{N-}$ and there are residues, not necessarily real, $\{\rho_n\}_{n\leq 2N-1}$ such that*

$$h(z) = \sum_{|n|\leq 2N-1} \frac{\rho_n}{z^n} + o\left(\frac{1}{\|z\|^{2N-1}}\right)$$

nontangentially.

Theorem 2.4 is also proven in several parts. (1) $\Leftrightarrow$ (2) follows directly from the Agler-McCarthy moment theory, specifically the their theorem given here as Theorem 2.8, in the light of Theorem 2.3. The implication (1) $\Leftrightarrow$ (3) is proven as Proposition 3.5.

2.1. **Some facts about moments.** In [3], Agler and McCarthy proved the following:

Theorem 2.5 (Agler, McCarthy [3]). *Let $\mathcal{H}$ be a Hilbert space, let $\alpha \in \mathcal{H}$ and assume that A and Y are operators acting on $\mathcal{H}$, with A self-adjoint and Y a positive contraction. The following conditions are equivalent.*

(i) *A has finite complex vector (Y, α)-moments to order $N-1$ and for each $l = 1, \ldots, N$ there exist vectors α_n, $|n| = l$ such that*

$$R_l(z) = \sum_{|n|=l} \frac{1}{z^n} \alpha_n$$

whenever $z \in \mathbb{C}^2 \setminus \{z \mid z_2 \neq 0, z_1/z_2 \notin (-\infty, 0]\}$.

(ii) *A has finite real vector (Y, α)-moments to order $N-1$ and for each $l = 1, \ldots, N$ there exist vectors α_n, $|n| = l$ such that*

$$R_l(b) = \sum_{|n|=l} \frac{1}{b^n} \alpha_n \tag{2.6}$$

whenever $b \in {\mathbb{R}^+}^2$.

We also define scalar moments.

Definition 2.7. *The kth real scalar moment is*

$$r_k(b) = \langle R_{\lceil k/2 \rceil}(b), A R_{\lfloor k/2 \rfloor}(b) \rangle.$$

We will use the following key result about scalar moments.

Theorem 2.8 (Agler, McCarthy [3]). *A function $h \in \mathcal{L}^N$ if and only if h has a type I representation*

$$h(z) = \langle (A - z_Y)^{-1} \alpha, \alpha \rangle,$$

such that A has polynomial vector (Y, α)-moments to order $N-1$. Moreover,

$$r_k(z) = -\sum_{|n|=k} \frac{\rho_n}{z^n}$$

where ρ_n are as in Definition 1.5.

The following telescoping lemma gives a formula that will let us prove the main results.

Lemma 2.9. *Let $h \in \mathcal{P}_2$ with type I representation*

$$h(z) = \langle (A - z_Y)^{-1} \alpha, \alpha \rangle,$$

be such that A has vector (Y, α)-moments to order $N-1$ and scalar moments up to order $2N-1$. Let $b \in (\mathbb{R}^+)^2$. Let $X_b = b_Y^{-1/2} A b_Y^{-1/2}$. Let $\beta_k = X_b^k b_Y^{-1/2} \alpha$. Then,

$$h(isb) + \sum_{k=1}^{2N-1} r_k(isb) = \frac{\langle [(X_b - is)^{-1} + (is)^{-1}] \beta_{N-1}, \beta_{N-1} \rangle}{(is)^{2(N-1)}}.$$

Proof. Note $b_Y^{-1/2}$ exists since b_Y is strictly positive. Note that the expressions for r_{2k-1} and r_{2k} in the notation of the lemma become:

$$r_{2k-1}(isb) = (is)^{-(2k-1)}\langle \beta_k, \beta_k\rangle$$
$$r_{2k}(isb) = (is)^{-2k}\langle \beta_{k-1}, \beta_k\rangle.$$

The proof will go by induction. When $N = 1$,

$$\begin{aligned} h(isb) + r_1(isb) &= \langle (A - isb_Y)^{-1}\alpha, \alpha\rangle + \langle (isb_Y)^{-1}\alpha, \alpha\rangle \\ &= \langle (X_b - is)^{-1}\beta_0, \beta_0\rangle + \langle (is)^{-1}\beta_0, \beta_0\rangle \\ &= \langle [(X_b - is)^{-1} + (is)^{-1}]\beta_0, \beta_0\rangle. \end{aligned}$$

