Int. Journal of Math. Analysis, Vol. 1, 2007, no. 17, 841 - 849

An Infeasible Interior Point Method for the Monotone Linear Complementarity Problem

Z. Kebbiche [1] **and A. Keraghel**

Department of Mathematics, Faculty of Sciences
University Ferhat Abbes, Setif, Algeria

A. Yassine

Laboratoire des Mathematiques Appliquees du Havre (LMAH)
Universite du Havre, France

Abstract. Linear complementarity problem noted (LCP) becames in present the subject of many reseach interest because it arises in many areas and it includes the two important domains in optimization:the linear programming (LP) and the convex quadratic (CQP) programming. So the researchers aims to extend the results obtained in (LP) and (CQP) to (LCP). Differents classes of methods are proposed to solve (LCP) inspired from (LP) and (CQP).
In this paper, we present an infeasible interior point method to solve the monotone linear complementarity problem. Comparative results of this method with feasible interior point method are reported.

Mathematics Subject Classifications: 90C33, 90C51

Keywords: Linear complementarity problem,Interior point method, quadratic programming, Newton method, Path-following methods, Feasible methods, Infeasible methods

1. Introduction

The monotone linear complementarity problem (PCL) is to find vector pair $(x, y) \in IR^n \times IR^n$ that satisfy the following conditions:
$y = Mx + q$, $(x, y) \geq 0$, $x^t y = 0$
where: $q \in IR^n$ and M is an $n \times n$ matrix supposed positive semidefinite (in this case, (PCL) is said monotone).
Linear complementarity problems arises in many areas such as variational inequalities, economic equilibria problems and bimatrix games. It is known

[1]kebbichez@yahoo.fr

that this problem trivially includes the two important domains in optimization: the linear programming (PL) and the convex quadratic programming (QP) in their usual formulations, then this problem became the subject of many reseach interest.The aim of researchers is to extend all results obtained in (LP) and (QP) to a more general class of problems known as monotone linear complementarity problems.Two classes of methods have been proposed for solving this problem:

- Direct methods based on the principle of the simplex method introduced by dantzing in 1951. The famous algorithm for (LCP) is Lemke's algorithm (1969).

- Iterative methods based on interior point methods which became efficient for solving many optimization problems after the introduction of Karmarkar's algorithm in 1984 for linear programming. Many authors tended to generalise this algorithm for the resolution of nonlinear optimization problems. Two approachs are proposed: feasible and infeasible interior point methods. For feasible approach, successive work to solve (LCP) have been described by a number of authors including Kojima, Mizuno and yoshize, ...

The most important class is the path following method , the algorithm start with a strictly feasible point neighbour the central path and generates a sequence of iterates which satisfy the same conditions.

these conditions are an expensive process in general. Most existing codes make use of a starting point that satisfy strict positivity but not the equality condition. For this reason, the authors tried to develop algorithms which start with any strict positive point not necessarly feasible.

More recently, algorithms that do not require feasible starting point have been the focus of active research (Zhang, Wright, Shanno, ...).These algorithms are called infeasible interior point algorithms.

The paper is organized as follow: In section 2, we describe the feasible path following methods. A presentation of an infeasible interior point algorithm is given in section 3. In section 4, we present some numerical and comparative results between the two algorithms. Concluding remarks are given in section 5.

2. Feasible interior point methods

2.1. **Description of the feasible method.** The linear complementarity problem (LCP) determinates a vector pair (x, y) satisfying:

$$(LCP)\begin{cases} y = Mx + q \\ (x,y) \geq 0 \qquad \text{............(1)} \\ x^t y = 0 \end{cases}$$

where M is an $n \times n$ matrix and q, x, y are vectors in IR^n .

The set of all the feasible solutions is defined by S:

$S = \{(x, y) \in IR^{2n} \ \backslash \ (x, y) \geq 0 \quad \text{et} \quad y = Mx + q\}$

and the set of strictly feasible points is S_{int} :
$S_{int} = \{(x, y) \in S \ \backslash \ (x, y) > 0\}$

and the set of solutions of (LCP) is $S(LCP)$:
$S(LCP) = \{(x, y) \in S \ \backslash \ x^t y = 0\}$

Given a vector x, X denotes as the diagonal matrix defined by the vector.
The symbol e represents the vector of all ones with dimension n.
$\|.\|$ denote the euclidean norm.

