# An inequality for warped product submanifolds of a locally product Riemannian manifold 

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#### Abstract

Recently, Sahin studied the warped product semi-slant submanifolds of locally product Riemannian manifolds. In this paper, we obtain some geometric properties of such submanifolds with an example. Also, we establish a sharp relationship between the squared norm of the second fundamental form and the warping function in terms of the slant angle. The equality case is also considered.


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## 1. Introduction

The idea of slant immersions is an increasing development in differential geometry which is given by Chen [6] in almost Hermitian setting. Recently, Sahin defined and studied these immersions for locally product Riemannian manifolds [10]. In [9], Papaghuic has extended this idea to semi-slant submanifolds of almost Hermitian manifolds which are the generalization of holomorphic, totally real and slant submanifolds. Recently, these submanifolds of locally product Riemannian manifolds were studied by Li and Li [7].

On the other hand, the idea of warped product manifolds is given by Bishop and O'Neill in [5]. They defined these manifolds as follows: Let $M_{1}$ and $M_{2}$ be two Riemannian manifolds with Riemannian metrics $g_{1}$ and $g_{2}$, respectively, and a positive differentiable function $f$ on $M_{1}$. Consider the product manifold $M_{1} \times M_{2}$ with its projections $\pi_{1}: M_{1} \times$ $M_{2} \rightarrow M_{1}$ and $\pi_{2}: M_{1} \times M_{2} \rightarrow M_{2}$. Then their warped product manifold $M=M_{1} \times_{f} M_{2}$ is the Riemannian manifold $M_{1} \times M_{2}=\left(M_{1} \times M_{2}, g\right)$ equipped with the Riemannian structure such that

$$
g(X, Y)=g_{1}\left(\pi_{1 \star} X, \pi_{1 \star} Y\right)+\left(f \circ \pi_{1}\right)^{2} g_{2}\left(\pi_{2 \star} X, \pi_{2 \star} Y\right)
$$

for any vector field $X, Y$ tangent to $M$, where $\star$ is the symbol for the tangent maps. A warped product manifold $M=M_{1} \times_{f} M_{2}$ is said to be trivial or simply Riemannian product manifold if the warping function $f$ is constant. Let $X$ be an unit vector field

[^0]tangent to $M_{1}$ and $U$ be an another unit vector field on $M_{2}$, then from Lemma 7.3 of [5], we have
\[

$$
\begin{equation*}
\nabla_{X} U=\nabla_{U} X=\left(\frac{X f}{f}\right) U \tag{1.1}
\end{equation*}
$$

\]

where $\nabla$ is the Levi-Civita connection on $M$. If $M=M_{1} \times_{f} M_{2}$ be a warped product manifold then $M_{1}$ is a totally geodesic submanifold of $M$ and $M_{2}$ is a totally umbilical submanifold of $M$ [5].

Recently, warped product submanifolds of locally product Riemannian manifolds were studied in $[1-3,11,12,15]$. Furthermore, the warped products of almost contact metric and almost Hermitian manifolds were appeared in $[13,14,16,17]$. In this paper, we study warped product semi-slant submanifolds of a locally product Riemannian manifold. We discuss some geometric properties of such submanifolds and give an example, our example fulfil the definition of a proper semi-slant submanifold i.e., the slant angle lies in the first quadrant. Also, we establish an inequality for the squared norm of second fundamental form in terms of the warping function and the slant angle. The equality case is also discussed.

## 2. Preliminaries

Let $\widetilde{M}$ be a $m$-dimensional differentiable manifold with a tensor field $F$ of type $(1,1)$ such that $F^{2}=I$ and $F \neq \pm I$, then we say that $\widetilde{M}$ is an almost product manifold with almost product structure $F$. If we set

$$
P=\frac{1}{2}(I+F), \quad Q=\frac{1}{2}(I-F)
$$

then we can easily see that

$$
P+Q=I, P^{2}=P, Q^{2}=Q, P Q=Q P=0, \text { and } P-Q=F
$$

Thus $P$ and $Q$ define two orthogonal complementary distributions $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$ on $\widetilde{M}$. If $\widetilde{M}$ admits a Riemannian metric $g$ such that

$$
\begin{equation*}
g(F X, F Y)=g(X, Y) \tag{2.1}
\end{equation*}
$$

for any $X, Y \in \Gamma(T \widetilde{M})$, then $\widetilde{M}$ is called an almost product Riemannian manifold [18], where $\Gamma(T \widetilde{M})$ denotes the set all vector fields of $\widetilde{M}$. Since $F^{2}=I$, we can easily see that the eigenvalues of $F$ are 1 or -1 . An eigenvector corresponding to the eigenvalue 1 associates with $P$ and the eigenvector corresponding to the eigenvalue -1 is associated with $Q$. Let $\widetilde{\nabla}$ denotes the Levi-Civita connection on $\widetilde{M}$ with respect to the Riemannian metric $g$. Then the covariant derivative of $F$ is defined by

