AN INEQUALITY OF THE HÖLDER TYPE, CONNECTED WITH STIELTJES INTEGRATION.

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1. In this note, I consider an inequality bearing a formal resemblance to that of Hölder, and I derive from it new conditions for the existence of a Stieltjes integral, and for passage to the limit under the integral sign. The conditions for limits under the integral sign differ from any previously known, in that, for the first time, absolute integrability is not required. They throw some light on problems of convergence of Fourier series.

The first proof of the inequality is due to Mr E. R. Love, who studied it at my suggestion. In a joint paper, elsewhere, we propose to consider further questions connected with it.

2. We begin with a simple *lemma*.

If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are two ordered sets of n complex numbers, and $p, q > 0$, then there is an index k $(0 < k \leq n)$, such that

$$|a_k b_k| \leq \left(\frac{1}{n}\sum_1^n |a_i|^p\right)^{1/p} \left(\frac{1}{n}\sum_1^n |b_i|^q\right)^{1/q}. \tag{2.1}$$

Proof. It will suffice to prove that the right hand side majorises the geometric mean of the n products $|a_k b_k|$, that is, the expression

$$|(a_1 b_1) \ldots (a_n b_n)|^{1/n} = [(|a_1|^p \ldots |a_n|^p)^{1/n}]^{1/p} [(|b_1|^q \ldots |b_n|^q)^{1/n}]^{1/q},$$

and for this purpose, we need only observe that, by the theorem of the arithmetic and geometric means[1], the expressions

[1] cf. Hardy, Littlewood, and Polya [5], (hereafter simply H. L. P.) and Bohr [2].

$$\left(\frac{1}{n}\sum_1^n |a_i|^p\right) \text{ and } \left(\frac{1}{n}\sum_1^n |b_i|^q\right)$$

majorise, respectively, the expressions

$$[(|a_1|^p \ldots |a_n|^p)^{1/n}] \text{ and } [(|b_1|^q \ldots |b_n|^q)^{1/n}].$$

3. *Hölder's inequality.* We remark in passing that the trivial lemma just proved may be used to give a simple proof of the well known inequality of Hölder

$$\sum_1^n |a_k b_k| \leq A \cdot \left(\sum_1^n |a_k|^p\right)^{1/p} \left(\sum_1^n |b_k|^q\right)^{1/q} \tag{3.1}$$

valid for $\frac{1}{p} + \frac{1}{q} \geq 1$, in which $A = 1$.

We suppose first $\frac{1}{p} + \frac{1}{q} > 1$.

Let the $|a_k b_k|$ be arranged in decreasing order. Then, by our lemma

$$|a_n b_n| \leq n^{-\left(\frac{1}{p}+\frac{1}{q}\right)} \left(\sum_1^n |a_i|^p\right)^{1/p} \left(\sum_1^n |b_i|^q\right)^{1/q}.$$

Similarly

$$|a_{n-1} b_{n-1}| \leq (n-1)^{-\left(\frac{1}{p}+\frac{1}{q}\right)} \left(\sum_1^{n-1} |a_i|^p\right)^{1/p} \left(\sum_1^{n-1} |b_i|^q\right)^{1/q}$$

$$\leq (n-1)^{-\left(\frac{1}{p}+\frac{1}{q}\right)} \left(\sum_1^n |a_i|^p\right)^{1/p} \left(\sum_1^n |b_i|^q\right)^{1/q}.$$

Proceeding in this way, we finally obtain, by addition,

$$\sum_1^n |a_k b_k| \leq \left[n^{-\left(\frac{1}{p}+\frac{1}{q}\right)} + (n-1)^{-\left(\frac{1}{p}+\frac{1}{q}\right)} + \cdots + 1\right] \cdot \left(\sum_1^n |a_k|^p\right)^{1/p} \left(\sum_1^n |b_k|^q\right)^{1/q},$$

and so (3.1), but only with $A = \zeta\left(\frac{1}{p} + \frac{1}{q}\right)$, — where $\zeta(s) = \Sigma n^{-s}$, and only for $\frac{1}{p} + \frac{1}{q} > 1$.

To make $A = 1$, still supposing $\frac{1}{p} + \frac{1}{q} > 1$, we argue as follows.[1] Applying (3. 1) in the form proved, to the double sum

$$\sum_1^n \sum_1^n | a_k a_l b_k b_l | = \left(\sum_1^n | a_k b_k | \right)^2,$$

we find that this is majorised by

$$A \cdot \left(\sum_1^n \sum_1^n | a_k a_l |^p \right)^{1/p} \left(\sum_1^n \sum_1^n | b_k b_l |^q \right)^{1/q},$$

that is, by

$$A \left[\left(\sum_1^n | a_k |^p \right)^2 \right]^{1/p} \left[\left(\sum_1^n | b_k |^q \right)^2 \right]^{1/q},$$

and, on taking the square root, we find our previous inequality for $\sum\limits_1^n | a_k b_k |$, *but with A replaced by $\sqrt{A}$.* The factor $\zeta\left(\frac{1}{p} + \frac{1}{q}\right)$ may therefore be replaced successively by its square root, its fourth root, its $2^{N\text{-th}}$ root, and making $N \to \infty$, the factor becomes 1.

Finally, with $A = 1$, both sides of (3. 1) are continuous in p, q for each fixed set of a's and b's and the inequality is therefore valid for $\frac{1}{p} + \frac{1}{q} \geq 1$.

4. Denoting, for a moment, by a, b the finite sequences of numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, Hölder's inequality states that a certain function of a, b is majorised by a product of the form $A\varphi(a)\,\psi(b)$.

In our main inequality, a similar state of affairs will occur. A certain function of a, b will be majorised by the largest of a finite number of such products, derived from one of them by a simple operation that we now describe.

The operation of replacing by

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[1] A favourite type of argument, cf Bohr [2].

separating consecutive terms of a finite sequence

$$a = (a_1, \ldots, a_n)$$

may be termed a *partition* P. The result of the operation is a finite sequence

$$Pa = x = (x_1, \ldots, x_m)$$

in which each x_k is a corresponding sum of a_i, and, of course, $m \leq n$. And if $\Phi(a, b)$ is a function of a pair of sequences $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ the expression

$$\Phi(Pa, Pb)$$

may be said to be derived from $\Phi(a, b)$ by the partition P.

It is with the expressions thus derived by partition from the product

$$A \cdot \left(\sum_1^n |a_i|^p\right)^{1/p} \cdot \left(\sum_1^n |b_i|^q\right)^{1/q}$$

that we shall be concerned.

5. *The inequality for finite sequences.* Let $S_{p,q}(a, b)$ be the largest of the values of the product

$$\left(\sum_1^m |x_k|^p\right)^{1/p} \cdot \left(\sum_1^m |y_k|^q\right)^{1/q}$$

for which $x_1, \ldots x_m$ and $y_1, \ldots, y_m$ are the result of a same partition applied to the finite sequences

$$a = a_1, \ldots, a_n \text{ and } b = b_1, \ldots, b_n.$$

We assert that, for $\frac{1}{p} + \frac{1}{q} > 1$ *and* $p, q > 0$,

$$(5.1) \qquad \left| \sum\sum_{0 < r \leq s \leq n} a_r b_s \right| \leq \left\{ 1 + \zeta\left(\frac{1}{p} + \frac{1}{q}\right) \right\} \cdot S_{p,q}(a, b).$$

Proof.[1] Consider the partition defined by changing the k^{th}

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[1] As already mentioned, the first proof was obtained by Mr E. R. Love. Mr Love's proof was on entirely different lines, and was not so simple as this one.

We have, for $0 < k \leq n - 1$,

$$x_r,\, y_r = \begin{cases} a_r,\ b_r \text{ if } r < k \\ a_{r+1},\ b_{r+1} \text{ if } k < r \leq n - 1 \end{cases} \text{ and } x_k,\, y_k = a_k + a_{k+1},\ b_k + b_{k+1},$$

so that

$$\sum_{0<s\leq n-1} (x_1 + \cdots + x_s) y_s = \sum_{0<s<k} (a_1 + \cdots + a_s) b_s + (a_1 + \cdots + a_{k+1})(b_k + b_{k+1}) +$$

$$+ \sum_{k<s\leq n-1} (a_1 + \cdots + a_{s+1}) b_{s+1} = a_{k+1} b_k + \sum_{0<s\leq n} (a_1 + \cdots + a_s) b_s.$$

Now, by the trivial lemma (2. 1), for some k, $(0 < k \leq n - 1)$, we have

$$|a_{k+1} b_k| \leq \left(\frac{1}{n-1} \sum_1^{n-1} |a_{i+1}|^p\right)^{1/p} \left(\frac{1}{n-1} \sum_1^{n-1} |b_i|^q\right)^{1/q}$$

which is certainly majorised by $(n-1)^{-\left(\frac{1}{p}+\frac{1}{q}\right)} \cdot S_{p,q}(a, b)$. Hence, with this value for k,

$$\left|\sum\sum_{0<r\leq s\leq n} a_r b_s\right| = \left|\sum_1^n (a_1 + \cdots + a_s) b_s\right| \leq |a_{k+1} b_k| + \left|\sum\sum_{0<r\leq s\leq n-1} x_r y_s\right|$$

$$\leq (n-1)^{-\left(\frac{1}{p}+\frac{1}{q}\right)} \cdot S_{p,q}(a, b) + \left|\sum\sum_{0<r\leq s\leq n-1} x_r y_s\right|.$$

A similar inequality applies to the sum $\sum\sum_{0<r\leq s\leq n-1} x_r y_s$, in terms of a sum of the same kind with $n - 2$ variables. Moreover, by definition, $S_{p,q}(a, b) \geq S_{p,q}(x, y)$. Proceeding in this way we therefore obtain finally,

$$\left|\sum\sum_{0<r\leq s\leq n} a_r b_s\right| \leq \left\{\left[(n-1)^{-\left(\frac{1}{p}+\frac{1}{q}\right)} + (n-2)^{-\left(\frac{1}{p}+\frac{1}{q}\right)} + \cdots + 1\right] + 1\right\} S_{p,q}(a, b),$$

and this implies (5. 1).

