

AN INEQUALITY SIMILAR TO OPIAL'S INEQUALITY

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In a recent paper [5], Willet gave a generalization of Opial's inequality [4]: If $y \in C^{(n)}[a, b]$ with $y^{(i)}(a) = 0$ for $i = 0, 1, \dots, n-1$, then

$$(1) \quad \int_a^b |yy^{(n)}| dx \leq \frac{1}{2} (b-a)^n \int_a^b |y^{(n)}|^2 dx.$$

(He employs (1) to prove uniqueness of the initial value problem for linear differential equations of order n .) Also there are generalizations of Opial's original inequality in other directions. (See, for instance, Calvert [2] and Yang [6].)

The purpose of this note is to obtain a sharper version of (1) and other generalizations.

THEOREM 1. Let $y \in C^{(n-1)}[a, b]$ be such that $y^{(i)}(a) = 0$ for $i = 0, 1, \dots, n-1$ where $n \geq 1$. Let $y^{(n-1)}$ be absolutely continuous and $\int_a^b |y^{(n)}|^2 dx < \infty$. Then,

$$(2) \quad \int_a^b |y(x)y^{(n)}(x)| dx \leq K(b-a)^n \int_a^b |y^{(n)}(x)|^2 dx,$$

where

$$(3) \quad K = \frac{1}{2(n!)} \left(\frac{n}{2n-1} \right)^{1/2}.$$

PROOF. In view of the assumptions on y , for any x in $[a, b]$

$$y(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} y^{(n)}(t) dt.$$

Therefore,

$$|y(x)y^{(n)}(x)| \leq \frac{|y^{(n)}(x)|}{(n-1)!} \int_a^x (x-t)^{n-1} |y^{(n)}(t)| dt,$$

and, by Schwarz inequality,

$$|y(x)y^{(n)}(x)| \leq \frac{|y^{(n)}(x)|}{(n-1)!} \frac{(x-a)^{n-1/2}}{(2n-1)^{1/2}} \left(\int_a^x |y^{(n)}(t)|^2 dt \right)^{1/2}.$$

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Thus, integrating from a to b and applying Schwarz inequality to the right-hand side again,

$$\int_a^b |y(x)y^{(n)}(x)| dx \leq \frac{1}{(n-1)!(2n-1)^{1/2}} \left(\int_a^b (x-a)^{2(n-1)} dx \right)^{1/2} \cdot \left(\int_a^b |y^{(n)}(x)|^2 \left(\int_a^x |y^{(n)}(t)|^2 dt \right) dx \right)^{1/2}.$$

The above is, in fact, (2). This completes the proof.

REMARK. For $n=1$, (2) is Opial's original result. For $n \geq 2$, (2) is evidently sharper than (1).

COROLLARY. Equality holds in (2) if and only if $n=1$ and $y^{(n)}(x) = \text{const}$.

PROOF. The "only if" part needs proof. Now, equality holds in (2) only if it holds in all intermediate steps. Therefore, it is necessary that

$$y^{(n)}(t) = C(x)(x-t)^{n-1}, \text{ for each } x, \text{ and } a \leq t \leq x.$$

Thus, unless $n=1$, $y^{(n)}(t)=0$, and if $n=1$, $y^{(n)}(t)=\text{const}$. Hence the assertion.

REMARK. If $y \in C[a, b]$ satisfy that $y', \dots, y^{(n-1)}$ are piecewise continuous, $y^{(n-1)}$ is absolutely continuous with $\int_a^b |y^{(n)}|^2 dx < \infty$, $y^{(i)}(a) = y^{(i)}(b) = 0$ for $i=0, \dots, n-1$, then

$$(4) \quad \int_a^b |y(x)y^{(n)}(x)| dx \leq K \left(\frac{b-a}{2} \right)^n \int_a^b |y^{(n)}(x)|^2 dx,$$

K as given in (3). Equality holds in (4) iff $n=1$ and

$$\begin{aligned} y(x) &= c(x-a)^n, & a \leq x \leq (a+b)/2, \\ &= c(b-x)^n, & (a+b)/2 \leq x \leq b. \end{aligned}$$

This is immediate on using (2) once on $[a, (a+b)/2]$ and again on $[(a+b)/2, b]$, where on the latter interval

$$y(x) = \frac{(-1)^n}{(n-1)!} \int_x^b (t-x)^{n-1} y^{(n)}(t) dt.$$

Following is a generalization of Theorem 1.