Through the paper, we assume that:
(**H1**) $\quad S_{int} \neq \emptyset$

(**H2**) $\quad M$ is positive semidefinite matrix

We know that (LCP) can be written as a convex quadratic programming:

$$\begin{cases} \min x^t y \\ y = Mx + q \\ (x, y) \geq 0 \end{cases} \quad \text{............(2)}$$

Most of interior point methods are motivated by the logarithmic barrier function technique, so we apply the logarithmic barrier function to obtain the following perturbed problem:

$$\begin{cases} \min \left[x^t y - \mu \sum\limits_{i=1}^{n} \log(x_i y_i) \right], \ \mu > 0 \\ y = Mx + q \\ (x, y) > 0 \end{cases} \quad \text{...............(3)}$$

The principle of these methods is to solve the system of KKT associated with $(LCP)_\mu$ by Newton method, so we have:
(x, y) is an optimal solution of $(LCP)_\mu$ if and only if (x, y) satisfy the following system:

$$\begin{cases} XYe - \mu e = 0 \\ y = Mx + q \\ (x, y) > 0 \end{cases} \quad \text{.............(4)}$$

We apply Newton method to solve the system of non linear equations, we gets the following linear system:

$$\begin{cases} Y\Delta x + X\Delta y = XYe - \mu e \\ \Delta y = M\Delta x \end{cases} \quad \text{...............(5)}$$

The new iterate is calculate as follow: $(x, y) = (x, y) - (\Delta x, \Delta y)$

The set of all solutions of the system (4) is called Central path following (CT), we note it by:

$$S(CT) = \{(x, y) \in S_{int} \ \backslash \ X\ y = \mu e,\ \mu > 0\}$$

The basic algorithm is described as follow:

2.2. **Basic Algorithm.** Let $\varepsilon > 0$ be a parameter and $(0 < \alpha \leq 0.1)$ is a given constant.

Initialisation: Given a strictly feasible pair $(x^0, y^0) \in S_{cen}(\alpha)$ where $S_{cen}(\alpha)$ is a neighbourhood of the central path,

$$S_{cen}(\alpha) = \left\{(x, y) \in S_{int} \ \backslash \ \left\| XYe - (\frac{x^t y}{n})e \right\| \leq (\frac{x^t y}{n})\alpha,\ \alpha > 0\right\}$$

$k = 0$

Iteration:

While: $(x^k)^t y^k \geq \varepsilon$ **then:**

- Compute: $\delta = \dfrac{\alpha}{1-\alpha}, \quad \mu = (1 - \dfrac{\delta}{\sqrt{n}})\dfrac{(x^k)^t y^k}{n}$

- Determinate the *Newton step* $(\Delta x^k, \Delta y^k)$ solution of the system (5).

- The new iterate: $(x^{k+1}, y^{k+1}) = (x^k, y^k) - (\Delta x^k, \Delta y^k), \ \ k = k + 1.$

End of While.
End of Algorithm.

Theorem 1.

Let $0 < \alpha \leq 0.1$ and $\delta = \dfrac{\alpha}{1-\alpha}$.

we assume that: $(x^k, y^k) \in S_{cen}(\alpha)$ and $\mu = (1 - \dfrac{\delta}{\sqrt{n}})\dfrac{(x^k)^t y^k}{n}$ then:

i) $(x^{k+1}, y^{k+1}) \in S_{cen}(\alpha)$

2.3. **ii)** $(x^{k+1})^t y^{k+1} \leq (1 - \dfrac{\delta}{6\sqrt{n}})(x^k)^t y^k$

Difficultis. The most expensive step in this algorithm is the initialisation: the algorithm start with a strictly feasible point which must be neighbour of the central path. These two conditions are difficults to ensure in practice. To overcome these problems, we attempt to start the algorithm with any positive point not necessarly feasible. These algorithms are called infeasible interior point algorithms.

3. Infeasible interior point methods

The algorithm in this case start with an initial positive point $(x^0, y^0) > 0$ and generate a sequence of iterates solutions of the system of nonlinear equalities:

$$\begin{cases} XYe = \mu e \\ y = Mx + q \\ (x, y) > 0 \end{cases} \quad ....(6)$$

The displacement direction $(\Delta x, \Delta y)$ is solution of the following system:

$$\begin{cases} Y\Delta x + X\Delta y = XYe - \mu e \\ \Delta y - M\Delta x = y - Mx - q \end{cases} \quad ....(7)$$

Then, the new iterate is $(x, y) = (x, y) - \alpha(\Delta x, \Delta y)$

To acheive feasibility and optimality, we introduce a merite function defined by:

$$\phi(x, y) = x^t y + \|Mx + q - y\|$$

The term $x^t y$ control optimality and the term $\|Mx + q - y\|$ measure feasibility.

α is introduced in order to ensure the positivity of iterates and to contol the reduction in ϕ at each step.