We shall see later (below § 7) that, contrary to the expectations raised by our treatment of Hölder's inequality, the factor involving $\zeta\left(\frac{1}{p} + \frac{1}{q}\right)$ cannot in (5. 1) be replaced by one remaining bounded for $\frac{1}{p} + \frac{1}{q} = 1$.

6. *The inequality for functions in an interval.* Let now $f(x)$, $g(x)$ be complex-valued functions defined in (x', x'').[1] We make a subdivision $\varkappa$,

$$x' = x_0 \le x_1 \le x_2 \le \cdots \le x_N = x'',$$

and form the sum

$$(6.1)\quad F(\varkappa) = \sum_{s=1}^{N} f(x_s)\{g(x_s) - g(x_{s-1})\} \equiv \sum_{0<r\le s\le N}\sum \varDelta_r f \cdot \varDelta_s g + f(x')[g(x'') - g(x')],$$

where $\varDelta_r \varphi$ denotes the difference $(\varphi(x_r) - \varphi(x_{r-1}))$ of a function $\varphi(x)$ at the ends of the r^{th} interval (x_{r-1}, x_r) of our subdivision.

Let us denote further by

$$S_{p,q}[x', x''] = S_{p,q}[x', x''; f, g]$$

the upper bound of the expression

$$\left(\sum_r |\varDelta_r f|^p\right)^{1/p}\left(\sum_r |\varDelta_r g|^q\right)^{1/q}$$

for every subdivision $\varkappa$ of (x', x''). Since a partition of the sequences of numbers $\varDelta_r f$, $\varDelta_r g$, $(r = 1, 2, \ldots, N)$, is a sequence of exactly the same form, corresponding to a subdivision of (x', x'') by a subset of the division points of $\varkappa$, we conclude from (5.1) and (6.1) that if ξ is the point x', and hence, more generally, *if* ξ *is a division point of* $\varkappa$ *in* (x', x''),

$$(6.2)\qquad |F(\varkappa) - f(\xi)[g(x'') - g(x')]| \le \left\{1 + \zeta\left(\frac{1}{p} + \frac{1}{q}\right)\right\} S_{p,q}[x', x''; f, g],$$

provided $p, q > 0$, $\dfrac{1}{p} + \dfrac{1}{q} > 1$.

For, this inequality, valid when $\xi = x'$, holds similarly (or by changing the sign of the variable) when $\xi = x''$. And, applying these two cases to the intervals (ξ, x'') and (x', ξ) respectively, we obtain the same inequality for any division point of $\varkappa$.

From (6.2), we now derive an inequality concerning sums of a more general kind. Let the points $\xi^{(k)}$, $x^{(k)}$ be a subset of the division points of $\varkappa$, such that for each k, $x^{(k-1)} \le \xi^{(k)} \le x^{(k)}$. Applying (6.2) to the interval $(x^{(k-1)}, x^{(k)})$ and adding, we find

[1] Always supposed closed.

$$(6.3)\quad \left| F(\varkappa) - \sum_k f(\xi_k)[g(x^{(k)}) - g(x^{(k-1)})] \right| \leq \left\{ 1 + \zeta\left(\frac{1}{p} + \frac{1}{q}\right)\right\} \sum_k S_{p,q}[x^{(k-1)}, x^{(k)}; f, g].$$

If now (x', x'') be divided in two ways into partial intervals of the form $(x^{(k-1)}, x^{(k)})$ and we select in each of these partial intervals a point $\xi^{(k)}$, it is always possible to find a subdivision $\varkappa$ including among its division points the $x^{(k)}$ and $\xi^{(k)}$ of both kinds. Hence, *if* $x' = x_0 \leq x_1 \leq \cdots \leq x_N = x''$ and $x' = x_0' \leq x_1' \leq \leq \cdots \leq x_{N'}' = x''$ *are two subdivisions and* $x_{r-1} \leq \xi_r \leq x_r$, $x_{s-1}' \leq \xi_s' \leq x_s'$, *then*

$$(6.4)\quad \left| \sum_r f(\xi_r)[g(x_r) - g(x_{r-1})] - \sum_s f(\xi_s')[g(x_s') - g(x_{s-1}')] \right|$$
$$\leq \left\{ 1 + \zeta\left(\frac{1}{p} + \frac{1}{q}\right)\right\} \left\{ \sum_r S_{p,q}[x_{r-1}, x_r] + \sum_s S_{p,q}[x_{s-1}', x_s'] \right\}$$

provided $p, q > 0$, $\dfrac{1}{p} + \dfrac{1}{q} > 1$.

7. *A »Gegenbeispiel» for the case* $\dfrac{1}{p} + \dfrac{1}{q} = 1$. We suppose, for simplicity, $p = q = 2$. Let $f(x)$ be the partial sum

$$\sum_{n=1}^{N} a^{-\frac{1}{2}n} e^{2\pi i a^n x}$$

of the complex Weierstrassian function, where a is an integer > 1, and x varies in $(0, 1)$. Let $g(x)$ be the conjugate of $f(x)$. We have[1]

$$|f(x+h) - f(x)| \leq \sum_{n=1}^{N} a^{-\frac{1}{2}n} |2 \sin(\pi a^n h)| \leq \sum_{1}^{n_0} a^{-\frac{1}{2}n} \cdot 2\pi |h| a^n + 2 \sum_{n_0+1}^{\infty} a^{-\frac{1}{2}n}$$
$$= \frac{\sqrt{|h|}}{\sqrt{a} - 1}\left\{ 2\pi \sqrt{a} \sqrt{a^{n_0}|h|} + \frac{2\sqrt{a}}{\sqrt{|h| a^{n_0+1}}} \right\} \leq \sqrt{|h|} \frac{4\sqrt{\pi a}}{\sqrt{a} - 1} \leq 32 \sqrt{|h|}$$

provided that n_0 is chosen so that $2\pi |h| a^{n_0} \leq 1 \leq 2\pi |h| a^{n_0 - 1}$. It follows that

$$S_{2,2}[0, 1; f, g] \leq (32)^2.$$

On the other hand, we can find a subdivision $\varkappa$ for which $F(\varkappa)$ differs by as little as we please, in modulus, from

[1] cf Hardy [3].

$$\int_0^1 f(x)\,dg(x) = \int_0^1 f(x)g'(x)\,dx = \sum_{n=1}^{N}\sum_{m=1}^{N} -2\pi i\, a^{-\frac{1}{2}(n+m)} a^m \int_0^1 e^{2\pi i x(a^n - a^m)}\,dx =$$

$$= \sum_{n=m} = \sum_1^N(-2\pi i) = -2\pi i N.$$

Since $g(1) = g(0)$, it follows that (6. 2) cannot hold when $\left\{1 + \zeta\left(\frac{1}{p} + \frac{1}{q}\right)\right\}$ is replaced by a factor remaining bounded for $\frac{1}{p} + \frac{1}{q} \geq 1$. The inequality (5. 1) is therefore not valid either, when such a change is made.

Other examples of the failure for $\frac{1}{p} + \frac{1}{q} = 1$ (proving rather more) are furnished by the Weierstrassian functions themselves[1], or by the simpler functions $\sqrt{x}\, e^{\pm 2\pi i/x}$. The examples may be adapted to any $p, q > 0$ subject to $\frac{1}{p} + \frac{1}{q} = 1$ and they may be further elaborated by introducing an oscillating factor $\Theta(n)$ or $\Theta(x)$ tending to zero sufficiently slowly, while its amplitude oscillates still more slowly.