THEOREM 2. Let p, q be both positive satisfying $p+q > 1$. Let y be as in Theorem 1 and let $\int_a^b |y^{(n)}|^{p+q} dx < \infty$. Then,

$$(5) \quad \int_a^b |y(x)|^p |y^{(n)}(x)|^q dx \leq K^*(b-a)^{(n)p} \int_a^b |y^{(n)}(x)|^{p+q} dx,$$

where

$$(6) \quad K^* = \alpha q^{\alpha} [n(1 - \alpha)/(n - \alpha)]^{p(1-\alpha)} (n!)^{-p}, \quad \alpha = (p + q)^{-1}.$$

PROOF. By Hölder's inequality with indices $p+q$ and its conjugate, from

$$|y(x)| \leq \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} |y^{(n)}(t)| dt$$

follows

$$|y(x)| \leq A(x-a)^{n-\alpha} \left(\int_a^x |y^{(n)}(t)|^{p+q} dt \right)^{\alpha},$$

where A is a constant.

Thus,

$$\begin{aligned} \int_a^b |y(x)|^p |y^{(n)}(x)|^q dx &\leq B \int_a^b (x-a)^{p(n-\alpha)} |y^{(n)}(x)|^q \\ &\quad \cdot \left(\int_a^x |y^{(n)}(t)|^{p+q} dt \right)^{p/(p+q)} dx. \end{aligned}$$

Applying Hölder's inequality with indices $(p+q)/p$ and $(p+q)/q$ to the integral on the right-hand side, one has

$$\begin{aligned} \int_a^b |y^{(k)}(x)|^p |y^{(n)}(x)|^q dx \\ \leq K^* (q\alpha)^{-q\alpha} \left(\int_a^b |y^{(n)}(x)|^{p+q} \left(\int_a^x |y^{(n)}(t)|^{p+q} dt \right)^{p/q} dx \right)^{q/(p+q)}, \end{aligned}$$

which is indeed (5).

REMARK 1. For $n=1$, (5) is a sharper result similar to Yang's.

REMARK 2. In case $p > 0$, $q > 0$ and $p+q=1$, one has

$$|y(x)| \leq \frac{1}{(n-1)!} (x-a)^{n-1} \int_a^x |y^{(n)}(t)| dt,$$

whence

$$\begin{aligned} \int_a^b |y(x)|^p |y^{(n)}(x)|^q dx \\ \leq \frac{1}{((n-1)!)^p} \int_a^b (x-a)^{p(n-1)} |y^{(n)}(x)|^q \left(\int_a^x |y^{(n)}(t)| dt \right)^p dx. \end{aligned}$$

Applying Hölder's inequality with indices $1/p$ and $1/q$ and simplifying

$$(7) \quad \int_a^b |y(x)|^p |y^{(n)}(x)|^q dx \leq \frac{q^q}{(n!)^p} (b-a)^{np} \int_a^b |y^{(n)}(x)| dx,$$

it is easy to check that K^* in (6) tends to $q^q(n!)^{-p}$ as $p+q \rightarrow 1$ from above. Therefore, (5) can be extended to include this case. In (7), equality holds if and only if $y^{(n)}(x) \equiv 0$ in (a, b) or $n=1$ and $y^{(n-1)}(x) = \text{const}(x-a)^q$. Indeed, if $y^{(n)}(x) \neq 0$ in (a, b) ,

$$\int_a^x (x-t)^{n-1} |y^{(n)}(t)| dt < \int_a^x (x-a)^{n-1} |y^{(n)}(t)| dt, \quad x > a, n > 1.$$

Thus case $n=1$ need only be considered. Equality holds in (7) only if

$$|y^{(n-1)}(x)| = \int_a^x |y^{(n)}(t)| dt,$$

and

$$|y^{(n)}(x)| \left(\int_a^x |y^{(n)}(t)| dt \right)^{p/q} = \text{const}.$$

The latter leads to

$$\int_a^x |y^{(n)}(t)| dt = \text{const} (x-a)^q,$$

whence, in view of the former, the assertion follows. (The integrals in (7) are to be interpreted as $\lim_{\epsilon \rightarrow 0^+} \int_{a+\epsilon}^b$.)

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