So we are done. Now suppose, by induction,

$$h(isb) + \sum_{k=1}^{2N-1} r_{2k-1}(isb) = \frac{\langle [(X_b - is)^{-1} + (is)^{-1}]\beta_{N-1}, \beta_{N-1}\rangle}{(is)^{2(N-1)}}$$

and additionally we have vector (Y, α)-moments to order N and scalar (Y, α)-moments to order $2N + 1$. So,

$$\begin{aligned} & h(isb) + \sum_{k=1}^{2N+1} r_k(isb) \\ &= \frac{\langle [(X_b - is)^{-1} + (is)^{-1}]\beta_{N-1}, \beta_{N-1}\rangle}{(is)^{2(N-1)}} + \frac{\langle \beta_{N-1}, \beta_N\rangle}{(is)^{2N}} + \frac{\langle \beta_N, \beta_N\rangle}{(is)^{2N+1}} \\ &= \frac{\langle [(X_b - is)^{-1} + (is)^{-1} + (is)^{-2}X_b]\beta_{N-1}, \beta_{N-1}\rangle}{(is)^{2(N-1)}} + \frac{\langle \beta_N, \beta_N\rangle}{(is)^{2N+1}} \\ &= \frac{\langle [(X_b - is)^{-1} + (is)^{-2}(X_b + is)]\beta_{N-1}, \beta_{N-1}\rangle}{(is)^{2(N-1)}} + \frac{\langle \beta_N, \beta_N\rangle}{(is)^{2N+1}} \\ &= \frac{\langle [(X_b - is)^{-1} + (is)^{-2}(X_b + is)(X_b - is)(X_b - is)^{-1}]\beta_{N-1}, \beta_{N-1}\rangle}{(is)^{2(N-1)}} + \frac{\langle \beta_N, \beta_N\rangle}{(is)^{2N+1}} \\ &= \frac{\langle [(X_b - is)^{-1} + ((is)^{-2}X_b^2 - 1)(X_b - is)^{-1}]\beta_{N-1}, \beta_{N-1}\rangle}{(is)^{2(N-1)}} + \frac{\langle \beta_N, \beta_N\rangle}{(is)^{2N+1}} \\ &= \frac{\langle [X_b^2(X_b - is)^{-1}]\beta_{N-1}, \beta_{N-1}\rangle}{(is)^{2N}} + \frac{\langle \beta_N, \beta_N\rangle}{(is)^{2N+1}} \\ &= \frac{\langle [(X_b - is)^{-1}]X_b\beta_{N-1}, X_b\beta_{N-1}\rangle}{(is)^{2N}} + \frac{\langle \beta_N, \beta_N\rangle}{(is)^{2N+1}} \\ &= \frac{\langle [(X_b - is)^{-1} + (is)^{-1}]\beta_N, X_b\beta_N\rangle}{(is)^{2N}}. \end{aligned}$$

This concludes the proof. □

3. Proofs of results

3.1. **Proof of operator theoretic results.** First we endeavor to prove the equivalence of (1) and (2) in Theorem 2.3. We separate the proof into two parts.

Proposition 3.1. *Let $h \in \mathcal{P}_2$ with type I representation*

$$h(z) = \langle (A - z_Y)^{-1}\alpha, \alpha\rangle.$$

If $h \in \mathcal{L}^{N-}$, then $h \in \mathcal{L}^{N-1}$ and A has vector (Y, α)-moments to order $N - 1$.

Proof. Suppose $h \in \mathcal{L}^{N-}$. Then, $h \in \mathcal{L}^{N-1}$. We will show A has real vector (Y, α)-moments to order $N-1$. This is sufficient by Theorem 2.5. By Theorem 2.8,

$$h(isb) + \sum_{k=1}^{2N-3} r_k(isb) = h(isb) - \sum_{|n| \leq 2N-3} \frac{\rho_n}{(isb)^n},$$

and A has (Y, α)-moments to order $N-2$. So by Lemma 2.9, adopting its notation,

$$(is)^{2N-1}\left[h(isb) - \sum_{|n|\leq 2N-2} \frac{\rho_n}{(isb)^n}\right] = (is)^3\langle[(X_b - is)^{-1} + (is)^{-1}]\beta_{N-1}, \beta_{N-1}\rangle - is \sum_{|n|=2N-2} \frac{\rho_n}{b^n}.$$

Since $h \in \mathcal{L}^{N-}$, for some $C > 0$,

$$\left|(is)^{2N-1}\left[h(isb) - \sum_{|n|\leq 2N-2} \frac{\rho_n}{(isb)^n}\right]\right| \leq C$$

So,

$$\left|(is)^3\langle[(X_b - is)^{-1} - (is)^{-1}]\beta_{N-2}, \beta_{N-2}\rangle - (is) \sum_{|n|=2N-2} \frac{\rho_n}{b^n}\right| \leq C.$$