8. *Higher mean variations of a function.* Following Wiener[2] (except for a slight change of notation), we associate with an $f(x)$ in (x', x'') and with $p > 0$, $\delta > 0$, the quantity

$$V_p^{(\delta)}(f) = V_p^{(\delta)}(f;\, x', x'') = \text{upper bound } \left(\sum_r |f(x_r) - f(x_{r-1})|^p\right)^{1/p}$$

for all subdivisions of (x', x'') into parts (x_{r-1}, x_r) of lengths each less than δ. The value of this quantity for a δ exceeding $(x'' - x')$ we write simply $V_p(f)$; it is then the upper bound of $V_p^{(\delta)}(f)$ as function of δ (evidently increasing). The limit of $V_p^{(\delta)}(f)$ as $\delta \to 0$ is the corresponding lower bound and we write it $V_p^*(f)$. We may call $V_p(f)$, *the mean variation of order* p.

(8. 1) *$V_p(f)$ is a decreasing function of p.* This is an immediate consequence of Jensen's inequality.[3]

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[1] i. e. the sums $\sum_1^\infty$.

[2] Wiener [11].

[3] H. L. P. p. 28 (Theorem 19).

Moreover[1], $\log \{ V_p^p(f) \}$ *is convex in* p, that is to say

$$\text{(8. 2)} \qquad \text{if } p_1 < p_2 < p_3 \text{ then } (V_{p_2}^{p_2}(f))^{p_3-p_1} \leq (V_{p_1}^{p_1}(f))^{p_3-p_2}(V_{p_3}^{p_3}(f))^{p_2-p_1},$$

by a familiar theorem.[2]

We write further

$$V_\infty^{(\delta)}(f) = \text{Osc.}\ (f, \delta)$$

for the upper bound of the difference $|f(x) - f(y)|$ when $|x - y| < \delta$, and $V_\infty(f) = \text{Osc.}\ (f) = \text{Osc.}\ (f;\ x', x'')$. These may be regarded as limiting cases for $p = \infty$, and the relations (8. 1), (8. 2) become

$$\text{(8. 1 a)} \qquad V_p(f) \geq \text{Osc.}\ (f)$$

$$\text{(8. 2 a)} \qquad V_{p_2}^{p_2}(f) \leq V_{p_1}^{p_1}(f)(\text{Osc.}\ f)^{p_2-p_1} \text{ provided } p_2 > p_1.$$

These are easily verified directly.

We shall say that $f(x)$ belongs to the Wiener class W_p

$$f(x) < W_p$$

if its p^{th} mean variation $V_p(f)$ is finite. The class of functions W_p evidently contains the Lipschitz class Lip $\left(\frac{1}{p}\right)$, which consists of the functions $f(x)$ such that, for all small $h > 0$,

$$\text{(8. 3)} \qquad |f(x + h) - f(x)| < A\, h^{1/p},$$

where A depends only on f. On the other hand W_p is contained in the Hardy-Littlewood[3] class Lip $\left(\frac{1}{p}, p\right)$, which consists of the functions satisfying the integrated condition

$$\text{(8. 4)} \qquad \int_a^{b-h} |f(x + h) - f(x)|^p dx < A\, h.$$

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[1] V_p^p denotes the p^{th} power of V_p.

[2] H. L. P. p. 28 theorem 18.

[3] Hardy, Littlewood [4].

This last condition is equivalent to

$$\int_0^h \sum_r |f(x_r + t) - f(x_{r-1} + t)|^p dt < Ah$$

where the x_r are the division points of a subdivision of $(a, b - h)$ into equal parts of length h, supposing as we may, without lose of generality, $b - a$ to be an integral multiple of h. And this is certainly satisfied when f belongs to W_p.

We shall require some simple properties of $V_p(f)$ connected with the inequalities of Minkowski and Hölder.

(8. 5) If $p > 1$, $V_p(f + g) \leq V_p(f) + V_p(g)$.

To prove this, we remark that, by the well-known inequality of Minkowski,

$$\Big(\sum_\Delta |\Delta f + \Delta g|^p\Big)^{1/p} \leq \big(\sum |\Delta f|^p\big)^{1/p} + \big(\sum |\Delta g|^p\big)^{1/p}$$

for any subdivision into intervals Δ (we denote by $\Delta\varphi$ the difference of a function φ at the ends of Δ), and (8. 5) follows by taking the upper bound on the left.

Since, for constant λ, $V_p(\lambda f) = |\lambda|\, V_p(f)$, we may also express (8. 5) by saying that *$V_p(f)$ is a convex function of f.*

$$(8.6) \qquad \Big\{\sum_r V_p^p(f;\, x_{r-1}, x_r)\Big\}^{1/p} \leq V_p^{(\delta)}(f)$$

for every subdivision of (x', x'') into (x_{r-1}, x_r), $(r = 1, \ldots, N)$, each of length less than δ. This is evident.

(8. 7) *If $p, q > 0$ and $\frac{1}{p} + \frac{1}{q} \geq 1$ then*

$$\sum_r V_p(f;\, x_{r-1}, x_r)\, V_q(g;\, x_{r-1}, x_r) \leq V_p^{(\delta)}(f)\, V_q^{(\delta)}(g)$$

for every subdivision of (x', x'') into x_{r-1}, x_r, $(r = 1, \ldots, N)$ *each of length* $< \delta$. This follows at once from (8. 6) since the left hand side, by Hölder's inequality, is at most

$$\Big\{\sum_r V_p^p(f;\, x_{r-1}, x_r)\Big\}^{1/p} \Big\{\sum_r V_q^q(g;\, x_{r-1}, x_r)\Big\}^{1/q}.$$

9. *The starred variation and the singular variation.* By analogy with the case $p = 1$ (bounded variation in the usual Jordan sense), we easily obtain various simple properties of functions of W_p, and the proofs require only trivial adaptation. Thus, an f of W_p has at most simple discontinuities, and these are enumerable.[1] This being so, we may introduce the symbols $x \pm 0$ in the usual way as arguments, and we see at once that when (x', x'') is divided at an increasing finite sequence of x_r^*, $(r = 0, 1, \ldots, N)$, with $x_0^* = x'$, $x_N^* = x''$ and the remaining divisions at places x or $x \pm 0$, then, if each (x_{r-1}^*, x_r^*) has length less than δ,

$$\left(\sum_{r=1}^{N} |f(x_r^*) - f(x_{r-1}^*)|^p\right)^{1/p} \leq V_p^{(\delta)}(f). \tag{9. 1}$$

We shall call *singular mean variation of order p*, the quantity $\mathfrak{S}_p(f)$ whose p^{th} power is the series, arising from the discontinuities of f, (that we suppose arranged as a sequence) and having for its general term, the greater of the two expressions

$$|f(x+0) - f(x-0)|^p, \ |f(x+0) - f(x)|^p + |f(x) - f(x-0)|^p.$$

It follows from (9. 1) that the series $\mathfrak{S}_p^p(f)$, that is to say

$$\sum_x \operatorname{Max} \{ |f(x+0) - f(x-0)|^p, \ |f(x+0) - f(x)|^p + |f(x) - f(x-0)|^p \} \tag{9. 2}$$

is majorised by $[V_p^{(\delta)}(f)]^p$ for every $\delta > 0$. Consequently

$$\mathfrak{S}_p(f) \leq V_p^*(f). \tag{9. 3}$$

Let us observe that the singular variation cannot in general be regarded as a variation of a »Singular function» of the type

$$\sum_{y < x} (f(y+0) - f(y-0)) + (f(x) - f(x-0)).$$

This series need not be convergent in any sense, and the singular part of f cannot therefore in general be detached from the remainder.

We shall say that $f(x)$ *belongs to the Wiener class* W_p^*, *if* $V_p^*(f)$ *is finite and equals* $\mathfrak{S}_p(f)$. Evidently, if $V_p^*(f)$ is finite so is $V_p^{(\delta)}(f)$ for all small δ, and

[1] Wiener [11] § 1.

therefore for all δ.[1] Hence *W_p^* is the subclass of W_p for which equality holds in* (9.3). It is easy to see that W_p^* includes the class $\mathrm{Lip}^*\left(\frac{1}{p}\right)$ obtained by strengthening the A in (8.3) to an ε_h that tends to zero with h. W_p^* is not, however, included in the class $\mathrm{Lip}^*\left(\frac{1}{p}, p\right)$ obtained by making the corresponding change in (8.4), unless we restrict ourselves to continous functions.

We shall now show that *if $p < p_1$ the class W_p is included in $W_{p_1}^*$, that is*

$$W_p < W_{p_1}^*, \text{ if } p < p_1. \tag{9.4}$$

This is implicitly contained in Wiener's discussion[2], but seems worthy of special mention.

Given ε, we choose discontinuities $\xi_1, \ldots, \xi_{n_0}$ so that the remainder of the series[3] for $\mathfrak{S}_{p_1}^{p_1}(f)$ is less than ε^{p_1}. It follows that, apart from the ξ_i, there are no points at which any of the quantities

$$|f(x+0) - f(x-0)| \text{ and } |f(x \pm 0) - f(x)|$$

exceed ε. It is therefore possible to choose a δ_0 less than the distance of every pair of the ξ_i, $(i = 1, \ldots, n_0)$, so that

$$|f(\beta) - f(\alpha)| < 2\varepsilon$$

whenever (α, β) is an interval of length less than δ_0, not containing a ξ_i. And since the limits $f(\xi_i \pm 0)$ exist, we can choose $\delta_1 \leq \delta_0$ so that, whenever α and β are separated by a ξ_i distant less than δ_1 from both, the differences

$$|f(\beta) - f(\alpha)|^{p_1} \text{ and } |f(\beta) - f(\xi_i)|^{p_1} + |f(\xi_i) - f(\alpha)|^{p_1}$$

exceed by at most ε/n_0 the corresponding expressions with α and β replaced by $\xi_i - 0$ and $\xi_i + 0$.