$$
\left(\widetilde{\nabla}_{X} F\right) Y=\widetilde{\nabla}_{X} F Y-F \widetilde{\nabla}_{X} Y
$$

for any $X, Y \in \Gamma(T \widetilde{M})$. If $\left(\widetilde{\nabla}_{X} F\right) Y=0$, the almost product Riemmannian manifold $\widetilde{M}$ is said to be a locally product Riemannian manifold.

Let $M$ be a Riemannian manifold isometrically immersed in $\widetilde{M}$ and we denote by the same symbol $g$ the Riemannian metric induced on $M$. Let $\Gamma(T M)$ be the Lie algebra of vector fields in $M$ and $\Gamma\left(T^{\perp} M\right)$, the set of all vector fields normal to $M$. Let $\nabla$ be the Levi-Civita connection on $M$. Then the Gauss and Weingarten formulas are respectively given by

$$
\begin{equation*}
\widetilde{\nabla}_{X} Y=\nabla_{X} Y+\sigma(X, Y), \quad \widetilde{\nabla}_{X} N=-A_{N} X+\nabla_{X}^{\perp} N \tag{2.2}
\end{equation*}
$$

for any $X, Y \in \Gamma(T M)$ and $N \in \Gamma\left(T^{\perp} M\right)$, where $\nabla^{\perp}$ is the normal connection in the normal bundle $\Gamma\left(T^{\perp} M\right)$ and $A_{N}$ is the shape operator of $M$ with respect to $N$. Moreover, $\sigma: T M \times T M \rightarrow T^{\perp} M$ is the second fundamental form of $M$ in $\widetilde{M}$. Furthermore, $A_{N}$ and $\sigma$ are related by

$$
\begin{equation*}
g(\sigma(X, Y), N)=g\left(A_{N} X, Y\right) \tag{2.3}
\end{equation*}
$$

for any $X, Y \in \Gamma(T M)$ and $N \in \Gamma\left(T^{\perp} M\right)$.
For any $X$ tanget to $M$, we write

$$
\begin{equation*}
F X=T X+\omega X, \tag{2.4}
\end{equation*}
$$

where $T X$ (resp. $\omega X$ ) denotes the tangential (resp. normal) component of $F X$. Then $T$ is an endomorphism of tangent bundle $T M$ and $\omega$ is a normal bundle valued 1-form on $T M$.

A sumanifold $M$ is said to be $F$-invariant if $\omega$ is identically zero, i.e., $F X \in \Gamma(T M)$, for any $X \in \Gamma(T M)$. On the other hand, $M$ is said to be $F$-anti-invariant if $T$ is identically zero i.e., $F X \in \Gamma\left(T^{\perp} M\right)$, for any $X \in \Gamma(T M)$. Moreover, from (2.1) and (2.4), we have $g(T X, Y)=g(X, T Y)$, for any $X, Y \in \Gamma(T M)$.

A submanifold $M$ of a locally product Riemnnian manifold $\widetilde{M}$ is said to be totally umbilical submanifold if $\sigma(X, Y)=g(X, Y) H$, for any $X, Y \in \Gamma(T M)$, where $H=$ $\frac{1}{n} \sum_{i=1}^{n} \sigma\left(e_{i}, e_{i}\right)$, the mean curvature vector of $M$. A submanifold $M$ is said to be totally geodesic if $\sigma(X, Y)=0$. Also, we set

$$
\sigma_{i j}^{r}=g\left(\sigma\left(e_{i}, e_{j}\right), e_{r}\right), \quad i, j=1, \cdots, n ; r=n+1, \cdots, m,
$$

and

$$
\begin{equation*}
\|\sigma\|^{2}=\sum_{i, j=1}^{n} g\left(\sigma\left(e_{i}, e_{j}\right), \sigma\left(e_{i}, e_{j}\right)\right) \tag{2.5}
\end{equation*}
$$

where $\left\{e_{1}, \cdots, e_{n}\right\}$ is an orthonormal basis of the tangent space $T_{p} M$, for any $p \in M$.
For a differentiable function $f$ on a $m$-dimensional manifold $\widetilde{M}$, the gradient $\vec{\nabla} f$ of $f$ is defined as $g(\vec{\nabla} f, X)=X f$, for any $X$ tangent to $\widetilde{M}$. As a consequence, we have

$$
\begin{equation*}
\|\vec{\nabla} f\|^{2}=\sum_{i=1}^{m}\left(e_{i}(f)\right)^{2} \tag{2.6}
\end{equation*}
$$

for an orthonormal frame $\left\{e_{1} \cdots, e_{m}\right\}$ on $\widetilde{M}$.
By the analogy with submanifolds in a Kaehler manifold, different classes of submanifolds in a locally product Riemannian manifold were considered.