Taking the real part preserves this inequality. Thus,

$$\left|\text{Re } (is)^3\langle[(X_b - is)^{-1} + (is)^{-1}]\beta_{N-2}, \beta_{N-2}\rangle\right| \leq C.$$

Simplifying,

$$\left|\text{Re } (is)^2\langle\frac{X_b}{X_b - is}\beta_{N-2}, \beta_{N-2}\rangle\right| \leq C$$

$$\left|\text{Re } s^2\langle\frac{X_b^2 + isX_b}{X_b^2 + s^2}\beta_{N-2}, \beta_{N-2}\rangle\right| \leq C$$

$$\left\langle\frac{s^2 X_b^2}{X_b^2 + s^2}\beta_{N-2}, \beta_{N-2}\right\rangle \leq C$$

By the spectral theorem, there is a measure μ so that,

$$\left\langle\frac{s^2 X_b^2}{X_b^2 + s^2}\beta_{N-2}, \beta_{N-2}\right\rangle = \int \frac{s^2 x^2}{x^2 + s^2} |\beta_{N-2}(x)|^2 \, d\mu(x).$$

Note the integrand is monotone increasing in s, so apply monotone convergence theorem to get

$$\int |x\beta_{N-2}(x)|^2 d\mu(x) = \int |\beta_{N-1}(x)|^2 d\mu(x)$$

exists and is finite. So $X_b\beta_{N-2} \in \text{Dom } X_b$. That is, $(b_Y^{-1/2} A b_Y^{-1/2})^{N-1} b_Y^{-1/2}\alpha \in \text{Dom } b_Y^{-1/2} A b_Y^{-1/2}$. So, $(Ab_Y^{-1})^{N-1} \in \text{Dom } A$. Thus, A has (Y, α)-moments to order $N-1$. □

The other direction goes as follows.

Proposition 3.2. *Let $h \in \mathcal{P}_2$ with type I representation*

$$h(z) = \langle (A - z_Y)^{-1}\alpha, \alpha\rangle.$$

Then, if $h \in \mathcal{L}^{N-1}$ and A has vector (Y, α)-moments to order $N-1$, then $h \in \mathcal{L}^{N-}$.

Proof. Suppose $h \in \mathcal{L}^{N-1}$ and h has (Y, α)-moments to order $N-1$. By Theorem 2.5,

$$z_Y^{-1}(Az_Y^{-1})^{N-2}\alpha = R_{N-1}(z) = \sum_{|n|=N-1} \frac{1}{z^n}\alpha_n.$$

Since we have (Y, α)-moments to order $N-1$,

$$(Az_Y^{-1})^{N-1}\alpha = AR_{N-1}(z) = A \sum_{|n|=N-1} \frac{1}{z^n}\alpha_n$$

is well defined. Note, by linear independence of monomials, each $\alpha_n \in \mathrm{Dom}A$. Thus,

$$(Az_Y^{-1})^{N-1}\alpha = AR_{N-1}(z) = \sum_{|n|=N-1} \frac{1}{z^n}A\alpha_n.$$

So,

$$\begin{aligned} r_{2N-2}(z) &= \langle AR_{N-1}(z), R_{N-1}(\overline{z})\rangle \\ &= \sum_{|m|=N-1}\sum_{|n|=N-1} \frac{1}{z^n z^m}\langle A\alpha_n, \alpha_m\rangle = \sum_{|n|=2N-2} \frac{\rho_n}{z^n}, \end{aligned}$$

where $\rho_n = \sum_{n+m=2N-2}\langle A\alpha_n, \alpha_m\rangle$. Note that if $b \in (\mathbb{R}^+)^2$,

$$r_{2N-2}(isb) = \frac{1}{(is)^{2N-2}} = \sum_{|n|=2N-2} \frac{\rho_n}{z^n}$$

is real valued. Thus, by linear independence of monomials, each ρ_n is real valued. So

$$h(z) - \sum_{l=1}^{2N-2} r_l(z) = \langle (A - z_Y)^{-1}AR_{N-1}(z), R_{N-1}(\overline{z})\rangle$$

$$\|z\|^{2N-1}(h(z) - \sum_{l=1}^{2N-2} r_l(z)) = \|z\|^{2N-1}\langle (A - z_Y)^{-1}AR_{N-1}(z), R_{N-1}(\overline{z})\rangle.$$

Now notice

$$\|z\|^{2N-1}(h(z) - \sum_{l=1}^{2N-2} r_l(z)) \leq \|z\|\|(A - z_Y)^{-1}\|\|z\|^{N-1}\|AR_{N-1}(z)\|\|z\|^{N-1}\|R_{N-1}(z)\|$$

is nontangentially bounded. So, $h \in \mathcal{L}^{N-}$. □

This concludes the proof of the equivalence of (1) and (2) in Theorem 2.3.