This being so, it is clear that a sum of the form

$$\sum_r |f(x_r) - f(x_{r-1})|^{p_1}$$

with $0 \leq x_r - x_{r-1} < \delta_1$, will be majorised by

[1] In particular the limits $f(x \pm 0)$ exist for all x and we can form the expression $\mathfrak{S}_p(f)$.

[2] Wiener [11] § 2.

[3] Convergent by (8.1) and (9.3).

$$(2\,\varepsilon)^{p_1-p} \sum_r |f(x_r) - f(x_{r-1})|^p + n_0 \cdot \varepsilon/n_0 + \mathfrak{S}_{p_1}^{p_1}(f),$$

so that (9. 4) must hold, as asserted.

10. *Existence of the Stieltjes integral.* We say that the Stieltjes integral

$$\int_{x'}^{x''} f(x)\,d\,g(x) \tag{10. 1}$$

exists in the Riemann sense with the value I, if the sum

$$\sum_{r=1}^{N} f(\xi_r)\,[g(x_r) - g(x_{r-1})], \tag{10. 2}$$

(in which, as usual, $x_1 = x_0 \leq \xi_1 \leq x_1 \leq \cdots \leq x_{N-1} \leq \xi_N \leq x_N = x''$) differs from I by at most ε_δ in modulus, as soon as all the (x_{r-1}, x_r) have lengths less than δ, where $\varepsilon_\delta \to 0$ as $\delta \to 0$.

The integral (10. 1) exists in the Moore-Pollard[1] sense with the value I, if there is, for each $\varepsilon > 0$, a finite set of points E such that the sum (10. 2) differs from I by at most ε in modulus as soon as the x_r include all points of E.

Finally, if the limits $f(x \pm 0)$ and $g(x \pm 0)$ exist for all x, we shall say that (10. 1) exists in the generalised Moore-Pollard sense, if the integral

$$\int_{x'}^{x''} f(x + 0)\,d\,g(x - 0) \tag{10. 3}$$

exists in the Moore-Pollard sense and the series

$$\sum_x (f(x) - f(x + 0))\,\{g(x + 0) - g(x - 0)\} \tag{10. 4}$$

summed over the (necessarily enumerable) set of common discontinuities of f and g is absolutely convergent. And we then assign, as value, to (10. 1), the sum of (10. 3) and (10. 4).

We observe that, for integrability in the Riemann sense, it is sufficient that the difference of any two sums (10. 2), for each of which the (x_{r-1}, x_r) have

[1] Pollard [9]. The idea is derived from various earlier papers by E. H. Moore.

lengths less than δ, be less than ε_δ in modulus. This will be the case, by (6. 4), if, for some $p, q > 0$ satisfying $\frac{1}{p} + \frac{1}{q} > 1$, we have

$$\sum_{r=1}^{N} S_{p,q}[x_{r-1}, x_r] < \varepsilon_\delta. \tag{10. 5}$$

Now it is clear from the definitions that, in any interval,

$$S_{p,q} \leq V_p(f) V_q(g). \tag{10. 6}$$

Thus, a sufficient condition for integrability in the Riemann sense is that there exist $p_1, q_1 > 0$ satisfying $\frac{1}{p_1} + \frac{1}{q_1} > 1$ such that

$$\sum_{r=1}^{N} V_{p_1}(f; x_{r-1}, x_r) V_{q_1}(g; x_{r-1}, x_r) < \varepsilon \tag{10. 7}$$

for all subdivisions into partial intervals (x_{r-1}, x_r) whose lengths are less than a certain δ depending on ε.

Similarly, in the Moore-Pollard sense, it is sufficient that (10. 7) hold for all subdivisions whose division points x_r include those of a certain finite set E depending on ε.

We shall have occasion, several times to use the following *lemma:*

(10. 8) *Let f and g belong respectively to the classes W_p and W_q, and suppose that, in each of the non overlapping intervals (α, β),*

$$\text{Osc. } f < \eta.$$

Then, for $p_1 > p > 0$, $q_1 \geq q > 0$, $\frac{1}{p_1} + \frac{1}{q_1} \geq 1$, we must have

$$\sum_{(\alpha,\beta)} V_{p_1}(f; \alpha, \beta) V_{q_1}(g; \alpha, \beta) \leq \eta^{(p_1 - p)/p_1} V_p^{p/p_1}(f) V_{q_1}(g).$$

The proof is immediate. By (8. 2 a) the left hand side is at most

$$\sum_{(\alpha,\beta)} [\eta^{p_1 - p} V_p^p(f; \alpha, \beta)]^{1/p_1} V_{q_1}(g; \alpha, \beta) \leq \eta^{(p_1-p)/p_1} \left[\sum_{(\alpha,\beta)} V_p^p(f; \alpha, \beta)\right]^{1/p_1} \cdot \left[\sum_{\alpha,\beta} V_{q_1}^{q_1}(g; \alpha, \beta)\right]^{1/q_1},$$

by Hölder's inequality. And this, as in (8, 6), is evidently at most equal to the quantity on the right of the inequality to be proved.

Theorem on Stieltjes integrability. *If an $f(x)$ of W_p and a $g(x)$ of W_q where*

$p, q > 0$, $\frac{1}{p} + \frac{1}{q} > 1$, *have no common discontinuities, their Stieltjes integral exists in the Riemann sense. If they have no common discontinuities on the right and no common discontinuities on the left, the integral exists in the Moore-Pollard sense. And in any event, it exists in the generalised sense.*

We determine, as in (9.4), n_0 discontinuities of f and n_0 discontinuities of g, and a positive δ_0, so that in every interval of length $< \delta_0$ not containing any of these first n_0 discontinuities of one of the functions, the function concerned has oscillation at most $2\varepsilon_0$, where $\varepsilon_0 > 0$ is arbitrarily chosen.

If f and g have no common discontinuities, we can determine $\delta_1 \leq \delta_0$, so that no interval of length less than δ_1 can contain together one of the first n_0 discontinuities of f and one of the first n_0 discontinuities of g. In that case, for any subdivision of (x', x'') into partial intervals (x_{r-1}, x_r) of lengths less than δ_1, in each (x_{r-1}, x_r) one of the two functions has oscillation less than $2\varepsilon_0$. Choosing $p_1 > p$, $q_1 > q$ so that $\frac{1}{p_1} + \frac{1}{q_1} > 1$, it follows from our lemma that the left hand side of (10.7) can be made arbitrarily small by choice of ε_0, and therefore of δ_1, by restricting the $x_r - x_{r-1}$ to be less than δ_1. This proves the first part.

Similarly if f and g have no common one-sided discontinuity, let E_1 be the combined set of the first n_0 discontinuities of each. On each side of a point ξ of E_1, one of our functions is continuous, and therefore we can find points ξ_- and ξ_+ on the two sides of ξ so that in each of the intervals (ξ_-, ξ), (ξ, ξ_+) one of our functions has oscillation less than $2\varepsilon_0$. Denoting by E a finite set of points, each distant less than δ_0 from its neighbours, that includes the ξ of E_1 together with the ξ_- and ξ_+, it is clear that in the interval determined by any two neighbouring points of E the oscillation of one of our functions will be less than $2\varepsilon_0$. From this, the second part, referring to the Moore-Pollard sense, follows at once by the argument of the first part.

Finally, if f and g are unrestricted in the classes W_p, W_q, the functions $f(x+0)$ and $g(x-0)$ belong to the same classes and have no common onesided discontinuity. Moreover the series (10.4) summed over the discontinuities common to f and g is absolutely convergent by Hölder's inequality, since

$$(\Sigma |f(x) - f(x+0)|^p)^{1/p} \quad \text{and} \quad (\Sigma |g(x+0) - g(x-0)|^q)^{1/q}$$

are majorised by $\mathfrak{S}_p(f)$ and $\mathfrak{S}_q(g)$ respectively. The integral (10.1) thus exists in the generalised Moore-Pollard sense in this case, and this completes the proof.

Remarks on the theorem. In the elementary Stieltjes integral, two conditions are used: continuity of one function, bounded variation of the other. The treatment introduced here consists essentially in dividing up the conditions of continuity and bounded variation between the two functions concerned. Instead of *one* function being W_1, *each* has to satisfy a weaker condition W_p or W_q where $\frac{1}{p} + \frac{1}{q} > 1$. Instead of *one* function being continuous at all x, *each* is continuous wherever the other is not, (with, in the Moore-Pollard sense, the further refinement each is continuous on one side, wherever the other is not continuous on that side). In the generalised Moore-Pollard sense, continuity has been abandonned, but the definition no longer corresponds to a unique limit in the Cauchy or in the Moore sense of a function of a subdivision, and is to be regarded as a modification, not of the elementary Stieltjes integral, but rather of the so-called Lebesgue-Stieltjes integral.