Let $\widetilde{M}$ be a locally product Riemannnian manifold and $M$ a submanifold of $\widetilde{M}$, then $M$ is said to be semi-invariant submanifold of $\widetilde{M}[4]$ if there exist a differentiable distribution $\mathcal{D}: p \rightarrow \mathcal{D}_{p} \subset T_{p} M$ such that $\mathcal{D}$ is invariant with respect to $F$ and the complementary distribution $\mathcal{D}^{\perp}$ is anti-invariant with respect to $F$ [8].

Let $M$ be a Riemannian submanifold of a locally product Riemannian manifold $\widetilde{M}$. For each nonzero vector field $X \in \Gamma(T M)$ at $p \in M$, the angle $\theta(X), 0 \leq \theta(X) \leq \frac{\pi}{2}$ between $F X$ and $T_{p} M$ is called the Wirtinger angle of $X$. If the angle $\theta(X)$ is constant, which is independent of the choice $p \in M$ and $X \in \Gamma(T M)$, then $M$ is called a slant submanifold of $\widetilde{M}$ and the angle $\theta$ is called the slant angle of the immersion. Thus, $F$ invariant immersion and $F$-anti-invariant immersion are slant immersions with slant angle $\theta=0$ and $\theta=\frac{\pi}{2}$, respectively. A slant immersion which is neither $F$-invariant nor $F$-antiinvariant is called proper slant immersion. It is easy to see that $M$ is slant submanifold of a locally product Riemannian manifold $\widetilde{M}$ if and only if

$$
\begin{equation*}
T^{2}=\lambda I \tag{2.7}
\end{equation*}
$$

for some real number $\lambda \in[0,1][10]$, where $I$ denotes the identity transformation of the tangent bundle $T M$ of the submanifold $M$. Moreover, if $M$ is a slant submanifold and $\theta$ is the slant angle of $M$, then $\lambda=\cos ^{2} \theta$. The following relations are consequences of (2.7)

$$
\begin{aligned}
g(T X, T Y) & =\cos ^{2} \theta g(X, Y) \\
g(\omega X, \omega Y) & =\sin ^{2} \theta g(X, Y)
\end{aligned}
$$

for any $X, Y \in \Gamma(T M)$.

Moreover, we say that $M$ is a semi-slant submanifold of $\widetilde{M}$, if there exist two orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^{\theta}$ such that
(i) $T M=\mathcal{D} \oplus \mathcal{D}^{\theta}$,
(ii) the distribution $\mathcal{D}$ is invariant, i.e. $F(\mathcal{D})=\mathcal{D}$,
(iii) the distribution $\mathcal{D}^{\theta}$ is slant with slant angle $\theta \neq 0$.

We will call $\mathcal{D}^{\theta}$ as a slant distribution. In particular, if $\theta=\frac{\pi}{2}$, then the semi-slant submanifold is a semi-invariant submanifold [3]. On the other hand, If we denote the dimension of $\mathcal{D}$ and $\mathcal{D}^{\theta}$ by $d_{1}$ and $d_{2}$, respectively. It is clear that if $d_{1}=0$ and $\theta \neq \frac{\pi}{2}$, then $M$ is a proper slant submanifold with slant angle $\theta$. Also, if $d_{2}=0$ then $M$ is $F$ invariant submanifold and if $d_{1}=0$ and $\theta=\frac{\pi}{2}$ then $M$ is $F$-anti-invariant submanifold. Furthermore, if neither $d_{1}=0$ nor $\theta=\frac{\pi}{2}$, then $M$ is a proper semi-slant submanifold.

## 3. Warped product semi-slant submanifolds

Warped product semi-slant submanifolds of locally product Riemannian manifolds were studied by Atceken and Sahin in [1] and [12]. They proved that the warped products of the type $M_{T} \times_{f} M_{\theta}$ do not exist. On the other hand, they provided some examples and a characterization on the existence of warped products $M_{\theta} \times{ }_{f} M_{T}$, where $M_{T}$ and $M_{\theta}$ are invariant and proper slant submanifolds of a locally product Riemannian manifold $\widetilde{M}$, respectively. In this section, we study the warped product semi-slant submanifolds of type $M_{\theta} \times_{f} M_{T}$ of locally product Riemannian manifolds which have not been considered in earlier studies. First, we prove the following results for later use.