3.1. **Basic algorithm (infeasible interior point algorithm).** Let $\varepsilon > 0$ be a parameter of précision, $\gamma \in (0, 1)$ is a factor of reduction.

Initialisation: start with: $\mu^0 > 0$, $(x^0, y^0) > 0$

$k = 0$, compute $\phi(x^k, y^k) = \phi^k$

Iteration:

While: $\phi^k \geq \varepsilon$ **then:**

- Compute: $\mu = \gamma \dfrac{(x^k)^t y^k}{n}$

- Determinate the displacement step $(\Delta x^k, \Delta y^k)$ solution of the linear system:

$$\begin{cases} Y\Delta x + X\Delta y = XYe - \mu e \\ \Delta y - M\Delta x = y - Mx - q \end{cases}$$

- The new iterate: $(x^{k+1}, y^{k+1}) = (x^k, y^k) - \alpha(\Delta x^k, \Delta y^k), \ \ k = k+1$

End of While.
End of Algorithm.

4. Numerical résults

In the implementation of the two algorithms, we have selected the displacement step α by a praticul procedure as follow:
$\alpha = \min(\alpha_x, \alpha_y)$

$$\alpha_x = \begin{cases} \min \dfrac{x_i}{\Delta x_i} \text{ if } \Delta x_i > 0 \\ \quad 1 \text{ else} \end{cases}$$

$$\alpha_y = \begin{cases} \min \dfrac{y_i}{\Delta y_i} \text{ if } \Delta y_i > 0 \\ \quad 1 \text{ else} \end{cases}$$

Remark 1. *For the initialisation step of the feasible interior point algorithm we have used Karmarkar's algorithm and Weighted logarithmic barrier function*

4.1. **Karmarkar's algorithm.** We use Karmarkar's algorithm to find a strict positive point $(x^0, y^0) > 0$ by solving the system: $\begin{cases} y = Mx + q \\ (x, y) > 0 \end{cases}$

This system can be written as: $Av = q, v \geq 0$..........(8), where $A = [I \ \ M]$ and $v = \left[x^T \ y^T\right]^T \in \mathbb{R}^{2n}$.

The last system is equivalent to the following linear program:

$$(1) \qquad \min_{\lambda} \lambda \ \text{ s.t. } Av + \lambda(q - Av^0) = q, \, v \geq 0, \, \lambda \geq 0 ....(9)$$

where $v^0 > 0,$ is chosen arbitrary in $\mathbb{R}^n_+$,and λ is an artificial real variable. Notice that the problem is always feasible, just take $v = v^0$ and $\lambda = 1,$ and we have:

If the problem (9) admits a solution $(v^T, \lambda)^T \in \mathbb{R}^{2n}$ such that $\lambda \leqslant \varepsilon_0$ with ε_0 sufficiently small. Then the problem (8) is feasible, otherwise the problem is not feasible.
To solve the problem (9), we use a variant of Karmarkar's algorithm.

4.2. **Weighted logarithmic barrier function.** To ensure that the initial point satisfy: $(x^0, y^0) \in S_{cen}(\alpha)$, we introduce a weighted logarithmic barrier function defined as:

$x^T y - \mu\left(\sum_{i=1}^{n} r_i \log x_i y_i\right), \ \mu > 0$

where $r = (r_1, r_2, ..., r_n)^T \in \mathbb{R}^n$, is a weighted vector associated to the logarithmic barrier function with $r_i > 0$, $i = 1, .., n$.

In this case, the algorithm is described as follows:

A primal-dual weighted (IPM) path-following algorithm

Initialization:

let $\varepsilon > 0$, be a given tolerance and $(x^0, y^0) \in S_{int}$, $\mu^0 = \dfrac{\|X^0 y^0\|}{\sqrt{n}}$, $r = \dfrac{X^0 y^0}{\mu^0}$, R =diag(r), and set $k = 0$;

Step 1:

while:$(x^k)^T R y^k \geq \varepsilon$ **do:**

- compute $\eta = \min\limits_{i=1}^{n} r_i$, $\delta = (1 - \dfrac{\sqrt{2}}{2})\eta$, $\mu = (1 - \dfrac{\delta}{\sqrt{n}})\dfrac{(x^k)^T R y^k}{n}$,
- compute a Newton step $(\Delta x^k, \Delta y^k)$ solution from (4).

Step 2:

- updates the iterates: let $x^{k+1} = x^k + \Delta x^k$, $y^{k+1} = y^k + \Delta y^k$,and let $k = k+1$; and go back to **step 1**.

End of While.