Let us remark further that, in whichever of the three senses our integral is taken[1], the inequality (6.2) gives at once, under the hypotheses of our theorem of existence of the integral

$$(10.9) \qquad \left| \int_{x'}^{x''} (f(x) - f(\xi))\, d g(x) \right| \leq \left\{ 1 + \zeta\left(\frac{1}{p} + \frac{1}{q}\right) \right\} S_{p,q}[x', x''; f, g]$$

where, on the right, $S_{p,q}[x', x''; f, g]$ may be replaced by the product

$$V_p(f; x', x'')\, V_q(g; x', x'').$$

11. *Integration of sequences.* By a *W_p-sequence* $\{f_n(x)\}$, we shall mean a sequence of functions $f_n(x)$, $n = 1, 2, \ldots$, for which $f_n(x')$ and $V_p(f_n)$ are bounded functions of n. Such a sequence will be said to be *densely convergent* in (x', x'') to a limit function $f(x)$, if $f_n(x)$ tends to $f(x)$ for each x of an everywhere dense set in (x', x''). When this is the case, it is always possible to define $f(x)$ outside this set in such a manner that $f(x)$ is of W_p.[2]

Besides the notion of dense convergence, we require that of uniform convergence. The latter is the closest analogue for a sequence of functions to the property of continuity for a single function. We shall say that *a sequenee $\{f_n(x)\}$ converges uniformly to $f(x)$ at x_0*, if given ε, there is an n_0 and a δ, such that, for all $n > n_0$ and all x distant less than δ from x_0,

[1] It is sufficient to observe that the integral is the limit of a suitable finite sum $F(x)$.

[2] Just as in the case $p = 1$ discussed by Helly [6].

$$|f_n(x) - f(x)| < \varepsilon.$$

We shall also speak of *uniform convergence on the right at* x_0 when the above holds for $x \geqq x_0$, the other conditions being the same, and, similarly, of *uniform convergence on the left at* x_0.

The notions of a W_p-sequence and of uniform convergence at a point, will play in term by term integration of a pair of sequences of functions, a part similar to that of the corresponding notions in the existence of the integral of a pair of functions. The classical theorems on term by term integration, all depend essentially on some condition of absolute integrability. In the Lebesgue theorems, the great generality achieved in other directions necessitates a particularly strong form of this condition. A theorem much closer to those that we shall be concerned with here, is due to Helly.[1] It is the analogue for sequences of the existence of the elementary Stieltjes integral, and it states, substantially, that a sufficient condition for

$$\lim_{n \to \infty} \int_{x'}^{x''} f_n(x)\, d\, g_n(x) = \int_{x'}^{x''} f(x)\, d\, g(x)$$

when $\{f_n(x)\}$ and $\{g_n(x)\}$ converge to finite functions $f(x), g(x)$ respectively, is that the $f_n(x)$ be uniformly (in n) continuous (in x) and the $g_n(x)$ uniformly (in n) of bounded variation in x. Our conditions for term by term integration may be regarded as derived from those of Helly by assigning the properties of uniform continuity, or uniform convergence, and of uniform bounded variation, partly to the sequence $\{f_n\}$ and partly to the sequence $\{g_n\}$.

We begin with a *lemma.*

Let f be a function belonging to the Wiener class W_p, and let $\{g_n\}$ be a W_q-sequence converging to zero densely in (x', x''). Suppose further that $p_1 > p$, $q_1 > q$ satisfy $\frac{1}{p_1} + \frac{1}{q_1} > 1$, and that (x', x'') can be divided into a finite number of partial intervals (x_{r-1}, x_r) in each of which either

$$\text{Osc. } f < \eta$$

or

$$\text{upper bound } |g_n(x)| < \frac{1}{2}\eta, \text{ for each large } n.$$

Then, for all large n,

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[1] Helly [6].

$$(11.1)\quad \left|\int_{x'}^{x''} g_n\, df\right| < \varepsilon_n + \left\{1 + \zeta\left(\frac{1}{p_1} + \frac{1}{q_1}\right)\right\} \eta^{(p_1-p)/p_1} V_p^{p/p_1}(f)\, V_{q_1}(g_n) + $$
$$+ \left\{1 + \zeta\left(\frac{1}{p_1} + \frac{1}{q_1}\right)\right\} \eta^{(q_1-q)/q_1} V_{p_1}(f)\, V_q^{q/q_1}(g_n),$$

where $\varepsilon_n \to 0$ *as* $n \to \infty$. *The same inequality applies also to* $\left|\int_{x'}^{x''} f\, dg_n\right|$, *provided that* g_n *tends to zero at* x' *and* x''.

The proof of this lemma is very simple. In each (x_{r-1}, x_r) we choose a point ξ_r at which $g_n \to 0$. We have

$$\int_{x'}^{x''} g_n\, df = \sum_r g_n(\xi_r)\,(f(x_r) - f(x_{r-1})) + \sum_r \int_{x_{r-1}}^{x_r} (g_n(x) - g_n(\xi_r))\, df.$$

The first sum on the right evidently tends to zero as $n \to \infty$. The second is majorised by

$$\left\{1 + \zeta\left(\frac{1}{p_1} + \frac{1}{q_1}\right)\right\} \sum_r V_{p_1}(f; x_{r-1}, x_r)\, V_{q_1}(g_n; x_{r-1}, x_r)$$

on account of (10.9). Making use of (10.8) we obtain (11.1). The corresponding inequality for $\int f\, dg_n$ may be obtained similarly, or by integration by parts.[1]

The following result, of some importance in applications, is an immediate consequence of the lemma just proved.

(11.2) *Let* f *be a function of* W_p *and* $\{g_n\}$ *a* W_q-*sequence, where* $p, q > 0$, $\frac{1}{p} + \frac{1}{q} > 1$. *Suppose that* $\{g_n\}$ *converges densely, and at the ends* x', x'', *to a function* g *of* W_q, *and that* $\{g_n\}$ *converges uniformly to* g *at each discontinuity of* f. *Then*

$$\int_{x'}^{x''} f\, dg_n \to \int_{x'}^{x''} f\, dg.$$

To see this, we determine, as on several previous occasions, a finite set of discontinuities of f, such that in any interval whose length does not exceed a

[1] Cf Pollard [9]

certain δ_0 and which contains no point of the finite set, the oscillation of f is less than η. Since $g_n - g$ converges uniformly to zero at each point of the finite set, we can determine a positive δ_1 not exceeding δ_0, and an integer n_1, so that for all $n > n_1$, the upper bound of $(g_n - g)$ in any interval containing a point of the finite set and having length less than δ_1, is less than $\frac{1}{2}\eta$. By choice of η, it follows from (11.1) that the upper limit, as $n \to \infty$, of $\left|\int f\, d(g_n - g)\right|$ is arbitrarily small and (11.2) follows at once.

Theorem on term by term integration. *Let $\{f_n\}$ be a W_p-sequence converging densely to an f of W_p and converging uniformly to f at each point of a set A. Let $\{g_n\}$ be a W_q-sequence converging densely, and at the ends x', x'', to a g of W_q, and converging uniformly to g at each point of a set B. Suppose further that $p, q > 0$, $\frac{1}{p} + \frac{1}{q} > 1$, and that A includes the discontinuities of g, B those of f, $A + B$ all points of (x', x''). Then*

$$\int_{x'}^{x''} f_n\, dg_n \to \int_{x'}^{x''} f\, dg$$

Proof. Given $\varepsilon > 0$, since $A + B$ includes all x of the (closed) interval (x', x''), each x is contained in a neighbourhood in which, for all large n, the upper bound of one of the expressions $|f_n - f|$, $|g_n - g|$ is less than $\frac{1}{2}\varepsilon$. By Borel's covering theorem we can divide (x', x'') into a finite number of intervals, separated by points at which $g_n \to g$, such that, in some of these intervals, the intervals (α, β) say, the upper bound of $|f_n - f|$ is less than $\frac{1}{2}\varepsilon$ for all n greater than a certain n_1, and the oscillation of $(f_n - f)$ therefore less than ε, while in the remainder, the intervals (γ, δ) say, the oscillation of $(g_n - g)$ is less than ε.

In an (α, β) we write

$$\int_\alpha^\beta f_n\, dg_n - \int_\alpha^\beta f\, dg = \int_\alpha^\beta (f_n - f)\, dg_n + \int_\alpha^\beta f\, d(g_n - g).$$

Since, at any discontinuity of f in (α, β), $g_n - g$ tends uniformly to zero, it follows, by the construction so often repeated, that (α, β) can be divided into a finite number of intervals, in each of which either Osc. $f < \varepsilon$, or, upper bound $|g_n - g| < \frac{1}{2}\varepsilon$.