Lemma 3.1. Let $M=M_{\theta} \times{ }_{f} M_{T}$ be a warped product semi-slant submanifold of a locally product Riemannian manifold $\widetilde{M}$. Then
(i) $g(\sigma(X, U), \omega V)=-g(\sigma(X, V), \omega U)$;
(ii) $g(\sigma(X, Y), \omega U)=-(U \ln f) g(X, F Y)+(T U \ln f) g(X, Y)$;
(iii) $g(\sigma(X, Y), \omega T U)=-(T U \ln f) g(X, F Y)+\cos ^{2} \theta(U \ln f) g(X, Y)$
for any $X, Y \in \Gamma\left(T M_{T}\right)$ and $U, V \in \Gamma\left(T M_{\theta}\right)$.
Proof. For any $X \in \Gamma\left(T M_{T}\right)$ and $U, V \in \Gamma\left(T M_{\theta}\right)$, we have

$$
\begin{aligned}
g(\sigma(X, U), \omega V) & =g\left(\widetilde{\nabla}_{X} U, F V\right)-g\left(\widetilde{\nabla}_{X} U, T V\right) \\
& =g\left(\left(\widetilde{\nabla}_{X} F\right) U, V\right)-g\left(\widetilde{\nabla}_{X} F U, V\right)-(U \ln f) g(X, T V)
\end{aligned}
$$

First and last terms in the right hand side are identically zero by using the structure of a locally product Riemannian manifold and the orthogonality of vector fields, then from (2.4), we obtain

$$
g(\sigma(X, U), \omega V)=g\left(\widetilde{\nabla}_{X} T U, V\right)+g\left(\widetilde{\nabla}_{X} \omega U, V\right)
$$

By using (2.2)-(2.3) and (1.1) in above relation, we get (i). For the second part, consider $X, Y \in \Gamma\left(T M_{T}\right)$ and $U \in \Gamma\left(T M_{\theta}\right)$, then we have

$$
\begin{aligned}
g(\sigma(X, Y), \omega U) & =g\left(\widetilde{\nabla}_{X} Y, \omega U\right) \\
& =g\left(\widetilde{\nabla}_{X} Y, F U\right)-g\left(\widetilde{\nabla}_{X} Y, T U\right) \\
& =g\left(\left(F \widetilde{\nabla}_{X} Y, U\right)+g\left(Y, \widetilde{\nabla}_{X} T U\right)\right. \\
& =g\left(\widetilde{\nabla}_{X} F Y, U\right)+(T U \ln f) g(X, Y) \\
& =-g\left(F Y, \widetilde{\nabla}_{X} U\right)+(T U \ln f) g(X, Y)
\end{aligned}
$$

Using (2.2) and (1.1), we obtain

$$
g(\sigma(X, Y), \omega U)=-(U \ln f) g(X, F Y)+(T U \ln f) g(X, Y)
$$

which is (ii). If we replace $U$ by $T U$ in (ii), and then using (2.7), we get (iii), which proves the lemma completely.

From the above lemma we can easily find the following relations by interchanging $X$ by $F X$ and $Y$ by $F Y$, for any $X, Y \in \Gamma\left(T M_{T}\right)$ in Lemma 3.1 (ii)-(iii)

$$
\begin{gather*}
g(\sigma(X, F Y), \omega U)=-(U \ln f) g(X, Y)+(T U \ln f) g(X, F Y)  \tag{3.1}\\
g(\sigma(F X, Y), \omega U)=-(U \ln f) g(X, Y)+(T U \ln f) g(F X, Y) \tag{3.2}
\end{gather*}
$$

and

$$
\begin{align*}
& g(\sigma(F X, Y), \omega T U)=-(T U \ln f) g(X, Y)+\cos ^{2} \theta(U \ln f) g(F X, Y)  \tag{3.3}\\
& g(\sigma(X, F Y), \omega T U)=-(T U \ln f) g(X, Y)+\cos ^{2} \theta(U \ln f) g(X, F Y) \tag{3.4}
\end{align*}
$$

Then using (2.1) in (3.1) and (3.2), we find

$$
\begin{equation*}
g(\sigma(X, F Y), \omega U)=g(\sigma(F X, Y), \omega U) \tag{3.5}
\end{equation*}
$$

