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An Inequality for the Weights of Two Families of Sets, Their Unions and Intersections

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1. Introduction

The object of this note is to prove

Theorem 1. *Let S be the family of all subsets of the set* $\{1, 2, \ldots, n\}$. *If* α, β, γ, δ *are non-negative real valued functions on S such that*

$$\alpha(a)\,\beta(b) \leqq \gamma(a \cup b)\,\delta(a \cap b) \quad \text{for all } a, b \in S, \tag{1}$$

then

$$\alpha(A)\,\beta(B) \leqq \gamma(A \vee B)\,\delta(A \wedge B) \quad \text{for all } A, B \subset S, \tag{2}$$

where $\alpha(A) = \sum(a \in A)\,\alpha(a)$ *and* $A \vee B = \{a \cup b;\, a \in A, b \in B\}$ *and* $A \wedge B = \{a \cap b;\, a \in A, b \in B\}$.

Since every distributive lattice can be embedded in the subsets of some set we get an immediate

Corollary. *If S is a distributive lattice and* (2) *holds whenever* A, B *each contain exactly one point of S then* (2) *always holds. Here* S, A, B *may be infinite.*

Our theorem contains as special cases results of Anderson, Daykin, Fortuin, Ginibre, Greene, Holley, Kasteleyn, Kleitman, Seymour, West and others[1]. We discovered it whilst guests at the Mathematisches Forschungsinstitut Oberwolfach and thank all concerned for their kindness to us.

2. The Proof

Case $n = 1$. Write 0, 1 for ϕ, $\{1\}$ respectively. The Conditions (1) are

$$\alpha(0)\,\beta(0) \leqq \gamma(0)\,\delta(0) \tag{3}$$

[1] This is explained in detail in the forthcoming paper "Inequalities for a pair of maps $S \times S \to S$ with S a finite set" by the same authors (submitted to Math. Zeitschrift). This paper contains several new inequalities also for other binary operations

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$$\alpha(0)\,\beta(1) \leqq \gamma(1)\,\delta(0) = \varepsilon \quad \text{say} \tag{4}$$

$$\alpha(1)\,\beta(0) \leqq \varepsilon \tag{5}$$

$$\alpha(1)\,\beta(1) \leqq \gamma(1)\,\delta(1). \tag{6}$$

When A and B both contain two elements the result (2) becomes

$$(\alpha(0)+\alpha(1))\,(\beta(0)+\beta(1)) \leqq (\gamma(0)+\gamma(1))\,(\delta(0)+\delta(1)). \tag{7}$$

Suppose $0<\varepsilon$ for otherwise (7) is trivial. We decrease $\gamma(0)$ and $\delta(1)$ until we get equality in (3) and (6). Then (7) simplifies to

$$0 \leqq (\alpha(0)\,\beta(1)-\varepsilon)\,(\alpha(1)\,\beta(0)-\varepsilon)/\varepsilon,$$

which holds by (4) and (5). The remaining choices for A, B hold by inspection, so case $n=1$ is verified.

Induction Step. Assume the result for $n=m \geqq 1$ and consider the case $n=m+1$. Write S, T, P for the family of all subsets of $\{1,2,\ldots,m+1\}$, $\{2,3,\ldots,m+1\}$, $\{1\}$ respectively.

Given $a \in S$ put $a^* = a \setminus \{1\} \in T$ and $^*a = a \setminus \{2,3,\ldots,m+1\} \in P$. Let α, β, γ, δ, A, B be chosen and fixed. Define α_2, β_2, γ_2, δ_2 on T by

$$\alpha_2(c) = \sum (a \in A,\ a^* = c)\, \alpha(a)$$

$$\gamma_2(c) = \sum (a \in A \vee B,\ a^* = c)\, \gamma(a)$$

with similar expressions for β_2 and δ_2. Then

$$\alpha(A) = \sum_{a \in A} \alpha(a) = \sum_{c \in T} \Bigg(\sum_{\substack{a \in A \\ a^* = c}} \alpha(a) \Bigg) = \sum_{c \in T} \alpha_2(c) = \alpha_2(T)$$

and similarly

$$\beta(B) = \beta_2(T), \qquad \gamma(A \vee B) = \gamma_2(T), \qquad \delta(A \wedge B) = \delta_2(T).$$

Assume for the moment that

$$\alpha_2(c)\,\beta_2(d) \leqq \gamma_2(c \cup d)\,\delta_2(c \cap d) \qquad \text{for all } c, d \in T. \tag{8}$$

Using $T \vee T = T$ and our induction hypothesis we get

$$\alpha(A)\,\beta(B) = \alpha_2(T)\,\beta_2(T) \leqq \gamma_2(T \vee T)\,\delta_2(T \wedge T) = \gamma(A \vee B)\,\delta(A \wedge B)$$

which is (2) as required. Hence it remains to prove (8).

Let $c, d \in T$ be fixed arbitrarily. Write $e = c \cup d$ and $f = c \cap d$. Define $\alpha_1, \beta_1, \gamma_1, \delta_1$ on P by

$$\alpha_1(p) = \begin{cases} \alpha(p \cup c) & \text{if } p \cup c \in A \\ 0 & \text{otherwise} \end{cases}$$

$$\beta_1(p) = \begin{cases} \beta(p \cup d) & \text{if } p \cup d \in B \\ 0 & \text{otherwise} \end{cases}$$

$$\gamma_1(p)=\begin{cases}\gamma(p\cup e) & \text{if } p\cup e\in A\vee B\\ 0 & \text{otherwise}\end{cases}$$

$$\delta_1(p)=\begin{cases}\delta(p\cup f) & \text{if } p\cup f\in A\wedge B\\ 0 & \text{otherwise}\end{cases}$$

Then

$$\alpha_2(c)=\sum_{\substack{a\in A\\ a^*=c}}\alpha(a)=\sum_{p\in P}\left(\sum_{\substack{a\in A\\ a^*=c\\ {}^*a=p}}\alpha(a)\right)=\sum_{p\in P}\alpha_1(p)=\alpha_1(P)$$

and similarly

$$\beta_2(d)=\beta_1(P), \qquad \gamma_2(e)=\gamma_1(P), \qquad \delta_2(f)=\delta_1(P).$$

Assume for the moment that

$$\alpha_1(p)\,\beta_1(q)\leqq\gamma_1(p\cup q)\,\delta_1(p\cap q) \qquad \text{for all } p,q\in P.$$

Then by the case $n=1$ we have

$$\alpha_1(P)\,\beta_1(P)\leqq\gamma_1(P\vee P)\,\delta_1(P\wedge P),$$

or in other words (8) holds. So it now remains for (9) to be proved.

The left hand side of (9) is zero unless $p\cup c\in A$ and $q\cup d\in B$ in which case it is $\alpha(p\cup c)\,\beta(a\cup d)$. We then have

$$(p\cup c)\cup(q\cup d)=(p\cup q)\cup e\in A\vee B \quad \text{and}$$
$$(p\cup c)\cap(q\cup d)=(p\cap q)\cup f\in A\wedge B.$$

Hence the right hand side of (9) is $\gamma((p\cup q)\cup e)\,\delta((p\cap q)\cup f)$ so (9) holds by hypothesis (1) and Theorem 1 follows inductively.

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