In a (γ, δ), we write

$$\int_\gamma^\delta f_n\,dg_n - \int_\gamma^\delta f\,dg = \int_\gamma^\delta f_n\,d(g_n - g) + \int_\gamma^\delta (f_n - f)\,dg$$

and remark similarly, that (γ, δ) can be divided into a finite number of parts in each of which either Osc. $g < \varepsilon$, or, upper bound $|f_n - f| < \frac{1}{2}\varepsilon$.

Now choose $p_1 > p, q_1 > q$ so that $\frac{1}{p_1} + \frac{1}{q_1} > 1$. We have

$$(11.3) \quad \left|\int_{x'}^{x''} f_n\,dg_n - \int_{x'}^{x''} f\,dg\right| \leq \sum_{(\alpha,\beta)} \left|\int_\alpha^\beta (f_n - f)\,dg_n\right| +$$

$$+ \sum_{(\gamma,\delta)} \left|\int_\gamma^\delta f_n\,d(g_n - g)\right| + \sum_{(\alpha,\beta)} \left|\int_\alpha^\beta f\,d(g_n - g)\right| + \sum_{(\gamma,\delta)} \left|\int_\gamma^\delta (f_n - f)\,dg\right|.$$

On account of (10.9) and lemma (10.8), the first two sums are majorised for all large n, by

$$\left\{1 + \zeta\left(\frac{1}{p_1} + \frac{1}{q_1}\right)\right\}\left[\varepsilon^{(p_1-p)/p_1} V_p^{p/p_1}(f_n - f)\, V_{q_1}(g_n) + \varepsilon^{(q_1-q)/q_1} V_{p_1}(f_n)\, V_q^{q/q_1}(g_n - g)\right],$$

while, on account of our lemma (11.1), the last two are majorised, for all large n, by

$$N\varepsilon_n + \left\{1 + \zeta\left(\frac{1}{p_1} + \frac{1}{q_1}\right)\right\}\left[\varepsilon^{(p_1-p)/p_1} V_p^{p/p_1}(f)\, V_{q_1}(g_n - g) + \varepsilon^{(q_1-q)/q_1} V_{p_1}(f)\, V_q^{q/q_1}(g_n - g) +\right.$$

$$\left. + \varepsilon^{(p_1-p)/p_1} V_p^{p/p_1}(f_n - f)\, V_{q_1}(g) + \varepsilon^{(q_1-q)/q_1} V_{p_1}(f_n - f)\, V_q^{q/q_1}(g)\right],$$

where N is the number of (α, β) and (γ, δ). Since, for all large n, $N\varepsilon_n < \varepsilon$, the left hand side of (11.3) is arbitrarily small for all large n, and this proves the theorem.

The theorem just proved, which corresponds to the existence theorem in the Riemann sense, can be slightly extended in accordance with the ideas of the

Moore-Pollard definition, and the proof requires only minor alterations. We shall content ourselves with stating the theorem in the extended form without proof.

Let $\{f_n\}$ be a W_p-sequence converging densely to an f of W_p, and converging to f uniformly on the right at each point of a set A_+, and uniformly on the left at each point of a set A_-. Let $\{g_n\}$ be a W_q-sequence converging densely and at the ends x', x'' to a g of W_q and converging to g uniformly on the right at each point of a set B_+ and uniformly on the left at each point of a set B_-. Suppose further that $p, q > 0$, $\frac{1}{p} + \frac{1}{q} > 1$, that A_+ includes the right hand discontinuities of g, A_- the left hand discontinuities of g, B_+ the right hand discontinuities of f, B_- the left hand discontinuities of f, and finally that $A_+ + A_- + B_+ + B_-$ includes all points of (x', x''). Then

$$\int_{x'}^{x''} f_n \, dg_n \to \int_{x'}^{x''} f \, dg.$$

12. *Fourier series.* It is well known that the depth of the convergence problem for Fourier series is largely due to the fact that, in the expression for the Fourier partial sums of a function $f(x)$,

$$S_n(x) = \varphi(x) + \frac{1}{\pi} \int_0^{\pi} \left\{ \frac{f(x+t) + f(x-t)}{2} - \varphi(x) \right\} d g_n(t), \tag{12.1}$$

(where $\varphi(x)$ is arbitrary), the functions

$$g_n(t) = \int_0^t \frac{\sin\left(n + \frac{1}{2}\right) t}{\sin \frac{1}{2} t} \, dt \tag{12.2}$$

do not satisfy uniformly the condition of bounded variation W_1 so essential to all classical theorems on term by term integration.

It is therefore of some interest, that, while the $g_n(t)$ of (12.2) do not fulfill the classical condition W_1, they have, neverless, for every $q > 1$, uniformly bounded mean variation of order q. Similar remarks apply to the functions $g_n^{(\gamma)}(t)$ arising from the corresponding expression for the Cesaro means of negative order $\gamma > -1$. These functions form a W_q-sequence for every $q > 1/(1+\gamma)$. It is trivial that, for Cesaro means of positive order, the corresponding $g_n^{(\gamma)}(t)$ have uniformly bounded variation, and it is not unnatural that something should remain of this property when $\gamma = 0$, the case of the partial sums $s_n(x)$, and even when $0 > \gamma > -1$. This »something» is W_q for $q > 1/(1+\gamma)$. It is curious that such a simple property should have escaped notice until now. The full result, together with similar information relating to the allied series, is as follows:

Theorem on mean variations of the Fourier kernels. *For $\gamma > -1$, the Cesaro means of order γ of the Fourier series of $f(x)$ may be written*

$$(12.3) \qquad S_n^{(\gamma)}(x) = \varphi(x) + \frac{1}{\pi}\int_0^\pi \left(\frac{f(x+t)+f(x-t)}{2} - \varphi(x)\right) d g_n^{(\gamma)}(t),$$

where the functions $g_n^{(\gamma)}(t)$ which vanish for $t = 0$ and tend uniformly to 1 at each $t \neq 0$, have, for every $q \geq 1$ satisfying $q(1+\gamma) > 1$, uniformly bounded mean variation of order q. Moreover, if $S_n^*(x)$ is the partial sum (to $2n+1$ terms) of the allied series, then

$$(12.4) \quad S_n^* - \frac{1}{2\pi}\int_{\pi/2n}^{\pi} (f(x+t) - f(x-t)) \cot\frac{1}{2} t\, dt = \frac{1}{2\pi}\int_0^\pi (f(x+t)-f(x-t))\, d g_n^*(t),$$

where the $g_n^*(t)$, which are bounded at $t = 0$, and tend uniformty to zero at each $t \neq 0$, have, for every $q > 1$, uniformly bounded mean variation of order q.

Proof. We make use of the known fact that (12.3) is valid with

$$g_n^{(\gamma)}(t) = \int_0^t \Omega(t)\, dt,$$

where[1], for $-1 < \gamma \leq 0$, (and indeed for $-1 < \gamma < 1$),

$$\Omega = \Omega_1 + \Omega_2, \quad |\Omega| \leq K_n, \quad |\Omega_2| \leq K/n t^2,$$

$$\Omega_1 = \frac{\Gamma(\gamma+1)\,\Gamma(n+1)}{\Gamma(n+\gamma+1)} \frac{\sin\left(\left(n+\frac{1}{2}\gamma+\frac{1}{2}\right)t - \frac{1}{2}\gamma\pi\right)}{2^\gamma\left(\sin\left(\frac{1}{2}t\right)\right)^{\gamma+1}},$$

and the K_s are independent of n, t.

[1] Kogbetliantz [6], Hardy and Littlewood [4].

And (12.4) is valid with $g_n^*(t) = -\int_t^\pi \Omega^*(t)\,dt$, where $\Omega^*(0) = 0$, $\Omega^*(t) = \sin nt + \cot\frac{1}{2}t(1 - \cos nt)$ for $0 < t \leq \pi/2n$, and $\Omega^*(t) = \sin nt - \cot\frac{1}{2}t\cos nt$ for $t > \pi/2n$, since (12.4) then reduces to the known formula[1]

$$s_n^*(x) = \frac{1}{2\pi}\int_0^\pi (f(x+t) - f(x-t))\left\{\sin nt + \cot\frac{1}{2}t\,(1 - \cos nt)\right\}dt.$$