Similarly, from (3.3) and (3.4), we obtain

$$
\begin{equation*}
g(\sigma(F X, Y), \omega T U)=g(\sigma(X, F Y), \omega T U) \tag{3.6}
\end{equation*}
$$

Also, if we interchange $X$ by $F X$ in (3.1) and $Y$ by $F Y$ in (3.3), we arrive at

$$
\begin{gather*}
g(\sigma(F X, F Y), \omega U)=-(U \ln f) g(F X, Y)+(T U \ln f) g(X, Y)  \tag{3.7}\\
g(\sigma(F X, F Y), \omega T U)=-(T U \ln f) g(F X, Y)+\cos ^{2} \theta(U \ln f) g(X, Y) \tag{3.8}
\end{gather*}
$$

Then from (3.7) and Lemma 3.1 (ii), we get

$$
\begin{equation*}
g(\sigma(F X, F Y), \omega U)=g(\sigma(X, Y), \omega U) \tag{3.9}
\end{equation*}
$$

and by Lemma 3.1 (iii) and (3.8), we derive

$$
\begin{equation*}
g(\sigma(F X, F Y), \omega T U)=g(\sigma(X, Y), \omega T U) \tag{3.10}
\end{equation*}
$$

Now, we construct the following example of warped product semi-slant submanifolds of a locally product Riemannian manifold.
Example 3.2. Let us consider the almost product manifold $\mathbb{R}^{7}=\mathbb{R}^{3} \times \mathbb{R}^{4}$ with coordinates $\left(x_{1}, x_{2}, x_{3}, y_{1}, y_{2}, y_{3}, y_{4}\right)$ and the product structure

$$
F\left(\frac{\partial}{\partial x_{i}}\right)=-\frac{\partial}{\partial x_{i}}, \quad F\left(\frac{\partial}{\partial y_{j}}\right)=\frac{\partial}{\partial y_{j}}, \quad i=1,2,3 \quad \text { and } \quad j=1,2,3,4
$$

Let $M$ be a submanifold of $\mathbb{R}^{7}$ given by

$$
\begin{aligned}
f(\theta, \varphi, v, u)= & (u \cos \theta, u \sin \theta, u+v, \sqrt{3} u-v, v, u \sin \varphi, u \cos \varphi) \\
& \text { with } u \neq 0, \quad v \neq 0 \text { and } \theta, \varphi \in\left(0, \frac{\pi}{2}\right)
\end{aligned}
$$

Then the tangent space $T M$ of $M$ is spanned by the following vector fields

$$
\begin{aligned}
Z_{1} & =-u \sin \theta \frac{\partial}{\partial x_{1}}+u \cos \theta \frac{\partial}{\partial x_{2}}, \quad Z_{2}=u \cos \varphi \frac{\partial}{\partial y_{3}}-u \sin \varphi \frac{\partial}{\partial y_{4}} \\
Z_{3} & =\frac{\partial}{\partial x_{3}}-\frac{\partial}{\partial y_{1}}+\frac{\partial}{\partial y_{2}} \\
Z_{4} & =\cos \theta \frac{\partial}{\partial x_{1}}+\sin \theta \frac{\partial}{\partial x_{2}}+\frac{\partial}{\partial x_{3}}+\sqrt{3} \frac{\partial}{\partial y_{1}}+\sin \varphi \frac{\partial}{\partial y_{3}}+\cos \varphi \frac{\partial}{\partial y_{4}}
\end{aligned}
$$

Then with respect to the Riemannian product structure $F$, we get

$$
\begin{aligned}
F Z_{1} & =-Z_{1}, \quad F Z_{2}=Z_{2}, \quad F Z_{3}=-\frac{\partial}{\partial x_{3}}-\frac{\partial}{\partial y_{1}}+\frac{\partial}{\partial y_{2}} \\
F Z_{4} & =-\cos \theta \frac{\partial}{\partial x_{1}}-\sin \theta \frac{\partial}{\partial x_{2}}-\frac{\partial}{\partial x_{3}}+\sqrt{3} \frac{\partial}{\partial y_{1}}+\sin \varphi \frac{\partial}{\partial y_{3}}+\cos \varphi \frac{\partial}{\partial y_{4}}
\end{aligned}
$$