3.2. **Proof of function theoretic results.** We now seek to prove the implication (1) ⇔ (3) in Theorem 2.3 and Theorem 2.4.

We begin with the following lemma which will allow us to prove (1) ⇔ (3) for Theorem 2.3.

Lemma 3.3. *Let $h \in \mathcal{P}_2$. Suppose $h \in \mathcal{L}^{N-1}$ and for each $b \in (\mathbb{R}^+)^2$, for large s,*

$$(is)^{2N-1} Im \left[h(isb) - \sum_{|n|\leq 2N-3} \frac{\rho_n}{(isb)^n} \right]$$

is bounded. Then

$$r_{2N-1}(b) = \lim_{s\to\infty} (is)^{2N-1} Im \left[h(isb) - \sum_{|n|\le 2N-3} \frac{\rho_n}{(isb)^n} \right].$$

Proof. Suppose $h \in \mathcal{L}^{N-1}$ and for each $b \in (\mathbb{R}^+)^2$,

$$J_b(s) := s^{2N-1}\text{Im} \left[h(isb) - \sum_{|n|\le 2N-3} \frac{\rho_n}{(isb)^n} \right]$$

is bounded. Let h have a type I representation $h(z) = \langle (A - z_Y)^{-1}\alpha, \alpha\rangle$. We will show A has vector (Y, α)-moments to order $N-1$ and apply the equivalence of 1 and 2 in Theorem 2.3. By Lemma 2.9, adopting its notation,

$$h(isb) - \sum_{|n|\le 2N-3} \frac{\rho_n}{(isb)^n} = (is)^{-2(N-2)}\langle[(X_b - is)^{-1} + (is)^{-1}]\beta_{N-2}, \beta_{N-2}\rangle.$$

With this substitution,

$$J_b(s) = (is)^{2N-1}\text{Im } (is)^{-2(N-2)}\langle[(X_b - is)^{-1} + (is)^{-1}]\beta_{N-2}, \beta_{N-2}\rangle.$$

Simplified, we obtain

$$J_b(s) = \langle \frac{-s^2X_b^2}{X_b^2 + s^2}\beta_{N-2}, \beta_{N-2}\rangle.$$

Applying the spectral theorem and monotone convergence theorem as in the proof of the equivalence on (1) and (2) in Theorem 2.3, we get

$$\lim_{s\to\infty} (is)^{2N-1}\text{Im} \left[h(isb) - \sum_{|n|\le 2N-3} \frac{\rho_n}{(isb)^n} \right] = -\|\beta_{N-1}\|^2 = r_{2N-1}(b).$$

□

We now prove (1) ⇔ (3) for Theorem 2.3.

Proposition 3.4. *Let $h \in \mathcal{P}_2$. Then, $h \in \mathcal{L}^{N-}$ if and only if $h \in \mathcal{L}^{N-1}$ and for each $b \in (\mathbb{R}^+)^2$, for large s,*

$$s^{2N-1} Im \left[h(isb) - \sum_{|n|\le 2N-3} \frac{\rho_n}{(isb)^n} \right]$$

is bounded.

Proof. Suppose $h \in \mathcal{L}^{N-}$. The term $\sum_{|n|=2N-2} \frac{\rho_n}{(isb)^n}$ is real. So,

$$s^{2N-1}\text{Im} \left[h(isb) - \sum_{|n|\le 2N-3} \frac{\rho_n}{(isb)^n} \right] = s^{2N-1}\text{Im} \left[h(isb) - \sum_{|n|\le 2N-2} \frac{\rho_n}{(isb)^n} \right],$$

which is bounded since $h \in \mathcal{L}^{N-}$.

On the other hand, suppose $h \in \mathcal{L}^{N-1}$ and for each $b \in (\mathbb{R}^+)^2$,

$$s^{2N-1}\text{Im} \left[h(isb) - \sum_{|n|\le 2N-3} \frac{\rho_n}{(isb)^n} \right]$$

is bounded. By Theorem 3.3, we have scalar moments to order $2N-1$ and thus vector (Y, α)-moments. So by the equivalence of (1) and (2) in Theorem 2.3, we are done. □

The following finishes the proof of Theorem 2.4 by showing that We now prove (1) $\Leftrightarrow$ (3).

Proposition 3.5. *Let $h \in \mathcal{P}_2$. Then, $h \in \mathcal{L}^N$ if and only if $h \in \mathcal{L}^{N-}$ and there are residues, not necessarily real, $\{\rho_n\}_{n \leq 2N-1}$ such that*

$$h(z) = \sum_{|n| \leq 2N-1} \frac{\rho_n}{z^n} + o\left(\frac{1}{\|z\|^{2N-1}}\right)$$

nontangentially.