Let now $a_m = \left(m + \frac{1}{2}\gamma\right)\pi\Big/\left(n + \frac{1}{2}\gamma + \frac{1}{2}\right)$ for $m = 1, 2, \ldots, n$, and $a_0 = 0$, $a_{n+1} = \pi$. From the expression for Ω_1, it follows, by the second mean value theorem, that

$$\int_\alpha^\beta \Omega_1(t)\,dt = \frac{\Gamma(\gamma+1)\,\Gamma(n+1)}{2^\gamma\,\Gamma(n+\gamma+1)}\cdot\left\{\frac{\int_\alpha^\xi \sin\left(\left(n + \frac{1}{2}\gamma + \frac{1}{2}\right)t - \frac{1}{2}\gamma\pi\right)dt}{\left(\sin\frac{1}{2}\alpha\right)^{\gamma+1}} + \frac{\int_\xi^\beta \sin\left(\left(n + \frac{1}{2}\gamma + \frac{1}{2}\right)t - \frac{1}{2}\gamma\pi\right)dt}{\left(\sin\frac{1}{2}\beta\right)^{\gamma+1}}\right\}$$

for some ξ in (α, β). If we perform the integrations on the right and then majorise crudely $\left[\text{the gamma factors are } O(n^{-\gamma}),\ |\pm\cos x| \leq 1, \text{ and } \sin\frac{1}{2}\beta \geq \sin\frac{1}{2}\alpha \geq \alpha/\pi \geq O((m+1)/n)\right]$ we get a majorant $K/(m+1)^{\gamma+1}$, with K depending only on γ. And combining this with the obvious inequality $\int_\alpha^\beta |\Omega_2(t)|\,dt \leq K/m \leq K/(m+1)^{\gamma+1}$, (for $-1 < \gamma \leq 0$), we find that, for (α, β) in (a_m, π), $m \neq 0$, and $-1 < \gamma \leq 0$,

$$\left|\int_\alpha^\beta \Omega_1(t)\,dt\right| + \int_\alpha^\beta |\Omega_2(t)|\,dt \leq K/(m+1)^{\gamma+1},$$

[1] Zygmund [12] p. 21.

a relation in which it will be remarked that, when $\beta \leq a_{m+1}$, the modulus of the first integral is the integral of $|\Omega_1|$, since the integrand is of constant sign. Moreover

$$\int_0^{a_1} |\Omega(t)|\, dt < K$$

since $a_1 < K/n$. Hence, for any (α, β) in (a_m, π), $m = 0, 1, \ldots, n+1$, we have, for $-1 < \gamma \leq 0$,

$$\left| \int_\alpha^\beta \Omega(t)\, dt \right| \leq K/(m+1)^{\gamma+1} \tag{12.5}$$

and for any (α, β) in (a_m, a_{m+1}), $(m = 0, 1, \ldots, n+1)$

$$\int_\alpha^\beta |\Omega(t)|\, dt \leq K/(m+1)^{\gamma+1}. \tag{12.6}$$

Similar inequalities hold for Ω^*, with $\gamma = 0$ and with $a_m = \left(m - \frac{1}{2}\right)\pi/n$ for $m = 1, \ldots, n$ and $a_0 = 0$, $a_{n+1} = \pi$. They are obtained in the same way.

Consider now any subdivision of $(0, \pi)$ by a finite increasing sequence of points of division x_r. The sum

$$\sum_r \left| \int_{x_{r-1}}^{x_r} \Omega(t)\, dt \right|^q$$

can be split up into $\overline{n+1}$ groups of terms for which (x_{r-1}, x_r) lie in the $\overline{n+1}$ portions (a_m, a_{m+1}), together with single terms arising from the values of r for which there is an inequality of the type $x_{r-1} < a_m < x_r$, at most one such r corresponding to each m. By (8.1) the groups of terms are majorised by the corresponding q^{th} powers of the right hand side of (12.6). The additional single terms are evidently majorised by the q^{th} powers of the right hand side of (12.5). Our sum is thus at most

$$K \sum_1^\infty m^{-q(\gamma+1)} = K\, \zeta(q(\gamma+1))$$

provided $q(\gamma+1)>1$. This is the same as our assertion of uniformly bounded mean variation of $g_n^{(\gamma)}(t)$ for $-1<\gamma\leq 0$. The case $\gamma>0$ is disposed of by trivial inequalities, while the corresponding assertion relating to the allied series is established in the same way as for $g_n^{(\gamma)}(t)$.

Finally, as regards convergence, we have $g_n^{(\gamma)}(0)=0$ and, since (12.3) is valid with $\varphi(x)$ arbitrary, $g_n^{(\gamma)}(\pi)=1$. By (12.5), $|g_n^{(\gamma)}(\pi)-g_n^{(\gamma)}(t)|$ tends uniformly to zero, when t exceeds a fixed positive δ, arbitrarily small, provided $-1<\gamma\leq 0$. The corresponding assertions for $\gamma>0$ and for the allied series are obtained similarly, and this completes the proof.

From the theorem just proved, we see that the notion of mean variation has a natural connection with Fourier series. We now consider some properties of the latter, for functions of W_p.

Theorem on the Fourier series of a function of W_p. (i) *If f belongs to W_p and has period 2π its Fourier coefficients are* $O\left(n^{\left(-\frac{1}{p}+\varepsilon\right)}\right)$.[1] (ii) *The Fourier series, together with its Cesaro means of order greater than* $-\frac{1}{p}$, *converges at each x to the value* $\frac{1}{2}(f(x+0)+f(x-0))$. (iii) *At a point x of continuity of f, the convergence is uniform and moreover the difference between the partial sum $s_n^*(x)$ of the allied series and the integral*

$$\frac{1}{2\pi}\int_{\pi/2n}^{\pi}(f(x+t)-f(x-t))\cot\frac{1}{2}t\,dt \tag{12.7}$$

converges uniformly to zero.

(i)[1] We have, (for n positive or negative), by (10.9)

$$|2\pi n c_n|=\left|\int_{-\pi}^{\pi}(f(t)-f(0))\,d(e^{int})\right|\leq K\,V_q(e^{int})$$

provided that $\frac{1}{p}+\frac{1}{q}>1$. It will therefore suffice to show that $V_q(e^{int})$ is $O(n^{1/q})$, that is to say, $V_q^q(e^{int})=O(n)$.

[1] The stronger result $O(n^{-1/p})$ is also true when f belongs to W_p and even when f belongs merely to Lip $\left(\frac{1}{p},p\right)$, *vide* Hardy and Littlewood (4). The two results are eqvivalent in the case (that usually arises) when f belongs to W_p for an open segment of values of p.

Now in forming the upper bound of

$$\sum_{(\alpha,\beta)} | e^{in\beta} - e^{in\alpha} |^q$$

for non-overlapping (α, β), we may suppose, by periodicity, $\beta - \alpha < 2\pi/n$. And the upper bound for non-overlapping (α, β) that lie in a same interval of length $2\pi/n$ is, by (8. 1) at most the total variation raised to the q^{th} power, and so is majorised by

$$\left(\frac{2\pi}{n} \max_x \left|\frac{d}{dx} e^{inx}\right|\right)^q = (2\pi)^q.$$

It follows immediately that, for $(-\pi, \pi)$, the upper bound is at most $n(2\pi)^q$ and this proves (i).

(ii) We have only to choose $\varphi(x) = \frac{1}{2}(f(x+0) + f(x-0))$ and observe that on the right of (12. 3) the integrand is continuous and zero as function of t for $t = 0$. The result then follows from (11. 2).

(iii) Uniformity of convergence is easily derived from (10. 9). We shall content ourselves with proving the part referring to $s_n^*(x)$. Given ε, we determine a neighbourhood of x and a small interval of t, in which (uniformly in x) the function of t, $f(x+t) - f(x-t)$, has oscillation less than ε, and therefore, by (8. 2 a), a mean variation of order p_1 not exceeding $K\varepsilon^{(p_1-p)/p_1}$, when $p_1 > p$. We choose such a p_1 with a $q_1 > 1$ for which $\frac{1}{p_1} + \frac{1}{q_1} > 1$. Since $V_{q_1}(g_n^*)$ is bounded, it follows at once from (10. 9) that the right hand side of (12. 4) tends uniformly to zero at the point x as required. This completes the proof of our theorem.

The convergence of the Fourier series of an f of W_p was proved by Wiener[1], when $p = 2$. But when $p \neq 2$, he obtained only the incomplete result that the Fourier series converges almost everywhere. For general p, the convergence of the Fourier series and its Cesaro means of order greater than $-1/p$, is implied in a theorem of Hardy and Littlewood[2], relating to the class of functions Lip $\left(\frac{1}{p}, p\right)$, which asserts summability $\left(C, -\frac{1}{p} + \varepsilon\right)$ whenever there is summability $(C, 1)$. The proofs of Wiener and of Hardy and Littlewood do not suggest,

[1] Wiener [11].

[2] Hardy and Littlewood [4] (Theorem 1).

what is evident from our discussion, that results of this kind are largely independent of the special nature of Fourier analysis.

In the part of our theorem that relates to the allied series, we assert nothing as to the convergence of this series by itself, for the integral (12.7) may diverge as $n \to \infty$, at a point x of continuity for which $f(x+t) - f(x)$ is, for instance, $O(1/\log t)$, and this may occur even when f is monotone. It is known, however, that, for almost all x, the integral converges under far more general conditions than we have supposed[1], and it follows that the allied series converges almost everywhere, a result that can also be inferred directly from a beautiful theorem of B. Kuttner[2], since the Fourier series converges. It will be remarked, however, that the only simple formula to which we may expect to sum the allied series, is the limit of an integral such as (12.7). A theorem of comparison between the series and the integral is therefore the natural analogue of the theorems of convergence, or summability, of a Fourier series, to a sum $f(x)$ or to a sum $\frac{1}{2}(f(x+0) + f(x-0))$.