Then, it is easy to see that the invariant and slant distributions are spanned by

$$
\mathcal{D}=\operatorname{span}\left\{Z_{1}, Z_{2}\right\} \quad \text { and } \quad \mathcal{D}^{\theta_{1}}=\operatorname{span}\left\{Z_{3}, Z_{4}\right\}
$$

with the slant angle say $\theta_{1}$. Then

$$
\theta_{1}=\arccos \left(\frac{g\left(F Z_{3}, Z_{3}\right)}{\left\|F Z_{3}\right\|\left\|Z_{3}\right\|}\right)=\arccos \left(\frac{g\left(F Z_{4}, Z_{4}\right)}{\left\|F Z_{4}\right\|\left\|Z_{4}\right\|}\right)=\arccos \left(\frac{1}{3}\right)=70^{\circ} 31^{\prime}
$$

Thus, $M$ is a proper semi-slant submanifold of $\mathbb{R}^{7}$. It is also easy to check that $\mathcal{D}$ and $\mathcal{D}^{\theta_{1}}$ are integrable. If we denote the integral manifolds of $\mathcal{D}$ and $\mathcal{D}^{\theta_{1}}$ by $M_{T}$ and $M_{\theta_{1}}$, respectively. Then the induced metric tensor on $M$ is given by

$$
g=\left(6 d u^{2}+3 d v^{2}\right)+u^{2}\left(d \theta^{2}+d \varphi^{2}\right)=g_{M_{\theta_{1}}}+u^{2} g_{M_{T}} .
$$

Thus $M$ is a warped product submanifold of the form $M=M_{\theta_{1}} \times f_{f_{1}} M_{T}$ with the warping function $f_{1}=u$.

If $M$ is a warped product semi-slant submanifold of the form $M=M_{\theta} \times_{f} M_{T}$ of a locally product Riemannian manifold $\widetilde{M}$ and if there is no $\mu$-components in the normal bundle of $M$, then $M$ is mixed totally geodesic (Proposition 4.1 [12]) i.e., $\sigma(X, U)=0$, for any $X \in \Gamma\left(T M_{T}\right)$ and $U \in \Gamma\left(T M_{\theta}\right)$.

Now, we construct the following frame field for an $n$-dimensional warped product semi-slant submanifold $M=M_{\theta} \times_{f} M_{T}$ of a $m$-dimensional locally product Riemannian manifold $\widetilde{M}$. Let us denote by $\mathcal{D}$ and $\mathcal{D}^{\theta}$ the tangent bundles of $M_{T}$ and $M_{\theta}$, respectively instead of $T M_{T}$ and $T M_{\theta}$. Also, we consider the $\operatorname{dim}\left(M_{T}\right)=t$ and $\operatorname{dim}\left(M_{\theta}\right)=q$, then the orthonormal frames of $\mathcal{D}$ and $\mathcal{D}^{\theta}$, respectively are given by $\left\{e_{1}=F e_{1}, \cdots, e_{k}=\right.$ $\left.F e_{k}, e_{k+1}=-F e_{k+1}, \cdots, e_{t}=-F e_{t}\right\}$ and $\left\{e_{t+1}=e_{1}^{*}=\sec \theta T e_{1}^{*}, \cdots, e_{t+q}=e_{q}^{*}=\right.$ $\left.\sec \theta T e_{q}^{*}\right\}$. Then the orthonormal frame fields of the normal subbundles of $\omega \mathcal{D}^{\theta}$ and $\mu$, respectively are $\left\{e_{n+1}=\widetilde{e}_{1}=\csc \theta \omega e_{1}^{*}, \cdots, e_{n+q}=\widetilde{e}_{q}=\csc \theta \omega e_{q}^{*}\right\}$ and $\left\{e_{n+q+1}=\right.$ $\left.\widetilde{e}_{q+1}, \cdots, e_{m}=\widetilde{e}_{m-n-q}\right\}$.

Now, we are able to construct the following inequality with the help of the above constructed frame fields and some previous formulas which we have obtained for warped product semi-slant submanifolds of a locally product Riemannian manifold.
Theorem 3.3. Let $M=M_{\theta} \times_{f} M_{T}$ be a proper warped product semi-slant submanifold of a locally product Riemannian manifold $\widetilde{M}$, where $M_{T}$ and $M_{\theta}$ are invariant and proper slant submanifolds of $\widetilde{M}$, respectively. Then
(i) The squared norm of the second fundamental form of the warped product immersion satisfies

$$
\|\sigma\|^{2} \geq t(\csc \theta-\cot \theta)^{2}\left\|\nabla^{\theta} \ln f\right\|^{2}
$$

where $t=\operatorname{dim} M_{T}$ and $\nabla^{\theta} \ln f$ is gradient of the function $\ln f$ along $M_{\theta}$.
(ii) If equality sign in (i) holds identically, then $M_{\theta}$ is totally geodesic in $\widetilde{M}$ and $M_{T}$ is a totally umbilical submanifold of $\widetilde{M}$. Furthermore, $M_{\theta} \times_{f} M_{T}$ is a mixed totally geodesic submanifold of $\widetilde{M}$
Proof. From (2.5), we have

$$
\|\sigma\|^{2}=\sum_{i, j=1}^{n} g\left(\sigma\left(e_{i}, e_{j}\right), \sigma\left(e_{i}, e_{j}\right)\right)=\sum_{r=n+1}^{m} \sum_{i, j=1}^{n} g\left(\sigma\left(e_{i}, e_{j}\right), e_{r}\right)^{2} .
$$