Proof. The forward direction is true by definition.

On the other hand, suppose $h \in \mathcal{L}^{N-}$ and the residues exist. Let $b \in (\mathbb{R}^+)^2$. Let h have a type I representation

$$h(z) = \langle (A - z_Y)^{-1} \alpha, \alpha \rangle.$$

By Lemma 2.9, adopting its notation,

$$h(isb) + \sum_{k=1}^{2N-1} r_k(isb) = (is)^{-2(N-1)} \langle [(X_b - is)^{-1} + (is)^{-1}] \beta_{N-1}, \beta_{N-1} \rangle.$$

Multiply through by $(is)^{2N-1}$.

$$(is)^{2N-1} \left[h(isb) + \sum_{k=1}^{2N-1} r_k(isb) \right] = \langle [is(X_b - is)^{-1} + 1] \beta_{N-1}, \beta_{N-1} \rangle.$$

So,

$$(is)^{2N-1} \left[h(isb) + \sum_{k=1}^{2N-1} r_k(isb) \right] = \langle [\frac{-s^2}{X_b^2 + s^2} + 1] \beta_{N-1}, \beta_{N-1} \rangle.$$

Applying the spectral theorem and taking limits gives

$$\lim_{s \to \infty} (is)^{2N-1} \left[h(isb) + \sum_{k=1}^{2N-1} r_k(isb) \right] = -\|\beta_{N-1}\|^2 + \|\beta_{N-1}\|^2 = 0.$$

Now

$$\lim_{s \to \infty} (is)^{2N-1} \left[h(isb) + \sum_{k=1}^{2N-1} r_k(isb) - h(isb) + \sum_{|n| \leq 2N-1} \frac{\rho_n}{(isb)^n} \right] = 0.$$

Applying Theorem 2.8,

$$\lim_{s \to \infty} (is)^{2N-1} \left[r_{2N-1}(isb) + \sum_{|n| = 2N-1} \frac{\rho_n}{(isb)^n} \right] = 0.$$

Simplifying,

$$\lim_{s \to \infty} r_{2N-1}(b) + \sum_{|n| = 2N-1} \frac{\rho_n}{b^n} = 0,$$

that is,

$$r_{2N-1}(b) = -\sum_{|n|=2N-1} \frac{\rho_n}{b^n}.$$

□

4. $\mathcal{L}^N \neq \mathcal{L}^{N-}$

Now we give an example that shows the hierarchy of Löwner classes in two variables at infinity does not collapse, that is, $\mathcal{L}^N \neq \mathcal{L}^{N-}$, which was shown for the case $N = 1$ in [4]. That $\mathcal{L}^N \neq \mathcal{L}^{N-}$ is in stark contrast to the theory in one variable where the classes are identical[11, 9].

Let $\mathcal{H} = l^2(Z_{2(n-1)})$, and $\pi : Z_{2(n-1)} \to B(l^2(Z_{2(n-1)}))$ a left regular representation i.e. $\pi(j)e_i = e_{j+i}$. Let $A = [\pi(1) + \pi(-1)]$ and Y be a diagonal matrix satisfying $Ye_i = e_i$, for $i \neq n$, and $Ye_{n-1} = te_{n-1}$. Let $\alpha = e_0$. Let f be the Pick function defined by

$$f(z) = \langle (A - z_Y)^{-1}\alpha, \alpha\rangle.$$

Recall $R_k(z) = (z_Y)^{-1}(Az_Y^{-1})^{k-1}e_0$. If $k < n$, it can be shown inductively that

$$R_k(z) = z_1^{-k} \sum_{l=0}^{k-1} \binom{k-1}{l} e_{-(k-1)+2l}.$$

Furthermore,

$$AR_{n-1}(z) = z_1^{-(n-1)} \sum_{l=0}^{n-1} \binom{n-1}{l} e_{-(n-1)+2l}$$

and

$$R_n(z) = \frac{1}{tz_1 + (1-t)z_2} z_1^{-(n-1)} e_{n-1} + z_1^{-n} \sum_{l=0}^{n-2} \binom{n-1}{l} e_{-(n-1)+2l}.$$

So, $r_{2n-1}(z) = \langle R_n(z), AR_{n-1}(\overline{z})\rangle$ is not a polynomial, but for $k < 2n-1$, r_k is a polynomial. That is, $f \in \mathcal{L}^{N-}$, but $f \notin \mathcal{L}^N$.

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