We conclude with a new form a Parseval's theorem, not included in the classical forms of this theorem, and not including them. The theorem is somewhat deeper than our preceeding ones on Fourier series, and in the proof we have combined our methods with those of Hardy and Littlewood.[3]

The »Parseval» equation. *Let the real periodic functions*

$$f(x) \sim \Sigma(a_n \cos nx + b_n \sin nx), \quad g(x) \sim \Sigma(a'_n \cos nx + b'_n \sin nx)$$

of period 2π belong respectively to W_p and W_q, where $p, q > 0$, $\frac{1}{p} + \frac{1}{q} > 1$. Then the series

$$\sum_1^\infty \pi n \begin{vmatrix} a_n & b_n \\ a'_n & b'_n \end{vmatrix} \tag{12.8}$$

converges and has the value

$$\int_{-\pi}^{\pi} f(x)\, dg(x),$$

[1] S. Pollard and R. C. Young [10]. The case of functions of integrable square (which amply suffices here), was treated much earlier by Besicovitch [1].

[2] Kuttner [8].

[3] Hardy and Littlewood [4].

if f and g have no common discontinuities, and, in the general case, the value

$$\int_{-\pi}^{\pi} \frac{1}{2}(f(x+0)+f(x-0))\,dg(x).$$

We obtain this theorem by considering the function

$$F(x)=\int_{-\pi}^{\pi} f(x+t)\,dg(t) \sim \Sigma(A_n \cos nx + B_n \sin nx). \tag{12.9}$$

Lemma (*i*). *When $q=1$ and $g(t)$ is continuous $V_p(F) \leq V_p(f)\,V_1(g)$.* To prove this, write $f_i(x)=f(x+t_i)$ and observe that (8.5) implies

$$V_p\left(\sum_i a_i f_i(x)\right) \leq \sum_i V_p(a_i f_i(x)) = \sum_i |a_i|\,V_p(f_i(x)) = V_p(f)\sum_i |a_i|$$

for any finite sum over i. Let now x_r be the points of division of any subdivision of $(-\pi,\pi)$ and t_i those of another. Then it follows that

$$\left(\sum_r \left|\sum_i (g(t_i)-g(t_{i-1}))\,(f(x_r+t_i)-f(x_{r-1}+t_i))\right|^p\right)^{1/p} \leq V_p(f)\sum_i |g(t_i)-g(t_{i-1})|.$$

Keeping the x_r fixed, we make the t_i everywhere dense and derive

$$\left(\sum_r \left|\int_{-\pi}^{\pi} (f(x_r+t)-f(x_{r-1}+t))\,dg(t)\right|^p\right)^{1/p} \leq V_p(f)\,V_1(g)$$

and replacing the left hand side by its upper bound, we have our assertion.

Lemma (*ii*). *If f belongs to W_p, the sequence $\sigma_n(f)$ of arithmetic means of its Fourier series form a W_p-sequence.* This follows at once from lemma (i) by expressing $\sigma_n(f)$ in the form of an $F(x)$ with for $g(t)$ the indefinite integral of $\sin^2 \frac{1}{2} nt \Big/ 2\pi n \sin^2 \frac{1}{2} t$, in which case the total variation of $g(t)$ is independant of n.

Lemma (*iii*). *With the hypotheses of the theorem, the function $F(x)$ of* (12.9) *has the Fourier coefficients $A_n = \pi n(a_n b'_n - a'_n b_n)$, $B_n = \pi n(a_n a'_n + b_n b'_n)$.* By lemma

(ii) and by (10.9) and (11.2), at every x for which $f(x+t)$ and $g(t)$ have no common discontinuities (that is, for almost all x), $F(x)$ is the bounded limit of the corresponding integral of $f(x)$ with respect to $\sigma_n(g)$.

Multiplying by $\cos nx$, $\sin nx$ and integrating term by term by the classical theorem of Lebesgue, we find for the Fourier constants the values stated.

Lemma (iv). The limits $F(x \pm 0)$ *exist and have the values*

$$\int_{-\pi}^{\pi} f(x+t \pm 0)\, dg(t).$$

In particular, $F(x)$ *is continuous for the values of* x *such that* $f(x+t)$ *and* $g(t)$ *have no common discontinuities as functions of* t.

This lemma follows at once from the obvious modification of (11.2) that corresponds to the Moore-Pollard order of ideas. Thus, when $h_n \to +0$,

$$\int_{-\pi}^{\pi} f(x+t+h_n)\, dg(t-0) \to \int_{-\pi}^{\pi} f(x+t+0)\, dg(t-0)$$

it being clear that $f(x+t+h_n)$ tends uniformly to $f(x+t+0)$ on the right at any point t, while $g(t-0)$ is continuous on the left. Observing that the integrals are unaffected in value if $g(t-0)$ is replaced by $g(t)$, the conclusion follows for $+0$, and similarly for -0,

Lemma (v). $F(x)$ *belongs, for some finite* $\lambda > 0$, *to the integrated Lipschitz class* $\mathrm{Lip}\left(\frac{1}{\lambda}, \lambda\right)$ *of Hardy and Littlewood.*[1]

It will suffice to show that, except for a set of x of measure less than $K|h|^{\alpha}$ (in which $F(x)$ is certainly bounded by (10.9)),

$$|F(x+h) - F(x)| < K|h|^{\beta}.$$

We may suppose $|h| < \pi$, and we determine h_1 and the integer N by

$$h_1 = 2\pi/N \geq |h| > 2\pi/(N+1) \geq \frac{1}{2} h_1.$$

We divide $(-\pi, \pi)$ into N equal parts (t_{r-1}, t_r) of length h_1 and select in each

[1] Cf. (8.4).

part a pair of points $\tau_r, \tau_r - h$ of distance $|h| \le h_1$. Choosing as usual $p_1 > p$, $q_1 > q$ so that $\frac{1}{p_1} + \frac{1}{q_1} > 1$, we have, by (10.9), for the difference

$$\left| F(x) - \sum_{r=1}^{N} f(x + \tau_r)(g(t_r) - g(t_{r-1})) \right|,$$

the majorant

$$\left\{1 + \zeta\left(\frac{1}{p_1} + \frac{1}{q_1}\right)\right\} \sum_{r=1}^{N} V_{p_1}(f; x + t_{r-1}, x + t_r)\, V_{q_1}(g; t_{r-1}, t_r),$$

and, since $(\tau_r - h)$ lies in (t_{r-1}, t_r), the same majorant holds for the difference

$$\left| F(x + h) - \sum_{r=1}^{N} f(x + \tau_r)(g(t_r) - g(t_{r-1})) \right|,$$

and therefore also for the expression

$$\frac{1}{2} | F(x + h) - F(x) | .$$

To prove our lemma, it will therefore suffice to show that

$$\sum_{r=1}^{N} V_{p_1}(f; x + t_{r-1}, x + t_r)\, V_{q_1}(g; t_{r-1}, t_r) < K h_1^{\beta}$$

except for a set of x of measure less than $K h_1^{\alpha}$. Let E_1 be the subset of the t_r for which

$$\text{Osc.}\,(f; t_r - 2h_1, t_r + 2h_1) > h_1^{\frac{1}{3p}} .$$

If n_1 is the number of such values of t_r, it is clear from the definition of $V_p(f)$ that

$$n_1 h_1^{1/3} < V_p^p(f; 0, 8\pi).$$

Similarly, if E_2 is the subset of the t_r for which

$$\text{Osc.}\,(g; t_r - 2h_1, t_r + 2h_1) > h_1^{1/3q}$$

the number n_2 of points of E_2 is at most $K h_1^{-1/3}$.

Let now E_0 be the set of the differences between points of E_1 and points of E_2. The number of points of E_0 is then

$$n_0 \leq n_1 n_2 < K h_1^{-2/3};$$

and if E is the set of points distant at most h_1 from points of E_0, the measure of E is at most $K h_1^{1/3}$, and for any x outside E and any r we must have either

$$\text{Osc.}\,(f;\, x + t_{r-1},\, x + t_r) < h^{1/3p}$$

or

$$\text{Osc.}\,(g;\, t_{r-1},\, t_r) < h^{1/3q}$$

and by lemma (10.8)

$$\Sigma\, V_{p_1}(f;\, x + t_{r-1},\, x + t_r)\, V_{q_1}(g;\, t_{r-1},\, t_r) < K h^{\beta}$$

with, for β, the smaller of the two values $(p_1 - p)/3pp_1$, $(q_1 - q)/3qq_1$.

This completes the proof of lemma (v).

To prove our theorem we observe that $\frac{1}{2}(F(+0) + F(-0))$ is the limit of the Cesaro mean of the Fourier series of $F(x)$ for $x = 0$ by a classical theorem.

By lemma (v) and the theorem of Hardy and Littlewood already referred to[1], this implies that the Fourier series of $F(x)$ converges for $x = 0$ and is indead summable $\left(C, -\frac{1}{\lambda} + \varepsilon\right)$.

By lemma (iii) this is equivalent to the statement of our theorem, and this completes the proof.

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