Then from the assumed frame fields of $\mathcal{D}$ and $\mathcal{D}^{\theta}$, we derive

$$
\begin{gather*}
\|\sigma\|^{2}=\sum_{r=n+1}^{m} \sum_{i, j=1}^{t} g\left(\sigma\left(e_{i}, e_{j}\right), e_{r}\right)^{2}+2 \sum_{r=n+1}^{m} \sum_{i=1}^{t} \sum_{j=t+1}^{n} g\left(\sigma\left(e_{i}, e_{j}\right), e_{r}\right)^{2} \\
+\sum_{r=n+1}^{m} \sum_{i, j=t+1}^{n} g\left(\sigma\left(e_{i}, e_{j}\right), e_{r}\right)^{2} . \tag{3.11}
\end{gather*}
$$

After leaving the second and third term in the right hand side of (3.11) and using the constructed frame fields, we find

$$
\begin{equation*}
\|\sigma\|^{2} \geq \sum_{r=1}^{q} \sum_{i, j=1}^{t} g\left(\sigma\left(e_{i}, e_{j}\right), \csc \theta \omega e_{r}^{*}\right)^{2}+\sum_{r=q+1}^{m-n-q} \sum_{i, j=1}^{t} g\left(\sigma\left(e_{i}, e_{j}\right), \widetilde{e}_{r}\right)^{2} \tag{3.12}
\end{equation*}
$$

The second term in the right hand side of the above expression has the $\mu$-components, therefore we shall leave this term and hence using Lemma 3.1, we get

$$
\begin{gather*}
\|\sigma\|^{2} \geq \csc ^{2} \theta \sum_{r=1}^{q} \sum_{i, j=1}^{t}\left[\left(e_{r}^{*} \ln f\right)^{2} g\left(e_{i}, F e_{j}\right)^{2}+\left(T e_{r}^{*} \ln f\right)^{2} g\left(e_{i}, e_{j}\right)^{2}\right. \\
\left.-2\left(T e_{r}^{*} \ln f\right) \cdot\left(e_{r}^{*} \ln f\right) g\left(e_{i}, e_{j}\right) g\left(e_{i}, F e_{j}\right)\right] \tag{3.13}
\end{gather*}
$$

Now, since $\nabla^{\theta} \ln f \in \Gamma\left(T M_{\theta}\right)$ and $e_{r}^{*}=\sec \theta T e_{r}^{*}$, then from (2.6) we have

$$
\begin{equation*}
\sum_{r=1}^{q}\left(T e_{r}^{*} \ln f\right) \cdot\left(e_{r}^{*} \ln f\right)=\sum_{r=1}^{q} \cos \theta\left(e_{r}^{*} \ln f\right)^{2}=\cos \theta\left\|\nabla^{\theta} \ln f\right\|^{2} \tag{3.14}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{r=1}^{q}\left(T e_{r}^{*} \ln f\right)^{2}=\sum_{r=1}^{q} g\left(T e_{r}^{*}, \nabla^{\theta} \ln f\right)^{2}=\cos ^{2} \theta\left\|\nabla^{\theta} \ln f\right\|^{2} \tag{3.15}
\end{equation*}
$$

Then, from (3.13), (3.14) and (3.15), we find

$$
\|\sigma\|^{2} \geq t \csc ^{2} \theta(1-\cos \theta)^{2}\left\|\nabla^{\theta} \ln f\right\|^{2}
$$

which is inequality (i). For the equality, from the remaining terms of (3.11), we find

$$
\begin{equation*}
\sigma\left(\mathcal{D}, \mathcal{D}^{\theta}\right)=0, \quad \text { and } \quad \sigma\left(\mathcal{D}^{\theta}, \mathcal{D}^{\theta}\right)=0 \tag{3.16}
\end{equation*}
$$