End of algorithm

4.3. **Numerical implementation.** The results in number of iterations and calculation time for some problems with different dimensions are illustrated in the following table:

n	**Iter1**	**Iter2**	**T1**	**T2**
2	**7**	**5**	**0.01**	**0.01**
5	**6**	**6**	**0.11**	**0.01**
8	**7**	**9**	**0.37**	**0.05**
12	**10**	**9**	**1.48**	**0.05**
15	**11**	**40**	**2.63**	**0.49**
20	**13**	**38**	**7.75**	**1.10**
20	**15**	**51**	**9.12**	**1.48**

n is the dimension of problems

Iter1 notes the number of iterations of the feasible algorithm

Iter2 notes the number of iterations of the infeasible algorithm

T1 is the calculation time of the feasible algorithm in seconds

T2 is the calculation time of the infeasible algorithm in seconds.

5. Concluding remarks

In this paper, we have presented an infeasible interior point method for solving the monotone linear complementarity problem. The feasible algorithm require the use of two procedures in the stape of initialisation (Karmarkar's algorithm and a weighted logarithmic barrier function to calculate a strictly feasible point which must be neighbour of the central path.). But in the case of infeasible algorithm we start the algorithm with any positive point.

References

[1] M. Achache. A weighted path-following method for the linear complementarity problem. *Studia Universitatis Babes-Bolyai. Series Informatica,* 48(1) 61- 73 (2004).

[2] S. Bazarra, H.D. Sherali and C.M. Shetty. Non linear programming, theory and algorithms. *Second edition* (1993).

[3] R.W. Cottle, J.S.Pang and R.E.Stone. The linear complementarity problem. *Academic press, New-York*,1992.

[4] Zs.Darvay. A weighted-path-following method for linear optimization. *Studia Universitatis Babes-Bolyai. Series Informatica, 47(1) 3-12,(2002).*

[5] J. Ding and T.Y. Li. An algorithm based on weighted logarithmic barrier function for linear complementarity problem. *The Arabian journal for Science and Engineering* $V15$, N $4B$, (1990).

[6] B. Jansen, C. Roos, and T. Terlaky. A family of polynomial affine scaling algorithms for positive semidefinite linear complementarity problems. *Technical Mathematics and computer science, Delfet University of Technology, Delft, The Netherlands* (1993).

[7] B. Jansen, C. Roos, T. Terlaky, and J.Ph. Vial. Primal-dual target following algorithms for linear programming. *Annals of operations Research*, 62, 197-231(1996).

[8] N. K. Karmarkar. A New polynomial time algorithm for linear programming. *Combinatorica* 4. 373-395.(1984),

[9] Z. Kebbiche**.** Mise en oeuvre d'une méthode de trajectoire centrale pour le problème complémentaire linéaire monotone. *Thèse de Magister, Université F.A. Sétif* (1997).

[10] M. Kojima, S. Mizuno, and A. Yoshize. A polynomial-time algorithm for a class of linear complementarity problems. *Mathematical Programming* 44**,** 1-2,(1989).

[11] M. Kojima, N. Meggido, T. Noma, and A. Yoshize. A unified approach to Interior Point Algorithms for Linear Complementarity problems. *volume* 538 *of Lecture Notes in Computer Science. Spring Verlag, Berlin, Germany.*(1991).

[12] M. Kojima, S. Mizuno and A.Yoshize**.** An $O(\sqrt{n}L)$ iteration potentiel reduction algorithm for linear complementarity problems. *Mathematical programming* 50 (1991), 331-344, *North Holland.S.J.*

[13] M. Kojima, N. Megiddo and Y.Ye. An interior point potentiel reduction algorithm for linear complementarity problems. *Mathematical programming* 54 (1992), 267-279, *North Holland*

[14] S. Mizuno. A new polynomial time method for a linear complementarity problems. *Mathematical programming* 56 (1992), 31-43, *North Holland.*

[15] K.G.Murty**.** Linear complementarity, linear and nonlinear programming. *Berlin: Helderman*, (1988).

[16] E. Simantiraki and D. Shanno**.** An infeasible interior point method for linear complementarity problems. *SIAM J. Optim. vol.* 7, *No.*3, *pp* 620-640, (1997).

[17] S. Wright and Y. Zhang. A superquadratic infeasible interior point method for linear complementarity problems. *Mathematical programming* 73 (1996), 269-289.

[18] S. Wright. An infeasible interior point algorithm for linear complementarity problems. *Mathematical programming* 67 (1994), 29-51.

[19] Wright. Primal-dual interior point methods, *SIAM, Philadelphia.*(1997).

[20] Y. Zhang. On the convergence of a class of infeasible interior point methods for the horizontal linear complementarity problems. *SIAM, Journal of Optimization*, 4, 53-67(1994).

Received: February 6, 2007