Also, from the remaining second term in the right hand side of (3.12), we observe that

$$
\begin{equation*}
\sigma(\mathcal{D}, \mathcal{D}) \perp \mu \quad \Rightarrow \quad \sigma(\mathcal{D}, \mathcal{D}) \in \Gamma\left(\omega \mathcal{D}^{\theta}\right) \tag{3.17}
\end{equation*}
$$

The second condition of (3.16) implies that $M_{\theta}$ is totally geodesic in $\widetilde{M}$ due to $M_{\theta}$ being totally geodesic in $M$ [5]. On the other hand, (3.17) implies that $M_{T}$ is totally umbilical in $\widetilde{M}$ with the fact that $M_{T}$ is totally umbilical in $M$ [5]. Moreover, all conditions of (3.16) imply that $M$ is a mixed totally geodesic submanifold of $\widetilde{M}$. Hence, the proof is complete.

From the above theorem, we have the following remark.
Remark 3.4. In Theorem 3.3, if we assume $\theta=\frac{\pi}{2}$, then the warped product becomes $M=$ $M_{\perp} \times{ }_{f} M_{T}$ in a locally product Riemannian manifold $\widetilde{M}$, where $M_{T}$ and $M_{\perp}$ are invariant and anti-invariant submanifolds of $\widetilde{M}$, respectively, which is a case of warped product semi-nivariant submanifolds which have been discussed in ([3], [11]). Thus, Theorem 4.2 of [11] and Theorem 4.1 of [3] are the special cases of Theorem 3.3.

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## References

[1] M. Atceken, Warped product semi-slant submanifolds in locally Riemannian product manifolds, Bull. Austral. Math. Soc. 77, 177-186, 2008.
[2] M. Atceken, Geometry of warped product semi-invariant submanifolds of a locally Riemannian product manifold, Serdica Math. J. 35, 273-286, 2009.
[3] M. Atceken, Warped product semi-invariant submanifolds in locally decomposable Riemannian manifolds, Hacet. J. Math. Stat. 40, 401-407, 2011.
[4] A. Bejancu, Semi-invariant submanifolds of locally Riemannian product manifolds, Ann. Univ. Timisoara S. Math. 22, 3-11, 1984.
[5] R.L. Bishop and B. O'Neill, Manifolds of Negative curvature, Trans. Amer. Math. Soc. 145, 1-49, 1969.
[6] B.Y. Chen, Slant immersions, Bull. Austral. Math. Soc. 41, 135-147, 1990.
[7] H. Li and X. Li, Semi-slant submanifolds of locally product manifolds, Georgian Math. J. 12, 273-282, 2005.
[8] K. Matsumoto, On submanifolds of locally product Riemannian manifolds, TRU Math. 18, 145-157, 1982.
[9] N. Papaghiuc, Semi-slant submanifolds of Kaehlerian manifold, Ann. St. Univ. Iasi, 9, 55-61, 1994.
[10] B. Sahin, Slant submanifolds of an almost product Riemannian manifold, J. Korean Math. Soc. 43, 717-732, 2006.
[11] B. Sahin, Warped product semi-invariant submanifolds of locally product Riemannian manifolds, Bull. Math. Soc. Sci. Math. Roumanie Tome, 49 (97), 383-394, 2006.
[12] B. Sahin, Warped product semi-slant submanifolds of a locally product Riemannian manifold, Studia Sci. Math. Hungar. 46, 169-184, 2009.
[13] S. Uddin and F.R. Al-Solamy, Warped product pseudo-slant submanifolds of cosymplectic manifolds, An. Ştiinţ. Univ. Al. I. Cuza Iaşi Mat. (N.S.) Tomul LXII, (f. 2, 3), 901-913, 2016.
[14] S. Uddin and F.R. Al-Solamy, Warped product pseudo-slant immersions in Sasakian manifolds, Publ. Math. Debrecen, 91 (3-4), 331-348, 2017.
[15] S. Uddin, A. Ali, N.M. Al-Asmari and W.A. Othman, Warped product pseudo-slant submanifolds in a locally product Riemannian manifold, Diff. Geometry Dyn. Syst. 18, 147-158, 2016.
[16] S. Uddin, B.Y. Chen and F.R. Al-Solamy, Warped product bi-slant im- mersions in Kaehler manifolds, Mediterr. J. Math. 14, Art. 95, 1-11, 2017.
[17] S. Uddin and A.Y.M. Chi, Warped product pseudo-slant submanifolds of nearly Kaehler manifolds, An. St. Univ. Ovidius Constanta, 19 (3), 195-204, 2011.
[18] K. Yano and M. Kon, Structures on manifolds, Series in Pure Mathematics, World Scientific Publishing Co., Singapore, 1984.


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